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Pure States as a Dual Object for C^*-Algebras

Frederic W. Shultz*
Wellesley College, Wellesley, MA 02181, USA

Abstract. We consider the set of pure states of a C^*-algebra as a uniform space equipped with transition probabilities and orientation, and show that the pure states with this structure determine the C^*-algebra up to *-isomorphism.

Introduction

For commutative unital C^*-algebras, it is well known that the set of pure states (as a topological space) determines the algebra. In fact, any two such algebras are isomorphic to $C(X)$ and $C(Y)$ for compact Hausdorff spaces X and Y. The pure states of $C(X)$ are just evaluation at each x in X, and every homeomorphism of Y onto X is induced by a *-isomorphism of $C(X)$ onto $C(Y)$. The Stone–Weierstrass theorem is a special case of this.

For general C^*-algebras it is clear that this result fails, e.g. not every homeomorphism of the pure states $P(B)$ onto $P(A)$ is induced by a *-isomorphism; $P(A)$ (as a topological space) does not determine A. The purpose of this paper is to show that $P(A)$ does determine A if given a suitable structure.

The roots of our investigation go back to the work of Kadison [12–14] and Wigner [19]. Kadison studied the representation of a C^*-algebra as continuous functions on $P(A)$ (or $P(A)^-$). He showed [13] that a homeomorphism of $P(B)^-$ onto $P(A)^-$ which carries A onto B is induced by a Jordan isomorphism. Wigner focused on transition probabilities between pure states. He showed that a bijection of the pure normal states of $B(H_2)$ onto those of $B(H_1)$ which preserves transition probabilities is induced by a Jordan isomorphism (in this case, a *-isomorphism or *-anti-isomorphism) of $B(H_1)$ onto $B(H_2)$. There have also been investigations of Stone–Weierstrass theorems for C^*-algebras, e.g. Kaplansky [16], Glimm [11], Sakai [17]; Akemann [1, 2], Giles and Kummer [9], and Effros [8].

The recent work from which this paper springs is joint work with Alfsen and Hanche-Olsen [5], in which the notion of orientation of a state space was introduced. It was shown there that an affine homeomorphism of state spaces is induced by a *-isomorphism iff the map preserves orientation.

Our work combines the structures of topology (or uniformity), transition

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probability and orientation. We show that a bijective map from $P(B) \cup \{0\}$ onto $P(A) \cup \{0\}$ taking 0 to 0 is induced by a *-isomorphism of A onto B iff the map (and its inverse) are uniformly continuous, preserve transition probabilities, and preserve orientation. We give an example to show that uniform continuity cannot be replaced by continuity.

Along the way we characterize (for separable C^*-algebras) those algebras for which the w^*-closure of the primary states contains only multiples of primary states. This corresponds to a result of Glimm [10] on pure states.

We remark finally that the results described above can be interpreted in terms of a Riemannian structure on the pure states. Recall that each unitary equivalence class of pure states can be identified in a natural way with complex projective space, at least if the dimension of the corresponding GNS representation is finite. Thus each such class can be made into a complex Riemannian manifold via the structure carried over from projective space. The condition that a bijective map of $P(B)$ onto $P(A)$ preserve transition probabilities and orientation (as used herein) is equivalent to the requirement that the map preserves equivalence and is a Riemannian isomorphism (i.e., biholomorphic isometry) on each equivalence class.

The Main Theorem

Throughout this paper A will be a C^*-algebra with state space K, $P(A)$ will denote the set of pure states, and $P(A)^-$ will denote the w^*-closure of $P(A)$ in the dual space A^*. We recall that the bidual A^{**} can be identified with the enveloping von Neumann algebra of A, and K can be identified with the normal state space of A^{**}.

Let A and B be C^*-algebras, and let $X \subseteq P(A)$ and $Y \subseteq P(B)$ be arbitrary subsets. A map $\Psi: Y \to X$ is said to be induced by a continuous linear map $\Phi: A \to B$ if the dual map Φ^* restricted to Y agrees with Ψ. Our goal in this paper is to characterize the maps from $P(B)$ to $P(A)$ which are induced by *-isomorphisms from A onto B.

A key requirement will be that Ψ must preserve transition probabilities. Recall that if x and y are unit vectors in a Hilbert space, the transition probability between the vector states ω_x and ω_y on $B(H)$ is defined to be $(\omega_x|\omega_y) = |(x, y)|^2$. More generally, if $\pi: A \to B(H)$ is an irreducible representation, then the transition probability between the pure states $\omega_x \circ \pi$ and $\omega_y \circ \pi$ is again defined to be $|(x, y)|^2$. If σ and τ are arbitrary pure states on A, let u_σ and u_τ be their support projections in A^{**}; we then define $(\sigma|\tau) = \langle u_\sigma, \tau \rangle = \langle u_\tau, \sigma \rangle$. Note that this agrees with the definition above if σ and τ are (unitarily) equivalent, and gives $(\sigma|\tau) = 0$ otherwise.

We can also define this notion in purely geometric terms. If σ and τ are equivalent, then the face of K they generate is a 3-dimensional ball, cf. [5]. (If they are inequivalent, the ball degenerates to the line segment $[\sigma, \tau]$.) Now u_σ restricted to this ball is the unique positive affine function with value 1 and σ and zero at the antipodal pure state; $(\sigma|\tau)$ is the value of this functional at τ. Note that every *-isomorphism (or *-anti-isomorphism) of C^*-algebras induces an affine isomorphism of their state spaces, which then preserves transition probabilities for pure states. The following result of Wigner [19] is a partial converse.

Theorem 1 (Wigner). *A bijective map from the vector states on $B(H_2)$ onto those of $B(H_1)$ which preserves transition probabilities extends to a unique affine isomorphism of the normal state spaces and is induced by a unique *-isomorphism or *-anti-isomorphism from $B(H_1)$ onto $B(H_2)$.*

Proof. Let Ψ be such a map of vector states. Extend Ψ to all normal states on $B(H_2)$ by defining $\Psi(\Sigma\lambda_i\sigma_i)=\Sigma\lambda_i\Psi(\sigma_i)$ for σ_i vector states and $0\leqq\lambda_i, \Sigma\lambda_i=1$. (Recall that every normal state on $B(H_2)$ is such a σ-convex combination.) To see that Ψ is well defined, suppose two σ-convex combinations of vector states on $B(H_2)$ agree, say $\Sigma\lambda_i\sigma_i=\Sigma\gamma_j\tau_j$. Then for each vector state σ on $B(H_2)$, using the fact that Ψ preserves transition probabilities gives

$$\begin{aligned}&\langle u_{\Psi(\sigma)}, \Sigma\lambda_i\Psi(\sigma_i)-\Sigma\gamma_j\Psi(\tau_j)\rangle\\ &=\Sigma\lambda_i(\Psi(\sigma)|\Psi(\sigma_i))-\Sigma\gamma_j(\Psi(\sigma)|\Psi(\tau_j))\\ &=\Sigma\lambda_i(\sigma|\sigma_i)-\Sigma\gamma_j(\sigma|\tau_j)\\ &=\langle u_\sigma, \Sigma\lambda_i\sigma_i-\Sigma\gamma_j\tau_j\rangle=0.\end{aligned}$$

As σ varies over the vector stages on $B(H_2)$, $u_{\Psi(\sigma)}$ varies through all minimal projections in $B(H_2)$, and thus $\Sigma\lambda_i\Psi(\sigma_i)=\Sigma\gamma_j\Psi(\tau_j)$, i.e. Ψ is well defined. It is then evident that Ψ is an affine isomorphism. We now identify $B(H_i)$ for $i=1,2$ with the space of bounded affine functions on its normal state space. We then define $\Phi: B(H_1)\to B(H_2)$ by $\langle\Phi(a),\sigma\rangle=\langle a,\Psi(\sigma)\rangle$ for $a\in B(H_1)$ and σ a normal state on $B(H_2)$. Note that Φ preserve self-adjointness, and Φ is a unital order isomorphism of the self adjoint (s.a.) part of $B(H_1)$ onto $B(H_2)_{\text{s.a.}}$ By [13, Corollary 5] Φ is a Jordan isomorphism, and by [15, Cor. 11] Φ is either a *-isomorphism or a *-anti-isomorphism and Φ induces Ψ. □

To distinguish *-isomorphisms from *-anti-isomorphisms, we will introduce a notion of orientation (based on that in [5]). Let S^2 denote the boundary of the unit ball E^3 in $\mathbb{R}^3$, equipped with transition probabilities (i.e. for $\sigma,\tau\in S^2$, $(\sigma|\tau)$ is the value at τ of the unique positive affine functional u_σ on E^3 such that u_σ is 1 at σ and zero at the antipodal point of E^3). Let A be a C^*-algebra, and σ,τ equivalent pure states. Recall that face (σ,τ) is affinely isomorphic to E^3; we will denote the set of pure states (i.e. the boundary) of face (σ,τ) by $S^2(\sigma,\tau)$. Let $\Psi_i: S^2\to S^2(\sigma,\tau)$ be a bijective map which preserves transition probabilities for $i=1,2$. Then $\Psi_2^{-1}\circ\Psi_1: S^2\to S^2$ preserves transition probabilities. As we will see below, S^2 can be identified with the set of pure states of $M_2(\mathbb{C})$, and so by Theorem 1, $\Psi_2^{-1}\circ\Psi_1$ can be extended to an affine automorphism of E^3, and then to an orthogonal transformation of $\mathbb{R}^3$. We say Ψ_1 and Ψ_2 are equivalent if this orthogonal transformation has determinant $+1$, and we refer to an equivalence class of such maps as an orientation of $S^2(\sigma,\tau)$.

We now single out a canonical orientation for $S^2(\sigma,\tau)$, still following [5]. Let $q\in A^{**}$ be the projection such that face $(\sigma,\tau)=q^{-1}(1)$; thus $S^2(\sigma,\tau)$ can be identified with the pure states of $qA^{**}q$. Let $\Phi: qA^{**}A\to M^2(\mathbb{C})$ be any *-isomorphism, and let Ψ be the affine isomorphism from E^3 onto the state space of $M_2(\mathbb{C})$ given by

$$\Psi(a,b,c)=\begin{pmatrix}\frac{1}{2}(1+a) & \frac{1}{2}(b+ic)\\ \frac{1}{2}(b-ic) & \frac{1}{2}(1-a)\end{pmatrix},$$

where we've identified the state space of $M_2(\mathbb{C})$ with the positive matrices of unit trace. Now $\Phi^* \circ \Psi$ is an affine isomorphism of E^3 onto face (σ, τ), and we define the canonical orientation of $S^2(\sigma, \tau)$ to be that given by the restriction of $\Phi^* \circ \Psi$ to S^2. (The equivalence class of $\Phi^* \circ \Psi$ does not depend on the choice of Φ.) We will refer to this collection of orientations of all 2-spheres $S^2(\sigma, \tau)$ as the canonical orientation of $P(A)$.

For $X \subseteq P(A)$ we write $X^\perp$ for $\{\sigma \in P(A) | (\sigma | \tau) = 0 \text{ for all } \tau \in X\}$. Note that for any $\sigma, \tau \in P(A)$, the set of pure states in the face generated by σ and τ is just $\{\sigma, \tau\}^{\perp\perp}$, and σ and τ are equivalent iff $\{\sigma, \tau\}^{\perp\perp}$ properly contains $\{\sigma, \tau\}$. Now let A_1 and A_2 be C^*-algebras, and Ψ: $P(A_2) \to P(A_1)$ a bijection which preserves transition probabilities. Then Ψ will preserve equivalence of pure states, and will map $S^2(\sigma, \tau)$ onto $S^2(\Psi(\sigma), \Psi(\tau))$. We say Ψ *preserves orientation* if Ψ carries the canonical orientation of $S^2(\sigma, \tau)$ onto that of $S^2(\Psi(\sigma), \Psi(\tau))$ for all pairs σ, τ of equivalent pure states. In [5] it is shown that an affine homeomorphism between state spaces of (unital) C^*-algebras is induced by a *-isomorphism iff it preserves orientation. If we ignore topology and consider maps defined only on the pure states, we have following result.

Proposition 2. *Let A_1 and A_2 be C^*-algebras. A bijective map $\Psi: P(A_2) \to P(A_1)$ is induced by a *-isomorphism of the atomic part of A_1^{**} onto the atomic part of A_1^{**} iff Ψ preserves transition probabilities and orientation.*

Proof. The atomic part of A_1^{**} is a direct sum of type I factors $c_i A_1^{**} \cong B(H_i)$. For each i, the pure states in $c_i^{-1}(1)$ are a maximal set of mutually equivalent pure states, and all such maximal sets occur in this way. It follows that Ψ carries the pure normal states of $c_i A_1^{**}$ onto those of some type I factor $d_i A_2^{**}$, a direct summand of A_2^{**}. By Theorem 1, there is a *-isomorphism or *-anti-isomorphism $\Phi_i: c_i A_1^{**} \to d_i A_2^{**}$ which induces $\Psi: d_i^{-1}(1) \to c_i^{-1}(1)$. By [5, Proposition 6.2] Φ_i is a *-isomorphism since Ψ preserves orientation. Now the direct sum $\Phi = \oplus \Phi_i$ will map the atomic part of A_1^{**} *-isomorphically onto that of A_2^{**}, and induces Ψ. The converse is clear. □

If we combine this with results of Akemann [1] and Giles and Kummer [9], we obtain one kind of structure on $P(A)$ which determines A up to isomorphism. If A is unital, define $X \subseteq P(A)$ to be q-closed if X consists of all pure states of some w^*-closed face of K.

Corollary 3. *Let A_1 and A_2 be unital C^*-algebras. A bijective map $\Psi: P(A_2) \to P(A_1)$ is induced by a *-isomorphism of A_1 onto A_2 iff Ψ preserves transition probabilities and orientation, and Ψ and Ψ^{-1} preserve q-closed sets.* □

The rest of this paper will be devoted to showing that the last condition of Corollary 3 can be replaced by the requirement that Ψ and Ψ^{-1} be uniformly continuous. We begin with a lemma relating convergence in $P(A)$ to pointwise convergence of representations. The lemma is a modification of a result of Fell's [7] relating convergence in $\hat{A}$ to convergence of representations.

Lemma 4. *Let A be a unital C^*-algebra, and let H be a Hilbert space whose dimension is greater than the cardinality of A. Let π be a cyclic representation of A on a closed subspace H_0 of H, with cyclic vector x, and define $\sigma = \omega_x \circ \pi$. Let $\{\sigma_i\}_{i \in I}$ be a net* *of pure states on A which converges (weak*) to σ. Then there exists a subnet $\{\sigma_j\}_{j\in J}$ and representations π_j on closed subspaces H_j of H containing x, such that for each $j\in J$, π_j is unitarily equivalent to the* GNS *representation associated with σ_j, and such that*

(i) $$\|\omega_x\circ\pi_j-\sigma_j\|\to 0,$$

(ii) *for each $y\in H$ and for all sufficiently large $j\in J$, H_j contains y and $\|\pi_j(b)y-\pi(b)y\|\to 0$ for all b in A.*

Proof. (Modelled after the proof in [6, Lemma 3.5.7].) Fix $\varepsilon>0$, and elements $a_1,\dots a_n$ in A of norm at most one, with $a_1=1$, and vectors $y_1,\dots y_m$ in H_0 with $y_1=x$. Choose $b_1,\dots b_m$ in A with $b_1=1$ and $\|\pi(b_k)x-y_k\|<\varepsilon/4$ for $1\leqq k\leqq m$. Let $M=\max\{\|b_k\|, 1\leqq k\leqq m\}$.

Now observe that with minor changes the proofs of [6, Lemma 3.5.7 and Proposition 3.5.9] show that for each integer $n\leqq\dim H_0$ there exists $i_0\in I$ such that $i\geqq i_0$ implies the dimension of the GNS representation for σ_i is at least n. In particular, we can choose i_0 so that $i\geqq i_0$ implies the dimension of the GNS representation for σ_i is at least as much as the dimension of the linear span of $\{\pi(a_jb_k)_x, 1\leqq j\leqq n, 1\leqq k\leqq m\}$. We may also arrange that for $i\geqq i_0$

$$|\langle b_j^*a_k^*a_lb_r,\sigma_i\rangle-\langle b_j^*a_k^*a_lb_r,\sigma\rangle|<\varepsilon_1 \tag{1}$$

for all j, k, l, r with ε_1 as specified below. Now for any fixed $i\geqq i_0$ let π_i' be a representation of A on a closed subspace H_i of H containing $\{\pi(a_jb_k)x, 1\leqq j\leqq n, 1\leqq k\leqq m$, such that x is a cyclic vector for π_i' and $\omega_x\circ\pi_i'=\sigma_i$. (Note that π_i' is unitarily equivalent to the GNS representation for σ_i [6, Prop. 2.4.1].) Now by (1), for all j, k

$$|(\pi_i'(a_jb_k)x,\pi_i'(a_jb_k)x)-(\pi(a_jb_k)x,\pi(a_jb_k)x)|<\varepsilon_1.$$

If ε_1 is suitably chosen, then by [6, Lemma 3.5.6] there is a unitary operator U on H_i such that

$$\|U\pi_i'(a_jb_k)x-\pi(a_jb_k)x\|\leqq\varepsilon/4(1+M) \tag{2}$$

for all j, k. In particular, since $x\in H_i$ then

$$\|Ux-x\|\leqq\varepsilon/4(1+M). \tag{3}$$

Now define $\pi_i\colon A\to B(H_i)$ by $\pi_i(b)=U\pi_i'(b)U^{-1}$.

Then for $b\in A$ with $\|b\|\leqq 1$

$$|(\pi_i(b)x,x)-(\pi_i'(b)x,x)|=|(\pi_i'(b)U^{-1}x,U^{-1}x)-(\pi_i'(b)x,x)|\leqq\frac{\varepsilon}{2(1+M)}.$$

Thus $\|\omega_x\circ\pi_i-\sigma_i\|\leqq\varepsilon/2(1+M)\leqq\varepsilon$. Also, from (2) and (3) we have

$$\begin{aligned}\|\pi_i(a_jb_k)x-\pi(a_jb_k)x\| &= \|U\pi'(a_jb_k)U^{-1}x-\pi(a_jb_k)x\|\\ &\leqq\|U\pi'(a_jb_k)x-\pi(a_jb_k)x\|+\|b_k\|\varepsilon/4(1+M)\\ &\leqq\varepsilon/4(1+M)+\|b_k\|\varepsilon/4(1+M)\leqq\varepsilon/4.\end{aligned} \tag{4}$$

For $j=1$ we get $\|\pi_i(b_k)x-\pi(b_k)x\|\leqq\varepsilon/4$. Now b_k was chosen to satisfy $\|\pi(b_k)x-$

$y_k\| \leqq \varepsilon/4$, and so $\|\pi_i(b_k)_x - y_k\| \leqq \varepsilon/2$. Combining this with (4) gives

$$\begin{aligned}&\|\pi_i(a_j)y_k - \pi(a_j)y_k\| \leqq \|\pi_i(a_j)y_k - \pi_i(a_jb_k)x\|\\ &\quad + \|\pi_i(a_jb_k)x - \pi(a_jb_k)x\| + \|\pi(a_jb_k)x - \pi(a_j)y_k\|\\ &\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Now if we choose a subnet $\{\pi_j\}$ indexed by members of I, finite subsets of A and H, and by ε, we can satisfy (i) and (ii). □

Lemma 5. *Let A be a C^*-algebra. Then the set B of elements $b \in A^{**}$ such that b, b^*b, and bb^* are continuous on $P(A)^-$ forms a C^*-subalgebra of A^{**}.*

Proof. We may assume A is unital. Let $\{\sigma_i\}_{i\in I}$ be a net in $P(A)$ converging to $\sigma \in P(A)^-$. Choose representations $\{\pi_j\}$, π in accordance with Lemma 4; we will use the notation of that Lemma. We will also denote by $\{\pi_j\}$, π the σ-weakly continuous extensions to A^{**}.

Note that by Lemma 4, for $a \in A^{**}$ $\langle a, \sigma_j\rangle \to \langle a, \sigma\rangle$ iff $(\pi_j(a)x, x) \to (\pi(a)x, x)$. Let C consist of those elements $c \in A^{**}$ such that

$$(\pi_j(c)y, y) \to (\pi(c)y, y) \text{ and } (\pi_j(c^*c)y, y) \to (\pi(c^*c)y, y) \quad \text{for all } y \in H_0. \tag{5}$$

Polarization of (5) gives

$$(\pi_j(c)y_1, y_2) \to (\pi(c)y_1, y_2) \text{ and } (\pi_j(c^*c)y_1, y_2) \to (\pi(c^*c)y_1, y_2) \quad \text{for all } y_1, y_2 \in H_0. \tag{6}$$

Now since

$$\|\pi_j(c)y - \pi(c)y\|^2 = (\pi_j(c^*c)y, y) - (\pi_j(c)y, \pi(c)y) - (\pi(c)y, \pi_j(c)y) + (\pi(c^*c)y, y),$$

by (6) we conclude that for $c \in C$

$$\|\pi_j(c)y - \pi(c)y\| \to 0 \quad \text{for all } y \in H_0. \tag{7}$$

Conversely, for $c \in A^{**}$ (7) implies (5). Now if a and b are in B, then (5) holds with $c = a$ or b, and so (7) holds for each. But then ab satisfies (7), so ab satisfies (5). It follows that $\langle ab, \sigma_j\rangle \to \langle ab, \sigma\rangle$. This argument applies to show that every subnet of $\{\sigma_i\}$ has a subnet on which ab converges to $\langle ab, \sigma\rangle$; it follows that $\langle ab, \sigma_i\rangle \to \langle ab, \sigma\rangle$. Thus ab is continuous on $P(A)^-$, showing that B is a subalgebra of A^{**}. It is clear that B is norm closed and closed under the $*$-operation, and so the proof is complete. □

In the rest of this paper, z will denote the central projection in A^{**} such that zA^{**} is the atomic part of A^{**}. (By definition zA^{**} is the direct sum of those direct summands of A^{**} which are type I factors.) The atomic part of the state space K of A is $z^{-1}(1)$, and can be identified with the normal state space of zA^{**}. It is the σ-convex hull of the pure states of A, cf. e.g. [4].

Proposition 6. *Let A be a C^*-algebra, and define B as in Lemma 5. Then $zA = zB$.*

Proof. We identify each state on A with its normal extension to A^{**}. Since the σ-convex hull of $P(A)$ can be identified with the normal state space of $zA^{**} \supset zB$, then the set of states $\{\sigma | zB, \sigma \in P(A)\}$ determines the ordering on $(zB)_{s.a.}$ Therefore the w^*-closure of this set of states of zB contains $P(zB)^-$, cf. [6, Lemma 3.4.1]. Now let σ and τ be functionals in $P(zB)^-$ which agree on $zA \subseteq zB$; we will show $\sigma = \tau$. Choose nets $\{\sigma_i\}_{i\in I}$ and $\{\tau_i\}_{i\in I}$ in $P(A)$ such that $\sigma_i \to \sigma$ on zB and $\tau_i \to \tau$ on zB. Let $\pi_z(a) = za$ for $a \in A$. Then $\{\sigma_i \circ \pi_z\}$ and $\{\tau_i \circ \pi_z\}$ converge on A to $\sigma \circ \pi_z = \tau \circ \pi_z$. Since each pure state of A annihilates $(1-z)A^{**}$, then $\sigma_i \circ \pi_z = \sigma_i$ and $\tau_i \circ \pi_z = \tau_i$, and so $\{\sigma_i\}$ and $\{\tau_i\}$ converge on A to the same state $\sigma \circ \pi_z = \tau \circ \pi_z$. By the definition of B, for each $b \in B$ the nets $\{\langle b, \sigma_i\rangle\}$ and $\{\langle b, \tau_i\rangle\}$ have the same limit, and thus so do $\{\langle zb, \sigma_i\rangle\}$ and $\{\langle zb, \tau_i\rangle\}$. By our choice of $\{\sigma_i\}$ and $\{\tau_i\}$, we conclude $\sigma = \tau$, and thus zA separates $P(zB)^-$. By Glimm's version of the Stone–Weierstrass theorem [11], $zA = zB$. □

If A is any C^*-algebra, we denote by A_u the set of elements $a \in zA^{**}$ such that a, a^*a, and aa^* are uniformly continuous on $P(A) \cup \{0\}$. We say A is weakly perfect if $zA = A_u$. For future use, we observe that if A does not have an identity, A is weakly perfect iff A with an identity adjoined is weakly perfect.

Lemma 7. *Let A be a C^*-algebra. Let $\{\sigma_i\}$ be a net of pure states converging to σ, a multiple of an atomic state. Then for each $a \in A_u$, $\langle a, \sigma_i\rangle \to \langle a, \sigma\rangle$.*

Proof. We may assume A is unital and so $\|\sigma\| = 1$. Let $\{\pi_j\}$, π be as in Lemma 4. We claim there exists a sequence $\{x_k\}$ in H_0 such that $x = \Sigma x_k$ and such that $\omega_{x_k} \circ \pi$ is a multiple of a pure state for each k. For suppose $\sigma = \Sigma\lambda_k\tau_k$ with $0 \leqq \lambda_k$, $\Sigma\lambda_k = 1$, $\tau_k \in P(A)$, and $(\tau_j | \tau_k) = \delta_{jk}$. Let (π_k, H_k, y_k) be the GNS representation associated with τ_k for each k, and let $\Psi = \oplus \pi_k : A \to B(\oplus H_k)$. Note that for all $b \in A$

$$(\Psi(b)(\Sigma\sqrt{\lambda_k}y_k), \Sigma\sqrt{\lambda_k}y_k) = \Sigma\lambda_k(\pi_k(b)y_k, y_k) = \langle b, \sigma\rangle.$$

Thus $(\Psi, \Psi(A)(\Sigma\sqrt{\lambda_k}y_k)^-, \Sigma\sqrt{\lambda_k}y_k)$ is unitarily equivalent to the GNS representation of A associated with σ, [6, 2.4.1]. Now let $\{x_k\} \subset H_0$ be the sequence of vectors corresponding to $\{\sqrt{\lambda_k}y_k\}$. (To see each y_j is in $\Psi(A)(\Sigma\sqrt{\lambda_k}y_k)^-$, given $\varepsilon > 0$ choose n so that $\sum_{n+1}^{\infty} \lambda_i < \varepsilon/2$, and choose $b \in A$ so that $\|b\| \leqq 2$ and $\pi_i(b)y_i = \delta_{ij}y_j$ for $i \leqq n$. Then $\|\Psi(b)(\Sigma\sqrt{\lambda_k}y_k) - \sqrt{\lambda_j}y_j\| < \varepsilon$). Now for each k and each $a \in A_u$, $\pi(a)x_k$ lies in H_0, and so for all $b \in A$, all $\lambda \in \mathbb{C}$, as $j \to \infty$:

$$(\pi_j(b)(x_k + \lambda\pi(a)x_k), x_k + \lambda\pi(a)x_k) \to (\pi(b)(x_k + \lambda\pi(a)x_k), x_k + \lambda\pi(a)x_k). \tag{8}$$

Each representation π_j is irreducible, so the map sending b to the left side of (8) is a multiple of a pure state. Since $\omega_{x_k} \circ \pi$ is a multiple of a pure state for each k, that $\pi(A)$ is irreducible on $(\pi(A)x_k)^-$, which includes $x_k + \lambda\pi(a)x_k$, so the map sending b to the right side of (8) is also a multiple of a pure state. Since A is unital, these functionals when normalized also converge (evaluate at the identity). Thus by the definition of A_u, (8) holds with $a \in A_u$ replacing b. If the resulting equation for suitable values of λ is polarized, the result is

$$(\pi_j(a)x_k, x_k) \to (\pi(a)x_k, x_k), \tag{9}$$

and

$$(\pi_j(a)x_k, \pi(a)x_k) \to (\pi(a)x_k, \pi(a)x_k). \tag{10}$$

Applying (9) and (10), and (9) with a^*a replacing a gives

$$\|(\pi_j(a)-\pi(a))x_k\|^2 = (\pi_j(a^*a)x_k, x_k) - (\pi_j(a)x_k, \pi(a)x_k) - (\pi(a)x_k, \pi_j(a)x_k) + (\pi(a^*a)x_k, x_k)$$

which approaches zero as $j \to \infty$.

Now given $\varepsilon > 0$ choose n so that $\|x - \sum_1^n x_k\| < \varepsilon/4$. Now given $a \in A$ choose j_0 so that $j \geqq j_0$ implies $\|\pi_j(a)\sum_1^n x_k - \pi(a)\sum_1^n x_k\| < \varepsilon/2$. Then for $\|a\| \leqq 1$

$$\|\pi_j(a)x - \pi(a)x\| \leqq \left\|\pi_j(a)\sum_1^n x_i - \pi(a)\sum_1^n x_i\right\| + \varepsilon/2 \leqq \varepsilon.$$

(Note we cannot use $\pi_j(a)\sum_1^\infty x_k = \sum_1^\infty \pi_j(a)x_k$ because for fixed j not all x_k may lie in H_j.) Now $(\pi_j(a)x, x) \to (\pi(a)x, x) = \langle a, \sigma\rangle$ and $\|\omega_x \circ \pi_j - \sigma_j\| \to 0$, so $\langle a, \sigma_j\rangle \to \langle a, \sigma\rangle$. Thus every subnet of $\{\sigma_i\}$ has a subnet on which a converges to $\langle a, \sigma\rangle$; it follows that $\langle a, \sigma_i\rangle \to \langle a, \sigma\rangle$. □

Lemma 8. *Let A be a C^*-algebra. If every element of $P(A)^-$ is a multiple of an atomic state, then A is weakly perfect.*

Proof. We may assume A is unital. Then under the hypotheses $P(A)^-$ consists entirely of atomic states. Now by lemma 7, for each $a \in A_u$ the elements a, a^*a, aa^* are continuous on $P(A)^-$, and so by Proposition 6, $A_u \subseteq zA$. Clearly $zA \subseteq A_u$, so $zA = A_u$. □

We now digress momentarily to discuss the geometry of primary states. Recall that a *split face* of a convex set K is a direct summand, i.e., one of a pair F, F' of faces of K such that every element σ of K can be written uniquely as a convex combination $\sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2$ with $\sigma_1 \in F$ and $\sigma_2 \in F'$. If σ is a state on the C^*-algebra A, and if π_σ is the corresponding GNS representation, then the σ-weakly continuous extension of π_σ to A^{**} maps $c_\sigma A^{**}$ $*$-isomorphically onto $\pi_\sigma(A)''$, where c_σ is the central support of σ in A^{**}. For any von Neumann algebra the map $c \to c^{-1}(1)$ gives a 1–1 correspondence of central projections and split faces of the normal state space. Thus $F(\sigma) = c_\sigma^{-1}(1)$ is a split face of K which can be identified with the normal state space $c_\sigma A^{**}$, and $F(\sigma)$ will be the smallest split face containing σ. Clearly $\pi_\sigma(A)'' \cong c_\sigma A^{**}$ is a factor iff $F(\sigma)$ contains no proper split faces. Thus *σ is primary iff the split face $F(\sigma)$ generated by σ is a minimal split face of K.*

The following result gives a criterion for the closure of primary states to consist of multiples of primary states. (A corresponding result for pure states was gives by Glimm [10]).

Proposition 9. *Let A be a C^*-algebra which is either separable or type I. Then these are equivalent:*

(i) *every limit point of primary states is a multiple of a primary state,*
(ii) *$\hat{A}$ is hausdorff.*

Proof. Assume (i) holds. Let $\sim$ be the relation of unitary equivalence on $P(A)$; to show $\hat{A}$ is Hausdorff it suffices to show $\sim$ is closed. Suppose $\sigma_i \to \sigma$ and $\tau_i \to \tau$ in $P(A)$, with $\sigma_i \sim \tau_i$ for all i. Then $(1/2)(\sigma_i + \tau_i)$ is a net of primary states converging to $(1/2)(\sigma + \tau)$, so the latter must be primary. Therefore $\sigma \sim \tau$, which establishes (ii).

Now assume (ii). Note that by [6, 9.5.3] A is CCR. Let $\{\sigma_i\}$ be a net of primary states on A converging to σ, and let $F(\sigma)$ denote the smallest split face of K containing σ_i. Similarly let $F(\sigma)$ be the smallest split face of $K_0 = \mathrm{co}(K \cup \{0\})$ containing σ. Let $F = F(\sigma)^- \subseteq K_0$. To establish (i) we will show $F \cap K = F(\tau)$ for some $\tau \in P(A)$.

Fix τ and τ' in $P(A) \cap F$; we claim τ and τ' are equivalent. To verify this, note that $F = F(\sigma)^- \subseteq \overline{\mathrm{co}} \cup \{F(\sigma_i), i \geqq i_0\}$ for each index i_0, since the closed convex hull of split faces of K is again a split face, e.g., cf. [3]. Since A is of type I, each $F(\sigma_i)$ is affinely isomorphic to the normal state space of a type I factor, and so is the σ-convex hull of its extreme points. It follows from Milman's theorem that τ and τ' are limits of nets $\{\tau_j\}$ and $\{\tau'_j\}$ with τ_j and τ'_j in $F(\sigma_j) \cap P(A)$ for each j and $\{\sigma_j\}_{j \in J}$ a subnet of $\{\sigma_i\}$. Since $\hat{A}$ is Hausdorff then the relation $\sim$ is closed, and so $\tau_j \sim \tau'_j$ for all j implies $\tau \sim \tau'$. Thus all pure states in F are equivalent.

We can identify $F \cap K$ with the state space of $A|J$, where J is the annihilator of F in A. (Note F is w^*-compact, so F is the annihilator of J in K_0.) Since all pure states of $A|J$ have been shown to be equivalent, then all irreducible representations of $A|J$ are equivalent, and so $A|J$ is simple. Since A is CCR, so is $A|J$, so $A|J =$ compact operators on some Hilbert space H. Now $F \cap K$ can be identified with the state space of $A|J$, and thus with the normal state space of $B(H)$. It follows that $F \cap K$ has no proper split faces, which shows that every member of F is a multiple of a primary state. □

Corollary 10. *Let A be a C^*-algebra. If A is* CCR *and $\hat{A}$ is Hausdorff, then A is weakly perfect.*

Proof. By Proposition 9, every element in $P(A)^-$ is a multiple of a primary state. Since A is CCR, every primary state is atomic. The corollary now follows from Lemma 8. □

For the next lemma, we say that $X \subseteq P(A)$ is saturated if any pure state equivalent to a member of X is in X.

Lemma 11. *Let A be a C^*-algebra, and X a saturated subset of $P(A)$. If $\sigma \in X^-$, then the w^*-closed face of $K_0 = \mathrm{co}(K \cup \{0\})$ generated by σ is contained in X^-.*

Proof. We may assume A is unital. Let $\{\sigma_i\}_{i \in I}$ be a net in X converging to σ, and let $\tau \in \mathrm{face}(\sigma)$. Choose a subnet $\{\sigma_j\}_{j \in J}$ and representations $\{\pi_j\}$, π as in Lemma 4. Since $\tau \in \mathrm{face}(\sigma)$, then τ is dominated by a multiple of σ. By [6, 2.5.1] there exists $c \in \pi(A)'$ such that $\tau = \omega_{cx} \circ \pi$, where $\sigma = \omega_x \circ \pi$. By Lemma 4, $\omega_{cx} \circ \pi_j$ converges to $\tau = \omega_{cx} \circ \pi$. Evaluating at the identity, the states $\tau_j = \| \omega_{cx} \circ \pi_j \|^{-1} \omega_{cx} \circ \pi_j$ converge to τ. Furthermore, each τ_j is a pure state equivalent to σ_j, and so is in X. Thus face $(\sigma) \subseteq X^-$.

Now let Y be a subset of X^- maximal among the convex subsets of X^- containing σ. Since face $(Y) = \cup \{\mathrm{face}\ \omega | \omega \in Y\}$, face (Y) is contained in X^- by the first paragraph. By maximality $Y =$ face (Y) is a closed face of K containing σ and contained in X^-. □

Lemma 12. *Let A be a C*-algebra, and let F be a closed face of $K_0 = \mathrm{co}(K \cup \{0\})$ contained in $P(A)^-$. Then for each $a \in A_u$ the unique continuous function a' on $P(A)^-$ which agrees with a on $P(A)$ is affine on F.*

Proof. We may assume A is unital. By Lemma 7, a' agrees with a on the atomic states in $P(A)^-$, and thus in particular on the convex hull of the pure states in F; thus a' is affine on $\mathrm{co}(\partial_e F)$. By the Krein–Milman theorem, this set is dense in F, and so by continuity a' is affine on F. □

Lemma 13. *Let A be a C*-algebra, and J a norm closed ideal of A such that J is weakly perfect. If $a \in A_u$ annihilates every pure state that annihilates J, then $a \in zJ$.*

Proof. Let F be the split face of K_0, which is the annihilator of J in K_0. Note that the state space of J can be identified with the split face $F' = \{\sigma \in K \mid \|\sigma | J\| = \|\sigma\| = 1\}$. Let a' denote the continuous extension to $P(A)^-$ of $a|P(A)$, and extend a' to $[0,1] \times P(A)^-$ by $\langle a', \lambda\sigma \rangle = \lambda\langle a', \sigma \rangle$. We will show a is uniformly continuous on $P(J) \cup \{0\}$ by showing that a' is continuous on $P(J)^-$ (for the uniformity and topology induced by J).

Let $\{\sigma_i\}_{i \in I}$ be a net in $P(J)$ which converges on J to $\lambda\tau$, where τ is a state on J and $0 \leqq \lambda \leqq 1$. By compactness of K_0 there is a subnet $\{\sigma_j\}_{j \in J}$ which converges on A to $\sigma \in K_0$, where $\sigma = \lambda\tau + (1-\lambda)\omega$, with $\omega \in F$. By construction, $\sigma \in (P(A) \cap F')^-$, *and so by Lemma* 11, τ *and* ω *are in* $(P(A) \cap F')^-$.

Let G be the closed face of K_0 generated by ω. Then $G \subseteq F$, and so $\partial_e G \subseteq \partial_e F$. Now by Lemmas 11 and 12, a' is continuous and affine on G. By assumption $a' = a = 0$ on $F \cap P(A)$, and so by Krein–Milman $a' = 0$ on G.

Now by continuity of a' on $P(A)^-$

$$\langle a', \sigma_j \rangle \to \langle a', \lambda\tau + (1-\lambda)\omega \rangle = \lambda\langle a', \tau \rangle,$$

where we've used Lemma 12 to know that a' is affine on face $(\sigma) \supseteq \{\tau, \omega\}$. Thus as σ_i converges on J to $\lambda\tau$, $\langle a', \sigma_j \rangle$ converges to $\langle a', \lambda\tau \rangle$. Thus every subnet of $\{\sigma_i\}$ has a subnet on which a' converges to $\langle a', \lambda\tau \rangle$; it follows that $\langle a', \sigma_i \rangle$ converges to $\langle a', \lambda\tau \rangle$. Thus a' is continuous on $P(J)^-$, and so a is uniformly continuous on $P(J) \cup \{0\}$. Since J is weakly perfect, there exists $j \in J$ such that a and j agree on $P(J)$. Both a and j are zero on the remaining pure states, and so $a = j$ on $P(A)$. Since $a \in zA^{**}$, *then* $a = zj$. □

Lemma 14. *Let A be a C*-algebra and J a norm closed ideal of A. If J and A/J are perfect, then A is weakly perfect.*

Proof. We may assume A is unital. Let $a \in A_u$; we'll show $a \in zA$. By Lemma 13, it suffices to reduce to the case where a is zero on the annihilator F of J in K. We can identify $P(A|J)$ with $P(A) \cap F$. Since $P(A|J)$ is closed (in the relative topology) in $P(A)$, then a is uniformly continuous on $P(A|J)$, as are a^*a and aa^*. Since $A|J$ is weakly perfect, a agrees on $P(A|J)$ with an element of $A|J$. Thus there exists $b \in A$ such that a and b agree on $P(A) \cap F$. Now $a - zb$ is in A_u, is zero on $P(A) \cap F$, and so by Lemma 13 is in zA. Thus $A_u \subseteq zA$. □

Corollary 15. *If a C*-algebra A is* GCR, *then A is weakly perfect.*

Proof. Let $\{J_i\}$ be a composition series for A such that the quotients J_{i+1}/J_i are

CCR with Housdorff spectrum, cf. [6, Theorem 4.5.5]. We proceed by transfinite induction. Assume J_i is weakly perfect. By Corollary 10, J_{i+1}/J_i is weakly perfect, and so by Lemma 14, J_{i+1} is weakly perfect.

Now let β be a limit ordinal such that J_i is weakly perfect for $i<\beta$. For simplicity of notation we assume $A=J_\beta$. Note that each algebra zJ_i for $i<\beta$ is an ideal in A_u. (If $a\in zJ_i$ and $b\in A_u$, then $ab\in A_u$ and ab is zero on those pure states annihilating J_i. By Lemma 13, ab agrees on $P(A)$ with an element of J_i, and thus $ab\in zJ_i$).

Thus $zA=(\cup zJ_i)^-$ is an ideal in A_u. If zA were properly contained in A_u, then there would exist a pure state σ of A_u which annihilates zA. Since $P(A)$ determines the order on $A_u\subseteq zA^{**}$, there would exist a net $\{\sigma_i\}\subseteq P(A)$ converging on A_u to σ. By hypothesis σ annihilates A, so $\{\sigma_i\}$ converges to zero on A. By the definition of A_u, this implies that each element of A_u converges to zero on $\{\sigma_i\}$, and thus $\sigma=0$ on A_u, a contradiction. Thus $zA=A_u$. □

Lemma 16. *Let A be a C^*-algebra with state space K. Let $\{F_i\}_{i\in I}$ be a collection of split faces of K, and assume for each $i\in I$ there is given an affine function a_i on F_i such that $\|a_i\|\leqq 1$ and $a_i=a_j$ on $F_i\cap F_j$. Then there exists $a\in A^{**}$ such that $\|a\|\leqq 1$ and $a=a_i$ on F_i for all $i\in I$.*

Proof. We may assume all a_i are positive. For each finite subset X of I let $F_X=\mathrm{co}\,\{F_i, i\in X\}$ and let a_X be the unique affine function on K which agrees with a_i on each F_i for $i\in X$, and is zero on the split face complementary to F_X. Note that if $X\subseteq Y$, then $a_X\leqq a_Y$ and a_Y restricted to F_X agrees with a_X. Furthermore, $\|a_X\|\leqq 1$ for all X. Thus $\{a_X\}_{X\subseteq I}$ is an increasing net bounded above, and so converges σ-weakly to its least upper bound. This l.u.b. is the desired element a. □

Theorem 17. *Every C^*-algebra A is weakly perfect.*

Proof. We may assume A is unital. By [6, Proposition 4.3.3] there is an ideal J of A such that J is GCR and $A|J$ is NGCR. By Corollary 15 and Lemma 14, it suffices to show $A|J$ is weakly perfect; thus without loss we may assume A is NGCR. By [6, Lemma 11.2.3], if $\sigma\in P(A)^-$ and π_σ is the associated GNS representation, then the annihilator $(\ker\pi_\sigma)^\perp$ of $\ker\pi_\sigma$ in K is contained in $P(A)^-$. The annihilator of $\ker\pi_\sigma$ is a split face of K, and so $P(A)^-$ is a union of split faces.

Let $a\in A_u$, and let a' be the continuous extension to $P(A)^-$ of $a\,|P(A)$. By Lemma 12, a' is affine on each split face $(\ker\pi_\sigma)^\perp$ for $\sigma\in P(A)^-$. Let a'' be a bounded affine extension of a' to all of K; such an extension exists by Lemma 16. Then a'' is continuous on $P(A)'$, as is $(a'')^*a''$ and $(a'')(a'')^*$, so by Proposition 6, $za''\in zA$. It follows that a agrees with an element of zA on $P(A)$, and thus $a\in zA$. □

Theorem 18. *Let A and B be C^*-algebras and $\Psi: P(B)\cup\{0\}\to P(A)\cup\{0\}$ a bijection with $\Psi(0)=0$. Then Ψ is induced by a $*$-isomorphism of A onto B iff Ψ and Ψ^{-1} are uniformly continuous and Ψ preserves orientation and transition probabilities.*

Proof. This follows at once from Proposition 2 and Theorem 17. □

A Counterexample. It can be shown that uniform continuity in Theorem 18 can be replaced by ordinary continuity if A and B are CCR with Hausdorff spectrum. However, this does not hold for all C^*-algebras, even those of type I.

To see this, let A be the sum of the algebra of compact operators on $H=L^2[0,1]$

with the algebra of multiplication operators for continuous functions on $[0, 1]$. The atomic part of A^{**} can be identified with $B(H) \oplus \ell^\infty [0, 1]$. Let $q \in B(H)$ be the projection on the closed linear span of $\{\sin 2^k \pi x, \cos 2^k \pi x, k = 1, 2, \dots\}$. Then q is continuous on $P(A)$, but is not uniformly continuous. The map $b \to (2q-1)b(2q-1)$ is a *-automorphism of the atomic part of A^{**} (it fixes $\ell^\infty [0, 1]$), and thus induces a bijective map of $P(A)$ onto $P(A)$ which preserves orientation, preserves transition probabilities, is a homeomorphism, but is not induced by a *-automorphism of A.

A Final Remark: Connections with the Stone-Weierstrass Conjecture

Let A be a C^*-algebra, and let A_c denote the set of elements b in the atomic part zA^{**} of A^{**} such that b, b^*b, and bb^* are continuous on $P(A) \cup \{0\}$. The proof of Lemma 5 applies without change to show that A_c is a C^*-subalgebra of zA^{**}. The map $a \to az$ imbeds A into A_c; let us say A is *perfect* if $zA = A_c$.

Perfect algebras are of interest because of their connection with the Stone–Weierstrass conjecture for C^*-algebras. Let us say that A has the Stone–Weierstrass property if whenever A is a C^*-subalgebra of B and separates $P(B) \cup \{0\}$, then $A = B$. If A is a C^*-subalgebra of B and separates $P(B) \cup \{0\}$, then the restriction map is a homeomorphism of $P(B) \cup \{0\}$ onto $P(A) \cup \{0\}$, and so there is a natural imbedding of B into the set $C(P(A) \cup \{0\})$ of continuous functions on $P(A) \cup \{0\}$. The image of B will contain that of A (or zA), and will be contained in the image of A_c in $C(P(A) \cup \{0\})$. Thus, if A is perfect then $A = B$ will follow, i.e., perfect C^*-algebras have the Stone–Weierstrass property.

The counterexample described previously shows that not all type I C^*-algebras are perfect. However, every C^*-algebra A can be imbedded in a perfect C^*-algebra (A_c), and if A is simple, then so is A_c. If A is perfect and q is a projection in A, then qAq is perfect. In the counterexample described above, it can be seen that qA_cq is isomorphic to $B(qH)$. Thus the C^*-algebra of all bounded operators on a Hilbert space is an example of a non-nuclear C^*-algebra which is perfect and thus has the Stone–Weierstrass property.

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