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Renormalization Group Study of a Critical Lattice Model

I. Convergence to the Line of Fixed Points

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Abstract. We start a nonperturbative study of the Wilson-Kadanoff renormalization group (RG) in weakly coupled massless lattice models. Nonlocal hierarchical models are introduced to mimic the infrared behaviour of the $\frac{1}{2}(\nabla\phi)^2+\lambda(\nabla\phi)^4$ model and the like. The RG is shown to drive these to the line of fixed points corresponding to the massless $\frac{1}{2}c_\infty(\lambda)(\nabla\phi)^2$ models.

1. Introduction

The present paper is a (self-contained) continuation of the program started by [11]. We aim at a rigorous theory of weakly coupled massless lattice models, a counterpart of the high and low temperature cluster expansions developed for the massive case. Our approach parallels other recent attempts of rigorously studying massless models like $\lambda(\nabla\phi)^4$, the dipole gas, the low temperature Coulomb gas or plane rotator [3–5, 7–9]. It is centered around the idea of the renormalization group.

In [11] we have exhibited the block spin structure of the free Gaussian model $\frac{1}{2}(\nabla\phi)^2$ in $d\geqq 2$ dimensions by writing

$$\nabla\phi_x=(\nabla QZ^0)_x+3^{-\frac{d}{2}}(\nabla\mathscr{A}_1QZ^1)_{\frac{x}{3}}+\ldots+3^{-\frac{dk}{2}}(\nabla\mathscr{A}_kQZ^k)_{\frac{x}{3^k}}+\ldots, \tag{1}$$

where the kernel $(\nabla\mathscr{A}_kQ)_{zy}$, $z\in 3^{-k}\mathbb{Z}^d$, $y\in\mathbb{Z}^d$, is concentrated around $z\sim y$ and decays exponentially for $|z-y|\to\infty$ uniformly in k. The Gaussian fluctuation fields Z^k are independent for different k and their covariances possess an exponential decay uniform in k.

Our hierarchical model is patterned on this structure. Here are the main simplifications we introduce when constructing it:

1. the number of random variables Z^k_y, $y\in\mathbb{Z}^d$, is reduced to one $Z^k_{y_0}$ for each block of 3^d sites centered around $3y_0$ (in the original model there were 3^d-1 variables),

2. all Z^k fields are taken as equally distributed,

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3. the Z_y^k variables are taken to be bounded uniformly in k and y,

4. each family of kernels $(\nabla_\mu \mathscr{A}_k Q)_{zy}$, $\mu=1, \ldots, d$, is replaced by one $\mathscr{A}(z-y)$ where $\mathscr{A}$ is a function on the lattice with vanishing mean supported in the block of 3^d sites around zero.

For the sake of generality we will also replace the rescaling factor 3 giving the scale of blocks by any odd number $L \geqq 3$.

Step 3 is an essential simplification. In the $\frac{1}{2}(\nabla\phi)^2$ model large values of the fluctuation field Z_k have small probability (because the Z^k-distribution is superstable, see [11]) and in fact cutting off the Z^k integrations does not change the critical behaviour of the model. With the perturbation $\lambda(\nabla\phi)^4$ we expect this to remain true. However, the removal of this restriction is nontrivial. We shall try to do it in the future basing on the methods developed here and for the standard (unbounded) hierarchical model [2]. The other simplifications are quite natural, especially in view of the analysis of [7].

In the hierarchical model the role of the Gaussian fields $\nabla_\mu\phi$, $\mu=1, \ldots, d$, is played by one field ϕ whose two-point function $\langle\phi_x\phi_y\rangle$ satisfies $\sum_y \langle\phi_x\phi_y\rangle=0$ and decays as $|x-y|^{-d}$, thus simulating the behaviour of $\langle\nabla\phi_x\nabla\phi_y\rangle$ in the massless Gaussian model.

The main aim of the paper is to study the model perturbed by means of, say, the $\lambda\phi^4$ interaction [this corresponds to the $\lambda(\nabla\phi)^4$ perturbation in the original model]. The interacting model is studied by means of the renormalization group transformations integrating out the fluctuations Z^k in turn, starting with Z^0. There is a line of fixed points for these transformations corresponding to the interactions $\frac{1}{2}c\phi^2$ (it mimics the Gaussian line $\frac{1}{2}c(\nabla\phi)^2$). The main result proven in the present paper is the convergence of the interacting model under the renormalization group transformations to one of the fixed point models for small λ (compare somewhat related studies of discrete spin models [13, 14, 16]). This will be the basis for the study of the long distance behaviour of correlations (governed by this fixed point) in the next paper. Because of pedagogical reasons and having in mind future generalizations to the case with unbounded fluctuations, we consider first the "local" model where all Z_y^k variables are independent. In this case the block spin transformations factorize and become transformations of functions on a compact interval. Taking the fourth order derivative is all one needs to prove the convergence to a fixed point.

Next we consider the "non-local" case where Z_y^k's are weakly coupled for different y's. The high temperature cluster expansion [18] for the Z^k integration becomes the main tool in the study of the renormalization group transformation. The main estimate needed is a bound on general truncated expectations in the high temperature Z^k state. The bound contains no factorials when no groups of variables, with respect to which the truncation occurs intersect, the factorial of the number of groups if they all coincide and interpolates properly between those two extreme cases. Such estimates were studied for lattice gases in [15]. For our model we prove one which, being not the strongest possible, is sufficient for our needs. This is the most technical part of the paper. We assume that the Z^k state produces a sufficiently strong exponential decay. In the second paper to appear we shall carry over the present construction to the $\frac{1}{2}(\nabla\phi)^2+\lambda(\nabla\phi)^4$ model simplified only by cutting off big fluctuations. There the fluctuation fields exhibit exponential decay for $\lambda=0$ but it cannot be made arbitrarily strong, so we have to perform the high temperature cluster expansion on a proper scale. Also, the marginal terms will have to be treated with special care.

The paper is organized as follows. Section 2 contains the description of the model together with some results about the free two-point function to be proved in the second part of the present study. In Sect. 3 the RG transformations are introduced. That they drive the model to the line of free fixed points is shown in Sect. 4 for the local case and in Sect. 5 for the nonlocal weakly coupled one. The main result of the paper is stated in the beginning of the section. Section 6 contains the proof of the truncated expectation bound used in Sect. 5. The bound is obtained by means of a technically involved but more or less standard high temperature cluster expansion. Finally, the Appendix contains the proof of simple results about shortest trees used in the text.

2. Description of the Model

We begin with the definition of our hierarchical model. To avoid the problems connected to the thermodynamical limit which shall be studied later, we work in finite volume using periodic boundary conditions. Let L be an odd integer, $L\geqq 3$, and $N=1,2,\ldots$. Take $\Lambda_N\equiv\mathbb{Z}^d_{L^N}$ as the periodic lattice. The obvious inclusion

$$\mathbb{Z}_{L^N}\to\left]-\tfrac{1}{2}L^N,\tfrac{1}{2}L^N\right[\cap\mathbb{Z}$$

allows us to identify Λ_N as a subset of $\mathbb{Z}^d$. Algebraic operations however as well as distance functions on periodic lattices will be taken as the periodic ones. Let b^j_y, $j=1,\ldots,N$, be a lattice block of L^{jd} sites in Λ_N, centered at L^jy:

$$b^j_y=\{x\in\Lambda_N: |x^\mu-y^\mu|<\tfrac{1}{2}L^j,\ y\in\Lambda_{N-j}\}\,. \tag{1}$$

Consider a function $\mathscr{A}$ on $\mathbb{Z}^d$, supported on b^1_0, with zero mean, $\mathscr{A}(0)\neq 0$ and nonconstant in $b^1_0\backslash\{0\}$. Introduce random variables Z^k_y labeled by $y\in\Lambda_{N-k-1}$, $k=0,\ldots,N-1$. For $x\in\mathbb{R}^d$ denote by $[x]$ the point with integral components closest to the components of x (for x for which we shall use $[x]$ this will be determined unambiguously).

The basic random field ϕ of the model labeled by $x\in\Lambda_N$ is given by

$$\begin{aligned}\phi_x&=\sum_{k=0}^{N-1}\sum_{y\in\Lambda_{N-k-1}}L^{-\frac{d}{2}k}\mathscr{A}([L^{-k}x]-Ly)Z^k_y\\&=\sum_{k=0}^{N-1}L^{-\frac{d}{2}k}\mathscr{A}([L^{-k}x]-L[L^{-k-1}x])Z^k_{[L^{-k-1}x]}\,.\end{aligned} \tag{2}$$

This mimics (1.1) showing how ϕ is built from the fluctuation fields Z^k.

It is also useful to introduce block spin fields ϕ^k labeled by $x\in\Lambda_{N-k}$:

$$\phi^k_x=L^{k\frac{d}{2}}L^{-kd}\sum_{y\in b^k_x}\phi_y=\sum_{j=k}^{N-1}L^{-\frac{d}{2}(j-k)}\mathscr{A}([L^{-j+k}x]-L[L^{-j+k-1}x])Z^j_{[L^{-j+k-1}x]}\,, \tag{3}$$

where we have used

$$\sum_{x \in \mathbb{Z}^d} \mathscr{A}(x) = 0 \tag{4}$$

following the definition (2) of $\mathscr{A}$. The factor $L^{\frac{kd}{2}}$ in (3) is the rescaling factor corresponding to the canonical dimension of the gradient of a scalar field. Notice that

$$\phi_x^k = L^{-\frac{d}{2}} \phi_{[L^{-1}x]}^{k+1} + \mathscr{A}(x - L[L^{-1}x]) Z_{[L^{-1}x]}^k, \tag{5}$$

which gives the decomposition of ϕ^k into the block spin field $L^{-\frac{d}{2}} \phi_{[L^{-1}x]}^{k+1}$ on the next scale L^{k+1} and the fluctuation on the scale L^k.

To specify fully our model the distribution $d\nu_k$ of the fluctuation fields Z^k has to be given. We shall consider several cases. The starting point will be

I. The "Free" (Noninteracting) Model

Here the random fields Z^k are independent for different k. Depending on whether Z_y^k's for different y's are independent or not the model will be called the *local* or the *nonlocal* one. Depending on whether all Z_y^k's are (almost surely) bounded uniformly in k and y or not we shall speak about the *bounded* or the *unbounded* case.

In the present paper we shall study only the bounded case. For the local model we shall take each Z_y^k to be distributed with the same compactly supported even probability measure $d\chi$.

To describe the nonlocal model suppose that we are given for each sequence $\bar{y} = (y_1, \ldots, y_{2m})$, $y_i \in \mathbb{Z}^d$, $m \geqq 1$, $U_{2m}(\bar{y})$ such that

$$\sum_{(y_2, \ldots, y_{2m})} \exp[A L(\underline{y})] |U_{2m}(\bar{y})| \leqq \tfrac{1}{2}(2m)! \kappa^{2m}, \tag{6}$$

where $A > 0$ is big enough, $\kappa > 0$ is small enough and $L(\underline{y})$ is the length of the shortest tree on the set $\underline{y}$ of points of the sequence $\bar{y}$ and possibly other (continuum) points. Throughout the paper we shall use the distance in $\mathbb{Z}^d$ which is the sum of the distances between the components of the vectors (and similarly for periodic tora). We shall also assume that

$$\begin{aligned} U_{2m}(y_1, \ldots, y_{2m}) &= U_{2m}(y_1 + a, \ldots, y_{2m} + a) \\ &= U_{2m}(y_{\pi(1)}, \ldots, y_{\pi(2m)}), \qquad a \in \mathbb{Z}^d, \\ U_{2m}(y, \ldots, y) &= 0. \end{aligned} \tag{7}$$

For $y_1, \ldots, y_{2m} \in \Lambda_k$ define $U_{2m}^k(y_1, \ldots, y_{2m})$ by averaging $U(y_1, \ldots, y_{2m})$ over identical periodic translations of all the variables. Put

$$U^{N-k-1}(Z^k) = \sum_{m=1}^{\infty} \sum_{y_i \in \Lambda_{N-k-1}} \frac{1}{(2m)!} U_{2m}^{N-k-1}(y_1, \ldots, y_{2m}) \prod_{i=1}^{2m} Z_{y_i}^k . \tag{8}$$

Notice that (6) guarantees the convergence of the series on the right hand side of (8), since the Z's are bounded.

In the nonlocal model the distribution of the Z^k field will now be given by

$$dv_k(Z^k) = \frac{1}{\mathcal{N}} \exp[-U^{N-k-1}(Z^k)] \prod_{y \in \Lambda_{N-k-1}} d\chi(Z_y^k), \tag{9}$$

where $\mathcal{N}$ stands for the normalization factor (as it always will). This slightly involved definition of the distributions of Z^k for the nonlocal case makes them almost coincident for different k in large volumes (in finite volume Z^k's live on different lattices).

Let us denote by $\langle - \rangle_0^N$ the expectation with respect to the above described total probability measure (in the bounded local or nonlocal case), i.e.

$$\langle - \rangle_0^N = \int (-) \prod_{k=0}^{N-1} dv_k(Z^k). \tag{10}$$

We gather here some elementary properties of the free two-point function, to be proved in [17], which show that the free model really mimics the free massless lattice field.

Proposition 1.

(1)
$$\sum_{x_2} \langle \phi_{x_1} \phi_{x_2} \rangle_0^N = 0. \tag{11}$$

2.
$$|\langle \phi_{x_1} \phi_{x_2} \rangle_0^N < C|x_1 - x_2|^{-d} \tag{12}$$

with C independent of N.

3.
$$\lim_{N \to \infty} \sum_{x_2} |\langle \phi_{x_1} \phi_{x_2} \rangle_0^N| = \infty. \tag{13}$$

In 2 and 3 we assume that A is sufficiently big and κ is sufficiently small for the nonlocal case.

Remark. (13) shows that (12) is the best polynomial bound. Recall that 1–3 are the properties of $\langle \nabla \phi_{x_1} \nabla \phi_{x_2} \rangle$ for ϕ being the massless free field.

For each free model we will consider

II. The Interacting Model

This is obtained from the free one by turning on interactions V, where V is a translation invariant functional of the field ϕ. Given such a V, define the expectation for the perturbed model to be

$$\langle - \rangle_V^N = \langle - \exp[-V(\phi)] \rangle_0^N / \langle \exp[-V(\phi)] \rangle_0^N. \tag{14}$$

One may take for example $V(\phi) = \lambda \sum_{x \in \Lambda_N} \phi_x^4$. Our main aim in this and the subsequent paper is to study the long distance behaviour of the bounded nonlocal interacting model.

3. The Renormalization Group Transformation

We shall examine the long distance behaviour of the interacting models via the renormalization group method. The first renormalization group transformation consists in generating the effective potential $T_1 V$ depending on the block spin field ϕ^1 [see (2.3) and (2.6)] by integrating out the fluctuation field Z^0 in the Gibbs factor:

$$\exp[-T_1 V(\phi^1)] = \text{const} \int \exp\left[-V\left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}\cdot]} + \mathscr{A}(\cdot - L[L^{-1}\cdot])Z^0_{[L^{-1}\cdot]}\right)\right] d\nu_0(Z^0). \tag{1}$$

The choice of the proportionality constant is to large extent arbitrary. We shall stick to the convention that potentials vanish at zero field:

$$V(0) = 0. \tag{2}$$

Then

$$\begin{aligned} T_1 V(\phi^1) = &-\log \int \exp\left[-V\left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}\cdot]} + \mathscr{Z}^0\right)\right] d\nu_0(Z^0) \\ &+ \log \int \exp[-V(\mathscr{Z}^0)] d\nu_0(Z^0), \end{aligned} \tag{3}$$

where we define

$$\mathscr{Z}^k_x = \mathscr{A}(x - L[L^{-1}x])Z^k_{[L^{-1}x]}. \tag{4}$$

The next renormalization group transformations are defined in a similar way:

$$\begin{aligned} T_k V(\phi^k) = &-\log \int \exp\left[-V\left(L^{-\frac{d}{2}}\phi^k_{[L^{-1}\cdot]} + \mathscr{Z}^{k-1}\right)\right] d\nu_{k-1}(Z^{k-1}) \\ &+ \log \int \exp[-V(\mathscr{Z}^{k-1})] d\nu_{k-1}(Z^{k-1}). \end{aligned} \tag{5}$$

Since they differ from the first transformation T_1 in fact only by the volume of the lattice, it is enough to study $T_1 \equiv T$. For simplicity we shall consider T only on even translation invariant potentials.

The first important property of T is that the potentials

$$V_c(\phi) = \frac{1}{2} c \sum_{x \in \Lambda_N} \phi_x^2 \tag{6}$$

constitute a one-parameter family of fixed points (i.e. are reproduced in the form). Indeed,

$$\begin{aligned} TV_c(\phi^1) = &-\log \int \exp\left[-\frac{1}{2} c \sum_{x \in \Lambda_N} \left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}x]} + \mathscr{Z}^0_x\right)^2\right] d\nu_0(Z^0) \\ &+ \log \int \exp\left[\frac{1}{2} c \sum_{x \in \Lambda_N} (\mathscr{Z}^0_x)^2\right] d\nu_0(Z^0) \\ = &\frac{c}{2} L^{-d} \sum_{x \in \Lambda_N} \phi^1_{[L^{-1}x]} = \frac{c}{2} \sum_{x \in \Lambda_{N-1}} (\phi^1_x)^2, \end{aligned} \tag{7}$$

where we have used

$$\sum_{x \in \Lambda_N} \phi^1_{[L^{-1}x]} \mathscr{Z}^0_x = \sum_x \phi^1_{[L^{-1}x]} \mathscr{A}(x - L[L^{-1}x]) Z^0_{[L^{-1}x]} = 0 \tag{8}$$

which follows from (2.4).

Similarly as in (8) one shows using (2.2) that

$$\sum_x \phi_x^2 = \left(\sum_y \mathscr{A}(y)^2\right) \sum_{k=0}^{N-1} \sum_{x\in\Lambda_{N-k-1}} (Z_x^k)^2 . \tag{9}$$

Hence the state $\langle - \rangle_{V_c}^N$ is again a free one but with changed one-spin distribution $d\chi$.

The key result of the present paper, as mentioned in the Introduction, consists in showing that for sufficiently small V the subsequent application of T drives V to a point V_{c_∞} on the line of fixed points. In other words the perturbed model becomes free at long distances.

4. Convergence to the Line of Fixed Points. The Local Case

In the local case, where $d\nu_0(Z^0) = \prod_{y\in\Lambda_{N-1}} d\chi(Z_y^0)$, T preserves the class of local potentials

$$V(\phi) = \sum_{x\in\Lambda_N} v(\phi_x), \tag{1}$$

i.e.

$$TV(\phi^1) = \sum_{x\in\Lambda_{N-1}} tv(\phi_x^1). \tag{2}$$

This is a result of the factorization of (3.4):

$$TV(\phi^1) = \sum_{y\in\Lambda_{N-1}} \left(-\log\int \exp\left[-\sum_{x\in b_y^1} v\left(L^{-\frac{d}{2}} \phi_y^1 + \mathscr{A}(x-Ly) Z_y^0 \right) \right] d\chi(Z_y^0) \right.$$
$$\left. + \log\int \exp\left[-\sum_x v(\mathscr{A}(x-Ly)Z_y^0) \right] d\chi(Z_y^0) \right).$$

Hence we write

$$tv(\phi) = -\log\int \exp\left[-\sum_{x\in b_0^1} v\left(L^{-\frac{d}{2}} \phi + \mathscr{A}(x)z \right) \right] d\chi(z)$$
$$+ \log\int \exp\left[-\sum_{x\in b_0^1} v(\mathscr{A}(x)z) \right] d\chi(z). \tag{3}$$

To define precisely the domain of t, notice that (2.2) and the uniform boundedness of Z_y^k's imply that ϕ_x^k are also (almost surely) uniformly bounded. Because of that we shall consider the potentials v defined only on a (sufficiently big) interval $[-\alpha, \alpha]$. The domain of t will be

$$\mathscr{D} = \{ v \in C^4([-\alpha,\alpha]) : v(0) = 0,\ v(-\phi) = -v(\phi) \}, \tag{4}$$

considered with the topology of uniform convergence with all derivatives up to order 4. It is easy to see that t maps $\mathscr{D}$ into itself.

Theorem 1. *Let $\mathscr{O}$ be a small enough neighbourhood of zero in $\mathscr{D}$. Then for each $v \in \mathscr{O}$ there exists $c_\infty \in \mathbb{R}^1$ such that*

$$t^n v \xrightarrow[n\to\infty]{} v_{c_\infty}, \tag{5}$$

where

$$v_c(\phi)=\tfrac{1}{2}c\phi^2\,. \tag{6}$$

Proof. Upon the iteration of t, v's are driven to zero in the directions transversal to the line of fixed points and to one of the fixed points along the line. The transversal directions are distinguished by means of the Taylor expansion up to the second order. Write

$$v(\phi)=\tfrac{1}{2}c\phi^2+\tilde{v}(\phi)\,, \tag{7}$$

$$v'(\phi)\equiv tv(\phi)=\tfrac{1}{2}c'\phi^2+\tilde{v}'(\phi)\,, \tag{8}$$

where

$$\frac{d^2\tilde{v}}{d\phi^2}(0)=\frac{d^2\tilde{v}'}{d\phi^2}(0)=0\,. \tag{9}$$

We shall prove the following

Lemma 1. *There exist* $0<\delta<1$ *and* $\alpha>0$ *such that for each* $0<\eta$ *small enough*

$$\left|\frac{d^4\tilde{v}}{d\phi^4}\right|\leqq\eta \tag{10}$$

implies that

$$\left|\frac{d^4\tilde{v}'}{d\phi^4}\right|\leqq\delta\eta \tag{11}$$

and

$$|c'-c|\leqq\alpha\eta\,. \tag{12}$$

Proof of Lemma 1. (3) yields

$$v'(\phi)=\tfrac{1}{2}c\phi^2+\tilde{\tilde{v}}'(\phi)-\tilde{\tilde{v}}'(0)\,, \tag{13}$$

where

$$\tilde{\tilde{v}}'(\phi)=-\log\int\exp\left[-\sum_{x\in b_0^1}\tilde{v}\left(L^{-\frac{d}{2}}\phi+\mathscr{A}(x)z\right)\right]d\chi_c(z) \tag{14}$$

and

$$d\chi_c(z)=\frac{1}{\mathscr{N}}\exp\left[-\frac{c}{2}\left(\sum_x\mathscr{A}(x)^2\right)z^2\right]d\chi(z)\,. \tag{15}$$

Notice that

$$\frac{d^4\tilde{\tilde{v}}'(\phi)}{d\phi^4}=L^{-2d}\sum_{k=1}^{4}(-1)^{k+1}\sum_{\{I_j\}_{j=1}^{k}}$$

$$\cdot\left\langle\sum_{x\in b_0^1}\frac{d^{|I_1|}\tilde{v}}{d\phi^{|I_1|}}\left(L^{-\frac{d}{2}}\phi+\mathscr{A}(x_1)z\right);\ldots;\sum_{x_k\in b_0^1}\frac{d^{|I_k|}\tilde{v}}{d\phi^{|I_k|}}\left(L^{-\frac{d}{2}}\phi+\mathscr{A}(x_k)z\right)\right\rangle^T, \tag{16}$$

where $\sum_{\{I_j\}}$ is the sum over the partitions of $\{1,2,3,4\}$ into k sets $I_j, \ldots, I_k$ and $\langle -, \ldots, - \rangle^T$ is the truncated expectation with respect to the state

$$\langle - \rangle = \frac{1}{\mathcal{N}} \int - \exp\left[- \sum_{x \in b_0^1} \tilde{v}\left(L^{-\frac{d}{2}} \phi + \mathscr{A}(x) z \right) \right] d\chi_c(z) . \tag{17}$$

Now since (10) implies that

$$\left| \frac{d^i \tilde{v}}{d\phi^i} \right| \leqq \mathcal{O}(\eta) \quad \text{for} \quad i = 1, 2, 3 , \tag{18}$$

(16) gives

$$\left| \frac{d^4 \tilde{\tilde{v}}'}{d\phi^4} \right| \leqq L^{-d} \eta + \mathcal{O}(\eta^2) \leqq \delta \eta . \tag{19}$$

Similarly

$$\left| \frac{d^2 \tilde{\tilde{v}}'(0)}{d\phi^2} \right| \leqq \mathcal{O}(\eta) . \tag{20}$$

Since

$$\tilde{v}'(\phi) = \tilde{\tilde{v}}'(\phi) - \tilde{\tilde{v}}'(0) - \frac{1}{2} \frac{d^2 \tilde{\tilde{v}}'(0)}{d\phi^2} \phi^2$$

and

$$c' = c + \frac{d^2 \tilde{\tilde{v}}'(0)}{d\phi^2} .$$

(19) and (20) yield (11) and (12). □

Theorem 1 follows immediately from Lemma 1 since the convergence of v's in $\mathscr{D}$ is equivalent to the uniform convergence of their fourth derivatives together with the convergence of c's. □

5. Convergence to the Line of Fixed Points. The Nonlocal Case

In the nonlocal case T does not preserve locality any more and so we have to consider general nonlocal potentials. Also, instead of the simple C^4 convergence in the local model, we have to deal with all the derivatives of V. To describe V in a uniform way for all volumes, let there be given kernels $V_{2m}(x_1, \ldots, x_{2m})$ defined for $x_i \in \mathbb{Z}^d$ satisfying

$$\sum_{x_2, \ldots, x_{2m} \in \mathbb{Z}^d} \exp\left[\frac{1}{L} AL(\underline{x}) \right] |\tilde{V}_{2m}(\bar{x})| \leqq (2m)!\, \eta^{2m} , \tag{1}$$

where we denoted $\bar{x} \equiv (x_1, \ldots, x_{2m})$, $\underline{x} \equiv \{x_1, \ldots, x_{2m}\}$, and η will be chosen sufficiently small. We also demand

$$\tilde{V}_{2m}(x_1, \ldots, x_{2m}) = \tilde{V}_{2m}(x_1 + a, \ldots, x_{2m} + a) = \tilde{V}_{2m}(x_{\pi(1)}, \ldots, x_{\pi(2m)}) \tag{2}$$

and

$$\tilde{V}_2(x,x)=0. \tag{3}$$

Let for $x_1, \ldots, x_{2m} \in \Lambda_N$ $V_{2m}^N(\bar{x})$ be defined averaging $V_{2m}(\bar{x})$ over identical periodic translations of all the variables. It is straightforward that $V_{2m}^N(\bar{x})$ satisfy the periodic versions of (1)–(3) [$L(\bar{x})$ is now the length of the shortest tree on the points x_i and possibly other (continuum) points lying on the torus].

Our potential V^N will now be

$$V^N(\phi)=V_c^N(\phi)+\tilde{V}^N(\phi), \tag{4}$$

where

$$V_c^N(\phi)=\frac{1}{2}c\sum_{x\in\Lambda_N}\phi_x^2 \tag{5}$$

and

$$\tilde{V}^N(\phi)=\sum_{m=1}^{\infty}\sum_{x_1,\ldots,x_{2m}\in\Lambda_N}\frac{1}{(2m)!}\tilde{V}_{2m}^N(\bar{x})\phi_{x_1}\ldots\phi_{x_{2m}}, \tag{6}$$

which is well defined when η is sufficiently small. Note that (1) defines a metric d_A on the space $\tilde{\mathscr{V}}^N$ of V^N's:

$$d_A(V_1,V_2)\equiv d_A(V_1-V_2,0)\equiv|V_1-V_2|_A,$$

where

$$|\tilde{V}^N|_A=\sup_m\left[\frac{1}{(2m)!}\sum_{x_2,\ldots,x_{2m}\in\Lambda_N}|\tilde{V}_{2m}^N(\bar{x})|e^{\frac{A}{L}L(\bar{x})}\right]^{\frac{1}{2m}} \tag{7}$$

($|\cdot|_A$ is not a norm). We use the notation $|V|_A$ also for V_{2m} and $x_i\in\mathbb{Z}^d$ in (7).

Our result is (compare with Theorem 1 of the local case)

Theorem 2. *Let V^N be of the form* (4), *with $|V|=\eta$. There are A_0, $\eta_0(A)>0$ such that for $A>A_0$, $\eta<\eta_0(A)$ TV^N can be written as*

$$TV^N=V_{c'}^{N-1}+\widetilde{TV}^N, \qquad \widetilde{TV}^N\in\tilde{\mathscr{V}}^{N-1} \tag{8}$$

and

$$|\widetilde{TV}^N|_A\leqq\delta|\tilde{V}^N|_A \qquad \delta<1, \tag{9}$$

$$|c'-c|\leqq\alpha|\tilde{V}^N|_A. \tag{10}$$

δ and α do not depend on N or C.

In [17] we will show (using Theorem 2) that the thermodynamic limit of our model exists provided A is large enough and η is small enough. That is, each pair $V\equiv(c,\{V_{2m}\})$ determines an ∞ volume Gibbs state $\langle-\rangle_V$ and

$$\langle-\rangle_V=\lim_{N\to\infty}\langle-\rangle_{V^N}^N$$

in the sense of convergence of correlations. Moreover there is a TV such that

$$\lim_{N\to\infty} \langle - \rangle_{TV^N}^{N-1} = \langle - \rangle_{TV}$$

and (8)–(10) hold in the limit $N\to\infty$.

We will also show that if $c_n \to c$ and $|V_{(n)} - V|_A \to 0$, then

$$\langle - \rangle_{V_{(n)}} \to \langle - \rangle_V$$

(in the sense of convergence of correlations). Thus we get

Theorem 3. *For $|\tilde{V}|_A$ sufficiently small*

$$\langle - \rangle_{T^n V} \xrightarrow[n\to\infty]{} \langle - \rangle_{V_{c_\infty}},$$

where

$$|c - c_\infty| \leqq \frac{\alpha}{1-\delta} |\tilde{V}|_A . \quad \square$$

We will now proceed with the proof of Theorem 2. We will suppress below the superscript N since all our estimates will be uniform in N. Let us start by computing V'. From (3.3) we get for $m>1$ or for $m=1$ and $x_1 \neq x_2$ that

$$\tilde{V}'_{2m}(x_1, \dots, x_{2m}) = L^{-dm} \sum_{k=1}^{2m} \sum_{\{I_j\}_{j=1}^k} \sum_{\substack{(y_1, \dots, y_{2m}) \\ [L^{-1}y_i] = x_i}} \left\langle \frac{\delta^{|I_1|}\tilde{V}}{\delta\phi_{\bar{y}_I}}(\mathscr{X}); \dots ; \frac{\delta^{|I_k|}\tilde{V}}{\delta\phi_{\bar{y}_I}}(\mathscr{X}) \right\rangle^T, \quad (11)$$

where $\sum_{\{I_j\}}$ is the sum over the partitions of $\{1, \dots, 2m\}$ into sets $I_1, \dots, I_k$, $\delta\phi_{\bar{y}_I} = \prod_{i\in I} \delta\phi_{y_i}$ and $\langle - ; \dots ; - \rangle^T$ is the truncated expectation with respect to the state

$$\langle - \rangle = \frac{1}{\mathscr{N}} \int - \exp[-V(\mathscr{X})] dv(z) . \quad (12)$$

But

$$\frac{\delta^{|I|}\tilde{V}}{\delta\phi_{\bar{y}_I}}(\mathscr{X}) = \sum_{m \geqq \frac{1}{2}|I|} \frac{1}{(2m-|I|)!} \sum_{(v_1, \dots, v_{2m-|I|})} \tilde{V}_{2m}(\bar{y}_I, \bar{v}) \mathscr{X}_{\bar{v}}, \quad (13)$$

where

$$\bar{y}_I = (y_j)_{j\in I}, \qquad \bar{v} = (v_1, \dots, v_{2m-|I|}), \quad (14)$$

$$\mathscr{X}_{\bar{v}} = \prod_{i=1}^{2m-|I|} \mathscr{X}_{v_i} . \quad (15)$$

Using (3.4) we may rewrite

$$\mathscr{X}_{\bar{v}} = \prod_{i=1}^{2m-|I|} \mathscr{A}(v_i - Lu_i) \prod_{i=1}^{2m-|I|} Z_{u_i} \equiv \mathscr{A}(\bar{v}, \bar{u}) Z_{\bar{u}}, \quad (16)$$

where

$$u_i = [L^{-1} v_i] . \quad (17)$$

Inserting (13) and (16) into (11) we obtain

$$\tilde{V}'_{2m}(\bar{x}) = L^{-dm} \sum_k (-1)^{k+1} \sum_{\{I_j\}} \sum_{\substack{\{m_j\} \\ 2m_j \geqq |I_j| + \min(1,k-1)}} \sum_{\bar{y}:[L^{-1}\bar{y}]=\bar{x}} \sum_{\{\bar{v}_j\}}$$

$$\cdot \prod_j \left(\frac{1}{(2m_j - |I_j|)!} \tilde{V}_{2m_j}(\bar{y}_{I_j}, \bar{v}_j) \mathscr{A}(\bar{v}_j, \bar{u}_j) \right) \langle Z_{\bar{u}_1}; \ldots; Z_{\bar{u}_k} \rangle^T . \tag{18}$$

The main input used in the proof will be the following result shown in the next section devoted to the cluster expansion.

Proposition 2. *Suppose that D, A are big enough, κ and $\eta = |\tilde{V}|_A$ are small enough (the bound on the next constant generally depending on the ones already fixed). Then uniformly in N*

$$|\langle Z_{\bar{u}_1}; \ldots; Z_{\bar{u}_k} \rangle^T| \leqq \prod_r M_r! \prod_{j=1}^k \exp[D(2m_j - |I_j| + L(\underline{u}_j))]$$

$$\cdot \exp[-\tfrac{1}{2} A L(\underline{u}_1, \ldots, \underline{u}_k)] , \tag{19}$$

where M_r are the numbers of the sequences $\bar{u}_1, \ldots, \bar{u}_k$ equal up to permutations and $L(\underline{u}_1; \ldots; \underline{u}_k)$ is the length of the shortest graph on the points of $\bigcup_j \underline{u}_j$ and possibly other points connected with respect to the groups $\underline{u}_j$.

The connected structure of the right hand side of (18) is now clear. Let us define $\tilde{W}_{2m}(\bar{x}) = \exp\left|\frac{A}{L} L(\bar{x})\right| V_{2m}(\bar{x})$ and similarly for $\tilde{W}'_{2m}$. From (18) and (19) it follows that

$$\sum_{(x_2, \ldots, x_{2m})} |W'_{2m}(\bar{x})|$$

$$\leqq L^{-dm} \sum_{m_1 \geqq m} \sum_{\bar{y}:[L^{-1}y_1]=x_1} \sum_{\bar{v}} \frac{c^{2(m_1-m)}}{(2(m_1-m))!} |\tilde{W}_{2m_1}(\bar{y}, \bar{v})| \exp\left[\frac{1}{L} AL(\underline{x}) - \frac{1}{L} AL(\underline{y})\right]$$

$$+ L^{-dm} \sum_{k \geqq 2, \{I_j\}, \{m_j\}} \sum_{\bar{y}:[L^{-1}y_1]=x_1} \sum_{\{\bar{v}_j\}} \prod_j \left(\frac{c^{2m_j - |I_j|}}{(2m_j - |I_j|)!} |\tilde{W}_{2m_j}(\bar{y}_{I_j}, \bar{v}_j)| \right)$$

$$\cdot \prod_r M_r! \exp\left[\frac{A}{L} L(\underline{x}) + D \sum_j L(\underline{u}_j) - \frac{A}{L} \sum_j L(\underline{y}_{I_j} \cup \underline{v}_j) - \tfrac{1}{2} AL(\underline{u}_1; \ldots; \underline{u}_k)\right] . \tag{20}$$

Notice that for $m=1$ in the right hand side of (20) we may sum over $\bar{y}$ such that $x_1 = [L^{-1}y_1] + [L^{-1}y_2]$ only. Denote by $\mathscr{I}$ a sequence $(i_1, \ldots, i_k)$ of integers, $1 \leqq i_j \leqq 2m_j - |I_j|$. For given $\bar{v}_j$ and $\mathscr{I}$ denote

$$\bar{v}_{\mathscr{I}} = (v_{i_1 1}, \ldots, v_{i_k k}), \qquad \bar{u}_{\mathscr{I}} = (u_{i_1 1}, \ldots, u_{i_k k}) \tag{21}$$

and by $\prod_s N_s!$ the product of the factorials of the multiplicities of occurrence of different points in $\bar{u}_{\mathscr{I}}$. Notice that

$$\prod_r M_r! \leqq \sum_{\mathscr{I}} \prod_s N_s! \tag{22}$$

We shall need the following results dealing with the shortest trees which are proven in the Appendix.

Lemma 2. *Let $\underline{t}$ and $\underline{s}$ be lattice subsets such that $[L^{-1}\underline{t}]=\underline{s}$. Then*

$$LL(\underline{s})-\tfrac{1}{2}(L-1)d|\underline{s}|\leqq L(\underline{t}), \tag{23}$$

$$L(\underline{s})\leqq L(\underline{t}), \tag{24}$$

$$(1+\varepsilon)L(\underline{s})-\tfrac{1}{2}(L-1)\leqq L(\underline{t}) \tag{25}$$

for sufficiently small $\varepsilon>0$ uniformly in $\underline{t}$.

From Lemma 2 it follows that

$$\begin{aligned}
&\frac{A}{L}\sum_j L(\underline{y}_{I_j}\cup\underline{v}_j)+\frac{A}{2}L(\underline{u}_1;\ldots;u_k)\\
&\geqq\frac{1+\varepsilon}{L}A\sum_j L(\underline{x}_{I_j}\cup\underline{u}_j)+\frac{A}{2}L(\underline{u}_1;\ldots;\underline{u}_k)-\tfrac{1}{2}Ak\\
&\geqq\frac{A}{L}L(\underline{x})+\sum_j\frac{\varepsilon A}{2L}L(\underline{u}_j)+\frac{\varepsilon A}{2L}L\Big(\bigcup_j u_j\Big)-\frac{Ak}{2}\\
&\geqq\frac{A}{L}L(\underline{x})+\sum_j\frac{\varepsilon A}{2L}L(\underline{u}_j)+\frac{\varepsilon A}{2L}L(u_{\mathscr{I}})-\tfrac{1}{2}Ak\,.
\end{aligned} \tag{26}$$

Using (22) and (26) we get a bound on the last part of (20)

$$\prod_r M_r!\exp[\ldots]\leqq\sum_{\mathscr{I}}\prod_s N_s!e^{\frac{1}{2}Ak}\exp\left[-\frac{\varepsilon A}{2L}L(\underline{u}_{\mathscr{I}})\right]. \tag{27}$$

But

Lemma 3.

$$\prod_s N_s!\,exp\left[-\frac{\varepsilon A}{2L}L(u_{\mathscr{I}})\right]\leqq 2^k\sum_\tau\exp[-\varepsilon' AL_\tau(\bar{u}_{\mathscr{I}})], \tag{28}$$

where $\sum_\tau$ is the sum over the trees on k points and $L_\tau(\bar{u}_{\mathscr{I}})$ is the length of the tree τ when the points are taken to be those of $\bar{u}_{\mathscr{I}}$.

Proof of Lemma 3. $\frac{\varepsilon}{2L}L(\underline{u}_{\mathscr{I}})>\varepsilon'\mathscr{L}(\underline{u}_{\mathscr{I}})$ where $\mathscr{L}(\underline{u}_{\mathscr{I}})$ is the length of the shortest tree on the points of $\underline{u}_{\mathscr{I}}$ and no other points (see [6, p. 197]). But

$$\begin{aligned}
\exp[-\varepsilon' A\mathscr{L}(\underline{u}_{\mathscr{I}})]&\leqq\prod_s N_s^{2-N_s}\sum_\tau\exp[-\varepsilon' AL_\tau(\bar{u}_{\mathscr{I}})]\\
&\leqq\prod_s(\tfrac{1}{2}N_s!)^{-1}\sum_\tau\exp[-\varepsilon' AL_\tau(\bar{u}_{\mathscr{I}})]
\end{aligned} \tag{29}$$

since $N_s^{N_s-2}$ is the number of trees on N_s (coinciding) points. □

Notice also that

$$L_\tau(\bar{u}_{\mathscr{I}})\geqq\frac{1}{L}L_\tau(\bar{v}_{\mathscr{I}})-\frac{1}{L}(L-1)d(k-1)\,. \tag{30}$$

(27), (28), and (30) yield

$$\prod_r M_r!\exp[\ldots]\leqq \sum_{\mathscr{I}} e^{\mathcal{O}(A)k}\sum_\tau \exp[-\varepsilon A L_\tau(\bar{v}_{\mathscr{I}})] \tag{31}$$

(we have omitted the "prime" at ε). Inserting (31) into (30) we obtain

$$\begin{aligned}\sum_{(x_2,\ldots,x_{2m})} |\tilde{W}'_{2m}(\bar{x})| \leqq L^{-dm} &\sum_{m_1\geqq m}\ \sum_{\bar{y}:[L^{-1}y_1]=x_1}\ \sum_{\bar{v}} \frac{c^{2m_1-2m}}{(2m_1-2m)!}\\ &\cdot |\tilde{W}_{2m_1}(\bar{y},\bar{v})| \exp\left[\frac{1}{L}AL(\underline{x})-\frac{1}{L}AL(\underline{y})\right]\\ &+L^{-dm}\sum_{k\geqq 2,\{I_j\},\{m_j\}}\ \prod_j \frac{C^{(2m_j-|I_j|)A}}{(2m_j-|I_j|)!}\sum_{\mathscr{I},\tau}\ \sum_{\bar{y},\{\bar{v}_j\}} \exp[-\varepsilon A L_\tau(\bar{v}_{\mathscr{I}})]\\ &\cdot \prod_j |\tilde{W}_{2m_j}(\bar{y}_{I_j},\bar{v}_j)|,\end{aligned} \tag{32}$$

where again in the first term A_1 on the right hand side for $m=1$ we may sum only over $\bar{y}=(y_1,y_2)$ such that $x_1=[L^{-1}y_1]\dotplus[L^{-1}y_2]$.

A. The $k=1$ Term

First we shall estimate A_1. Denoting $|V|_A$ by η (7) gives

$$\begin{aligned} L^{-dm}&\sum_{m_1>m}\ \sum_{\bar{y}:[L^{-1}y_1]=x_1}\ \sum_{\bar{v}} \frac{c^{m_1-m}}{(2m_1-2m)!}|\tilde{W}_{2m_1}(\bar{y},\bar{v})|\exp\left[\frac{A}{L}L(\underline{x})-\frac{A}{L}L(\underline{y})\right]\\ &\leqq L^{-d(m-1)}\sum_{m_1}(2m_1)!\,\eta^{2m_1}\frac{c^{m_1-m}}{(2m_1-2m)!}\leqq L^{-d(m-1)}(2m)!\,\eta^{2m}\sum_{\ell=1}^{\infty}\binom{2m+\ell}{2m}(c\eta)^{\ell}\\ &=L^{-d(m-1)}(2m)!\,\eta^{2m}[(1-c\eta)^{-2m-1}-1]\\ &\leqq L^{-d(m-1)}(2m)!\,\eta^{2m}(e^{\mathcal{O}(\eta)m}-1).\end{aligned} \tag{33}$$

For $m_1=m>1$

$$L^{-dm}\sum_{\bar{y}}|\tilde{W}_{2m}(\bar{y})|\exp\left[\frac{A}{L}L(\underline{x})-\frac{A}{L}L(\underline{y})\right]\leqq L^{-d(m-1)}(2m)!\,\eta^{2m}. \tag{34}$$

Hence for $m>1$

$$A_1\leqq L^{-d(m-1)}(2m)!\,\eta^{2m}e^{\mathcal{O}(\eta)m}\leqq(2m)!\,(\delta\eta)^{2m} \tag{35}$$

for some $0<\delta<1$, provided η is small enough.

For $m_1=m=1$

$$\begin{aligned}L^{-d}&\sum_{(y_1,y_2):[L^{-1}y_1]=x_1\dotplus[L^{-1}y_2]}|\tilde{W}_2(\bar{y})|\exp\left[\frac{A}{L}L(\underline{x})-\frac{A}{L}L(\underline{y})\right]\\ &\leqq 2\eta^2L^{-d}(L^d-(L-2)^d)+2\eta^2L^{-d}(L-2)^d\exp\left[-\frac{A}{L}\right]\leqq 2(\delta\eta)^2\end{aligned} \tag{36}$$

for some $0<\delta<1$ since for $L^d-(L-2)^d$ choices of y_1 there exist choices of y_2 such that $L(\underline{x})=L(\underline{y})$. For other $(L-2)^d$ choices of y_1 $L(\underline{x})-L(\underline{y})\leqq -1$. Hence putting together (33) and (36) we get for $m=1$

$$A_1 \leqq 2(\delta\eta)^2 \tag{37}$$

for some $0<\delta<1$ provided η is small enough.

B. The $k>2$ Terms

We pass to estimation of the second term A_2 on the right hand side of (32). We shall use the following

Lemma 4.

$$\begin{aligned}&\sum_{\bar{y}:[L^{-1}y_1]=x_1} \sum_{\{\bar{v}_j\}} \exp[-\varepsilon A L_\tau(\bar{v}_{\mathscr{I}})] \prod_j |\tilde{W}_{2m_j}(\bar{y}_{I_j}, \bar{v}_j)| \\ &\leqq L^d c^{k-1} \prod_j (2m_j)!\,\eta^{2m_j}.\end{aligned} \tag{38}$$

Proof of Lemma 4. Fix y_1. Note that $W_{2m}(\bar{x})$ is translation invariant. Write Fourier series

$$|\tilde{W}_{2m}(\bar{x})| = L^{-d(2m-1)} \sum_{\bar{p}:\sum p_i=0} e^{i\bar{p}\bar{x}}\bar{W}(\bar{p})$$

$$e^{-\varepsilon A|x|} = L^{-d}\sum_p e^{ipx}\ell(p).$$

Then by the tree structure of (42)

$$\begin{aligned}\sum_{\bar{y}:y_1=\text{fixed}} \ldots &= \ell(0)^{k-1}\prod_j \bar{\mathrm{W}}_{2m_j}(0) \\ &= \left[\sum_x \mathrm{e}^{-\varepsilon A|x|}\right]^{k-1} \prod_j \left(\sum_{x_2,\ldots,x_{2m_j}} |\tilde{W}_{2m_j}(\bar{x})|\right).\end{aligned}$$

The claim follows by virtue of (11) since there are L^d choices of y_1. $\square$

With Lemma 4 we obtain

$$A_2 \leqq L^{-d(m-1)} \sum_{k\geqq 2,\{I_j\},\{m_j\},\mathscr{I},\tau} \prod_j \frac{c^{A(2m_j-|I_j|)}}{(2m_j-|I_j|)!}(2m_j)!\,\eta^{2m_j}. \tag{39}$$

Now use

$$\sum_{\mathscr{I},\tau} 1 = k^{k-2}\prod_j (2m_j-|I_j|) \leqq k!\prod_j c^{2m_j-|I_j|} \tag{40}$$

and estimate

$$\begin{aligned}A_2 &\leqq L^{-d(m-1)} \sum_{k\geqq 2,\{I_j\},\{m_j\}} k!\prod_j \frac{c^{A(2m_j-|I_j|)}}{(2m_j-|I_j|)!}(2m_j)!\,\eta^{2m_j} \\ &\leqq L^{-d(m-1)} \sum_{k=2}^{2m} \sum_{\substack{(n_1,\ldots,n_k) \\ \sum n_j=2m,\, n_j>0}} \frac{(2m)!}{\prod n_j!} \sum_{\substack{\{m_j\} \\ 2m_j>n_j}} \prod_j \frac{c^{A(2m_j-n_j)}}{(2m_j-n_j)!}(2m_j)!\,\eta^{2m_j} \\ &= L^{-d(m-1)}(2m)!\,\eta^{2m} \sum_{k=2}^{2m} \sum_{(n_1,\ldots,n_k)} \prod_j \sum_{\ell=1}^{\infty} \binom{n_j+\ell}{n_j}(c^A\eta)^\ell \\ &\leqq L^{-d(m-1)}(2m)!\,\eta^{2m} \sum_{\sum n_j=2m} \prod_j (e^{\mathcal{O}(\eta)n_j}-1),\end{aligned} \tag{41}$$

where $\mathcal{O}(\eta)$ is A dependent. But

$$\prod_j (e^{\mathcal{O}(\eta)n_j}-1) \leqq \prod_j \mathcal{O}(\eta^{1/2}) \exp[\mathcal{O}(\eta^{1/2})n_j] \leqq \mathcal{O}(\eta^{1/2})^k \exp \mathcal{O}(\eta^{1/2})m. \tag{42}$$

Hence

$$\begin{aligned} A_2 &\leqq L^{-d(m-1)}(2m)!\,\eta^{2m} e^{\mathcal{O}(\eta^{1/2})m} \sum_{k=2}^{2m} \binom{2m-1}{k-1} \mathcal{O}(\eta^{1/2})^k \\ &\leqq \mathcal{O}(\eta^{1/2})(2m)!\,(\delta\eta)^{2m} \end{aligned} \tag{43}$$

for some $0<\delta<1$ provided η is small enough (the bound on η depending on A).

Gathering (35), (37), and (43) we obtain

$$\sum_{(x_2,\dots,x_{2m})} |\tilde{W}'_{2m}(\bar{x})| \leqq (2m)!\,(\delta\eta)^{2m} \tag{44}$$

for some $0<\delta<1$ provided η is small enough.

The first part of Theorem 2 is thus proven.

We still have to estimate the coefficient c' multiplying the term $\frac{1}{2}\sum_x \phi_x^2$ in V'. Notice that $\frac{1}{2}(c'-c)$ is given by the right hand side of (18) for $m=1$ and $x_1=x_2\equiv x$. Thus $\frac{1}{2}|c'-c|$ may be bounded by the right hand side of (32) for $m=1$ with the sum in the first term A_1 over all $\bar{y}=(y_1,y_2)$ such that $[L^{-1}y_1]=x$. Now (37) must be replaced by

$$A_1 \leqq 2\eta e^{\mathcal{O}(\eta)}. \tag{45}$$

Combining (44) and (45) we obtain

$$|c'-c| \leqq \alpha\eta \tag{46}$$

and the proof of Theorem 2 is completed. □

6. The Cluster Expansion

In this section we shall prove the basic Proposition 2.

A. The Properties of the Measure

Let us begin with the following result about the measure $d\nu(Z)$ given by (5.12)

Lemma 6. *Let η be small enough (the bound on η depending on A and κ). Then*

$$d\tilde{\nu}(Z) = \frac{1}{\mathcal{N}} \exp\left[-\sum_{m=1}^{\infty} \sum_{\bar{y}} \frac{1}{(2m)!} I_{2m}(\bar{y}) Z_{\bar{y}}\right] \prod_y d\tilde{\chi}(Z_y), \tag{1}$$

where I_{2m} are translation and permutation invariant, $I_{2m}(y,\dots,y)=0$ and the even probability measure $d\tilde{\chi}$ has the same support as $d\chi$. Moreover,

$$\sum_{(y_2,\dots,y_{2m})} \exp[AL(\underline{y})]\,|I_{2m}(\bar{y})| \leqq (2m)!\,\kappa^{2m}. \tag{2}$$

Proof of Lemma 6. By (5.12)

$$d\tilde{v}(Z)=\frac{1}{\mathcal{N}}\exp[-V(\mathscr{Z})+U(\mathscr{Z})]\prod_{y} d\chi(Z_y), \tag{3}$$

where we have omitted the sub- and superscripts indicating the volume. Notice that due to (5.4) and (5.16)

$$V(\mathscr{Z})=\frac{c}{2}\sum_{v}\mathscr{A}(v-Ly)^2Z_y^2+\sum_{m}\sum_{\bar{v}}\frac{1}{(2m)!}\tilde{V}_{2m}(\bar{v})\mathscr{A}(\bar{v},\bar{y})Z_{\bar{y}}, \tag{4}$$

where $\bar{y}=[L^{-1}\bar{v}]$. But by virtue of (5.23) and $|V|_A\leqq\eta$

$$\begin{aligned}&\sum_{(y_2,\dots,y_{2m})}\exp[AL(\underline{y})]\left|\sum_{\bar{v}:[L^{-1}\bar{v}]=\bar{y}}\tilde{V}_{2m}(\bar{v})\mathscr{A}(\bar{v},\bar{y})\right|\\ &\quad\leqq C_A^{2m}\sum_{\bar{v}:[L^{-1}\bar{v}]=\bar{y}}\exp\left[\frac{A}{L}L(\bar{v})\right]|\tilde{V}_{2m}(\bar{v})|\leqq(2m)!(C_A\eta)^{2m}.\end{aligned} \tag{5}$$

The terms of (4) local in the field Z will be used to define $d\tilde{\chi}$. The other terms together with the terms coming from $U(Z)$ [see (2.8)] build up the $\frac{1}{(2m)!}I_{2m}(\bar{y})Z_{\bar{y}}$ terms in (1). (2.6) and (5) show that (2) holds if $(C_A\eta)^{2m}\leqq\frac{1}{2}\kappa^{2m}$. □

From now on we assume that η satisfies the assumption of Lemma 6. It is preferable to rewrite (1) by introducing for lattice subsets $\underline{y}$ the random variables

$$\mathscr{I}(\underline{y})\equiv\sum_{m,\bar{y}:\underline{y}\text{ fixed}}\frac{1}{(2m)!}I_{2m}(\bar{y})Z_{\bar{y}}. \tag{6}$$

Then

$$d\tilde{v}(Z)=\frac{1}{\mathcal{N}}\exp\left[-\sum_{\underline{y}:|\underline{y}|\geqq 2}\mathscr{I}(\underline{y})\right]\prod_{y} d\tilde{\chi}(Z_y). \tag{7}$$

The following estimate will be needed:

Lemma 7. *For κ sufficiently small*

$$\sum_{\underline{y}\ni 0}\kappa^{-\frac{1}{2}|\underline{y}|}\exp[AL(\underline{y})]|\mathscr{I}(\underline{y})|\leqq\mathcal{O}(\kappa). \tag{8}$$

Proof of Lemma 7. By (6) and (2)

$$\begin{aligned}&\sum_{\underline{y}\ni 0}\kappa^{-\frac{1}{2}|\underline{y}|}\exp[AL(\underline{y})]|\mathscr{I}(\underline{y})|\\ &\quad\leqq\sum_{m}\sum_{\bar{y}}\kappa^{-\frac{1}{2}|\underline{y}|}\frac{1}{(2m)!}\exp[AL(\underline{y})]|I_{2m}(\bar{y})||Z_{\bar{y}}|\\ &\quad\leqq\sum_{m}\frac{(C\kappa^{-1/2})^{2m}}{(2m-1)!}\sum_{\bar{y}=(0,y_2,\dots,y_{2m})}\exp[AL(\underline{y})]|I_{2m}(\bar{y})|\\ &\quad\leqq\sum_{m\geqq 1}2m(C\kappa^{1/2})^{2m}\leqq\mathcal{O}(\kappa).\quad\square\end{aligned}$$

B. The Expansion

Our main aim is to estimate $\langle Z_{\bar{u}_1};\ldots;Z_{\bar{u}_k}\rangle^T$, where $\langle -;\ldots;-\rangle^T$ is the truncated expectation with respect to $d\tilde{v}(Z)$. Without loss of generality we may assume that the order of points in the sequences $\bar{u}_1,\ldots,\bar{u}_k$ agrees with some fixed ordering of the lattice points. First we shall obtain for $\langle Z_{\bar{u}_1};\ldots;Z_{\bar{u}_k}\rangle^T$ a cluster expansion formula. Write

$$\begin{aligned}\int\prod_{j=1}^{k} Z_{\bar{u}_j}d\tilde{v}(Z)&=\frac{1}{\mathcal{N}}\int\prod_j Z_{\bar{u}_j}\exp\Big[-\sum_{\underline{y}}\mathscr{I}(\underline{y})\Big]\prod_y d\tilde{\chi}(Z_y)\\&=\frac{1}{\mathcal{N}}\int\prod_j Z_{\bar{u}_j}\sum_\gamma\prod_{\underline{y}\in\gamma}(\exp[-\mathscr{I}(\underline{y})]-1)\prod_y d\tilde{\chi}(Z_y),\end{aligned}\tag{9}$$

where γ is a family of lattice subsets $\underline{y}$ [$|\underline{y}|\geqq 2$ since otherwise $\mathscr{I}(\underline{y})=0$].

Given γ, consider the set

$$\left(\bigcup_{j=1}^{k}\underline{u}_j\right)\cup\left(\bigcup_{\underline{y}\in\gamma}\underline{y}\right)$$

and its finest partition with the property that

$$\text{each } \underline{u}_j \text{ and each } \underline{y}\in\gamma \text{ are in the same subset of the partition.}\tag{10}$$

Denote by X_α, $\alpha=1,\ldots,A_1$ the sets of the partition containing $\underline{u}_j$'s and by Y_β, $\beta=1,\ldots,B_1$ the other ones. In (9) we shall fix first the clusters $\{X_\alpha\}$ and $\{Y_\beta\}$ performing the rest of the summation and only then shall we sum over $\{X_\alpha\}$ and $\{Y_\beta\}$.

For $H\subset\{1,\ldots,k\}$ and X being a subset of the lattice such that for each $j\in H$ $\underline{u}_j\subset X$, denote

$$(H,X)\equiv\mathbb{X},\tag{11}$$

and define

$$\varrho(\mathbb{X})=\exp\Big[-D\sum_{j\in H}(\ell_j+L(\underline{u}_j))\Big]\sum_{\gamma_c}\int\prod_{j\in H}Z_{\bar{u}_j}\prod_{\underline{y}\in\gamma_c}(c^{-\mathscr{I}(\underline{y})}-1)\prod_{y\in X}d\tilde{\chi}(Z_y),\tag{12}$$

where ℓ_j is the length of the sequence $\bar{u}_j$ and $\sum\limits_\gamma$ is the sum over the families γ_c of $\underline{y}$, $\underline{y}\subset Z$ ($|\underline{y}|\geqq 2$), such that X cannot be divided into two subsets so that each $\underline{u}_j$, $j\in H$, and each $\underline{y}\in\gamma_c$ is in one of the subsets. Similarly for a lattice subset Y, $|Y|\geqq 2$, put $\varrho(Y)=\varrho(\mathbb{Y})$ where $\mathbb{Y}=(\phi, Y)$, i.e.

$$\varrho(Y)=\sum_{\gamma_c}\int\prod_{\underline{y}\in\gamma_c}(\exp[-\mathscr{I}(\underline{y})]-1)\prod_{y\in Y}d\tilde{\chi}(Z_y).\tag{13}$$

Now we may rewrite (9) as

$$\begin{aligned}\Big\langle\prod_j Z_{\bar{u}_j}\Big\rangle&=\frac{1}{\mathcal{N}}\exp\Big[D\sum_{j=1}^{k}(\ell_j+L(\bar{u}_j))\Big]\\&\quad\cdot\sum_{\{X_\alpha\}_{\alpha=1}^{A}}{}'\;\sum_{\{Y_\beta\}_{\beta=1}^{B}}{}'\;\prod_\alpha\varrho(\mathbb{X}_\alpha)\prod_\beta\varrho(Y_\beta),\end{aligned}\tag{14}$$

where $\sum\limits_{\{X_\alpha\}}{}'$ runs through the sets $\{X_\alpha\}$ such that $\mathbb{X}_\alpha=(H_\alpha,X_\alpha)$ with

$$\phi\neq H_\alpha:=\{j:\underline{u}_j\subset X\},$$

X_α are pairwise disjoint and each $\underline{u}_j$ is in some X_α. $\sum_{\{Y_\beta\}}'$ runs through the sets $\{Y_\beta\}$ (the empty set included) of lattice subsets $Y_\beta, |Y_\beta| \geqq 2$, disjoint among themselves and with X_α's.

Next we shall transform (14) so as to exhibit the cancellations between the numerator on the right hand side and the denominator

$$\mathcal{N} = \sum_{\{Y_\beta\}_{\beta=1}^B} \prod_\beta \varrho(Y_\beta). \tag{15}$$

To this end introduce for two lattice subsets V_1 and V_2

$$U(V_1, V_2) = \begin{cases} 0 & \text{if} \quad V_1 \cap V_2 \neq \emptyset, \\ 1 & \text{if} \quad V_1 \cap V_2 = \emptyset. \end{cases} \tag{16}$$

Using these symbols we may write

$$\left\langle \prod_j Z_{\bar{u}_j} \right\rangle = \frac{1}{\mathcal{N}} \exp\left[D \sum_j (\ell_j + L(\underline{u}_j)\right]$$

$$\cdot \sum_{\substack{\text{partitions } \pi = \{H_\alpha\}_{\alpha=1}^A \\ \text{of } \{1, \ldots, k\}}} \; \sum_{\substack{\{\mathbb{X}_\alpha\}_{\alpha=1}^A \\ \mathbb{X}_\alpha = (H_\alpha, X_\alpha)}} \; \sum_{(Y_1, \ldots, Y_B)} \frac{1}{B!} \prod_{\mathcal{L}} U(\mathcal{L}) \prod_\alpha \varrho(\mathbb{X}_\alpha) \prod_\varrho \varrho(Y_\beta), \tag{17}$$

where in $\prod_{\mathcal{L}}$ $\mathcal{L}$ runs through the pairs $(X_{\alpha_1}, X_{\alpha_2})$, $\alpha_1 < \alpha_2$, (X_α, X_β), and $(Y_{\beta_1}, Y_{\beta_2})$, $\beta_1 < \beta_2$, with all X_α and Y_β treated as different elements. We shall call $\mathcal{L}$ a line on elements X_α, Y_β. Now standard transformations (see [1, Chap. II]) yield

$$\begin{aligned} \left\langle \prod_j Z_{\bar{u}_j} \right\rangle &= \frac{1}{\mathcal{N}} \exp\left[D \sum_j (\ell_j + L(\underline{u}_j))\right] \\ &\quad \sum_{\pi = \{H_\alpha\}} \; \sum_{\substack{\{\mathbb{X}_\alpha\} \\ \mathbb{X}_\alpha = (H_\alpha, X_\alpha)}} \; \sum_{(Y_1, \ldots, Y_B)} \frac{1}{B!} \sum_\Gamma \prod_{\mathcal{L} \in \Gamma} A(\mathcal{L}) \prod_\alpha \varrho(\mathbb{X}_\alpha) \prod_\beta \varrho(Y_\beta) \\ &= \exp\left[D \sum_j (\ell_j + L(\underline{u}_j))\right] \\ &\quad \cdot \sum_{\pi = \{H_\alpha\}} \; \sum_{\substack{\{\mathbb{X}_\alpha\} \\ \mathbb{X}_\alpha = (H_\alpha, X_\alpha)}} \; \sum_{(Y_1, \ldots, Y_B)} \frac{1}{B!} \sum_\Gamma' \prod_{\mathcal{L} \in \Gamma} A(\mathcal{L}) \prod_\alpha \varrho(\mathbb{X}_\alpha) \prod_\beta \varrho(Y_\beta), \end{aligned} \tag{18}$$

where $\sum_\Gamma$ is the sum over the sets of lines $\mathcal{L}$, $\sum_\Gamma'$ is the sum over the sets of lines $\mathcal{L}$ forming a graph on X_α, Y_β connected with respect to the elements Y_β and the group of elements composed of X_α's. $A(\mathcal{L}) := U(\mathcal{L}) - 1$.

Since by the definition of the truncated expectation

$$\left\langle \prod_{j=1}^k Z_{\bar{u}_j} \right\rangle = \sum_{\substack{\text{partitions } \pi = \{K_\gamma\} \\ \text{of } (1, \ldots, k)}} \prod_\gamma \left\langle \prod_{j \in K_\gamma} Z_{\bar{u}_j} \right\rangle^T, \tag{19}$$

we easily read off from (18) the following cluster expansion formula:

Proposition 5.

$$\langle Z_{\bar{u}_1};\ldots;Z_{\bar{u}_k}\rangle^T=\exp\left[D\sum_{j=1}^{k}(\ell_j+L(\underline{u}_j))\right]$$
$$\cdot\sum_{\pi=\{H_\alpha\}}\sum_{\substack{\{\mathbb{X}_\alpha\}\\ \mathbb{X}_\alpha=(H_\alpha,X_\alpha)}}\sum_{(Y_1,\ldots,Y_B)}\frac{1}{B!}\sum_{\Gamma_c}\prod_{\mathscr{L}\in\Gamma_c}A(\mathscr{L})\prod_\alpha\varrho(\mathbb{X}_\alpha)\prod_\beta\varrho(Y_\beta),\tag{20}$$

where $\sum_\Gamma$ is the sum over the sets of lines $\mathscr{L}$ forming a connected graph on the elements X_α, Y_β (treated as different).

Proof. With this expression for the truncated expectation we easily obtain (19), first fixing on the right hand side of (18) the partition $\{K_\gamma\}$ of $(1,\ldots,k)$, each K_γ composed of points of H_α's such that the corresponding X_α's enter into a single connected component of Γ, and then summing over $\{K_\gamma\}$. □

Remark. We shall call the sum on the right hand side of (20) the external sum, as opposed to the internal sum contained in the ϱ terms.

(20) will be the starting point for the estimation for which we shall also need appropriate bounds on $\varrho(\mathbb{X})$ and $\varrho(Y)$.

C. Estimation of the ϱ Terms

Proposition 6. *For D sufficiently big, κ sufficiently small and some $C\geqq 1$*

$$|\varrho(Y)|\leqq(C\kappa^{1/2})^{|Y|-1}\exp[-AL(Y)],\tag{21}$$

$$|\varrho(\mathbb{X})|\leqq\exp\left[-\tfrac{1}{2}D\sum_{j\in H}(\ell_j+L(\underline{u}_j))\right](C\kappa^{1/2})^{\left|X\setminus\bigcup\underline{u}_j\right|}$$
$$\cdot\exp\left[-AL\left((\underline{u}_j)_{j\in H};X\setminus\bigcup_{j\in H}\underline{u}_j\right)\right],\tag{22}$$

where $\mathbb{X}=(H,X)$ and $L\left((\underline{u}_j)_{j\in H};X\setminus\bigcup_{j\in H}\underline{u}_j\right)$ is the length of the shortest graph on the points of X and possibly other points connected with respect to the sets $\underline{u}_j$, $j\in H$, and the points of $X\setminus\bigcup_{j\in H}\underline{u}_j$.

Proof. Define for $X_1,\ldots,X_a$, Y being disjoint lattice subsets, $X_1,\ldots,X_a\neq 0$,

$$\tilde{\varrho}(X_1,\ldots,X_a;Y)=\sum_{\gamma_c}\prod_{\underline{y}\in\gamma_c}(\exp[-\mathscr{I}(\underline{y})]-1),\tag{23}$$

where $\sum_\gamma$ runs through the collections γ_c of $\underline{y}\subset\left(\bigcup_{X=1}X_\alpha\right)\cup Y$, $|y|\geqq 2$, such that $(\cup X_\alpha)\cup Y$ cannot be split into two subsets so that each X_α and each $\underline{y}\in\gamma_c$ is in some set of the partition.

We shall estimate $\tilde{\varrho}$ using the standard Kirkwood-Salzburg equation method (see e.g. [10]).

Lemma 8. *For $x_1\in X_1$, $X_1':=X_1\setminus\{x_1\}$*

$$\tilde{\varrho}(x_1,\ldots,x_a;Y)=\exp\left[-\sum_{\substack{\underline{y}\\ X_1\in\underline{y}\subset X_1}}\mathscr{I}(\underline{y})\right]\sum_{S\subset X_2\cup\ldots\cup X_a\cup Y}K(X_1,S)$$
$$\cdot\tilde{\varrho}\left(X_1'\cup\left(\bigcup_{\alpha:S\cap X_\alpha\neq\emptyset}X_\alpha\right)\cup S,(X_\alpha)_{\alpha:S\cap X_\alpha=\emptyset};Y\setminus S\right),\tag{24}$$

where for $S \neq \emptyset$

$$K(X_1, S) = \sum_{\substack{\{y_1, \ldots, y_b\} \\ y_\beta \neq \emptyset, \cup y_\beta = S}} \prod_\beta \left(\exp\left[- \sum_{\substack{y \\ x_1 \in y \subset X_1}} \mathscr{I}(\underline{y} \cup \underline{y}_\beta) \right] - 1 \right) \tag{25}$$

and

$$K(X_1, \emptyset) = 1 . \tag{26}$$

Proof of Lemma 8. Let us consider in γ_c in (23) those $\underline{y}$ for which $x_1 \in \underline{y} \cap X_1 \neq \underline{y}$ taking $S = \cup(\underline{y} \backslash X_1)$ and $\{\underline{y}_1, \ldots, \underline{y}_b\} = \{\underline{y} \backslash X_1\}$. The set γ' of those $\underline{y}$ is described by giving $\{\underline{y}_1, \ldots, \underline{y}_b\}$ and the sets $\underline{y}_{\beta\gamma}$, $x_1 \in \underline{y}_{\beta\gamma} \subset X_1$ for fixed β giving $\{\underline{y} \cap X_1 : \underline{y} \in \gamma'$ and $\underline{y} \backslash X_1 = \underline{y}_\beta\}$. For $\underline{y} \in \gamma_c \backslash \gamma'$, $x_1 \in \bar{y}$ we have $\underline{y} \subset X_1$. Hence we may write

$$\begin{aligned} \tilde{\varrho}(X_1, \ldots, X_a; Y) = & \sum_{\{\underline{y}_\gamma : x_1 \in \underline{y}_\gamma \subset X_1\}} \prod_\gamma (\exp[-\mathscr{I}(\underline{y}_\gamma)] - 1) \\ & \cdot \sum_{S \subset X_2 \cup \ldots \cup X_a \cup Y} \; \sum_{\substack{\{y_1, \ldots, y_b\} \\ y_\beta \neq \emptyset, \cup y_\beta = S}} \prod_\beta \sum_{\{y_{\beta\gamma}\}} \prod_\gamma (\exp[-\mathscr{I}(\underline{y}_{\beta\gamma} \cup \underline{y}_\beta)] - 1) \\ & \cdot \tilde{\varrho}\left(X_1' \cup \left(\bigcup_{\alpha : S \cap X_\alpha \neq \emptyset} X_\alpha \right) \cup S, (X_\alpha)_{\alpha : S \cap X_\alpha = \emptyset} ; Y \backslash S \right). \end{aligned} \tag{27}$$

(27) easily yields (24). □

We shall need the following estimate for the kernels $K(X_1, S)$:

Lemma 9. *For* κ *sufficiently small*

$$\sum_{S : S \cap X_1 = \emptyset} \kappa^{-\frac{1}{2}|S|} \exp[AL(\{X_1\} \cup S)] |K(X_1, S)| \leqq \exp[\mathcal{O}(\kappa)] . \tag{28}$$

Proof of Lemma 9. By virtue of Lemma 7 for κ sufficiently small

$$\begin{aligned} & \sum_{S : S \cap X_1 = \emptyset} \kappa^{-\frac{1}{2}|S|} \exp[AL(\{X_1\} \cup S)] |K(X_1, S)| \\ & \leqq \sum_{\substack{\{\underline{y}_1, \ldots, \underline{y}_b\} \\ \underline{y}_\beta \cap X_1 = \emptyset}} \prod_\beta \kappa^{-\frac{1}{2}|\underline{y}_\beta|} \exp[AL(\{x_1\} \cup \underline{Y}_\beta)] \left| \exp\left[- \sum_{\underline{y} : x_1 \in \underline{y} \subset X_1} \mathscr{I}(\underline{y} \cup \underline{y}_\beta) \right] - 1 \right| \\ & \leqq \sum_{b=0}^{\infty} \frac{1}{b!} \sum_{\substack{(\underline{y}_1, \ldots, \underline{y}_b) \\ \underline{y}_\beta \cap X_1 = \emptyset}} \prod_\beta C \kappa^{-\frac{1}{2}|y_\beta|} \sum_{\underline{y} : X_1 \in \underline{y} \subset X_1} \exp[AL(\underline{y} \cup \underline{y}_\beta)] |\mathscr{I}(\underline{y} \cup \underline{y}_\beta)| \\ & \leqq \sum_{b=0}^{\infty} \frac{1}{b!} \left(C \sum_{\underline{y} \ni X_1} \kappa^{-\frac{1}{2}|\underline{y}|} \exp[AL(\underline{y})] |\mathscr{I}(\underline{y})| \right)^b \leqq \exp(\mathcal{O}(\kappa)) . \quad \square \end{aligned}$$

The Kirkwood-Salzburg equation (24) together with Lemma 9 allow us to prove

Lemma 10. *For* κ *sufficiently small*

$$|\tilde{\varrho}(X_1, \ldots, X_a; Y)| \leqq C^{\sum_{i=1}^{a} |X_i| - 1} (C\kappa^{1/2})^{|Y|} \exp[-AL((x_\alpha); Y)] \tag{29}$$

uniformly in Z *(for almost all* Z*).*

Proof of Lemma 10. We shall proceed by induction over $M := |X_1| + \ldots + |X_a| + |Y|$. For $M = 1$ $\tilde{\varrho}(\{x_1\}; \phi) = 1$ so that (29) holds. Now suppose that we have proven (29) for $M \leqq M_0$. Then for $M = M_0 + 1$

$$\begin{aligned}
|\tilde{\varrho}(X_1, \ldots, X_a; Y)| &\leqq \exp[\mathcal{O}(\kappa)] \sum_{S \subset X_2 \cup \ldots \cup X_a \cup Y} |K(X_1, S)| C^{\sum_1^a |X_i| + |S| - 2} \\
&\quad \cdot (C\kappa^{1/2})^{|Y \setminus S|} \exp\left[AL\left(X_1' \cup \left(\bigcup_{\alpha: S \cap X_\alpha \neq \emptyset} X_\alpha\right) \cup S, (X_\alpha)_{\alpha: S \cap X_\alpha = \emptyset}; Y \setminus S\right)\right] \\
&\leqq \exp[\mathcal{O}(\kappa)] C^{\sum_1^a |X_i| - 2} (C\kappa)^{\frac{1}{2}|Y|} \exp[AL((X_\alpha); Y] \\
&\quad \cdot \sum_{S: S \cap X_1 = \emptyset} \kappa^{-\frac{1}{2}|S|} \exp[AL(\{X_1\} \cup S)] |K(X_1, S)|,
\end{aligned} \tag{30}$$

where we have used

$$L(\{X_1\} \cup S) + L\left(X_1' \cup \left(\bigcup_{\alpha: S \cap X_\alpha \neq \emptyset} X_\alpha\right) \cup S, (X_\alpha)_{\alpha: S \cap X_\alpha = \emptyset}; Y \setminus S\right) \geqq L((X_\alpha); Y).$$

By Lemma 9, (30) yields (29). □

Since for $\mathbb{X} = (H, X)$

$$\varrho(\mathbb{X}) = \exp\left[-D \sum_{j \in H} (\ell_j + L(\underline{u}_j))\right] \int \prod_{j \in H} Z_{\bar{u}_j} \tilde{\varrho}\left(X_1, \ldots, X_a; X \setminus \left(\bigcup_\alpha X_\alpha\right)\right) \prod_{y \in X} d\tilde{\chi}(Z_y),$$

where $X_1, \ldots, X_a$ are the sets of the finest partition of $\bigcup_{j \in H} \underline{u}_j$ such that each $\underline{u}_j$, $j \in H$, is in one subset of the partition and since

$$\varrho(Y) = \int \tilde{\varrho}(\{y_1\}; Y \setminus \{y_1\}) \prod_{y \in Y} d\tilde{\chi}(Z_y)$$

for some $y_1 \in Y$, Lemma 10 yields (21) and (22). □

D. Estimation of the External Sum

Let among the sequences $\bar{u}_j$, $j = 1, \ldots, k$, there be s different ones $\bar{u}_{j_1}, \ldots, \bar{u}_{j_s}$ occurring with multiplicities $M_1, \ldots, M_s$, $\sum_r M_r = k$. For $H \subset \{1, \ldots, k\}$ let $\underline{H} := (k_1, \ldots, k_s)$ be the sequence of the corresponding multiplicities ($k_r \geqq 0$). Let for $\mathbb{X} = (H, X)$, $\underline{\mathbb{X}} := (\underline{H}, X)$. Notice that $\varrho(\mathbb{X})$ depends only on $\underline{\mathbb{X}}$. Denote by $\mathbb{V}$ either $\underline{\mathbb{X}}$ or Y and put

$$\varrho(\mathbb{V}) = \begin{cases} \varrho(\mathbb{X}) & \text{if } \mathbb{V} = \underline{\mathbb{X}} \\ \varrho(Y) & \text{if } \mathbb{V} = Y, \end{cases} \tag{31}$$

$$L(\mathbb{V}) = \begin{cases} L\left((\underline{u}_j)_{j \in H}; X \Big\backslash \bigcup_{j \in H} \underline{u}_j\right) & \text{if } \mathbb{V} = \underline{\mathbb{X}} \\ L(Y) & \text{if } \mathbb{V} = Y \end{cases} \tag{32}$$

and $V = X$ or Y respectively. Consider for $m \geqq 1$, $n \geqq 0$

$$\begin{aligned}
\phi(\mathbb{V}_1, \ldots, \mathbb{V}_m; \mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n}) &\equiv \sum_\Gamma{}' \prod_{\mathcal{L} \in \Gamma} A(\mathcal{L}) \prod_{i=1}^{m+n} \varrho(\mathbb{V}_i), \\
\phi(\mathbb{V}_1; \emptyset) &= \varrho(\mathbb{V}_1),
\end{aligned} \tag{33}$$

where $\sum_\Gamma{}'$ runs through the diagrams on the elements V_i, $1 \leqq i \leqq m+n$, (treated as different ones for different i's) connected with respect to the group $\{V_i\}$, $1 \leqq i \leqq m$, and the elements V_i, $m+1 \leqq i \leqq m+n$.

Lemma 11. *For D and A big enough, κ small enough and some C* $(n \geqq 1)$

$$\sum_{\substack{(\mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n}) \\ \sum_{i=m+1}^{m+n} L(\mathbb{V}_i) \geqq E}} |\phi(\mathbb{V}_1, \ldots, \mathbb{V}_m; \mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n})| \leqq n!2^{-n} \exp\left[-\frac{AE}{2}\right] \prod_{i=1}^{m} (C^{|V_i|}|\varrho(\mathbb{V}_i)|). \tag{34}$$

Proof of Lemma 11. (Compare [1, Lemma II, 4].) We have the following Kirkwood-Salzburg equation obtained by studying the lines of the graphs Γ ending on V_1:

$$\phi(\mathbb{V}_1, \ldots, \mathbb{V}_m; \mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n}) = \varrho(\mathbb{V}_1) \prod_{i=2}^{m} U(V_1, V_i)$$
$$\cdot \sum_{\Omega \subset \{m+1, \ldots, m+n\}} \prod_{t_1 \in \Omega} A(V_1, V_{t_1}) \phi(\mathbb{V}_2, \ldots, \mathbb{V}_m, \{\mathbb{V}_{t_1}\}_{t_1 \in \Omega}; \{\mathbb{V}_{t_2}\}_{t_2 \notin \Omega}), \tag{35}$$

where by definition $\phi(\emptyset; \mathbb{V}, \ldots) = 0$.

We shall use induction on $m+n$. Suppose that (34) has been proven for $m+n \leqq M_0$ [it holds for $m+n=2$ since $|\phi(\mathbb{V}_1; \mathbb{V}_2)| = |\varrho(\mathbb{V}_1)\varrho(\mathbb{V}_2)|$ and we can use (21) and (22)]. Then for $m+n = M_0+1$

$$\sum_{\substack{(\mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n}) \\ \sum L(\mathbb{V}_i) \geqq E}} |\phi(\mathbb{V}_1, \ldots, \mathbb{V}_m; \mathbb{V}_{m+1}, \ldots, \mathbb{V}_{m+n})| \leqq |\varrho(\mathbb{V}_1)| \sum_{p=0}^{n} \binom{n}{p}$$
$$\cdot \sum_{\substack{(\mathbb{V}_{t_1})_{m<t_1 \leqq m+p} \\ V_{t_1} \cap V_1 \neq \emptyset}} \sum_{\substack{(V_{t_2})_{t_2 > m+p} \\ \sum_{t_2 > m+p} L(\mathbb{V}_{t_2}) \geqq E - \sum_{m < t_1 \leqq m+p} L(\mathbb{V}_{t_1})}} |\phi(\mathbb{V}_2; \ldots; \mathbb{V}_{m+p}; \mathbb{V}_{m+p+1}, \ldots, \mathbb{V}_{m+n})|$$
$$\leqq |\varrho(\mathbb{V}_1)| \sum_{p=0}^{n-1} \binom{n}{p} \sum_{\substack{(\mathbb{V}_{t_1}) \\ V_{t_1} \cap V_1 \neq \emptyset}} (n-p)! 2^{-n+p} \exp\left[-\tfrac{1}{2}A\left(E - \sum_{t_1} L(\mathbb{V}_{t_1})\right)\right]$$
$$\cdot \prod_{i=2}^{m+p} (C^{|V_i|}|\varrho(\mathbb{V}_i)|) + \prod_{i=1}^{n+1} |\varrho(\mathbb{V}_i)|$$
$$\leqq n! 2^{-n} \exp[-\tfrac{1}{2}AE] \prod_{i=2}^{m} (C^{|V_i|}|\varrho(\mathbb{V}_i)|)\varrho(\mathbb{V}_1)$$
$$\cdot \sum_{p=0}^{\infty} \frac{1}{p!} \left(\sum_{\substack{\mathbb{V} \\ V \cap V_1 \neq \emptyset}} 2C^{|V|} |\varrho(\mathbb{V})| \exp[\tfrac{1}{2}AL(\mathbb{V})] \right)^p, \tag{36}$$

where we used $|\phi(\mathbb{V}_1, \ldots, \mathbb{V}_m; \phi)| \leqq \prod_{i=1}^{m} |\varrho(\mathbb{V}_i)|$, following from (35).

We shall estimate the sum over $\mathbb{V}$ by the method of combinatorial coefficients [12]:

$$\sum_{\beta} |A_\beta| \leqq \sup_{\beta} |C_\beta A_\beta| \quad \text{if} \quad \sum_{\beta} |C_\beta|^{-1} \leqq 1.$$

Given V one may construct $\mathbb{V}$ in the following way:

1. Choose $Q \subset \bigcup_{j=1}^{k} \underline{u}_j$ which will contain a point in each $\underline{u}_{j_r}$ such that $h_r > 0$ ($Q = \phi$ will correspond to $\mathbb{V} = Y$).

2. Choose the number $h=\sum_{r=1}^{s} h_r$.

3. For each $q\in Q$ choose $n_q \geqq 1$. n_q will be equal to the sum of $h_r>0$ over those r for which q will be taken as the point of $\underline{u}_{j_r}$. $\sum_{q\in Q} n_q = h$.

4. For each $q\in Q$ choose n_q times the sets $\underline{u}_{j_r}$ containing q.

5. For each set $\underline{u}_{j_r}$ just chosen pick a corresponding sequence $\bar{u}_{j_r}$.

Notice that in Step 5 we only have to specify the multiplicities of the points of $\underline{u}_{j_r}$ since, as we have assumed, points in $\bar{u}_j$ appear in fixed order.

We take account of the procedure described above by choosing proper combinatorial coefficients for each step:

Ad 1. $2^{|V|}$,

Ad 2. $2^{\sum h_r}$,

Ad 3. $\binom{\sum h_r - 1}{|Q|-1} < 2^{\sum h_r}$,

Ad 4. $\exp\left[\mathcal{O}(1)\sum_{r=1}^{s} h_r(\ell_{j_r} + L(\underline{u}_{j_r}))\right]$ for $\mathcal{O}(1)$ sufficiently big (see below),

Ad 5. $2^{\sum h_r \ell_{j_r}}$.

Only Step 4 is non-trivial. Estimate

$$\begin{aligned}\sum_{\underline{u}\ni q} \exp[-\mathcal{O}(1)(|\underline{u}|+L(\underline{u}))] &\leqq \sum_{\ell=1}^{\infty} \frac{1}{(\ell-1)!} \sum_{(u_2,\dots,u_\ell)} \exp[-\mathcal{O}(1)\ell] \\ &\quad \cdot \exp[-\mathcal{O}(1)\varepsilon\mathcal{L}(\{q,u_2,\dots,u_\ell\})] \\ &\leqq \sum_{\ell=1}^{\infty}\sum_{\tau} \frac{1}{(\ell-1)!}\exp[-\mathcal{O}(1)\ell] \sum_{(u_2,\dots,u_\ell)} \exp[-\mathcal{O}(1)\varepsilon L_\tau(q,u_2,\dots,u_\ell)] \\ &\leqq \sum_{\ell=1}^{\infty} \frac{\ell^{\ell-2}}{(\ell-1)!}\exp[-\mathcal{O}(1)\varepsilon\ell] \leqq 1 \end{aligned} \tag{37}$$

if $\mathcal{O}(1)$ is big enough. In (37) we have used $L(\underline{u}) \geqq \varepsilon\mathcal{L}(\underline{u})$ where $\mathcal{L}(\underline{u})$ is the length of the shortest tree on $\underline{u}$ (and no other points) (see [6, p. 197]) and $\sum_\tau 1 = \ell^{\ell-2}$ where $\sum_\tau$ runs through the trees on ℓ points.

(37) shows that we have chosen a correct coefficient for Step 4.

Altogether we may bound the product of the coefficients just chosen by

$$2^{|V|}\exp\left[\mathcal{O}(1)\sum_{r=1}^{s} h_r(\ell_{j_r} + L(\underline{u}_{j_r}))\right] \tag{38}$$

for $\mathcal{O}(1)$ sufficiently big. Hence

$$\begin{aligned}\sum_{V\cap V_1 \neq \emptyset} 2C^{|V|}|\varrho(\mathbb{V})|\exp[\tfrac{1}{2}AL(\mathbb{V})] &\leqq \sum_{V:V\cap V_1\neq\emptyset} 2(2C)^{|V|}|\varrho(\mathbb{V})| \\ &\quad \cdot\exp\left[\mathcal{O}(1)\sum_r h_r(\ell_{j_r} + L(\underline{u}_{j_r}))\right]\exp[\tfrac{1}{2}AL(\mathbb{V})] \\ &\leqq \sum_{V:V\cap V_1\neq\emptyset} \exp[-\mathcal{O}(1)(|V|+L(V))] \leqq |V_1| \end{aligned} \tag{39}$$

if only D, A and κ^{-1} are big enough. We have used (21), (22), (31), (32) and in the last estimate also (37).

Substituting (39) into (36) we complete the proof of (34) for $m+n=M_0+1$ and thus altogether. $\square$

Now we come back to (20). Denoting $\underline{H}!:=\prod_{r=1}^{s} k_r!$ and using (33) we may rewrite (20) as

$$
\begin{aligned}
|\langle Z_{\bar{u}_1};\ldots;Z_{\bar{u}_k}\rangle^T| &\leqq \prod_{j=1}^{k}\exp[D(\ell_j+L(\underline{u}_j)] \sum_{\substack{(H_\alpha)_{\alpha=1}^{A} \\ \Sigma \underline{H}_\alpha=(M_1,\ldots,M_s)}} \frac{1}{A!}\frac{M_1!\ldots M_s!}{\prod_\alpha \underline{H}_\alpha!} \\
&\quad\cdot \sum_{(\mathbb{X}_\alpha)_{\alpha=1}^{A}} \sum_{(Y_\beta)_{\beta=1}^{B}} \frac{1}{B!}|\phi(\underline{\mathbb{X}}_1;\underline{\mathbb{X}}_2,\ldots,\underline{\mathbb{X}}_A, Y_1,\ldots,Y_B)| \\
&\leqq \prod_i M_i!\prod_j \exp[D(\ell_j+L(\underline{u}_j))] \sum_{\mathbb{X}_1:x_0\in X_1} \sum_{(\mathbb{X}_\alpha)_2^A} \frac{1}{(A-1)!} \\
&\quad\cdot \sum_{(Y_\beta)_{\beta=1}^{B}} \frac{1}{B!}|\phi(\underline{\mathbb{X}}_1;\underline{\mathbb{X}}_2,\ldots,\underline{\mathbb{X}}_A, Y_1,\ldots,Y_B)|,
\end{aligned}
\tag{40}
$$

where x_0 is a fixed point of $\bigcup_{j=1}^{k} \underline{u}_j$. Notice that

$$
\sum_{\alpha=1}^{A} L(\underline{\mathbb{X}}_\alpha)+\sum_{\beta=1}^{B} L(Y_\beta)\geqq L(\underline{u}_1,\ldots,\underline{u}_k)
\tag{41}
$$

for non-vanishing terms ϕ. Hence

$$
\begin{aligned}
|\langle Z_{\bar{u}_1};\ldots;Z_{\bar{u}_k}\rangle^T| &\leqq \prod_i M_i!\prod_j \exp[D(\ell_j+L(\underline{u}_j))] \\
&\quad\cdot \sum_{\mathbb{X}_1:x_0\in X_1} \sum_{A+B\geqq 1} \frac{1}{(A+B-1)!} \sum_{\substack{(\mathbb{V}_2,\ldots,\mathbb{V}_{A+B}) \\ \sum_{i=2}^{A+B} L(\mathbb{V}_i)\geqq L(u_1,\ldots,u_k)-L(\mathbb{X}_1)}} |\phi(\underline{\mathbb{X}}_1;\mathbb{V}_2,\ldots,\mathbb{V}_{A+B})| \\
&\leqq \prod M_i!\prod_j \exp[D(\ell_j+L(\underline{u}_j))] \sum_{\mathbb{X}_1:x_0\in X_1} \\
&\quad\cdot \sum_{A+B\geqq 1} 2^{-(A+B-1)}\exp[-\tfrac{1}{2}A(L(\underline{u}_1;\ldots;\underline{u}_k)-L(\underline{\mathbb{X}}_1))]C^{|X_1|}|\varrho(\underline{\mathbb{X}}_1)| \\
&\leqq \prod_j M_i!\prod_j \exp[D(\ell_j+L(\underline{u}_j))]\exp[-\tfrac{1}{2}AL(\underline{u}_1;\ldots;\underline{u}_k)] \\
&\quad\cdot \sum_{X_1\ni x_0} \exp[-\mathcal{O}(1)(|X_1|+L(X_1))]
\end{aligned}
\tag{42}
$$

with arbitrarily big $\mathcal{O}(1)$, provided that D, A, κ^{-1} are big enough. We have used in turn (34), (22) and the argument with the combinatorial coefficient (38) to bound the sum over $\mathbb{X}_1$ with fixed X_1. (42) together with (37) imply (5.13) completing the proof of Proposition 2. $\square$

Appendix

Proof of Lemma 2. Take the shortest tree τ on the points of $\underline{t}$ and possibly other points. Consider the set $L\underline{s}$. For each point Ls of $L\underline{s}$ which is not in $\underline{t}$ or in the set of the other points, add to τ a line joining this point to one of the points t of $\underline{t}$ such that $L^{-1}t=s$. We obtain a tree on Ls and possibly other points of the length $\leqq L(\underline{t}) + |\underline{s}|\left(\frac{L-1}{2}\right)d$ (recall that we work with the distance being the sum of coordinate distances). This gives (5.23).

Now consider a map $\kappa:\mathbb{R}^1\to\mathbb{R}^1$

$$\kappa(x):=\begin{cases}[L^{-1}x] & \text{if} \quad |x-L[L^{-1}x]|<\frac{1}{2}(L-1),\\ x-(L-1)u-\frac{1}{2}(L-1) & \text{if} \quad Lu+\frac{1}{2}(L-1)<x<Lu+\frac{1}{2}(L+1).\end{cases}$$

κ shrinks the balls of radius $\frac{1}{2}(L-1)$ around the points of $L\mathbb{Z}^d$ to points. Notice that

$$|\kappa(x_1)-\kappa(x_2)|\leqq|x_1-x_2|.$$

Consider the d^{th} Cartesian power of $\kappa, \kappa^d:\mathbb{R}^d\to\mathbb{R}^d$. κ^d also decreases the distance (in the periodic case $\mathbb{R}^d$ must be replaced by tora). But $\kappa^d(\underline{t})=[L^{-1}\underline{t}]=\underline{s}$. For τ as above application of κ^d produces a tree on $\underline{s}$, and possibly other points, which is not larger than κ. Hence (5.24) follows. Under κ^d acting on the endpoints, the length of the lines of τ is preserved only if the endpoints lie in one of the $1\times \ldots \times 1$ cubes which are not shrunk by κ^d. Any line joining points in two different cubes gets shortened by at least $L-1$ (here we use the non-Pythagorean form of the distance). It is possible to follow along τ in a continuous way running through each line at most two times. Suppose that doing this we cross the points of $\underline{t}$ in the order $(t_1, \ldots, t_{|t|})$. Since a $1\times \ldots \times 1$ cube contains at most 2^d (lattice) points in $\underline{t}$, when following along τ we have to jump to another cube at least $2^{-d}|\underline{t}|-1$ times. Thus the length of τ gets shortened by at least $\frac{1}{2}(L-1)(2^{-d}|t|-1)$ under κ^d. Hence

$$L(\underline{s})\leqq L(\underline{t})-\tfrac{1}{2}(L-1)2^{-d}|\underline{t}|+\tfrac{1}{2}(L-1).$$

This together with (5.23) yields (5.25). □

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