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On the Regular Holonomic Character of the S-Matrix and Microlocal Analysis of Unitarity-Type Integrals

Takahiro Kawai*, [1] and Henry P. Stapp**, [2]

1 Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139 USA and Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan
2 Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720 USA

Abstract. The previously proved results that every analytically renormalized Feynman integral is a regular holonomic function suggests that the S-matrix should be locally expressible as an infinite sum of regular holonomic functions. A regularity property R is formulated that expresses the condition that the S-matrix be locally expressible near each physical point p as a convergent sum of regular holonomic functions, with each term enjoying some of the regularity properties of a corresponding Feynman integral. This property R holds at every physical point p that has yet been analyzed by the methods of axiomatic field theory or S-matrix theory. Some analyticity properties of unitarity-type integrals are then examined under the assumption that the S-matrix satisfies property R and a weak integrability condition. These results rest heavily on some recently proved properties of regular holonomic functions.

1. Introduction

Sato [1] has conjectured that the S-matrix satisfies a holonomic system of (micro)-differential equations with characteristic variety determined by the Landau equations. Support for this conjecture has been adduced by Kashiwara and Kawai [2], who have shown that the analytically renormalized Feynman function $F_G(p)$ associated with any Feynman graph G satisfies such a system of equations with characteristic variety confined to the extended Landau variety $\tilde{\mathscr{L}}(G)^{\mathbb{C}}$.

The Feynman functions enjoy an important additional property: they are regular holonomic functions. A regular holonomic function is, by definition, a hyperfunction that satisfies a holonomic system of linear differential equations with regular singularities. Kashiwara and Kawai [3] have developed a microlocal

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theory of holonomic systems with regular singularities, and have shown, as an immediate by-product of their theory, that the Feynman functions $F_G(p)$ are all Nilsson class functions. This fact had been believed previously, but the proof had been blocked by technical difficulties. (Private communication to T. K. from Professor J. Lascoux and Professor F. Pham.)

The fact that every Feynman function is regular holonomic suggests that the S-matrix may be expressible as an infinite sum of regular holonomic functions. Indeed, Kawai and Stapp [4] have shown on the basis of the general S-matrix discontinuity formulas and weak analyticity requirements that each point P in a large part of the physical region has a complex mass-shell neighborhood $\Omega(P)$ such that the kernel of the connected part of the S-matrix restricted to $\Omega(P)$ can be expressed in the form

$$S_P(p)=\sum_{G\in\mathscr{G}_P} a_{G,P}(p)F_G(p), \tag{1.1}$$

where the functions $a_{G,P}(p)$ are holomorphic in $\Omega(P)$, and $\mathscr{G}_P$ is the collection of connected graphs G such that P lies on the positive-α Landau surface $\bar{L}_1^+(G)$. This result immediately entails the weaker condition that $S_P(p)$ can be expressed in the form

$$S_P(p)=\sum_{G\in\mathscr{G}_P} S_{G,P}(p), \tag{1.2a}$$

where $S_{G,P}(p)$ satisfies on $\Omega(p)$ a holonomic system with regular singularities whose characteristic variety $\mathrm{Ch}(S_{G,P}(p))$ is confined to the characteristic variety $\mathrm{Ch}(F_G(p))$ of the system that $F_G(p)$ satisfies,

$$\mathrm{Ch}(S_{G,P}(p))\subset\mathrm{Ch}(F_G(p)), \tag{1.2b}$$

and whose singularity spectrum (SS) is confined to the singularity spectrum of $F_G(p)$,

$$\mathrm{SS}(S_{G,P}))\subset\mathrm{SS}(F_G(p)). \tag{1.2c}$$

At the previously examined points P, only a finite number of nonvanishing terms occur in the sums (1.1) and (1.2a), and hence no convergence problem arises. But for any point P lying, for example, on a three-particle threshold, any equations of the form (1.1) or (1.2) must contain an infinite number of nonzero terms. Hence the question of convergence must in general be considered.

A formulation of property (1.2) that incorporates an appropriate convergence condition is provided by the following definition. Let P be any point in the original real domain of definition of the S-matrix. Then the regularity property R_P consists of the following four conditions:

1) There exists a complex product-neighborhood $\Omega_i(P)\times\Omega_f(P)$ of P and a set of bounded operator S_P and $S_{G,P}$ (for all $G\in\mathscr{G}_P$) that transform square integrable functions defined over the initial real domain $\Omega_i^{\mathbb{R}}(P)$ into square integrable functions defined over the final real domain $\Omega_f^{\mathbb{R}}(P)$, where $\Omega_j^{\mathbb{R}}(P)$ is the restriction of $\Omega_j(P)$ to the real mass shell.

2) The sum $\sum_{G\in\mathscr{G}_P} S_{G,P}$ converges absolutely to S_P in the sense that

$$\left|S_P - \sum_{G\in\mathscr{G}_P} S_{G,P}\right| \to 0 \tag{1.3a}$$

and

$$\sum_{G\in\mathscr{G}_P} |S_{G,P}| \to B_P < \infty\,. \tag{1.3b}$$

3) The kernel of $S_{G,P}$ considered as a hyperfunction $S_{G,P}(p)$ defined over $\Omega_i^{\mathbb{R}}(P)\times\Omega_f^{\mathbb{R}}(P)$ is regular holonomic and satisfies (1.2b) and (1.2c).

4) The kernel $S_P(p)$ of S_P is the restriction to $\Omega_i(P)\times\Omega_f(P)$ of the kernel $S_c(p)$ of the connected part of the S-matrix.

The appropriateness of the convergence condition specified in R_P is discussed in Sect. 2.

The purpose of this paper is to derive conditions on the singularity structure of unitarity-type integrals under the condition that the scattering functions appearing in the integrand enjoy the regularity property R_P for certain critical points P in the domain of integration. These critical points are the critical points associated with the so-called $u=0$ points of the integral. Subject to the validity of property R_P at these critical points, our result extends the earlier result [5] on the singularity spectra of unitarity-type integrals at $u\neq 0$ points to many $u=0$ points.

This extension to $u=0$ points constitutes a significant improvement over the earlier $u\neq 0$ results. Indeed, there are many unitarity-type integrals for which the $u=0$ points cover the entire domain of definition. For these integrals the earlier $u\neq 0$ result entails no domain of analyticity at all, whereas our result, when applicable, restricts the singularities to well-defined codimension-one subvarieties.

The property R_P required for our result has, as noted above, been derived at many points P from S-matrix arguments. In fact, the stronger property with $S_{G,P}$ replaced by $a_{G,P}F_G$ has been obtained at these points. Similar results have been obtained also from axiomatic field theory [6]. These stronger results are in accord with Landau's suggestion [7] that the singularity structure of the S-matrix is given correctly by the Feynman integrals, quite apart from the validity or nonvalidity of the perturbation theory in which they first arose.

The condition that property R_P holds for every physical point P is called property R. This property can be regarded as a specific and precise formulation of Landau's suggestion.

The special examples mentioned above yield instead of property R the stronger property R_S, which is R with $S_{G,P}$ replaced by $a_{G,P}F_G$. Thus one might wish to regard R_S as the precise formulation of Landau's suggestion. However, this stronger property R_S is not compatible with the convergence condition (1.3). This will be explained in Sect. 2. The essential point is that the conditions (1.2b) and (1.2c) on $S_{G,P}(p)$ hold not only for $a_{G,P}(p)F_G(p)$ but also for the similar functions associated with the contractions of G, and moreover, for any finite linear combination of such functions. This flexibility is needed to maintain the convergence property (1.3): the analogous convergence condition does not hold for the expansion (1.1).

Property R is also a specific and precise formulation of Sato's conjecture. It adds to Sato's general holonomicity requirement an appropriate convergence condition, and also the requirement that the singularities of the holonomic systems be regular.

Property R is, lastly, a very reasonable ansatz for the physical-region part of the maximal analyticity property of S-matrix theory [8]. For this property R is compatible with the stringent requirements of macrocausality. Moreover, the previous studies [4, 9, 10] suggest that the S-matrix requirements of unitarity, macrocausality, and Lorentz invariance require the presence of no singularities other than those allowed by R. Furthermore, they suggest that if only those singularities permitted by R are allowed then all these singularities must in fact be present, provided no special selection rules intervene.

To prove that property R is in fact compatible with the general S-matrix (or field-theoretic) principles one must know the singularity structure of unitarity-type integrals under the condition that property R holds. Sections 3 and 4 address this problem, and in particular the preliminary problem of extending with the aid of property R the earlier $u \neq 0$ results on the singularity structure of unitarity-type integrals to the more delicate $u=0$ points. Our earlier works have made clear that some analyticity property beyond that provided by macrocausality is needed to cope with these $u=0$ points.

An alternative approach to the $u=0$ problem has been developed by Iagolnitzer [11]. It is based on a different assumed regularity property. Whereas the present approach is within the general framework of maximal analyticity, where the ultimate aim is to impose the strongest analyticity assumption compatible with the other general principles, Iagolnitzer's approach is based rather on a strengthened formulation of the macrocausality principle. In both approaches one is faced with the task of verifying the compatibility of the assumption with the other general principles. Our assumption is known to be compatible with all cases that have yet been studied, and also with the possibility that the S-matrix be locally expressible as a sum of renormalized Feynman functions with analytic coefficients. Iagolnitzer's property has not yet been shown to be compatible with the well-understood analyticity properties near the leading two-particle threshold. If that property can be shown to be compatible with this and the other detailed results so far derived from field theory and S-matrix theory, then we would expect Iagolnitzer's approach to be complementary to our own.

We conclude this introduction with a brief review of some terminology connected with Landau surfaces.

A *Landau graph* G is an oriented graph each edge (or line) l of which is associated with a particle-type label t_l. The graph G is completely specified by giving for each edge l of G the corresponding particle-type label $t_l(G)$ (which fixes the mass $m_l \equiv m(t_l)$, and distinguish a particle from its antiparticle) and for each edge l and vertex j of G the incidence matrix element $\varepsilon_{lj}(G)=[j:l]$, which is $+1$, -1, 0 according to whether the edge l terminates, on, originates on, or is not incident upon vertex j. A vertex of G is an internal vertex if more than one edge is incident upon it and is an external vertex if exactly one edge is incident upon it. An external vertex j is an initial or final vertex according to whether the one edge incident upon j originates or terminates on j. The edge incident upon an initial or final vertex is called an initial or final line (or edge) respectively. The initial and final lines are called the external lines, and the others are called the internal lines. Vertices with no edges incident upon them are excluded. Vertices with exactly two lines incident upon them are called trivial vertices, and are excluded unless otherwise stated.

A *Landau diagram* D is a spacetime diagram obtained by assigning to each vertex j of some corresponding Landau graph $G=G(D)$ a spacetime point x_j and assigning to each edge l of $G=G(D)$ an oriented spacetime line segment that runs from point $x_{j^-(l)}$ to point $x_{j^+(l)}$, where $[j^{\pm}(l):l]=\pm 1$. Each line segment l of D is required to have positive Lorentz norm $|x_{j^+(l)}-x_{j^-(l)}|>0$, and is associated with a momentum-energy vector p_l that is defined by the conditions that p_l be parallel to $x_{j^+(l)}-x_{j^-(l)}$ and satisfy the mass-shell and positive-energy conditions $p_l^2=m_l^2$ and $p_{l,0}>0$. The final condition on D is that momentum-energy is conserved at each internal vertex j:

$$\sum_l [j:l]p_l=0. \tag{1.4}$$

The unique graph $G(D)$ associated with any spacetime diagram D is constructed by extracting from it the incidence matrix and the set of particle-type labels.

The $4n$-vector formed from the n four-vectors p_l associated with the n external lines of D is denoted by $p_{\text{ext}}(D)$. The Landau surface $L_1(G)$ consists of the set of the vectors $p_{\text{ext}}(D)$ for all D such that $G(D)=G$:

$$L_1(G)\equiv\{p_{\text{ext}}(D);G(D)=G\}. \tag{1.5}$$

The positive-α surface $L_1^+(G)$ is the subset of $L_1(G)$ obtained by imposing on the diagrams D in (1.5) the condition that for each line l of D the vector $x_{j^+(l)}-x_{j^-(l)}$ has a positive time component: $x^0_{j^+(l)}>x^0_{j^-(l)}$. The Landau surface $\bar{L}_1^+(G)$ is the closure of $L_1^+(G)$.

These geometric definitions will be supplanted in Sect. 3 by equivalent algebraic ones.

2. Convergence

We begin the discussion of convergence by considering a simple example in which there is only one kind of particle, which is a spinless particle, and in which all connected parts involving less than six particles vanish. Then the only contributing graphs G that give positive-α Landau surfaces that intersect the three-particle normal-threshold surface in a 3-to-3 amplitude are the graphs G^n of the kind shown in Fig. 1.

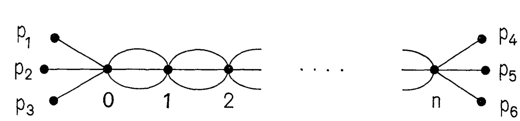

Fig. 1. The $(n+1)$-vertex three-particle-threshold graphs G^n

In this example the formula for the discontinuity around the three-particle threshold asserts that in some real neighborhood of any three-particle threshold point $P=(P_1, \ldots, P_6)$ one has

$$S_c^+(p_f, p_i) - S_c^-(P_f, p_i) = \int S_c^+(p_f, p_m) S_c^-(p_m, p_i) dp_m , \tag{2.1}$$

where $p_f=(p_1, p_2, p_3)$, $p_i=(p_4, p_5, p_6)$, $p_m=(p_7, p_8, p_9)$, and the functions $S_c^+(p_f, p_m)$ and $S_c^-(p_m, p_f)$ are, respectively, the limits of the kernel of the connected part of the S-matrix from above and below a cut placed on the positive real axis $\mathrm{Re}\, z \equiv x \geqq 0$ in the variable

$$z(p) = (p_1 + p_2 + p_3)^2 - 9m^2 . \tag{2.2}$$

The p_j for $j \in \{1, \ldots, 9\}$ are mass-shell four-vectors with $p_{j,0} = \sqrt{m^2 + \mathbf{p}_j}$, and

$$dp_m = \left(\prod_{j=7}^{9} \frac{d^3 \mathbf{p}_j}{(2\pi)^3 2p_{j,0}} \right) . \tag{2.3}$$

For real p_f one has

$$\int (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - p_7 - p_8 - p_9) dp_m = \omega(z(p_f)) z(p_f)^2 \, Y(z(p_f)) , \tag{2.4}$$

where $\omega(z)$ is analytic in z, and nonzero near $z=0$, and Y is the Heaviside function. Then the function

$$f(z) = \omega(z) z^2 \left[\frac{i}{2\pi} \log(z e^{-\pi i}) \right] \tag{2.5}$$

is analytic near the origin of the cut z plane, with the cut again placed along $x \geqq 0$, and the boundary values $f^+(x)$ and $f^-(x)$ of $f(z)$ from above and below this cut satisfy

$$f^+(x) - f^-(x) = \omega(x)^2 Y(x) . \tag{2.6}$$

For any Lorentz-invariant function $a(p)$ analytic near the three-particle threshold point P the function

$$S_c(p) = \left[\sum_{n=0}^{\infty} a^{[n+1]}(p) f^n(z(p_f)) \right] (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - p_4 - p_5 - p_6) \tag{2.7}$$

is a solution of (2.1) near P by virtue of the identity

$$(f^+)^n - (f^-)^n = \sum_{m=0}^{n-1} (f^+)^m (f^+ - f^-)(f^-)^{n-m-1} . \tag{2.8}$$

The function $a^{[n+1]}(p)$ is defined only on the restricted mass shell, and hence is a function of the variables $z(p_f)$, $\Omega(p_f)$, and $\Omega(p_i)$, where $\Omega(p_f)$ and $\Omega(p_i)$ are z-independent "angular" variables. It is defined by the equation

$$a^{[n+1]}(p) \equiv \langle \Omega(p_f) | a_{z(pf)}^{n+1} | \Omega(p_i) \rangle ,$$

where the operator a_z^{n+1} is the $(n+1)$-th power of the operator a_z, which is defined by the above equation for the special case $a^{[1]}(p) \equiv a(p)$.

Solution (2.7) has the general form

$$\left[\sum_{n=0}^{\infty} a_n(p) f^n(z(p_f))\right](2\pi)^4 \delta^4(p_1+p_2+p_3-p_4-p_5-p_6), \tag{2.9}$$

where the functions $a_n(p)$ are holomorphic near P. This general form holds also for the Feynman function F_G corresponding to any contributing graph G (i.e., any connected graph G each vertex of which connects at least six lines). Thus Landau's suggestion is naturally interpreted in our example as the condition that the function $S_c(p)$ should have this form (2.9).

Accepting this condition and substituting the form (2.9) of $S_c(p)$ into (2.1), and using the fact that the functions $f^n(z)$ for $n=(0,1,2,\ldots)$ are linearly independent, in the sense that no nontrivial linear combination of these functions multiplied by analytic functions vanishes, one obtains, using (2.8), recursion relations that imply that (2.7) is the unique solution of (2.1) on the restricted mass-shell variety

$$\{p_j; p_j^2=m_j^2, p_1+p_2+p_3=p_4+p_5+p_6\}.$$

We shall not attempt to derive the uniqueness of solution (2.7) from the weaker property R, but rather accept on the basis of Landau's suggestion that solution (2.7) is the physically appropriate solution.

Solution (2.7) provides a simple example of property R_P. The connection is made by identifying the term $a^{[n+1]}f^n(2\pi)^4\delta^4$ of (2.7) with the term $S_{G,P}$ of (1.2) for $G=G^n$. Then in some sufficiently small real product neighborhood $\Omega(P)$ of P the sum (1.2) is absolutely convergent in the operator sense (1.3) due to the decreasing factors $(z^2)^n$. On the other hand, if this same sum (2.7) were to be arranged in the form (1.1), i.e. as $\sum a_{G,P}(p)F_G(p)$, then it would not in general converge. The convergence in the form (2.7) is due to the fact that each independent function f^n occurs just once in (2.7), as compared to an infinite number of times in the rearranged form (1.1). The parsimonious arrangement (2.7) avoids the divergence associated with the infinite multiple counting of like terms.

In the context of the Landau condition that $S_c(p)$ should be formally representable in the form (1.1) one may describe the rearrangement that converts the divergent formal sum (1.1) to the convergent sum (2.7) as follows: for each Feynman function $F_G(p)$ one exhibits a "leading part" $F^L_{G,P}(p)$ by subtracting from $F_G(p)$ a sum of products of analytic functions times functions $F_{G'}(p)$, where the G' are contractions of G:

$$F^L_{G,P}(p)=F_G(p)-\sum a_{G',G,P}(p)F_{G'}(p). \tag{2.10}$$

This equation permits a formal rearrangement of (1.1) into the form

$$\sum_{G\in\mathscr{G}_P} a^L_{G,P}(p)F^L_{G,P}(p)=\sum_{G\in\mathscr{G}_P} S_{G,P}(p), \tag{2.11}$$

where the $S_{G,P}(p)$ satisfy (1.2b) and (1.2c). However, this sum (2.11) converges only if the leading parts $F^L_{G,P}(p)$ are appropriately defined.

In our example the leading part of $F_{G^n}(p)$ is $F^L_{G^n}(p)=(f^+(z(p)))^n$. This function is characterized by the close connection of its behavior near the point P to that of the phase-space integral associated with G^n: both functions have, up to logarithmic factors, the same power-law fall off x^{4n} when the point $x=0$ is approached along the positive real axis.

The close connection between singularities of Feynman integrals and those of phase-space integrals is not accidental. It is demanded by the constraints imposed on the singularity structure of the S-matrix by unitarity. In particular the S-matrix is required to have singularities that cancel the explicit singularities arising from the phase-space factors that occur in the unitarity equation, and in the more complex equations that arise by combining the cluster decomposition property of the S-matrix with multiple applications of unitarity [9]. The explicit singularities of phase-space integrals thus become the "driving terms" that force the S-matrix to have singularities [9]. And these S-matrix singularities must be of such a form as to be able to cancel the explicit singularities associated with the phase-space factors.

In our example the function $S_{G,P}(p)$ can be identified as the part of $S_c(p)$ that exactly cancels the purely positive-α part of the singularities arising from the phase-space integral associated with G. This can be seen as follows. Iteration of (2.1) gives the expression

$$S_c^+ = \sum_{n=1}^{\infty} (S_c^-)^n, \tag{2.12}$$

which converges absolutely in some sufficiently small real product neighborhood of P in the operator sense that the sum of the norms of the terms on the right-hand side of (2.12) converges. Inserting solution (2.7) into the left-hand side of (2.12), and the analogous solution with f^- replacing f^+ into the factors S_c^- on the right-hand side, and using the representation $(f^+ - f^-)$ for the phase-space factors (2.4) occurring on the right, one obtains an identity: every term on the right-hand side containing a factor f^- can be paired with an identical term of opposite sign, leaving precisely the sum (2.7). Moreover, each term $S^n_{G,P} = a^{[n+1]}(f^+)^n(2\pi)^4\delta^4$ of this remaining sum (2.7) enters only once on the right-hand side, and appears in precisely that term that has the phase-space factor $(f^+ - f^-)^n$ corresponding to G^n.

There is a generalization of the expansion (2.12) that expresses any connected part S_c^+ as an infinite sum of unitarity-type integrals involving only the functions S_c^-. ([9]) In this sum there is for each $G \in \mathscr{G}_P$ precisely one term that corresponds to a bubble diagram that reduces to G when each bubble is replaced by a point vertex. And in the unique unitarity-type integral there is precisely one point $K(P)$ in the domain of integration that gives a contribution to the integral at point P. The pair $(P, K(P))$ defines a point in the domain of definition of each of the functions S_c^- occurring in the integrand of this unitarity-type integral. At these points each S_c^- has an analytic background term. The constant parts of these background terms combined with the conservation law and mass-shell constraints give a contribution to the integral that is a multiple of the phase-space integral corresponding to G. There must be a contribution to S_c^+ that cancels that positive-α part of the singularity associated with this phase-space factor. Our example shows that the logarithmic factors associated with $S_{G,P}$ can be different from those associated with the phase-space factor, which at least in its simplest form, has no logarithmic factors at all. However, the remaining power-law behavior of $S_{G,P}$ is the same as that of the associated phase-space factor.

This equality of the power-law parts suggests that one can formulate in the following way a small part of the idea that $S_{G,P}$ is that part of S_c that is generated by the unitarity-induced phase-space factor associated with graph G: Suppose for some $r_0>0$ the restrictions to real multispherical domains $|p_j - P_j| < r_0$ (all j) of the phase-space integrals corresponding to the graphs $G \in \mathscr{G}_P$ have norms bounded by expressions of the form

$$D_P(C_P)^{n'_G} r^{b_{G,P}}, \quad \text{for all} \quad r < r_0, \tag{2.13a}$$

where n'_G is the number of vertices of G. Then the norms of the similarly restricted operators $S_{G,P}$ have bounds of the form

$$D_P(AC'_P)^{n'_G} r^{b_{G,P}} (\log r)^{N_G} \tag{2.13b}$$

for all $r < r'_0$, for some constants C'_P and $r'_0 > 0$, where N_G is the number of internal lines of G and A is the maximum value of the analytic parts of the scattering functions $S_c^- / (2\pi)^4 \delta^4$ in the relevant domains.

This bound on $|S_{G,P}|$ is far stronger than what is needed to prove the convergence property R_P. To see this observe first that the result of [12] implies that for any P the set of graphs $\mathscr{G}_P$ corresponds to a set of space-time diagrams $\mathscr{D}_P$ that consists of a *finite* set of space-time diagrams $\mathscr{D}'_P$ together with the diagrams that can be formed by taking some diagram D of $\mathscr{D}'_P$ and inserting extra vertices on *special* subsets of space-time lines of D. Each of these special subsets consist of a set of space-time lines of D that are all parallel. Thus the inserted vertices correspond to zero-energy processes in which all the initial and final particles are at rest in some frame, namely that frame defined by the set of parallel space-time lines. The insertion of these zero-energy vertices does not alter the kinematics, and hence infinite numbers of them can be inserted. The convergence problem arises only because of the infinite sets of diagrams that can be formed in this way by the insertion of zero-energy vertices. However, these zero-energy vertices correspond to operators whose norms fall-off as some power of ϱ, the radius of the real multispherical domain centered on the zero-energy point defined by $(P, K(P))$. For the simplest case of a 2-to-2 vertex the fall-off is according to the first power of ϱ, and for the general n-to-n vertex the fall-off is like ϱ^{3n-5}.

There is no essential loss of generality in considering the case in which $\mathscr{D}'_P$ consists of just one diagram, and in which this diagram has just one set of parallel lines: the modifications needed to pass to the general case are simple. Suppose the set of parallel lines consists of N' lines. It is then sufficient for our purpose to use a single bound of the form $C_1 \varrho^{\varepsilon_1}$ for all n-to-n vertices with $n \leqq N'$ and for all $\varrho < \varrho_1$, where $\varrho_1 > 0$ and $\varepsilon_1 > 0$.

Suppose K is the number of ways in which a vertex can connect some subset of the set of N' lines. Then for some constant C_2 and sufficiently small $\varrho_2 > 0$ one has

$$\sum_{n=1}^{\infty} (KC_1 \varrho^{\varepsilon_1})^n < C_2 \varrho^{\varepsilon_1}$$

for all $\varrho < \varrho_2$. The sum on the left-hand is a sum of bounds on the norms of the infinite set of operators corresponding to the infinite set of ways in which the zero-energy vertices can be inserted into the set of N' parallel lines.

Let r be the radius of the real multi-spherical domain $|p_j - P_j| < r$ (all j) centered on P. The condition $r=0$ forces $\varrho=0$. Hence it follows – from the Łojasiewicz inequality – that $\varrho<(Cr)^f$ holds for some C and $f>0$, in the domain $r<r_0''$ for some $r_0''>0$. Thus for some sufficiently small r_0 our condition (2.13a) on the norms of the phase-space factors associated with the phase space factors $G\in\mathscr{G}_P$ holds, with each zero-energy vertex contributing a factor $C'r^\varepsilon$ for some $\varepsilon>0$. Then the resultant condition (2.13b) on the norms of the $|S_{G,P}|$ give the required convergence property R_P, since a contraction of the domain $r<r_0$ to a domain $r<r_0'=r_0/\lambda$ converts $r^\varepsilon(\log r)^{N'}$ to $\left(\frac{r}{\lambda}\right)^\varepsilon\left(\log\frac{r}{\lambda}\right)^{N'}$, which for any fixed $\varepsilon>0$ and N' is smaller than the original value for some sufficiently large λ. Note that the number of lines N_G can increase no faster than N' times the number of vertices.

Within the framework of perturbation theory the argument given above is merely heuristic. Indeed, since the questions at issue involve the convergence of sums of diagrammatic contributions, the perturbation theoretic framework presumably fails to provide an adequate basis for rigorous analysis. The essential point of the above argument is to show that if the analytic background parts are finite, as they must be for a sensible theory, then the terms in any infinite sequence of singular diagrammatic contributions should, if these terms are properly organized, appear with geometric factors that fall off fast enough to ensure convergence. This result should, in principle, be demonstrable within the framework of axiomatic field theory. Indeed, recent developments within that discipline should place within grasp a proof of the result in a neighborhood of the three-particle threshold. That proof should exhibit in this concrete case the relationships described in a general context in the above argument. A general proof within the framework of axiomatic field theory lies at present out of reach. Within the framework of S-matrix theory, where the aim is to impose the strongest possible analyticity assumptions compatible with the other general principles, it is reasonable to take the regularity property R as an ansatz and then examine its compatibility with the other principles. This program leads immediately to the question considered in the following sections, namely that of the analytic properties of unitarity-type integrals, under the assumption that the S-matrix enjoys the regularity property R.

The present work is based principally on the property R_P. However, the above arguments suggest the likely validity of a stronger property R_P^L that includes also the condition $S_{G,P}=a_{G,P}^L F_{G,P}^L$, where the leading part $F_{G,P}^L$ of F_G has the form (2.10), and is subject to norm conditions of the kind (2.13). This property is considerably stronger than R_P, since it specifies not only the locations of the singularities of $S_{G,P}$ but also their nature, to the extent that the nature of the singularities of the leading parts $F_{G,P}^L$ are specified.

Landau's original suggestion [7] included the idea that the Feynman functions should determine both the location and the nature of the singularities of the S-matrix. Thus the property R^L can be regarded as a precise formulation of Landau's suggestion that incorporates an appropriate convergence condition.

This property R^L can also be regarded as a very reasonable ansatz in the framework of analytic S-matrix theory, which can be regarded as a development of

Landau's suggestion. For the notion of maximal analyticity is essentially an instruction to impose the most stringent analyticity properties compatible with unitarity, macrocausality, and Lorentz invariance. Property R^L is much more stringent than R, yet it appears, on the bases of the many studies done between the time of Landau's 1959 paper and now, to be fully compatible with these S-matrix conditions. Further studies like those of [4] examining in detail the compatibility of R^L with unitarity are needed. These demand an understanding of the singularity structure of unitarity integrals, under the assumption that R^L holds.

Although the present work rests largely on R_P we do require also the local integrability of the relevant integrands. This local integrability property appears plausible in its own right. Yet it does not follow immediately from R_P, which says nothing about the nature of the singularities. In our examples the required local integrability is shown to follow from the results of [4], or, alternatively, from the Landau postulate R^L.

A complete formulation of R^L would demand the specification of the leading parts $F^L_{G,P}$ of all Feynman functions F_G. Since only very limited use is made here of R^L we shall not develop the general theory but will be content to specify $F^L_{G,P}$ in a few simple cases. These cases cover those that occur in our examples.

Suppose the Feynman function $f_G \equiv F_G/\delta^4$ has, near P, the form

$$f_G(p) = a(p)(\psi(p) + \sqrt{-1}\,0)^\alpha + b(p) \qquad (2.14a)$$
$$(\alpha \text{ non-integer})$$

or

$$f_G(p) = a(p)\psi(p)^\nu \log(\psi(p) + \sqrt{-1}\,0) + b(p) \qquad (2.14b)$$
$$(\nu \text{ non-negative integer}).$$

Then the leading part of $f^L_{G,P}(p)$ is this same function with b set equal to zero.

If a connected graph G can be cut into two connected parts G' and G'' by cutting through a single vertex then

$$f^L_{G,P} = f^L_{G',P'} \cdot f^L_{G'',P''}, \qquad (2.15)$$

where P' and P'' are the parts of P that refer to G' and G'', respectively. Equation (2.15) can be used iteratively to obtain the leading parts of the Feynman functions corresponding to iterated graphs such as those occurring in Fig. 1.

For a triangle graph G, and a point P lying on the intersection of the triangle singularity surface $L_1^+(G)$ and the two-particle normal threshold surface $L_1^+(G')$, where G' is a contraction of G, the analysis of [4] [see Eq. (4.2)] suggests that $F^L_{G,P}$ should be a non-vanishing analytic function times $\log((x_1 + \sqrt{-1}\,0) + x_2)$, where x_1 and x_2 are the variables discussed there.

Added Note. After this work was completed we received a communication from D. Iagolnitzer kindly informing us that our assumption that the general solutions should have the form (2.9), which we extracted from Landau's suggestion, and which follows also from R^L, is entailed in a field theoretic context by a consideration of the Bethe Salpeter equation. Details can be found in a forthcoming paper by J. Bros, D. Iagolnitzer, and D. Pesenti entitled "Non-Holonomic

Singularities of the S-matrix and Greens Functions". (Saclay Preprint Dph-5/81/8 Submitted to Commun. Math. Phys.) In that work these authors have independently examined in great detail the model discussed from a slightly different viewpoint in beginning of this section. The present work deals explicitly with the fact, stressed by those authors, that the S-matrix cannot be a single holonomic function: we assume only that it is locally a convergent sum of regular holonomic functions.

3. Micro-Local Analyticity of Bubble Diagram Functions

To fix the notations, we first recall the definition of Landau equations associated with the signed Landau graph G having n external lines, n' internal vertices and N internal lines. Each internal line L_l carries a sign σ_l ($=+1$ or -1), which is distinct from its orientation. In what follows we label each external vertex by the same index r that labels the (external) line incident upon it. The graph G is assumed to be partially ordered and connected.

Definition 3.1. A set $(p_1, \ldots, p_n; u_1, \ldots, u_n) \underset{\text{def}}{=} (p; u)$ consisting of n real four-vectors p_r and n real four-vectors u_r is said to be a real solution of the Landau equations associated with G if and only if there are sets of real four-vectors k_l ($l=1, \ldots, N$) and v_j ($j=1, \ldots, n'$) and real scalars α_l ($l=1, \ldots, N$) and β_r ($r=1, \ldots, n$) such that the following relations (3.1a)$\sim$(3.1f) are satisfied:

$$\sum_{r=1}^{n} [j:r]p_r + \sum_{l=1}^{N} [j:l]k_l = 0, \quad j=1, \ldots, n', \tag{3.1a}$$

$$p_r^2 = k_r^2, \quad p_{r,0} > 0, \quad r=1, \ldots, n, \tag{3.1b}$$

$$k_l^2 - m_l^2 = 0, \quad k_{l,0} > 0, \quad l=1, \ldots, N, \tag{3.1c}$$

$$v_{j^+(l)} - v_{j^-(l)} = \alpha_l k_l, \quad l=1, \ldots, N, \tag{3.1d}$$

$$u_r = -[j(r):r](v_{j(r)} - \beta_r p_r), \quad r=1, \ldots, n, \tag{3.1e}$$

$$\sigma_l \alpha_l \geqq 0, \quad l=1, \ldots, N. \tag{3.1f}$$

The relations (3.1a)$\sim$(3.1f) are called Landau equations. The set of vectors $(p; \sqrt{-1}u)$, where $(p; u)$ is a real solution of Landau equations is denoted by $\mathscr{L}(G)$.

Remark 3.2. We regard $\mathscr{L}(G)$ as a subset of $\sqrt{-1}T^*\mathbb{R}^{4n}$. That is, u is regarded as a cotangent vector at p.

Definition 3.3. The projection $\pi(\mathscr{L}(G))$ of $\mathscr{L}(G)$ to $\mathbb{R}^{4n}$ is denoted by $L(G)$, where π is the canonical projection from $\sqrt{-1}T^*\mathbb{R}^{4n}$ to $\mathbb{R}^{4n}$.

Definition 3.4. $[\mathscr{L}(G)]^{\mathbb{C}}$ denotes the set of all complex vectors $(p; u)$ that satisfy the relations (3.1) except for the inequalities, and $(L(G))^{\mathbb{C}}$ is its projection onto $\mathbb{C}^{4n}$.

Definition 3.5. The set of equations obtained by replacing the condition (3.1d) with the following conditions (3.1d′) is called the set of pre-Landau equations:

$$v_{j^+(l)} - v_{j^-(l)} - \alpha_l k_l = w_l. \tag{3.1d′}$$

Here w_l is a real four-vector. The set of all vectors $(p, k; \sqrt{-1}(u, w))$, where $(p, k; u, w)$ is a solution of the pre-Landau equations, denoted by $\mathscr{K}(G)$. This set $\mathscr{K}(G)$ is called the pre-Landau variety associated with G.

Definition 3.6. If $(p; u) = (p; 0)$ satisfies the Landau equations with some $\alpha_l \neq 0$, then p is called a $u = 0$ point for the graph G.

If $(p, k; u, w) = (p, k; 0, 0)$ satisfies the pre-Landau equations (and hence the Landau equations) with some $\alpha_l \neq 0$, then such (p, k) is called a $u = 0$ solution for the graph G.

The set $L(G)$ is contained in the reduced mass-shell variety

$$\mathscr{M}_r \underset{\text{def}}{=} \left\{ p \in \mathbb{R}^{4n};\ \sum_{r=1}^{n} \varepsilon_r p_r = 0,\ p_r^2 = 0 \text{ and } p_{r,0} > 0 \ (r = 1, \ldots, n) \right\}.$$

Here and in what follows ε_r denotes $[j(r) : r]$. Furthermore $\mathscr{M}_r$ is non-singular outside $\mathscr{M}_{\text{exc}} \underset{\text{def}}{=} \{p \in \mathscr{M}_r;$ all p_r's are parallel$\}$. Hence, if we denote $\mathscr{M}_r - \mathscr{M}_{\text{exc}}$ by $\mathscr{M}'$, then we may regard the Landau equations (3.1a)∼(3.1f) as defining a subset of $\sqrt{-1}T^*\mathscr{M}'$ under the convention that $(p; \sqrt{-1}u)$ and $(p'; \sqrt{-1}u')$ define the same point in $\sqrt{-1}T^*\mathscr{M}'$ if and only if both

$$p = p' \tag{3.2a}$$

and

$$u_r - u'_r = -\varepsilon_r a - \gamma_r p_r \qquad (r = 1, \ldots, n) \tag{3.2b}$$

hold for some real four-vector a and real scalars γ_r $(r = 1, \ldots, n)$. (See e.g. [13, p. 115] for the detailed arguments.)

To state our main results (Proposition 3.16 and Theorem 3.17) we fix our notations concerning the bubble diagram function $F_B(p)$, and the bubble diagram amplitude $f_B(p)$. This latter function is obtained from $F_B(p)$ by factorizing out the over-all energy-momentum conservation δ-function factor $\delta^4\left(\sum_r [j(r) : r] p_r\right)$: $F_B(p) = f_B(p)\delta^4(\sum [j(r) : r] p_r)$. See [4] for the definition of $F_B(p)$ and the notations which are not explained here.

Each bubble of the bubble diagram B is labelled by an index b $(1 \leqq b \leqq b_0)$, and each (explicit) internal line of B is labelled by an index i $(1 \leqq i \leqq i_0)$. The mass and the energy-momentum four-vector associated with the i-th internal line of B are denoted by μ'_i and q_i, respectively. We denote by $G(B)$ the Landau graph obtained from B by replacing each bubble with a point. For any set of Landau graphs G_b $(1 \leqq b \leqq b_0)$ we denote by $\bigotimes_B G_b$ the Landau graph obtained by inserting G_b into the b-th bubble of B. And define $\sigma(b)$ to be $+$ or $-$ according to whether the b-th bubble is a plus-bubble or a minus-bubble. Each graph G_b must have the same set of external lines as bubble b. We denote by $\mathscr{G}(B)$ the set of all sets $\{G_b\}_{b=1}^{b_0}$ that fit into B. For $\{G_b\}_{b=1}^{b_0}$ in $\mathscr{G}(B)$ we denote by $\mathscr{C}(\{G_b\})$ the set of Landau graphs $\{G'_b\}_{b=1}^{b_0}$, where G'_b is G_b or its contraction.

Example of B, $G(B)$, and $\bigotimes_B G_b$:

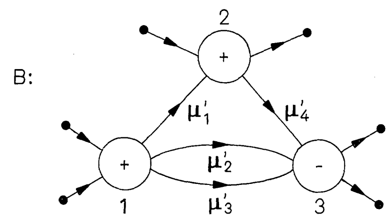

Fig. 2. A bubble diagram B

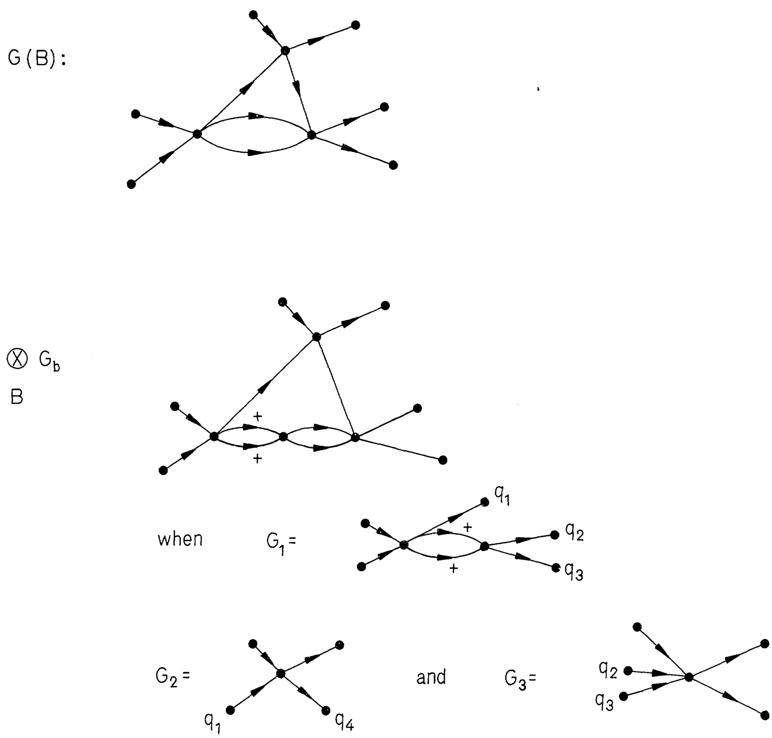

Fig. 3. Some Landau graphs

In what follows we denote by $\Phi_{G(B)}(p,q)$ the integrand of the phase-space integral associated with $G(B)$, with the over-all energy-momentum conservation δ-function being factorized out:

$$\Phi_{G(B)}(p,q) = \prod_{j=2}^{j_0} \delta^4\left(\sum_r [j:r]p_r + \sum_i [j:i]q_i\right) \prod_{i=1}^{i_0} \delta^+(q_i^2 - \mu_i'^2), \tag{3.3}$$

where p lies in the reduced mass-shell variety $\mathscr{M}_r$ and j_0 denotes the number of internal vertices of $G(B)$. We denote by $\mathscr{M}_{G(B)}$ the subvariety of $\mathscr{M}_r \times \mathbb{R}^{4i_0}$ outside of which $\Phi_{G(B)}(p,q)$ vanishes. The complexification of $\mathscr{M}_{G(B)}$ is denoted by $\mathscr{M}_{G(B)}^{\mathbb{C}}$. If a point (p, q) in $\mathscr{M}_{G(B)}$ is not a $u=0$ point for $G(B)$, then $\mathscr{M}_{G(B)}$ is non-singular near the point. We denote by $\mathscr{M}_{G(B),\mathrm{reg}}$ the set of all such points. If a point p in $\mathscr{M}_r$ is not a $u=0$ point for $G(B)$, then the bubble diagram amplitude $f_B(p)$ takes the following form by definition:

$$f_B(p)=\int \prod_{b=1}^{b_0} s_b(p,q)\Phi_{G(B)}(p,q)\prod_{i=1}^{i_0} d^4q_i\,, \tag{3.4}$$

where $s_b(p,q)$ denotes the scattering amplitude [or its complex conjugate if $\varepsilon(b)=-1$] associated with the b-th bubble of B and the product $\prod_{b=1}^{b_0} s_b(p,q)$ is a distribution on $\mathscr{M}_{G(B)}$.

Before beginning the study of the singularity structure of the bubble diagram functions, we introduce some notations concerning a bubble diagram B and present some preparatory results.

Definition 3.7. (i) $R(b) \underset{\mathrm{def}}{=} \{r;\, 1\leqq r\leqq n,\ [b:r]\neq 0\}$.

(ii) $I(b) \underset{\mathrm{def}}{=} \{i;\, 1\leqq i\leqq i_0,\ [b:i]\neq 0\}$.

(iii) $p(b)$: the $4(\#R(b))$ vector obtained from $(p_1,\dots,p_n)$ by deleting those p_r such that $[b:r]=0$. Here $(\#R(b))$ denotes the number of elements in $R(b)$.

(iv) $q(b)$: the $4(\#I(b))$ vector obtained from $(q_1,\dots,q_{i_0})$ by deleting those q_i such that $[b:i]=0$.

(v) Let $\varpi(b)$ denote the map from $\mathbb{C}^{4(n+i_0)}_{(p,q)}$ to $\mathbb{C}^{4(\#R(b)+\#I(b))}$ defined by assigning $(p(b),q(b))$ to (p,q).

Let $\mathscr{M}_r^{\mathbb{C}}(b)$ denote the subvariety of $\mathbb{C}^{4(n+i_0)}_{(p,q)}$ defined by

$$\sum_r [b:r]p_r+\sum_i [b:i]q_i=0\,, \tag{3.5a}$$

$$p_r^2=m_r^2 \quad \text{for } r \text{ such that} \quad [b:r]\neq 0\,, \tag{3.5b}$$

$$q_i^2=\mu_i'^2 \quad \text{for } i \text{ such that} \quad [b:i]\neq 0\,. \tag{3.5c}$$

We denote $\varpi(b)\mathscr{M}_r^{\mathbb{C}}(b)$ by $\mathscr{M}_{\mathrm{red}}^{\mathbb{C}}(b)$. It follows immediately from the definition that $\mathscr{M}_{G(B)}^{\mathbb{C}}=\bigcap_{b=1}^{b_0}\mathscr{M}_r^{\mathbb{C}}(b)$ holds. Throughout this section we always assume that the bubble diagram B satisfies the following additional condition:

$$\text{For each bubble } b \text{ of } B\text{, there are at least two incoming lines and at least two outgoing lines incident upon } b\,. \tag{3.6}$$

Lemma 3.8. *Let B be a bubble diagram and let P_0 be a real point in $\mathscr{M}_r^{\mathbb{C}}(b_1)$ for a bubble b_1 of B. Let $\mathscr{U}$ be a sufficiently small neighborhood of P_0, and let $\phi(p,q)$ be a holomorphic function defined on $\mathscr{U}$. Assume that ϕ has the form $\phi(p(b_1),q(b_1))$ and that $L(b_1)\underset{\mathrm{def}}{=}\phi^{-1}(0)\cap\mathscr{U}\cap\mathscr{M}_r^{\mathbb{C}}(b_1)$ is a hypersurface of $\mathscr{U}\cap\mathscr{M}_r^{\mathbb{C}}(b_1)$. Then $L(b_1)\cap\left(\bigcap_{b\neq b_1}\mathscr{M}_r^{\mathbb{C}}(b)\right)$ does not contain an open subset of $\mathscr{M}_{G(B)}^{\mathbb{C}}$.*

Proof. It suffices to show the following property P: there exist an open neighborhood $\mathscr{U}(b_1)$ of $\varpi(b_1)(P_0)$ in $\mathscr{M}^{\mathbb{C}}_{\mathrm{red}}(b_1)$ and a continuous map $f(b_1)$ from $\mathscr{U}(b_1)$ to $\mathscr{M}^{\mathbb{C}}_{G(B)}$ such that $\varpi(b_1)f(b_1)$ is the identity map id. For the relation $\varpi(b_1)f(b_1)=\mathrm{id}$ implies that set $\varpi(b_1)(L(b_1))$ contains $f(b_1)^{-1}(L(b_1)\cap\mathscr{M}^{\mathbb{C}}_{G(B)})$, while the continuity of $f(b_1)$ implies that $f(b_1)^{-1}L(b_1)\cap\mathscr{M}^{\mathbb{C}}_{G(B)}$ contains an open set of $\mathscr{U}(b_1)$ if $L(b_1)\cap\mathscr{M}^{\mathbb{C}}_{G(B)}$ contains an open set of $\mathscr{M}^{\mathbb{C}}_{G(B)}$. Since $\varpi(b_1)(L(b_1))$ is a proper analytic subset of $\mathscr{M}^{\mathbb{C}}_{\mathrm{red}}(b_1)$ the theorem follows from property P by contradiction.

In what follows, we say a bubble b of B is downstream (respectively, upstream) from b_1 if b can be reached from b_1 by moving in $G(B)$ in the direction (respectively, anti-direction) of the lines of $G(B)$. The map $f(b_1)$ is constructed by just allowing any change in the final (=outgoing) p_r's and q_i's of b_1 to propagate downstream through the bubbles b of B, and allowing any change in the initial (=incoming) p_r's and q_i's of b_1 to propagate upstream through the bubbles b of B. To show that this propagation is possible we allow all of the change of energy-momentum coming into each bubble b that is downstream from b_1 to go into the energy-momentum vectors associated with some two preferred lines outgoing from b. The existence of such lines is guaranteed by (3.6). Then what we have to show is the existence of a solution (p,p') of the following Eqs. (3.7a)$\sim$(3.7d) that depends continuously on $(E,\mathbf{P})$ for $(E,\mathbf{P})$ sufficiently close to some original value, which by a suitable choice of coordinate system can be taken to be $(\mu,0)$ with μ a strictly positive number. In the following equations m_1 and m_2 denote the relevant masses.

$$p_0^2-\mathbf{p}^2=m_1^2, \tag{3.7a}$$

$$p_0'^2-\mathbf{p}'^2=m_2^2, \tag{3.7b}$$

$$p_0+p_0'=E, \tag{3.7c}$$

$$\mathbf{p}+\mathbf{p}'=\mathbf{P}. \tag{3.7d}$$

Denote $m_1^2+p_2^2+p_3^2$ [respectively, $m_2^2+(P_2-p_2)^2+(P_3-p_3)^2$] by A (respectively B). Then it suffices to show the existence of a continuous solution (p_0,p_1) of the Eqs. (3.8) and (3.9) below, where a continuous solution is required to depend continuously on the parameters (A,B,E,P_1):

$$p_0^2-p_1^2=A, \tag{3.8}$$

$$(E-p_0)^2-(P_1-p_1)^2=B. \tag{3.9}$$

From (3.8) and (3.9) one obtains

$$4(P_1^2-E^2)p_1^2+4P_1(E^2-P_1^2+A-B)p_1+(E^2-P_1^2+A-B)^2-4E^2A=0, \tag{3.10}$$

$$p_0^2=p_1^2+A. \tag{3.11}$$

Since $P_1^2 \neq E^2$ holds on sufficiently small $\mathscr{U}$, the existence of a continuous solution of (3.10) and (3.11) follows. This implies the existence of the required $f(b_1)$. Q.E.D.

The following lemma is a variant of Theorem 2.8 of [14], designed to be suitable for our purpose.

Lemma 3.9. *Let $\mathscr{U}^{\mathbb{C}}$ be an open subset of $\mathbb{C}^l$ and let $\mathscr{U}$ be defined by $\mathscr{U}^{\mathbb{C}} \cap \mathbb{R}^l$. Let $\chi_j(x)$ $(j=1, \ldots, m)$ be holomorphic functions defined on $\mathscr{U}^{\mathbb{C}}$ which satisfy the following two conditions:*

$$\chi_j(x) \text{ is real-valued on } \mathscr{U}, \tag{3.12}$$

$$\begin{aligned} &\operatorname{grad}\chi_j(x)|_{\mathscr{M}^{\mathbb{C}}} (j=1, \ldots, m) \text{ are linearly independent at each point} \\ &\text{in } \mathscr{M}^{\mathbb{C}} \underset{\text{def}}{=} \{x \in \mathscr{U}^{\mathbb{C}};\ \chi_j(x)=0\ (j=1, \ldots, m)\}. \end{aligned} \tag{3.13}$$

Let $\mathscr{M}$ be the manifold given by $\{x \in \mathscr{U};\chi_j(x)=0\ (j=1, \ldots, m\}$ and let f be a regular holonomic hyperfunction on $\mathscr{M}$ which is locally summable and with characteristic variety Λ. Let $\phi(x)\,(\not\equiv 0)$ be a holomorphic function defined on $\mathscr{U}^{\mathbb{C}}$ which is real-valued on $\mathscr{U}$ and vanishes on $\pi(\Lambda)$. Let $\mathscr{N}$ be a submanifold of $\mathscr{U}$ defined by $\{x \in \mathscr{U};$ $\chi_1(x)= \ldots =\chi_d(x)=0\ \ (d<m)\}$. Then $g \underset{\text{def}}{=} f\delta(\chi_{d+1}(x)) \ldots \delta(\chi_m(x))$ is a well-defined hyperfunction on $\mathscr{N}$ and its singularity spectrum is confined to the following set: $\{(x;\xi) \in \sqrt{-1}S^*\mathscr{N}$; there exist a sequence $x(v)$ in $\mathscr{M}^{\mathbb{C}}$, $c(v)$ in $\mathbb{C}$ and $c_j(v)\,(j=1, \ldots, m)$ in $\mathbb{C}^m$ which satisfy the following:

$$x(v) \to x, \tag{3.14a}$$

$$c(v)\phi(x(v)) \to 0, \tag{3.14b}$$

$$c(v)\operatorname{grad}\phi(x(v)) + \sum_{j=1}^{m} c_j(v)\operatorname{grad}\chi_j(x(v)) \to \xi. \tag{3.14c}$$

The vector ξ in (3.14c) is identified with a cotangent vectors of $\mathscr{N}$ at x by the usual rule, namely, by being considered modulo vectors of the form $\sum_{j=1}^{d} a_j \operatorname{grad}\chi_j(x)$ with a_j in $\mathbb{R}$.

This lemma follows immediately from Theorem 2.8 of [14]. In case $\pi(\Lambda)$ is of codimension 1 in $\mathscr{U}^{\mathbb{C}}$ the function ϕ can be taken to be a defining function of $\pi(\Lambda)$. Then the result is independent of the choice of ϕ.

In what follows we choose $\mathscr{M}_{G(B),\,\mathrm{reg}}$ as $\mathscr{M}$ and $\mathscr{M}' \times \mathbb{R}^{4i_0}$ as $\mathscr{N}$. Thus we choose $l=4(n+i_0)$, (p, q) as x,

$$d=n+4m=n+i_0+4b_0,$$

$$\chi_r(x)=p_r^2-m_r^2 \quad (r=1, \ldots, n),$$

$$\chi_{n+j}(x)=\sum_{r=1}^{n} \varepsilon_r p_{r,\,j-1} \quad (j=1,2,3,4),$$

$$\chi_{d+i}(x)=q_i^2-\mu_i'^2 \quad (i=1, \ldots, i_0)$$

and

$$\chi_{d+i_0+(b-1)j}(\alpha)\sum_r [b:r]p_{r,\,j-1}+\sum_i [b:i]q_{i,\,j-1} \quad (b=2, \ldots, b_0, j=1,2,3,4).$$

Definition 3.10. For a bubble diagram B and a set of Landau graphs $\{G_b\}_{b=1}^{b_0}$ that fits into B, $K_0(\{G_b\})$ is, by definition, the following subset of $\sqrt{-1}S^*(\mathscr{M}' \times \mathbb{R}^{4i_0})$:

$$\{(p, q; \sqrt{-1}(u, w) \in \sqrt{-1}S^*(\mathscr{M}' \times \mathbb{R}^{4i_0});$$

(i) $(p, q) \in \mathscr{M}_{G(B), \mathrm{reg}}$.

(ii) (p, q) is a $u=0$ point for some $\bigotimes_B G'_b$ with $\{G'_b\}_{b=1}^{b_0}$ in $\mathscr{C}(\{G_b\})$.

(iii) For any function ϕ that

(a) is holomorphic but not identically zero on a complex neighborhood $\Omega^{\mathbb{C}}$ of (p, q) in $\mathscr{M}^{\mathbb{C}}_{G(B)}$,

(b) is real valued on $\mathscr{M}_{G(B), \mathrm{reg}} \cap \Omega^{\mathbb{C}}$, and

(c) vanishes on

$$\mathscr{S} \underset{\mathrm{def}}{=} \bigcup_{b=1}^{b_0} [(\pi(\mathrm{Ch}(f_{G_b}) \cap \mathscr{M}^{\mathbb{C}}_{G(B)}] \cap \Omega^{\mathbb{C}},$$

where f_G is the Feynman function F_G with the conservation δ^4 factored out, there exists a sequence of complex numbers $c(v)$, $\beta_r(v)$ $(r=1, \ldots, n)$, $\alpha_i(v)$ $(i=1, \ldots, i_0)$, complex four-vectors $v_b(v)$ $(b=1, \ldots, b_0)$ and complex vectors $(p(v), q(v))$ which satisfy the following conditions (3.15α)~(3.15ε).}

$$(p(v), q(v)) \in \mathscr{M}^{\mathbb{C}}_{G(B)}, \tag{3.15α}$$

$$(p(v), q(v)) \to (p, q), \tag{3.15β}$$

$$c(v)\,(p(v), q(v)) \to 0, \tag{3.15γ}$$

$$\sum_b [b:r] v_b(v) + \beta_r(v) p_r(v) + c(v) \frac{\partial \phi}{\partial p_r}(p(v), q(v)) \to u_r, \quad (r=1, \ldots, n), \tag{3.15δ}$$

$$\sum_b [b:r] v_b(v) + \alpha_i(v) q_i(v) + c(v) \frac{\partial \phi}{\partial q_i}(p(v), q(v)) \to w_i, \quad (i=1, \ldots, i_0). \tag{3.15ε}$$

Remark on (iii). Since $\pi(\mathrm{Ch}(f_{G_b}))$ is a proper analytic subset of $\mathscr{M}^{\mathbb{C}}_r(b)$, Lemma 3.8 guarantees the existence of functions ϕ that vanish on $\mathscr{S}$ and are not identically zero on $\Omega^{\mathbb{C}} \cap \mathscr{M}^{\mathbb{C}}_{G(B)}$. If for any such ϕ the reality condition is satisfied and (3.15) cannot be satisfied then $(p, q; \sqrt{-1}(u, w))$ does not belong to $K_0(\{G_b\})$.

Definition 3.11. For a bubble diagram B and a set of Landau graphs $\{G_b\}_{b=1}^{b_0}$ that fits into B, $K_1(\{G_b\})$ denotes the set of all points (p, q) that satisfy the following two conditions:

The point (p, q) is a $u=0$ point for $\bigotimes_B G'_b$ with

$$\{G'_b\}_{b=1}^{b_0} \text{ in } \mathscr{C}(\{G_b\}). \tag{3.16}$$

There is no open neighborhood ω of (p, q) in $\mathscr{M}_{G(B)}$ such that the product $\prod_{b=1}^{b_0} s_{b, G_b, (p, q)}$ is an integrable function on ω. Here $s_{b, G_b, (p, q)}$ is the function obtained by factorizing out the conservation δ-function from the function associated with G_b and point (p, q) that appears in the expansion (1.2a) of s_b, the scattering amplitude, or of its Hermitian conjugate if

$$\sigma(b) = -1. \tag{3.17}$$

Definition 3.12. A point (p,q) in $\mathscr{M}_{G(B)}$ is called a *tame point* with respect to a bubble diagram B, if it belongs to $\mathscr{M}_{G(B),\mathrm{reg}}$ and if it is not contained in $K_1(\{G_b\})$ for any $\{G_b\}_{b=1}^{b_0}$ in $\mathscr{G}(B)$. We donote by $\mathscr{M}_{G(B),\mathrm{tame}}$ the set of all tame points with respect to B.

Definition 3.13. $\mathscr{M}_{\mathrm{good}}(B) \underset{\mathrm{def}}{=} \{p\in\mathscr{M}'$; p is not a $u=0$ point for $G(B)$, and for each q such that (p,q) is in $\mathscr{M}_{G(B)}$, the point (p,q) is contained in $\mathscr{M}_{G(B),\mathrm{tame}}\}$.

Definition 3.14. For a bubble diagram B, $K(B)$ is, by definition, the closure of

$$\bigcup_{\{G_b\}\in G(B)}\left[\mathscr{K}\left(\bigotimes_B G_b\right)\cup K_0(\{G_b\})\right]$$

in $\sqrt{-1}S^*(\mathscr{M}\times\mathbb{R}^{4i_0})$.

Definition 3.15. For a bubble diagram B, $\Lambda(B)$ denotes the subset of $\sqrt{-1}S^*\mathscr{M}'$ given by $\{(p;\sqrt{-1}u)\in\sqrt{-1}S^*\mathscr{M}'$; there exists $q\in\mathbb{R}^{4i_0}$ such that $(p,q;\sqrt{-1}(u,0))$ belongs to $K(B).\}$

Now our main results are stated as follows:

Proposition 3.16. *Let B be a bubble diagram that satisfies the condition* (3.6), *and let $\{G_b\}_{b=1}^{b_0}$ in $\mathscr{G}(B)$. Let (p,q) be a point in $\mathscr{M}_{G(B),\mathrm{reg}}$ that is not contained in $K_1(\{G_b\})$. Then $\left(\prod_{b=1}^{b_0} s_{b,G_b,(p,q)}\right)\Phi_{G(B)}$ is zero as a microfunction on*

$$\pi^{-1}(\omega)-\left[\bigcup_{\{G'_b\}\in\mathscr{C}(\{G_b\})}\mathscr{K}\left(\bigotimes_B G'_b\right)\cup K_0(\{G_b\}\right]$$

for an open neighborhood ω of (p,q) that does not intersect $K_1(\{G_b\})$. Here π denotes the projection from $\sqrt{-1}S^*(\mathscr{M}'\times\mathbb{R}^{4i_0})$ to $\mathscr{M}'\times\mathbb{R}^{4i_0}$.

Theorem 3.17. *Let B be a bubble diagram that satisfies the condition* (3.6). *Then, on the condition that the scattering amplitude has the property R, $f_B(p)$ regarded as a microfunction is zero on $\pi^{-1}\mathscr{M}_{\mathrm{good}}(B)-\Lambda(B)$, where π denotes the projection from $\sqrt{-1}S^*\mathscr{M}'$ to $\mathscr{M}'$.*

Proof of Proposition 3.16. First consider a point $(\tilde{p},\tilde{q})$ close to (p,q) and that is not a $u=0$ point for any $\bigotimes_B G'_b$ with $\{G'_b\}_{b=1}^{b_0}$ in $\mathscr{C}(\{G_b\})$. It then follows from the definition that $(\tilde{p},\tilde{q})$ is not a $u=0$ point for any G'_b. Hence

$$\mathrm{SS}\, s_{b,G_b,(p,q)}\subset\bigcup_{G'_b\in\mathscr{C}(G_b)}\mathscr{L}(G'_b)$$

holds in a neighborhood of $(\tilde{p},\tilde{q})$. In this case, by the general theory of microfunctions [15, Chap. I], or, essential support [16], we can conclude that $\mathrm{SS}\left[\left(\prod_{b=1}^{b_0} s_{b,G_b,(p,q)}\right)\Phi_{G(B)}\right]$ is confined $\bigcup_{\{G'_b\}\in\mathscr{C}(\{G_b\})}\mathscr{K}\left(\bigotimes_B G'_b\right)$ in a neighborhood of $(\tilde{p},\tilde{q})$. (See Iagolnitzer [17] and Kawai and Stapp [4] for the detailed argument in this case.) Next consider a point $(\tilde{p},\tilde{q})$ which is a $u=0$ point for some $\bigotimes_B G'_b$ with $\{G'_b\}_{b=1}^{b_0}$ in $\mathscr{C}(\{G_b\})$, but is not in $K_1(\{G_b\})$. Lemma 3.9 applies to this point. That is, the singularity spectrum of $\left(\prod_b s_{b,G_b,(p,q)}\right)\Phi_{G(B)}$ is confined to $K_0(\{G_b\})$ at that point. Thus we have verified that $(\prod s_{b,G_b,(p,q)})\Phi_{G(B)}$ is zero as a microfunction outside

$$\bigcup_{\{G'_b\}\in\mathscr{G}(\{G_b\})}\mathscr{K}\left(\bigotimes_B G'_b\right)\cup K_0(\{G_b\}).$$

This completes the proof of Proposition 3.16.

Proof of Theorem 3.17. Since p is not a $u=0$ point for $G(B)$, $f_B(p)$ has the form

$$\int\left(\prod_b s_b(p,q)\right)\Phi_{G(B)}(p,q)\prod_i d^{4q_i}.$$

Since we may change the order of summation of absolutely convergent series, we may assume, on the basis of the property R, that $\prod_b s_b\Phi_{G(B)}$ has the form

$$\sum_{(k_b)}\prod_b s_{b,G_b(k_b),(p,q)}\Phi_{G(B)}$$

in a neighborhood of a point (p,q) in $\mathscr{M}_{G(B)}$. Since (p,q) is a tame point with respect to B by the assumption, Proposition 3.16 implies that

$$\sum_{(k_b)}\prod_b s_{b,G_b(k_b),(p,q)}\Phi_{G(B)}$$

is zero as a microfunction outside $K(B)$. Then it follows from the general result on the integration of microfunctions [15, Chap. I, Theorem 2.3.1] that $f_B(p)$ regarded as a microfunction is zero outside $\Lambda(B)$. This completes the proof of Theorem 3.17.

Remark 3.18. The above proof shows that what is needed is not the full property R, but merely $R_{(p,q)}$ at each point (p,q) in $\mathscr{M}_{G(B)}$ that is a $u=0$ point for some $\bigotimes_B G_b$ with $\{G_b\}_{b=1}^{b_0}$ in $\mathscr{G}(B)$.

We do not presently have much detailed knowledge about the geometry of $\Lambda(B)$, particularly because of the need to consider the closure of the union of infinitely many varieties. Note, however, that only finitely many terms are needed in the expansion (1.2a) if no $n(\geqq 3)$-particle threshold is relevant at the point in question. (Zimmermann [18], cf. [4] and references cited there.) In such circumstances the singularities of $f_B(p)$ can be attributed to each Landau graph $\bigotimes_B G_b$, on the supposition that the scattering amplitudes satisfy the property R.

The following examples illustrate the effectiveness of our results in resolving $u=0$ problems. A geometric study of $\Lambda(B)$ will be given in Sect. 4.

Example 3.19. Let B denote the following bubble diagram.

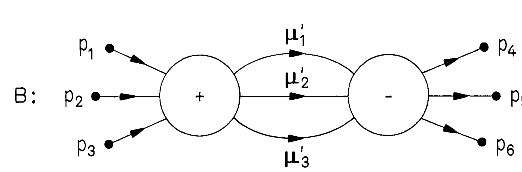

Fig. 4. A bubble diagram B

Let G_1 and G_2 be the following Landau graphs.

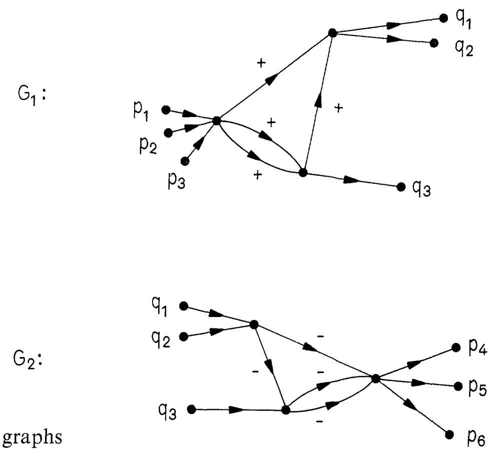

Fig. 5. Several Landau graphs

Suppose that the masses associated with $p_r (r=1, \ldots, 6)$ and the internal lines of G_1 and G_2 are all equal to $m (>0)$. Suppose further the following conditions on $\mu'_i (i=1,2,3)$ and m

$$\mu'_1 = \mu'_2 > m, \tag{3.18}$$

$$\mu'_3 < m, \tag{3.19}$$

$$9m^2 - \mu'^2_3 < 8\mu'^2_1. \tag{3.20}$$

Define M by μ'_1 $(=\mu'_2)$ and μ by μ'_3, respectively. Since $q_3^2 = \mu^2$ holds, the sets $L(G_1)$ [and $L(G_2)$] can be described in the (s, σ)-plane as below (e.g. [19, p. 60]). Here

$$s = (p_1 + p_2 + p_3)^2 = (p_4 + p_5 + p_6)^2 \quad \text{and} \quad \sigma = (q_1 + q_2)^2.$$

In Fig. 6, $L_1^+(G_1)$ $(=L_1^+(G_2))$ denotes the leading positive-α Landau surface (i.e., all α_l are strictly positive) corresponding to G_1 and the coordinates of the points $A, B, \ldots, F$ are given as follows:

$$\begin{aligned} &A: (9m^2, 4m^2), && B: (10m^2 - \mu^2, 4m^2), \\ &C: (9m^2, (9m^2 - \mu^2)/2), && D: (9m^2, 4M^2), \\ &E: (s_0, 4M^2), && F: ((2M + \mu)^2, 4m^2). \end{aligned}$$

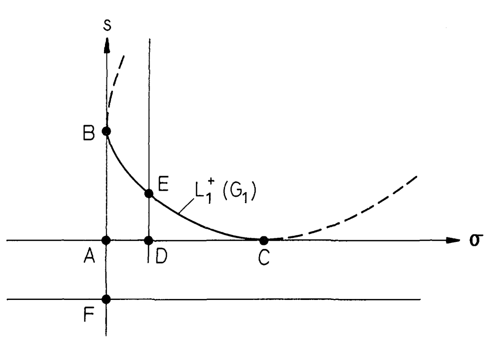

Fig. 6. Singularity surfaces

Here s_0 is the smaller root of the following equation in s:

$$(s+5m^2)^2+\alpha\beta(s+5m^2)/m^2+\alpha^2+4\beta^2-16m^4=0, \tag{3.21}$$

where $\alpha=\mu^2-5m^2$ and $\beta=4M^2-2m^2$. Here we note the following:

(i) The condition (3.19) guarantees that the s-coordinate of B is greater than that of A.

(ii) The condition (3.20) guarantees that D is located in the segment $\overline{AC}$ and that the s-coordinate of F is smaller than that of A.

Now let us consider the analyticity of the function f defined below in the domain $\Omega=\{p\in\mathscr{M}_r; 9m^2<s<s_0\}$:

$$f \underset{\text{def}}{=} \int s_{G_1}(p',q)\bar{s}_{G_2}(q,p'')\Phi_B(p',p'',q)dq, \tag{3.22}$$

where $p'=(p_1,p_2,p_3)$ and $p''=(p_4,p_5,p_6)$. Note that every point in Ω is a $u=0$ point for the graph $G_1 \underset{B}{\otimes} G_2$. Define N by

$$\Omega\times\{q=(q_1,q_2,q_3)\in\mathbb{R}^{12}; q_i^2=\mu_i'^2 \text{ and } q_{i,0}>0\ (i=1,2,3)\}$$

and define N_{par} by $\{(p,q)\in N;\ q_1$ is parallel to $q_2\}$. If (p,q) belongs to N_{par}, then (p',q) must lie on the open segment $\overline{DE}$. Hence (p',q) does not lie either on $L_1^+(G_1)$ or on the half line $\{(s,\sigma); s=9m^2, \sigma\geqq 4m^2\}$ or on $\{(s,\sigma); s\geqq 9m^2, \sigma=4m^2\}$. In other words, (p',q) does not lie on the singularities of f_{G_1} or $\overline{f_{G_2}}$, the complex conjugate of f_{G_2}. Therefore these points (p',q) can contribute to singularities of the integral f only at $s=(2M+\mu)^2$. However, $(2M+\mu)^2$ is smaller than $9m^2$ [by (ii)]. Thus this singularity of f lies outside the domain Ω.

Next consider the case where (p',q) does not belong to N_{par}, but lies on singularities of the integrand that can lead to $u=0$ points of the integrand. In this case (p',q) belongs to the open curve $\widehat{EC}$. Since the singularities of f_{G_i} [$i=1$ or 2] are contained in the nonsingular hypersurface $L_1^+(G_i)$, the set $\pi(\text{Ch}(f_{G_i}))$ is confined to the complexification $L^{\mathbb{C}}(G_i)$ of $L_1^+(G_i)$, in a complex neighborhood of $\widehat{EC}$. Furthermore, the function s_{G_i} has near $\widehat{EC}$ the form $a(p',q)(\phi(p',q)+\sqrt{-1}0)^{3/2}+b(p',q)$ for some holomorphic functions a and b, where ϕ is a defining function of $L_1^+(G_i)$ [4, Corollary in p. 222]. Hence the point (p,q) in question is a tame point: i.e., the local integrability requirement is satisfied. Since $\text{grad}_{\mathbf{q}}\phi(\mathbf{p}',\mathbf{q})$ never vanishes on the open curve $\widehat{EC}$, $\Lambda(B)$ is void. Thus we have verified that f is analytic in Ω.

Although we have used here a result of [4], which is based on the discontinuity formula, in order to guarantee that the point (p,q) in question is a tame point (i.e. that the local integrability requirement is satisfied), we could have used the property $R^L_{G_i,P}(P\in L_1^+(G_i))$ with the definition of $F^L_{G_i,P}$ given at the end of Sect. 2. That is, we do not need to use the result in [4] if we accept $R^L_{G_1,P}$.

Example 3.20. Let B be the following bubble diagram and suppose that all the relevant masses are equal to m.

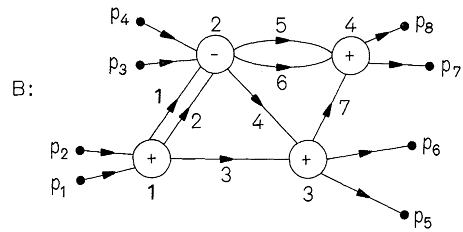

Fig. 7. A bubble diagram B

Let G_b $(b=1, 2, 3, 4)$ be the following Landau graphs.

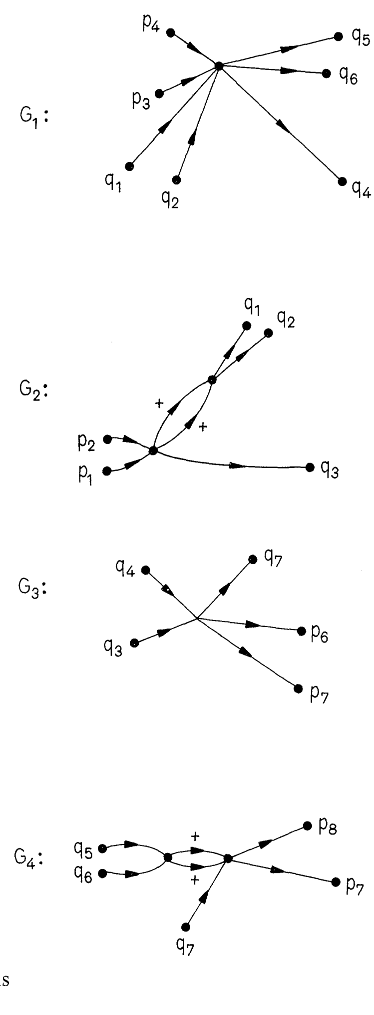

Fig. 8. Several Landau graphs

Let p be a point such that some point (p, q) belongs to $\mathcal{M}_{G(B)}$. Then p is a $u=0$ point for $\bigotimes_B G_b$. However, if the local integrability requirement is satisfied, then Theorem 3.17 and property R ensure that the singularities of $\int \prod_{b=1}^{4} s_{G_b} \Phi_{G(B)}(p, q) dq$ are restricted to a hypersurface H of $\mathcal{M}_r$.

The validity of the local integrability requirement (and also of R_P at the relevant points P) is ensured by the results of [4, Sect. 3.1], or by Zimmermann's result [18], or by assumption $R^L_{G,P}$ applied to the two-particle threshold graph G and the two particle threshold points P.

Example 3.21. Let B be the same bubble diagram as in Example 3.20. Let G_b $(b=2, 3, 4)$ be the same Landau graphs as in Example 3.20 and let G_1 be the following.

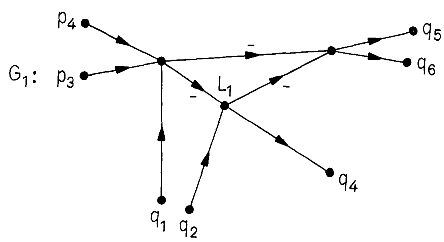

Fig. 9. A Landau graph

Let G'_1 denote the graph obtained from G_1 by contracting out the internal line L_1. Again each point (p, q) in $\mathcal{M}_{G(B)}$ gives a $u=0$ point for $\bigotimes_B G_b$. In this case, f_{G_1} is a locally integrable function near $L^+_1(G'_1) \cap \overline{L^+_1(G_1)}$, and the functions f_{G_b} $(b=2, 3, 4)$ are bounded. The form of s_{G_1} near the point in question has the form demanded by R^L (see [4, Eq. (4.2) and Eqs. (2.13) and (2.14) of the present work]). And again the results either of [4] or alternatively, of the Landau postulate R^L, ensure the validity of the local integrability requirement. Then Theorem 3.17 again shows that the singularities of $\int \prod_{b=1}^{4} s_{G_b} \Phi_{G(B)} dq$ associated with the indicated graphs are confined to a hypersurface of $\mathcal{M}_r$.

4. The relation Between $\Lambda(B)$ and the Extended Landau Variety

The purpose of this section is to study in the simplest case the relationship between the set $\Lambda(B)$ (Definition 3.15) and the set $\tilde{\mathcal{L}}\left(\bigotimes_B G_b\right)$ introduced in [13, (1.50), p. 114].[1]

1 [13] uses the notation $\tilde{\mathcal{L}}(D)$

In what follows, we consider exclusively a bubble diagram B such that $\# R(b) \geqq 2$ holds for every bubble b of B. For simplicity we consider the problem in the subset $\mathscr{M}_1$ of $\mathscr{M}'$ where the following condition is satisfied:

$$\text{For each bubble } b \text{ of } B \text{ there exist two external non-parallel energy-momentum four-vectors touching upon } b. \tag{4.1}$$

It is readily verified that no point of $\mathscr{M}_1$ is a $u=0$ point for $G(B)$. Furthermore, under the assumption (4.1) we can choose a local coordinate system on $\mathscr{M}_{G(B)}$ in an explicit manner as follows: Let (p^0, q^0) be a point in $\mathscr{M}_{G(B)}$ such that p^0 is in $\mathscr{M}_1$. Then there exists a neighborhood ω_1 of (p^0, q^0) where the following condition is satisfied:

There exist $r(b)$ and $\tilde{r}(b)$ $(b=1, \dots, b_0)$ such that $p_{r(b)}$ and $p_{\tilde{r}(b)}$ are not parallel on ω_1.

By shrinking ω_1, if necessary, we may assume further that

$$p_{r(b), v(b)}/p_{r(b), 0} \neq p_{\tilde{r}(b), v(b)}/p_{\tilde{r}(b), 0}$$

holds in ω for some $v(b)(=1, 2 \text{ or } 3)$. Define R_0 by $\{r; r \neq r(b), \tilde{r}(b) \text{ for } b=1, \dots, b_0\}$. Let $\mathbf{p}_{r(b)}^{(2)}$ $(b=1, \dots, b_0)$ denote the two-vector

$$(p_{r(b), v_1(b)}, p_{r(b), v_2(b)}) \qquad [v_1(b) < v_2(b),\ v_1(b), v_2(b) \neq 0, v(b)].$$

Let $\mathbf{p}_r$ and $\mathbf{q}_i$ denote the three-momentum-vector part of p_r and q_i, respectively. Define $\mathbf{p}'$ to be the $(3n-4b_0)$-vector obtained from $(p_1, \dots, p_n)$ by deleting $p_{\tilde{r}(b)}$, replacing $p_{r(b)}$ with $\mathbf{p}_{r(b)}^{(2)}$ and replacing p_r $(r \notin R_0)$ with $\mathbf{p}_r$. Then we can choose $(\mathbf{p}', \mathbf{q}) (=(\mathbf{p}', \mathbf{q}_1, \dots, \mathbf{q}_{i_0}))$ as a local coordinate system on ω_1. We call it a preferred local coordinate system. The corresponding cotangent vector is denoted by $(\mathbf{u}', \mathbf{w})$. This is a $[3(n+i_0)-4b_0]$-vector and, as usual, it can be identified with a representative of the $4(n+i_0)$-vector (u, w) modulo vectors of the form

$$N(v, \beta, \alpha) \underset{\text{def}}{=} \left(\sum_{b=1}^{b_0} [b:r] v_b + \beta_r p_r, \sum_{b=1}^{b_0} [b:i] v_b + \alpha_i q_{i_0} \right),$$
$$1 \leqq r \leqq n, \quad 1 \leqq i \leqq i_0 \tag{4.2}$$

for some four vectors v_b and real numbers β_r and α_i. In what follows we denote by $(0, \mathbf{u}', 0, \mathbf{w})$ the $4(n+i_0)$-vector which is canonically assigned to $(\mathbf{u}', \mathbf{w})$ by setting to zero the components $u_{r,0}$, $u_{\tilde{r}(b)}$, $u_{r(b), v(b)}$, and $w_{i,0}$ of the $4(n+i_0)$-vector (u, w).

We note that the above procedure for constructing a preferred local coordinate system works equally well for the construction of an explicit local coordinate system on $\mathscr{M}'$. Such a local coordinate system is also called preferred local coordinate system and is denoted by $\mathbf{p}'$.

We now discuss, under the assumption (U) given below, the relationship between $\tilde{\mathscr{L}}\left(\bigotimes_B G_b\right)$ and the part $\Lambda(\{G_b\})$ of $\Lambda(B)$ to which $\bigotimes_B G_b$ contributes. This part $\Lambda(\{G_b\})$ is, by definition, the following subset of $\sqrt{-1}S^*\mathscr{M}'$: $\{(p; \sqrt{-1}u) \in \sqrt{-1}S^*\mathscr{M}'$:

(i) There exists $q \in \mathbb{R}^{4i_0}$ such that (p, q) is a $u=0$ point for $\bigotimes_B G_b$.

(ii) For any point (p, q) that is a $u=0$ point for $\bigotimes_B G_b$, $(p, q; \sqrt{-1}(u, 0))$ belongs to $K_0(\{G_b\})\}$.

We shall examine cases that satisfy the following assumption:

(U) If a point (p_0, q_0) is a $u=0$ point for $\bigotimes_B G_b$, then (p_0, q_0) is in $L_0(G_b^{\sigma(b)})^2$ for each b $(1 \leqq b \leqq b_0)$ and the variety $\pi(\mathrm{Ch}(f_{G_b}))$ is contained in $L(G_b)^{\mathbb{C}}$ in a complex neighborhood of (p_0, q_0) for each b $(1 \leqq b \leqq b_0)$.

The situations considered in Examples 3.19 and 3.20 are simple examples that satisfy the assumption (U).

Now we show that $\Lambda(\{G_b\})$ is contained in $\tilde{\mathscr{L}}\left(\bigotimes_B G_b\right)$. We begin our discussion by preparing a geometric result on Landau surfaces. Until the end of the proof of Lemma 4.2 we abbreviate G_b by G, for the sake of simplicity of notation. Further we denote G by $G(m_l^2)$ to emphasize its dependence on the mass m_l^2 associated with some particular internal line L_l. As a mathematical device we allow m_l to be a complex number.

In what follows we use a preferred local coordinate system $(\mathbf{p}')$ on $\mathscr{M}'$. Note that its dual vector $(\mathbf{u}')$ is in a one-to-one correspondence with a $4n$-vector u modulo vectors of the form $-[j(r):r]a-\beta_r p_r$ $(a \in \mathbb{R}^4, \beta_r \in \mathbb{R})$.

Lemma 4.1. *Let* $\mathbf{p}'(v)$ *be any sequence of* $(3n-4)$*-vectors converging to a point* P *in* $L_0(G^+) \subset \mathscr{M}'$ *and let* L_1 *be any single specified internal line of* G. *Then there exists a sequence of complex numbers* $m_1(v)$ *that converges to* m_1 *and that is such that the point* $\mathbf{p}'(v)$ *is contained in* $L(G(m_1(v)^2))^{\mathbb{C}}$. *Furthermore, we can find a complex neighborhood* ω *of* P *and a holomorphic function* $f(\mathbf{p}', m^2)$ *defined on* $\omega \times \{m \in \mathbb{C};\ |m-m_1| < \varepsilon\}$ *so that*

$$L(G(m^2))^{\mathbb{C}} \cap \omega = \{\mathbf{p}' \in \omega;\ f(\mathbf{p}', m^2) = 0\} \qquad (|m-m_1| < \varepsilon) \tag{4.4}$$

and

$$\mathrm{grad}_{(\mathbf{p}', m^2)} f(\mathbf{p}', m^2) \neq 0 \quad \text{on} \quad L(G(m^2)) \cap \omega. \tag{4.5}$$

Proof. Let α be the set of Landau constants corresponding to the solution P of the Landau equations. It follows from the definition of $L_0(G^+)$ that α_1 is strictly positive. Now, a result of [20, Theorem 6] (see also [4, pp. 197–198]) guarantees the existence of a holomorphic function $f(\mathbf{p}', m^2)$ defined in a complex neighborhood of (P, m_1^2) which satisfies conditions (4.4) and (4.5). Furthermore, it follows from the definition of the Landau equations that

$$\frac{\partial f}{\partial m^2}(p_0', m_0^2) = 2d(\mathbf{p}_0', m_0^2)\alpha_{l,0}, \tag{4.6}$$

where $\alpha_{l,0}$ is the Landau constant corresponding to a solution $\mathbf{p}_0'$ of the Landau equations associated with $G(m_0^2)$ and $d(\mathbf{p}', m^2)$ is a holomorphic function that does not vanish in a neighborhood of (P, m_l^2). In particular, $\frac{\partial f}{\partial m_1^2}(P, m_1^2) \neq 0$ holds. Hence the implicit function theorem guarantees the existence of required $m_1(v)^2$ for $\mathbf{p}'(v)$ sufficiently close to P. Let $k_l(v)$, $v_j(v)$, $u_r(v)$, $\alpha_l(v)$, and $\beta_r(v)$ denote the quantities giving the solution $\mathbf{p}'(v)$ of the Landau equations associated with $G(m(v)^2)$. We now

2 See Chandler and Stapp [20] for the definition of $L_0(G^+)$. The definition of $L_0(G^-)$ is the same except for a change of sign of all α's. Chandler and Stapp use a script L

use a $(3n-4)$-vector $\mathbf{u}'$ to represent a $4n$-vector u, so that $\mathbf{u}'$ may be the dual vector of $\mathbf{p}'$. Since we are concerned with quantities on $S^*\mathscr{M}'$, we may further normalize $\mathbf{u}'$ by imposing a normalization condition $|\mathbf{u}'|=1$. Under this normalization condition the quantities $\alpha_l(v)$ $(l=1,\dots,N)$ and $\mathbf{u}'(v)$ converge to α_l and $\mathbf{u}'$, respectively. In accordance with this normalization we normalize $f(\mathbf{p}',m^2)$ so that

$$|\partial f/\partial \mathbf{p}'|=1 \tag{4.7}$$

holds. Furthermore we have the following

Lemma 4.2. *Let $f(\mathbf{p}',m^2)$ be the function given by the preceding lemma. Let $c(v)$ be a sequence of complex numbers which satisfy the following:*

$$c(v)f(\mathbf{p}'(v),m_1^2)\to 0. \tag{4.8}$$

Then we have the following:

$$c(v)\alpha_1(v)(k_1(v)^2-m_1^2)\to 0, \tag{4.9}$$

$$c(v)(\mathrm{grad}_{\mathbf{p}'}f(\mathbf{p}'(v),m_1^2)-\mathrm{grad}_{\mathbf{p}'}f(\mathbf{p}'(v),m_1(v)^2))\to 0. \tag{4.10}$$

Proof. Let us first prove (4.9). Since $k_1(v)^2=m_1(v)^2$ holds by definition, it suffices to show

$$c(v)\alpha_1(v)(m_1(v)^2-m_1^2)\to 0. \tag{4.11}$$

On the other hand, by the Taylor expansion of f, we find

$$\begin{aligned} f(\mathbf{p}'(v),m_1^2)-f(\mathbf{p}'(v),m_1(v)^2)=&\frac{\partial f}{\partial m_1^2}(\mathbf{p}'(v),m_1(v)^2)(m_1^2-m_1(v)^2)\\ &+g(\mathbf{p}'(v),m_1^2,m_1(v)^2), \end{aligned} \tag{4.12}$$

where

$$|g|/(m_1^2-m_1(v)^2)\to 0.$$

Since

$$\frac{\partial f}{\partial m_1^2}(\mathbf{p}'(v),m_1(v)^2)\to 2d(P)\alpha_1\neq 0,$$

and since $f(\mathbf{p}'(v),m_1(v)^2)=0$ holds, (4.12) combined with (4.8) entails

$$c(v)\frac{\partial f}{\partial m_1^2}(\mathbf{p}'(v),m_1(v)^2)(m_1^2-m_1(v)^2)\to 0. \tag{4.13}$$

In view of (4.6), we obtain the required relation (4.11) from (4.13). This completes the proof of (4.9).

We next show (4.10). For that purpose we first note that (4.11) actually implies

$$c(v)(m_1(v)^2-m_1^2)\to 0, \tag{4.14}$$

because the limiting value of $\alpha_1(v)$, i.e. α_1, is different from zero. Again, by the Taylor expansion, we find

$$\begin{aligned} &c(v)(\mathrm{grad}_{\mathbf{p}'}f(\mathbf{p}'(v),m_1^2)-\mathrm{grad}_{\mathbf{p}'}f(\mathbf{p}'(v),m_1(v)^2)\\ &=c(v)h(\mathbf{p}'(v),m_1^2,m_1(v)^2(m_1^2-m_1(v)^2) \end{aligned} \tag{4.15}$$

with a vector h of holomorphic functions. Then (4.10) immediately follows from (4.14) and (4.15). Q.E.D.

Let us now reinstate the index b of G_b and denote by f_b the corresponding f given in Lemma 4.1. Let $f_b^0(\mathbf{p}', \mathbf{q})$ denote $f_b(\mathbf{p}', \mathbf{q}, m_{l(b)}^2)$, where $L_{l(b)}$ is an internal line of G_b. Then we may take the function ϕ in Definition 3.10 to be $\prod_b f_b^0$. Using the set of numbers $c(v)$ given there, we define $c_b(v)$ by

$$c_b(v) = c(v) \prod_{b' \neq b} f_{b'}^0(\mathbf{p}'(v), \mathbf{q}(v)). \tag{4.16}$$

Then the condition (3.15γ) implies that

$$c_b(v) f_b^0(\mathbf{p}'(v), \mathbf{q}(v)) \to 0.$$

Thus the condition (4.8) of Lemma 4.2 is satisfied for the pair $(c_b(v), f_b^0(\mathbf{p}'(v), \mathbf{q}(v)))$. Then (4.10) guarantees that we may replace $\mathrm{grad}_{(\mathbf{p}', \mathbf{q})} f_b^0(\mathbf{p}'(v), \mathbf{q}(v))$ by $\mathrm{grad}_{(\mathbf{p}', \mathbf{q})} f_b(\mathbf{p}'(v), \mathbf{q}(v),$
$m_{l(b)}(v)^2)$ in (3.15δ) and (3.15ε) without changing the limiting point $(\mathbf{u}', \mathbf{w})$. Here we have used the fact that

$$c(v)\,\mathrm{grad} \prod_b f_b^0(\mathbf{p}'(v), \mathbf{q}(v)) = \sum_b c_b(v)\,\mathrm{grad} f_b^0(\mathbf{p}'(v), \mathbf{q}(v)).$$

Now let us denote by $p_r^b(v)$, $k_l^b(v)$, $\alpha_l^b(v)$, $\beta_r^b(v)$, $v_j^b(v)$, and $u_r^b(v)$ the corresponding quantities which appear in the Landau equations associated with $G_b(m_{l(b)}(v)^2)$. We expand $u_r^b(v)$ to a $4(n+i_0)$-vector by setting the components irrelevant to G_b to zero. For the i-th explicit internal line of B, there exists a unique $b^+(i)$ [respectively, $b^-(i)$] such that $[b^+(i):i] = +1$ (respectively, $[b^-(i):i] = -1$). Denote by $j^+(i)$ [respectively, $j^-(i)$] the unique vertex of $G_{b^+(i)}$ (respectively, $G_{b^-(i)}$) that L_i terminates upon (respectively, starts from). Then it follows from the definition of f_b^0 and the normalization (4.7) that (see [20, Theorem 6])

$$(0, \mathrm{grad}_{\mathbf{q}_i} f_{b^+(i)}^0(\mathbf{p}', \mathbf{q})) = v_{j^+(i)} + \beta_{i,+}(v) q_i(v)) \tag{4.16}$$

and

$$(0, \mathrm{grad}_{\mathbf{q}_i} f_{b^-(i)}^0(\mathbf{p}', \mathbf{q})) = (v_{j^-(i)} + \beta_{i,-}(v) q_i(v)) \tag{4.17}$$

hold with some constants $\beta_{i,+}(v)$ and $\beta_{i,-}(v)$. Here $(0, \mathrm{grad}_{\mathbf{q}_i} f_{b^+(i)}^0(\mathbf{p}', \mathbf{q}))$ denotes the four-vector

$$\left(0, \frac{\partial f_{b^\pm(i)}^0}{\partial q_{i,1}}, \frac{\partial f_{b^\pm(i)}^0}{\partial q_{i,2}}, \frac{\partial f_{b^\pm(i)}^0}{\partial q_{i,3}}\right), \quad \text{and} \quad q_i(v) = (\sqrt{\mathbf{q}_i(v)^2 + \mu_i'^2}, \mathbf{q}_i(v)).$$

Since

$$\begin{aligned}\sum_b c_b(v)\,\mathrm{grad}_{\mathbf{q}_i} f_b^0(\mathbf{p}'(v), \mathbf{q}(v)) = {} & c_{b^+(i)}(v)\,\mathrm{grad}_{\mathbf{q}_i} f_{b^+(i)}^0(\mathbf{p}'(v), \mathbf{q}(v)) \\ & + c_{b^-(i)}(v)\,\mathrm{grad}_{\mathbf{q}_i} f_{b^-(i)}^0(\mathbf{p}'(v), \mathbf{q}(v))\end{aligned}$$

holds, (4.16) and (4.17) entail

$$\left(0, \sum_b c_b(v)\,\mathrm{grad}_{\mathbf{q}_i} f_b^0(\mathbf{p}'(v), \mathbf{q}(v)\right) = (c_{b^+(i)}(v))(v_{j^+(i)}(v) + \beta_{i,+}(v)q_i(v)) \\ - (c_{b^-(i)}(v))(v_{j^-(i)}(v) + \beta_{i,-}(v)q_i(v)). \quad (4.18)$$

We now note that

$$(c_{b^+(i)}(v)\beta_{i,+}(v) - c_{b^-(i)}(v)\beta_{i,-}(v))(q_i(v)^2 - \mu_i'^2) = 0 \quad (4.19)$$

holds, because $q_i(v)^2 - \mu_i'^2 = 0$ holds by the definition of $q_i(v)$. Note also that the left-hand side of (4.18) tends to zero by the definition of $\Lambda(B)$. Thus we assign $(q_i(v), c_{b^+(i)}(v)\beta_{i,+}(v) - c_{b^-(i)}(v)\beta_{i,-}(v))$, $(k_l(v), c_b(v)\alpha_l(v))$, $(c_b(v)v_j(v))$, and $(c_b(v)u_r^b(v))$, respectively, to the i-th explicit internal line of B, the internal line L_l, the vertex V_j of G_b, and the external line L_r^e of G_b that is not an external line of B, and obtain, by virtue of (4.9), (4.18), and (4.19), a sequence needed to define $\tilde{\mathscr{L}}(\otimes G_b^0)$ [13, p. 114, (1.50) and p. 115 (1.50h.1) and (1.50h.2)]. This proves that $\Lambda(B) - \bigcup_{\mathscr{G}(B)} \mathscr{L}\left(\bigotimes_B G_b\right)$ is contained in $\tilde{\mathscr{L}}\left(\bigotimes_B G_b\right)$ on the assumption ($\mathscr{U}$). This is what we wanted to prove.

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