

Integrable Nonlinear Equations and Liouville's Theorem, II

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Abstract. A symplectic structure is constructed and the Liouville integration carried out for a stationary Lax equation $[L, P] = 0$, where L is a scalar differential operator of an arbitrary order. n^{th} order operators are included into the variety of first-order matrix operators, and properties of this inclusion are studied.

This article is in fact the third part of the work ([1, 2]) although the last two parts are independent of the first. Here we deal with equations arising from an n^{th} order linear differential operator. For simplicity we consider the scalar case only, this restriction is not of great importance. The integration of such equations was carried out first by Krichever ([3, 4]) and his work relates to this article as our previous paper relates to the work of Dubrovin ([5, 6]). But this time the connection is much weaker. Our method does not resemble Krichever's, in particular since we use different variables, it is even difficult to compare the results. We use the reduction of an n^{th} order differential operator to a first order matrix operator (in an Appendix we discuss this reduction in more details than are needed for this article). After this reduction, the further development is close to that of the previous article, however with essential differences. We pay more attention to these differences, as often as possible replacing detailed proofs by references to [2].

1. We start off with the equation

$$-Q' + [U + \zeta A, Q] = 0 \quad (1)$$

where Q, U, A are $n \times n$ matrices, $\zeta = z^n$ a complex parameter, A and U have the form

$$A = \begin{pmatrix} 0 & & & & \\ \vdots & \ddots & & & \\ & & \ddots & & \\ 1 & \dots & 0 & & \end{pmatrix} \quad U = \begin{pmatrix} 0 & 1 & . & . & . & . & . & . & . & . \\ \vdots & & & & & & & & & \\ & & & & & & & & & \\ -u_0, \dots, -u_{n-2}, 0 \end{pmatrix}.$$

u_0, \dots, u_{n-2} will be taken as independent generators of a differential algebra \mathcal{A} (which consists of polynomials in $u_i^{(k)}$ with complex coefficients). The matrix Q is a solution we are looking for. We shall numerate rows and columns from 0 to $n - 1$.

We give another form of the equation, in another basis. Letting Z be a matrix $(z^i \delta_{ij})$ put $P = Z^{-1} Q Z$. For P we have

$$-P' + [V + zB, P] = 0, \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ 1 & \dots & 0 & \end{pmatrix} \quad V = \begin{pmatrix} 0 & & & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -z^{-n+1}u_0, \dots, -z^{-1}u_{n-2}, 0 \end{pmatrix}.$$

Only the last row of the matrix V is nonzero. Finally we can transform B to the diagonal form. If ε is a primitive root of 1, $\varepsilon^n = 1$, let K be the matrix $\frac{1}{\sqrt[n]{n}}(\varepsilon^{ij})$. Then for $C = K^{-1}BK, R = K^{-1}PK, W = K^{-1}VK$ we have

$$C_{ij} = \varepsilon^i \delta_{ij}, W_{ij} = -\frac{1}{n(\varepsilon^i z)^{n-1}} \sum_{\alpha=0}^{n-2} u_{\alpha}(\varepsilon^j z)^{\alpha}$$

and

$$-R' + [W + zC, R] = 0. \tag{3}$$

By comparison with [2], we see that this form of the equation resembles the main equation (1) of [2], because the matrix C is diagonal and nondegenerate. The distinction (besides an insignificant difference in the sign) consists of the special form of the matrix W which incorporates the spectral parameter z in negative powers.

It is convenient to introduce a grading in the algebra \mathcal{A} . Namely let us take the number $n - i + k$ as the weight of a factor $u_i^{(k)}$. The weight of a product will be the sum of the weights of factors. The weight of the operator $\partial = d/dx$ and the weight of z will both be 1. In what follows all formulas are homogeneous with respect to this weight.

2. We seek the solutions as formal series $R = \sum_0^{\infty} R_r z^{-r}, (R)_{ij} \in \mathcal{A}$ and corresponding $P = \sum_0^{\infty} P_r z^{-r}, Q = \sum_{r=-n+1}^{\infty} Q_r z^{-r}$. We can formulate and prove some simple assertions similar to those in [2].

Proposition 1. *The solutions form a ring.*

Proposition 2. R_0 is a diagonal matrix with constant coefficients.

Proposition 3. A solution is uniquely determined by the constants in all $(R_r)_{ij}$.

(Instead of R we could write P or Q here).

Further solutions $R^\alpha, \alpha = 0, \dots, n - 1$ will be constructed for which

$$R_0^\alpha = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix}$$

(unity stands on the α^{th} place) and without constants in other $R_r, r > 0$.

Proposition 4.

$$R^\alpha R^\beta = \delta_{\alpha\beta} R^\alpha, \sum_{\alpha=0}^{n-1} R^\alpha = I.$$

Proposition 5. *The general form of a solution is $\sum_{\alpha=0}^{n-1} w_\alpha(z)R^\alpha$, where $w_\alpha(z)$ are formal series in z^{-1} with constant coefficients. Matrices R^α are spectral projectors of an arbitrary solution; all the solutions commute. The corresponding P, Q will be denoted as P^α, Q^α . $(P_0^\alpha)_{ij} = \frac{1}{n} \varepsilon^{\alpha(i-j)}$ holds.*

3. The construction of R^α is the same as in [2]. We seek the solution of a vector equation

$$-\varphi^{*'} + (W + zC)\varphi^* = \lambda\varphi^*, \tag{4}$$

where $\varphi^* = (\varphi_i^*), i = 0, \dots, n - 1$ is a row vector, λ is a scalar, in the form

$$\varphi_i^* = \sum_{r=0}^{\infty} \varphi_{i,r}^* z^{-r}, \quad \lambda = \sum_{r=0}^{\infty} \lambda_r z^{-r+1}. \tag{5}$$

Note that the matrices W_{ij} and C_{ij} are formally defined for all i, j and $W_{i+n,j+n} = W_{ij}, C_{i+n,j+n} = C_{ij}$. Therefore it is convenient to consider φ_i^* as defined for all i and periodic: $\varphi_{i+n}^* = \varphi_i^*$.

We can take $\varphi_{i,0}^* = \delta_{i\alpha}; \varphi_{\alpha,r}^* = 0, r > 0; \lambda_0 = \varepsilon^\alpha$ for arbitrary α and then find all other $\varphi_{i,r}^* \in \mathcal{A}$ and $\lambda_r \in \mathcal{A}$, in the same way as in [2]. The insignificant distinction is that now the matrix W depends on z . Its expansion includes, however, only negative powers of z , which does not spoil the recurrence procedure. The obtained solution will be denoted as $\varphi^{*\alpha}, \lambda^\alpha$.

Proposition 1. *The dependence of $\varphi_{i,r}^{*\alpha}, \lambda_r^\alpha$ on α is as follows*

$$\varphi_{i,r}^{*\alpha} = \varphi_{i-\alpha,r}^{*0} \varepsilon^{-\alpha r}, \lambda_r^\alpha = \lambda_r^0 \varepsilon^{-\alpha(r-1)}. \tag{6}$$

Proof. The matrices W and zC enjoy the property $W_{ij}(z) = W_{i+1,j+1}(\varepsilon^{-1}z), zC_{ij} = \varepsilon^{-1}zC_{i+1,j+1}$. Therefore if $\{\varphi_i^*(z), \lambda(z)\}$ is a solution of (4) so is $\{\varphi_{i+1}^*(\varepsilon^{-1}z), \lambda(\varepsilon^{-1}z)\}$. Thus it is easy to see that $\{\varphi_{i+1}^{*\alpha+1}(\varepsilon^{-1}z), \lambda^{\alpha+1}(\varepsilon^{-1}z)\}$ is the same as $\{\varphi_i^{*\alpha}(z), \lambda^\alpha(z)\}$. We have

$$\begin{aligned} \varphi_{i+1,r}^{*\alpha+1} \varepsilon^r &= \varphi_{i,r}^{*\alpha} = \dots = \varphi_{i-\alpha,r}^{*0} \varepsilon^{-\alpha r}, \text{ and } \lambda_r^{\alpha+1} \varepsilon^{(r-1)} \\ &= \lambda_r^\alpha = \dots = \lambda_r^0 \varepsilon^{-\alpha(r-1)} \end{aligned} \quad \blacksquare$$

Proposition 2. *The weights of $\varphi_{i,r}^{*\alpha}$ and λ_r^α are r .*

Proof. Equation (4) is homogeneous in weight if we consider λ as having the weight 1. Hence the expansions of $\varphi^{*\alpha}$ and λ^α in z^{-1} must be homogeneous. Taking into account that $\varphi_{i,0}^{*\alpha}$ and λ_0^α have the weight 0 and that z has the weight 1, we obtain the required result. ■

Let Φ^* be a matrix $(\varphi_i^{*\alpha})$ (we consider α as the number of a column, i that of

a row) and $\Psi^* = \Phi^{*-1} = \{\psi_\alpha^{*i}\}$ (α is the number of a row). The row vectors $\psi_\alpha^* = \{\psi_\alpha^{*i}\}$ satisfy the adjoint equation of (4):

$$\psi^{*'} + \psi^*(W + zC) = \lambda\psi^*. \tag{7}$$

The coefficients $\psi_{\alpha,r}^{*i}$ satisfy the same relations (6) as $\varphi_{i,r}^{*\alpha}$. The projection operators R^α can be built as $R_{ij}^\alpha = \varphi_i^{*\alpha}\psi_\alpha^{*j}$, that is $R^\alpha = \varphi^{*\alpha}\cdot\psi_\alpha^*$. The dependence of R^α on α is as follows:

$$R_{ij,r}^\alpha = \varepsilon^{-\alpha r} R_{i-\alpha,j-\alpha,r}^0. \tag{8}$$

Now we return to the first basis. Letting φ^α be $ZK\varphi^{*\alpha}$ and ψ_α be $\psi_\alpha^*K^{-1}Z^{-1}$ we obtain $Q^\alpha = \varphi^\alpha\cdot\psi_\alpha$ where φ^α and ψ_α satisfy equations

$$-\varphi' + (U + \zeta A)\varphi = \lambda\varphi, \tag{9}$$

$$\psi' + \psi(U + \zeta A) = \lambda\psi. \tag{10}$$

The dependence of $Q_{ij,r}^\alpha$ on α is:

$$Q_{ij,r}^\alpha = \varepsilon^{-\alpha r} Q_{ij,r}^0, \tag{11}$$

so the expansion of Q^α has a form

$$Q_{ij}^\alpha = \sum_{r=j-i}^{\infty} Q_{ij,r}^0 (\varepsilon^\alpha z)^{-r}. \tag{12}$$

Thus Q^α as functions of ζ are branches of one n -valued function $Q(\zeta)$.

The weight of the first coefficient $Q_{ij,j-i}^0$ is zero, every next coefficient has the weight of 1 more than the previous one, hence the weight of $Q_{ij,r}^0$ is $r + i - j$.

4. Proposition 1. *The identity*

$$\delta \operatorname{tr} AQ^\alpha = -\operatorname{tr}(\delta U Q_\zeta^\alpha) + \partial \operatorname{tr}(\delta\varphi^\alpha\cdot\psi_{\alpha,\zeta} - \varphi_\zeta^\alpha\cdot\delta\psi_\alpha) \tag{13}$$

holds where the subscript ζ denotes the derivative with respect to ζ .

This proposition has the same proof as the similar proposition in [2].

Corollary.

$$\frac{\delta}{\delta u_i} \operatorname{tr} AQ^\alpha \equiv \frac{\delta}{\delta u_i} Q_{0,n-1}^\alpha = \frac{\partial}{\partial \zeta} Q_{i,n-1}^\alpha.$$

Now let us take

$$\mathcal{L} = \operatorname{tr} AQ_{m+2n} = Q_{0,n-1;m+2n}$$

as a Lagrangian. We have omitted α in Q_{m+2n}^α since it does not play any role on account of (11). The number m is arbitrary.

The set of equations whose integration is our main object is

$$\frac{\delta \mathcal{L}}{\delta u_i} = 0, \quad i = 0, \dots, n - 2, \tag{14}$$

that is

$$Q_{i,n-1;m+n} = 0, \quad i = 0, \dots, n - 2. \tag{15}$$

Note the following distinction between this case and [2]. In that case we have had to take as Lagrangians combinations of R_m^α with different nonzero coefficients to obtain nondegenerate equations. Now we take Q_{m+2n}^α with only one (arbitrary) α since all of them are proportional.

Let us rewrite (14) distinguishing linear terms of the highest weight

$$\begin{aligned}
 0 &= \frac{\delta \mathcal{L}}{\delta u_0} \equiv a_{00} u_0^{(m-n+1)} + \dots + a_{0,n-2} u_{n-2}^{(m-1)} + \dots, \dots, \\
 0 &= \frac{\delta \mathcal{L}}{\delta u_{n-2}} \equiv a_{n-2,0} u_0^{(m-1)} + \dots + a_{n-2,n-2} u_{n-2}^{(m+n-3)} + \dots
 \end{aligned}
 \tag{16}$$

Other terms contain derivatives of lower orders. The coefficients a_{ij} are constants.

Proposition 2. *If m and n are mutually prime numbers then $\det(a_{ij}) \neq 0$.*

Proof. The matrix a_{ij} can be found explicitly. We seek $\varphi^{*\alpha}$ from (4). If we follow only the terms linear in u_k it can easily be found

$$\varphi_{i,r}^{*\alpha} = \delta_{i\beta} + \sum_{\beta=0}^{n-2} \frac{\varepsilon^{i-\alpha-r}}{(\varepsilon^{i-\alpha} - 1)^{\beta+r-n+1}} u_\beta^{(\beta+r-n)} + (\text{nonlinear terms}).$$

Thus

$$\psi_{\alpha,r}^{*j} = \delta_{\alpha j} + \sum_{\beta=0}^{n-2} \frac{\varepsilon^{-j-r}}{(\varepsilon^{\alpha-j} - 1)^{\beta+r-n+1}} u_\beta^{(\beta+r-n)} + (\text{nonlinear terms}).$$

After some obvious calculations we obtain

$$Q_{i,n-1,m+n}^0 = \sum_{j=0}^{n-1} \sum_{\gamma=1}^{n-1} \frac{\varepsilon^{(i+1)\gamma} - \varepsilon^{\gamma(i+1+m)}}{(\varepsilon^\gamma - 1)^{i+j+m-n+2}} u_j^{(i+j+m-n+1)},$$

whence

$$a_{ij} = \sum_{\gamma=1}^{n-1} \frac{\varepsilon^{(i+1)\gamma} - \varepsilon^{\gamma(i+1+m)}}{(\varepsilon^\gamma - 1)^{i+j+m-n+2}}; \quad i, j = 0, \dots, n-2.$$

The calculation of the determinant seems to be rather awkward but in fact it can be carried out surprisingly readily. First of all we replace the matrix a_{ij} by a matrix

with the same determinant $b_{ij} = \sum_{\alpha=0}^i \binom{i}{\alpha} (-1)^{i-\alpha} a_{\alpha j}$.

$$\begin{aligned}
 b_{ij} &= \sum_{\gamma=1}^{n-1} \sum_{\alpha=0}^i \binom{i}{\alpha} (-1)^{i-\alpha} \frac{\varepsilon^{\alpha\gamma}}{(\varepsilon^\gamma - 1)^\alpha} \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{j+m-n+2}} \\
 &= \sum_{\gamma=1}^{n-1} \left(\frac{\varepsilon^\gamma}{\varepsilon^\gamma - 1} - 1 \right)^i \cdot \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{j+m-n+2}} = \sum_{\gamma=1}^{n-1} \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{i+j+m-n+2}}.
 \end{aligned}$$

Let us construct a quadratic form with coefficients b_{ij} :

$$\sum_{i,j=0}^{n-2} b_{ij} \xi_i \xi_j = \sum_{\gamma=1}^{n-1} \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{m-n+2}} \cdot \sum_{i,j=0}^{n-2} \frac{\xi_i \xi_j}{(\varepsilon^\gamma - 1)^{i+j}} = \sum_{\gamma=1}^{n-1} c_\gamma \eta_\gamma^2,$$

where

$$c_\gamma = \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{m-n+2}}, \quad \eta_\gamma = \sum_{i=0}^{n-2} \frac{\xi_i}{(\varepsilon^\gamma - 1)^i}.$$

The determinant of the last quadratic form is $\prod c_\gamma$. Now we note that if m and n are mutually simple, then all c_γ do not vanish which proves our statement. Nevertheless we give the exact expression for $\det(a_{ij})$:

$$\det(a_{ij}) = \prod_{\gamma=1}^{n-1} \frac{\varepsilon^\gamma (1 - \varepsilon^{\gamma m})}{(\varepsilon^\gamma - 1)^{m+n-2}} \cdot \prod_{\substack{\gamma, \delta=1 \\ (\gamma > \delta)}}^{n-1} (\varepsilon^\gamma - \varepsilon^\delta)^2 = n^{-m+1} \quad \blacksquare$$

This proposition was first conjectured in [7] then proved by Veselov [8] who used quite other technical means.

Corollary. *There exists a sequence of nonvanishing determinants*

$$a_{0j_0} \neq 0, \quad \begin{vmatrix} a_{0j_0} & a_{0j_1} \\ a_{1j_0} & a_{1j_1} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{0j_0} & a_{0j_1} & a_{0j_2} \\ a_{1j_0} & a_{1j_1} & a_{1j_2} \\ a_{2j_0} & a_{2j_1} & a_{2j_2} \end{vmatrix} \neq 0, \dots,$$

where j_0, j_1, \dots, j_{n-2} is a permutation of the numbers $0, 1, \dots, n-2$.

Proposition 3. *All $u_j^{(p)}$ can be expressed as polynomials in $\partial^k(\delta\mathcal{L}/\delta u_i)$ and the following $u_j^{(p)}$ which will be called “phase variables”:*

$$u_{j_0}^{(p)}, \quad p \leq m - n + j_0; \quad u_{j_1}^{(p)}, \quad p \leq m - n + 1 + j_1; \quad u_{j_2}^{(p)}, \quad p \leq m - n + 2 + j_2; \\ \dots; \quad u_{j_{n-2}}^{(p)}, \quad p \leq m - n + (n - 2) + j_{n-2}.$$

Proof. From the first of the Eqs. (16) we express $u_{j_0}^{(m-n+j_0+1)}$ as required. From the set of two equations, the second, and the first equation differentiated we express $u_{j_0}^{(m-n+j_0+2)}$ and $u_{j_1}^{(m-n+j_1+1)}$ etc. \blacksquare

Corollary. *The order of the set of Eqs. (14) is $(n - 1)(m - 1)$.*

Indeed, the equations make it possible to express all the derivatives $u_i^{(p)}$ as polynomials in phase variables (since $\partial^k(\delta\mathcal{L}/\delta u_i) = 0$). The whole number of phase variables is

$$(m - n + 1 + j_0) + (m - n + 2 + j_1) + \dots + (m - n + n - 1 + j_{n-2}) = (n - 1)(m - 1).$$

From the nondegeneracy of Eqs. (14) it follows (see [7]) that this set of equations represents a Hamiltonian system with a nondegenerate symplectic form. We shall find this form.

Proposition 4. *The symplectic form corresponding to the Hamiltonian system (14) is*

$$\omega^{(2)} = \text{tr } \delta\varphi \wedge \delta\psi|_{m+n},$$

where the subscript $m + n$ denotes the coefficient by z^{-m-n} .

The proof is the same as in [2]. From this proposition it is clear that the

1-form ω corresponding to the system can be taken as

$$\omega = \text{tr } \varphi \delta \psi \Big|_{m+n}.$$

As in [2] we can transform this expression (on adding an exact differential) into

$$\omega = \frac{1}{Q_{jl}} (\delta Q \cdot Q)_{jl} \Big|_{m+n} \tag{16}$$

where j, l are arbitrary integers from 0 to $n - 1$.

5. Now we are going to find the first integrals of (14).

Proposition 1. *The set of equations (14) is equivalent to*

$$[A, Q_{m+n}] = 0. \tag{17}$$

Proof. This is evident in one direction: from (17) it follows in particular that $Q_{i,n-1;m+n} = 0, i = 0, \dots, n - 2$. The inverse assertion follows from the fact proved in the Appendix that $Q_{i,n-1;m+n} = 0, i = 0, \dots, n - 2$ implies that the matrix Q_{m+n} is strictly triangular, $Q_{ij;m+n} = 0$ when $i \leq j$ ■

We denote

$$\tilde{Q} = \sum_{s=0}^{s_1} Q_{m-ns} \zeta^s, \quad s_1 = \left[\frac{m-1}{n} \right] + 1. \tag{18}$$

Proposition 2. *Functions $u_i(x)$ satisfy (17) if and only if \tilde{Q} satisfy (1) where $u_i(x)$ and their derivatives are substituted for the letters $u_i^{(p)}$ into the differential polynomials $Q_{ij;m-ns}$.*

Proof. Equation (1) is equivalent to the recurrence relation

$$-Q'_r + [U, Q_r] + [A, Q_{r+n}] = 0. \tag{19}$$

Whence

$$-\tilde{Q}' + [U, \tilde{Q}] = -[A, Q_{m+n}].$$

The rest is plain ■

The essential difference between expression (18) and the corresponding expression for \tilde{P} in [2] is that the latter includes all the coefficients P_r with $r \leq m$ while \tilde{Q} contains only a subsequence of Q_r ; this is connected with the peculiarity of Eq. (1) that it involves z in the form of $\zeta = z^n$ and therefore the recurrence formula relates Q_r with Q_{r+n} instead of Q_{r+1} .

We write $m = ns_1 - \mu, 0 < \mu \leq n - 1$. Then $Q = Q_{-\mu} \zeta^{s_1} + Q_{-\mu+n} \zeta^{s_1-1} + \dots + Q_m$.

Proposition 3. *The matrices $Q_{-\mu}, Q_{-\mu+n}$ have such structure*

$$Q_{-\mu} = \underbrace{\begin{pmatrix} 0 & 0 \\ * & \dots & 0 \\ * & * & 0 \end{pmatrix}}_{\text{column } \mu} Q_{-\mu+n} = \overbrace{\begin{pmatrix} * & \dots & * & 0 \\ \dots & \dots & \dots & * \\ * & \dots & \dots & * \end{pmatrix}}^{\text{column } n - \mu}. \tag{20}$$

Proof. The expansion of P in z^{-1} begins with z^0 . Taking into account that $Q = ZPZ^{-1}$, we see that the expansion of Q_{ij} begins with z^{i-j} . The power z^μ can occur in this expansion if $i - j \geq \mu$, the power $z^{-\mu+n}$ if $i - j \geq \mu - n$ ■

Proposition 4. *The coefficients of the polynomials in ζ*

$$\text{tr } \tilde{Q}^k$$

are first integrals of (14). There are $(m - 1)(n - 1)/2$ such nontrivial first integrals.

Proof. Only the second assertion is not evident. Let us put $Q^* = \sum_{s=-\infty}^{s_1} Q_{m-ns} \zeta^s$.

This is a solution of (1) which can be expressed in terms of Q^α :

$$Q^* = \frac{1}{n} z^n \sum_{\alpha=0}^{n-1} \varepsilon^{-\alpha\mu} Q^\alpha. \tag{21}$$

The coefficients of $\text{tr} Q^{*k}$ for every k are identically constant. The coefficients of $\text{tr } \tilde{Q}^k$ coincide with corresponding coefficients of $\text{tr } Q^{*k}$, therefore they are trivial first integrals. Moreover for every k the coefficients of the expansion of $\text{tr } \tilde{Q}^k$ which coincide with those of $\text{tr } Q^{*k}$ are trivial first integrals. We must find out how many coefficients of $\text{tr } \tilde{Q}^k$ are different from those of $\text{tr } Q^{*k}$.

\tilde{Q}^k is formally a polynomial of degree ks_1 . However a few highest coefficients may vanish since $Q_{-\mu}$ is a nilpotent matrix (see (20)). It follows from (21) that

$$Q^{*k} = \frac{1}{n^k} z^{mk} \sum_{\alpha=0}^{n-1} \varepsilon^{-\alpha\mu k} Q^\alpha.$$

Putting $k = \left[\frac{\mu k}{n} \right] n + \rho$ we have

$$Q^{*k} = n^{-k+1} z^{mk} (Q_{-\rho} z^\rho + Q_{-\rho+n} z^{\rho-n} + \dots)$$

The highest power is $z^{mk+\rho} = z^{mk+\mu k - [\mu k/n]n} = z^{n(s_1 k - [\mu k/n])} = z^{n(s_1 k - [\mu k/n])} = \zeta^{s_1 k - [\mu k/n]}$.

Put $\tilde{Q} = Q^* + q$, i.e. $q = - \sum_{s=-\infty}^{-1} Q_{m-ns} \zeta^s$. We have $\text{tr } \tilde{Q}^k = \text{tr } Q^{*k} + k \text{tr } Q^{*k-1} R + \dots$. What is the highest power of ζ in $\text{tr } Q^{*k-1} R$? In Q^{*k-1} the highest power is $\zeta^{s_1(k-1) - [\mu(k-1)/n]}$ as we have seen. Its coefficient is a lower triangular matrix. The second factor R has the highest term with ζ^{-1} . Its coefficients is Q_{m+n} which is a strictly triangular matrix (see the proof of proposition 1). The product of these two matrices is a strictly triangular matrix whose trace vanishes. Thus the highest

nonvanishing term in $\text{tr } \tilde{Q}^k$ is of degree $s_1(k-1) - \left[\frac{\mu(k-1)}{n} \right] - 2$.

We obtain the whole number of

$$\sum_{k=2}^n \left(s_1(k-1) - \left[\frac{\mu(k-1)}{n} \right] - 1 \right)$$

nontrivial first integrals. To calculate this sum we remark that $\left[\frac{\mu(k-1)}{n} \right] =$

$\frac{\mu(k-1)}{n} - \left\{ \frac{\mu(k-1)}{n} \right\}, \sum_{k=2}^n \left\{ \frac{\mu(k-1)}{n} \right\} = \sum_1^{n-1} \frac{k}{n} = \frac{n(n-1)}{2n}$ (since μ and n are mutually simple). Now it is easy to finish up the calculation and to obtain the required number of first integrals ■

Note that the number of the known first integrals is half of the dimension of the phase space.

The coefficients of the characteristic polynomial

$$f(\zeta, w) = \det(\tilde{Q} - wI) = \sum_{k=0}^n \sum_{l=0}^{l_0(k)} J_{kl} w^{n-k} \zeta^l \tag{22}$$

$\left(l_0(k) = ks_1 - \left[\frac{\mu k}{n} \right] \right)$ may be taken as first integrals, instead of the coefficients of $\text{tr } \tilde{Q}^k$. The highest nontrivial coefficient J_{kl} with given k occurs for $l = (k-1)s_1 - \left[\frac{\mu(k-1)}{n} \right] - 2$.

6. The equation $f(\zeta, w) = 0$ specifies an algebraic function $w(\zeta)$. Its Riemann surface is n -sheeted. The branch points are those where $f_w = 0$ and also the point $\zeta = \infty$.

Proposition 1. *The behaviour of $w(\zeta)$ at infinity is as follows*

$$w = n^{-1} \zeta^{m/n} + O(\zeta^{-1}).$$

Proof. We have $\tilde{Q} = n^{-1} z^m \sum_{\alpha=0}^{n-1} \varepsilon^{-\alpha\mu} Q^\alpha + O(z^{-m})$ (see (21)). The eigenvalues of the first term are exactly $n^{-1} \varepsilon^{-\alpha\mu} z^m$, i.e. they are branches of the multivalued function $n^{-1} \zeta^{m/n}$. Note that $w(\zeta)$ are eigenvalues of $\tilde{Q}(\zeta)$ ■

Proposition 2. *The number of branch points in the finite part of the Riemann surface is $m(n-1)$ (in general they are of the second order). The genus of the Riemann surface is $(n-1)(m-1)/2$.*

Proof. The first assertion follows from the fact that the discriminant $\Delta = \prod (w_i - w_j)$ which is a polynomial in ζ behaves at infinity as $\zeta^{m(n-1)}$ (since $w_i \sim \zeta^{m/n}$, $i \neq j$). The genus ρ can be calculated according to the formula $2\rho = \sum (j_k - 1) - 2n + 2$ where j_k is the degree of the branch point, and n is the number of sheets. We have $m(n-1)$ branch points of the second order and one point of the n^{th} order, $2\rho = m(n-1) + n - 1 - 2n + 2 = (m-1)(n-1)$ ■

Note that the genus is equal to half of the phase space dimension.

We now introduce spectral projection operators of \tilde{Q} . If $f(\zeta, w) = \sum_{t=0}^n J_t(\zeta) w^t$ then the projection operator attached to the point $P = (\zeta, w)$ of the Riemann surface is

$$g(P) = (f_w)^{-1} \sum_{l=1}^n J_l(\zeta) \sum_{k=0}^{l-1} w^k \tilde{Q}^{l-1-k}. \tag{23}$$

space and

$$|d_i^e| = |d^{n-1-i,e}| = (2\rho + i + n - 1 - (n - 1 - i))/2 = \rho + i$$

as stated ■

A question arises whether the extension of the theory leads to a new class of integrable equations. We do not think so since for new equations the number of known first integrals remains the same as before while the dimension of the phase space has increased almost twice. Anyway this is an interesting point.

7. Now we are in a position to succeed in our main object, to integrate the 1-form and obtain the angle variables corresponding to action variables J_{kl} .

Proposition. *The 1-form ω can be written as*

$$\omega = n \frac{w(P)(\delta g \cdot g)_{jl}}{g_{jl}} \Big|_1$$

where the subscript 1 denotes the coefficient in ζ^{-1} in the asymptotical expansion at expansion at infinity.

Proof. This follows from (16) and proposition 1 Sect. 6, g is asymptotically equal to $Q(\zeta)$ ■

The obtained expression can be interpreted as the residue of the differential form $nw(P)(\delta g \cdot g)_{jl} \cdot (g_{jl})^{-1} d\zeta$ at the point $\zeta = \infty$ of the Riemann surface. This can be calculated in the same way as in [2], and we have

Theorem 1. *The 1-form can be written as*

$$\omega = \sum_{P \in d_i} w(P) d\zeta_P. \tag{24}$$

There is, however, an essential difference between this case and that in [2]. The points of the divisor d_0 (but not of $d_i, i = 0$) can be taken as independent coordinates in the phase space (together with J_{kl}); if $i > 0$ then $|d_i| > \rho$, the amount of $P \in d_i$ is too great and these points must be dependent. (The points of the divisor d^{n-1} could also be taken.)

Theorem 2. *The angle variables θ_{kl} corresponding to the action variables J_{kl} are given by the Abel mapping of the divisor d_0 ;*

$$\theta_{kl} = \sum_{P \in d_0} \int \frac{w^{n-k} \zeta^l}{f_w} d\zeta. \tag{25}$$

The proof is the same as in [2].

We restrict ourself to this theorem. It remains to establish relatively standard things: to express, as in [2], the Hamiltonian in terms of J_{kl} and thus to find the dependence of θ_{kl} on x , then using the Jacobi–Riemann method of Abel mapping inversion to find $g_{0,n-1}$ and from its asymptotics at infinity to obtain u_i .

Appendix. The connection between Eq. (1) and the resolvent of an n^{th} order differential operator.

Equation (1) is a convenient tool of studying the resolvent of an n^{th} order differential operator. (The formal resolvents were introduced in [9–11].)

Here we need a construction of a ring of operators over \mathcal{A} (differential and Volterra integral). This construction has been given many times: by Gelfand and the author, Manin [12], and Adler [13]. Differential operators are as usual

$\sum_{i=0}^m a_i \partial^i (a_i \in \mathcal{A}, \partial = d/dx)$. The commutation rule between ∂ and any $a \in \mathcal{A}$ is $\partial a = a\partial + a'$. The differential operator is written here in the right-hand form, it can be also written in the left-hand form $\sum \partial^i a_i$. The integral operator can be defined in various ways: in terms of symbols of pseudo-differential operators, with the help of formal kernels ([10]) or as formal series $\sum_{i=0}^{\infty} a_i \partial^{-i-1}$ and $\sum_{i=0}^{\infty} \partial^{-i-1} a_i$ (right- and left-hand form). The commutation rules are¹

$$a\partial^{-1} = \partial^{-1}a + \partial^{-2}a' + \partial^{-3}a'' + \dots, \partial^{-1}a = a\partial^{-1} - a'\partial^{-2} + a''\partial^{-3} + \dots$$

In more general form the coefficients a_i of an integral or a differential operator

may belong to a ring $\mathcal{A}[z^{-1}]$ of formal Laurent series $\sum_{r=r_0}^{\infty} a_r z^{-r}$. The arbitrary operator is a sum of a differential and an integral operator. Let R be the ring of operators, R_+ and R_- its subrings of differential and correspondingly integral operators.

If $A = \sum_k^{\infty} a_r \partial^{-r-1} (k < 0)$ then we denote $A_+ = \sum_k^{-1} a_r \partial^{-r-1} \in R_+, A_- = \sum_0^{\infty} a_r \partial^{-r-1} \in R_-$.

Let R_-^n be $R_-/\partial^{-n}R_-$. The elements of this space are classes of integral operators $\sum_0^{\infty} a_r \partial^{-r-1}$ with the same $a_r, r \leq n - 1$. Two such operators we call $n - 1$ -equivalent. R_-^n is a n -dimensional module over \mathcal{A} (or $\mathcal{A}[z^{-1}]$).

The coefficient in ∂^{-1} will be called the residue of the operator. The definition does not depend on which form, right- or left-hand, the operator is written. $\text{Res} \sum a_r \partial^{-r-1} = a_0, \text{Res} \sum \partial^{-r-1} b_r = b_0$.

We can regard R_+ as the dual space of R_- with respect to a coupling: for $A \in R_+, B \in R_-$ put $(A, B) = \text{Res} AB$ (the order of operators is here important, i.e. $\text{Res} \partial a \cdot \partial^{-1} = a' \neq \text{Res} \partial^{-1} \cdot \partial a = 0$). The dual space of R_-^n is R_+^n , the space of differential operators of orders not higher than $n - 1$.

Matrices of the n^{th} order, for example the matrix Q of Sect. 1 will be regarded as matrices of linear transformations of R_-^n in the left basis $\partial^{-r-1}, r = 0, \dots, n - 1$.

The column vectors are the elements of $R_-^n : Q^j = \sum_{i=0}^{n-1} \partial^{-i-1} Q_{ij}$, row vectors are

1 To connect this definition of the integral operator with formal kernels, note that ∂^{-1-p} corresponds to a kernel $X(\xi, \eta) = \sum \frac{(\xi - x)^k (\eta - x)^l}{k!l!} X_{kl}$ (see [10]) with $X_{kl} = (-1)^l$ when $k + l = p$ and $X_{kl} = 0$ when $k + l \neq p$

the elements of $R_+^n : Q_i = \sum_{j=0}^{n-1} Q_{ij} \partial^j$. The product of two matrices can be calculated as

$$(QR)_i = \sum_j \text{Res}(Q_i R^j) \partial^j.$$

The matrix $U + \zeta A = \hat{U}$ from Sect. 1 has the following row vectors

$$\hat{U}_i = \begin{cases} \partial^{i+1}, & i < n-1 \\ -\hat{L} + \partial^n, & i = n-1 \end{cases}; \hat{L} = L - \zeta = \sum_{k=0}^n u_k \partial^k - \zeta; u_n = 1, u_{n-1} = 0. \quad (I)$$

One can give another expression for \hat{U} as a linear transformation of R_-^n , in an invariant, that is independent of the basis, form:

$$Y \in R_-^n \mapsto \hat{U}(Y) = [\partial Y - \partial^{-n+1}(\hat{L}Y)_-]_- \quad (II)$$

Here $\hat{U}(Y)$ is well defined: the arbitrary representative of the $n-1$ -equivalency class may be chosen as Y .

A mapping $X \in R_-^n \mapsto Q_X \in \text{End } R_-^n$ will be introduced:

$$Q_X(Y) = (X\hat{L}Y)_+ - (X\hat{L})_- Y_- \quad (III)$$

This mapping is also well defined.

Proposition 1. *The row vectors of the transformation (III) are*

$$(Q_X)_i = \partial^i (X\hat{L})_+ - (\partial^i X)_+ \hat{L}. \quad (IV)$$

Column vectors of the transformation are given by the formula

$$(Q_X)^j = [- (X\hat{L})_- \partial^{-j-1} + X(\hat{L}\partial^{-j-1})_+]_- \quad (V)$$

Proof. Let us make sure that the transformation with row vectors (IV) coincides with (III). The action of a transformation with row vectors Q_i on a $Y \in R_-^n$ can be calculated via the formula

$$Q(Y) = \sum_{i=0}^{n-1} \partial^{-i-1} \text{Res}(Q_i Y).$$

For (IV) this yields

$$Q_X(Y) = \sum_{i=0}^{n-1} \partial^{-i-1} \text{Res}[\partial^i (X\hat{L})_+ Y - (\partial^i X)_+ \hat{L}Y].$$

Then we note that $\text{Res}(\partial^i X)_+ \hat{L}Y = \text{Res}(\partial^i X)_+ (\hat{L}Y)_- = \text{Res } \partial^i X(\hat{L}Y)_-$ hence

$$\begin{aligned} Q_X(Y) &= \sum_{i=0}^{n-1} \partial^{-i-1} \text{Res } \partial^i [(X\hat{L})_+ Y - X(\hat{L}Y)_-] = \sum_{i=0}^{n-1} \partial^{-i-1} \\ &\cdot \text{Res } \partial^i [(X\hat{L})_+ Y - X(\hat{L}Y)_-]_- = [(X\hat{L})_+ Y - X(\hat{L}Y)_-]_- = - [(X\hat{L})_- Y \\ &- X(\hat{L}Y)_+]_- \end{aligned}$$

The second formula can be checked in a similar way ■

In particular the last column of the matrix Q_X is X , the first row is $(X\hat{L})_+$.

We denote as Q' the transformation with the matrix (Q'_{ij}) , in other words $Q' = \partial Q - Q\partial$.

Proposition 2. *The relation*

$$-Q'_X + [\hat{U}, Q_X](Y) = \partial^{-n+1} \{ [(\hat{L}X)_+ \hat{L} - \hat{L}(X\hat{L})_+] Y \}_-. \tag{VI}$$

holds.

Proof. The formula can be verified directly, whether in the invariant form using (II) and (III) or in the matrix form using (I), (IV) and (V) ■

Corollary. Q_X satisfies (I) if and only if X satisfies

$$(\hat{L}X)_+ \hat{L} - \hat{L}(X\hat{L})_+ = 0. \tag{VIII}$$

Proposition 3. *An arbitrary solution Q of equation (1) is Q_X for some $X \in R^n_-$ which satisfies (VIII).*

Proof. It is clear that for X we choose the last column vector of $Q: X = \sum \partial^{-i-1} Q_{i,n-1}$. It is easy to check that equation (1) is equivalent to the set of equations for operators $Q_i = \sum Q_{ij} \partial^j$

$$Q_{i+1} = \partial Q_i - Q_{i,n-1} \hat{L}, \quad i = 0, \dots, n-1, \tag{IX}$$

where Q_n is defined by

$$\sum_{i=0}^n u_i Q_i - \zeta Q_0 = 0, \tag{X}$$

whence

$$Q_i = \partial^i Q_0 - \sum_{\alpha=0}^{i-1} \partial^{i-1-\alpha} Q_{\alpha,n-1} \hat{L}, \quad i = 1, \dots, n. \tag{XI}$$

In particular

$$Q_n = \partial^n \left[Q_0 - \sum_{\alpha=0}^{i-1} \partial^{-1-\alpha} Q_{\alpha,n-1} \hat{L} \right] = \partial^n (Q_0 - X\hat{L}).$$

In the left-hand side of the equation there is an operator of the $n-1$ th order. The right-hand side can be such an operator if $(Q_0 - X\hat{L})_+ = 0$, that is $Q_0 = (X\hat{L})_+$. Then (XI) yields

$$Q_i = \partial^i (X\hat{L})_+ - (\partial^i X)_+ \hat{L}.$$

According to proposition 1 this means that $Q = Q_X$. ■

The significance of these propositions is that Eq. (1) is completely equivalent to Eq. (VIII) for the last column vector of the matrix Q . This equation is none other than n th order differential equation resolvent equation. The form (VIII) for this equation was suggested by Adler [13].

Now it is not difficult to make sure that “the variational theorem” (corollary of proposition 1 in Sect. 4) coincides with the corresponding theorem in [7, 10, 11]. Thus the equations we have considered are in fact the stationary equations connected with n th order differential operators.

We now prove a proposition that has been used in Sect. 5.

Proposition 4. *If Q satisfies (1) and for some m the coefficient Q_m of the expansion of Q in powers of z^{-1} is such that $(Q_m)_{i,n-1} = 0$ for $i = 0, \dots, n-2$ and Q_m does not contain constants then Q_m is a strictly lower triangular matrix.*

Proof. Let X be the operator $\sum \partial^{-\alpha-1} Q_{\alpha,n-1}$. Expanding it into a series in z^{-1} we get $X|_m = \partial^{-n} Q_{n-1,n-1;m}$. Then (IV) shows that $Q_i|_m$ are operators of orders no more than i , and coefficients in ∂^i are equal to $Q_{n-1,n-1;m}$. The matrix Q_m is triangular with equal diagonal elements. It remains to note that $\text{tr } Q = \text{const}$ (it holds for all solutions of (1)) hence $\text{tr } Q_m = 0$ since Q_m does not contain constants; Q_m is strictly triangular as required \blacksquare

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Communicated by A. Jaffe

Received June 2, 1980, in revised form January 30, 1981

