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Phase Diagrams and Cluster Expansions for Low Temperature $\mathscr{P}(\phi)_2$ Models*

I. The Phase Diagram

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Abstract. Low temperature phase diagrams of two-dimensional $\mathscr{P}(\phi)$ quantum field models are constructed. Let $\mathscr{P}$ lie in an $(r-1)$-dimensional space of perturbations of a polynomial with r degenerate minima. Perform a scaling $\mathscr{P}(\phi)\to\lambda^{-2}\mathscr{P}(\lambda\phi)$ and assume $\lambda \ll 1$. We construct k distinct states on $\binom{r}{k}$ hypersurfaces of codimension $k-1$ in the space of perturbations. An expansion is used to exhibit exponential clustering of the Schwinger functions of each of these states. At the core of the construction is a general technique for finding the thermodynamically stable phases from a collection of competing minima. We draw on ideas of Pirogov and Sinai [24] for this problem.

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I. Introduction

1.1. Background and Main Results

This paper presents a detailed analysis of low-temperature $\mathscr{P}(\phi)_2$ quantum fields models. The last several years have seen considerable development of constructive field theory techniques for studying multiphase models. However, the range of models that can be treated remains rather limited when compared with the large class of polynomials to which expansion techniques ought to apply. We close this gap by giving cluster expansions for the Schwinger functions of essentially arbitrary low-temperature $\mathscr{P}(\phi)_2$ models. Moreover, we construct coexisting states on various hypersurfaces of a phase diagram homeomorphic to the classical (zero coupling) phase diagram. A unique feature of this investigation is the fundamental role expansions play in determining the stable phases of a theory.

The study of phase transitions in quantum fields models was initiated by Glimm, Jaffe and Spencer [18] with their proof that two phases exist in the double-well $(\lambda\phi^4-\frac{1}{4}\phi^2)_2$ model. Subsequent results include existence of phase transitions for models without a symmetry [9] or with continuous symmetry [11, 12], and absence of symmetry breakdown for vector models in two dimensions [2].

Gawędzki [13] proved that the parameters in a ϕ^6 polynomial can be adjusted so as to achieve three distinct phases at one point. Summers [28, 29] established that two phases coexist on lines leading up to that point. These results justified classical ideas about the structure of the phase diagram of that model.

Of course, there are a number of methods available to construct models corresponding to general polynomials [14, 10]. But these methods do not lead directly to results on multiplicity of phases or phase transitions.

When dealing with a model that is a small perturbation of a massive Gaussian, cluster expansions usually give the most detailed information, e.g. on the relation between the Schwinger functions and their perturbation series [6, 8] or on cluster properties [16, 17]. This has been the case for multiphase theories also. Glimm et al. [19] developed a convergent expansion for the Schwinger functions of the $(\lambda\phi^4-\frac{1}{4}\phi^2-\mu\phi)_2$ model (with $|\mu|\leqq\lambda^2\ll 1$), establishing also the mass gap of the theory. Their mean field expansion has formed a basis for studies of other models with deep, widely separated minima. The technique was used for the Coulomb gas [3,4], for a two-component ϕ^4 model with three phases [5], and for the pseudoscalar Yukawa model in the two-phase region [1]. Summers [28] used it in the ϕ^6 model mentioned above, although his expansion did not converge in a neighborhood of the asymmetric phase transition lines. (He was nevertheless able to prove asymptoticity of perturbation theory across the phase transition lines [30].)

The work cited above goes very far in the description of quantum fields at or near first-order phase transitions. However, all the results depend in an essential way on special properties of the models considered. In all of the models, the space of parameters in the interaction has an *a priori* reduction to 0 or 1 dimensions with the use of symmetries or correlation inequalities. Thus all work has depended on either

(i) the existence of a preferred point in parameter space at which the phase transition should occur (e.g., a symmetric point) or

(ii) the existence of a preferred direction in parameter space along which one can pass to obtain the phase transition.

The Glimm-Jaffe-Spencer analysis of $(\lambda\phi^4-\frac{1}{4}\phi^2-\mu\phi)_2$ [19], and Fröhlich's proof of phase transitions without symmetry breaking [9] rely on the external field as the preferred direction. Gawędzki's proof of the existence of three phases [13] uses both (i) and (ii): a $\phi\rightarrow-\phi$ symmetry prevailed all along the direction of variation of a quadratic term in the polynomial. Summers's analysis of the phase diagram of that model [28, 29] also used the quadratic coefficient as the preferred direction, even though the diagram was two-dimensional. Bałaban and Gawędzki's work on the two-phase Yukawa model [1] used a boson symmetry $\phi\rightarrow-\phi$. Brydges and Federbush [3,4] and Constantinescu and Ruck [5] also used symmetries in their multiphase expansions.

The limitations (i) and (ii) rule out a wide range of situations where one expects phase transitions. A general polynomial will have many local minima. The process of adjusting the coefficients of the polynomial to achieve many phases at once involves many parameters nontrivially. One of the main contributions of this paper is a technique for finding the hypersurfaces on which various phases of a polynomial coexist. If r phases are involved, this involves a search through an $(r-1)$-dimensional space of parameters, with no preferred directions and no symmetries.

We also obtain detailed properties of each of the phases by giving a cluster expansion which converges throughout the space of parameters. This represents a considerable advance in expansion techniques, since up till now all cluster expansions have relied on symmetries [19, 3–5, 1], correlation inequalities [19, 20, 28], or large differences in classical energy densities [28]. When correlation inequalities are used, one is in case (ii) above. A large difference in classical energy densities necessarily excludes neighborhoods of phase transition hypersurfaces from consideration.

The main difficulty with applying expansion techniques to general polynomials has been in obtaining ratio of partition function bounds. These bounds are absolutely crucial in getting the boundary conditions to select a stable phase.

Except in situations (i) and (ii) above, it is not clear which minima are stable and which ones are unstable at a particular choice of parameters. Quantum fluctuations can suppress some minima relative to others, making the classical polynomial an unreliable guide. One can only hope to obtain good bounds on ratios of partition functions when a stable partition function is in the denominator. Thus the difficulty is compounded by the fact that one does not know which ratios of partition functions one should attempt to bound.

This problem has been solved in lattice statistical mechanics where spins take values in a finite set. Pirogov and Sinai [23, 24] have developed a powerful technique to give a complete analysis of the phase diagrams of such models at low temperatures. They show that in the $(r-1)$-dimensional space of perturbations of a Hamiltonian with r degenerate ground states, there exist $\binom{r}{k}$ hypersurfaces of codimension $k-1$ at which k phases coexist.

We solve the ratio of partition function problem by drawing on the ideas of Pirogov and Sinai. Quantum field models, and unbounded spin systems in general, have a number of properties which make a straightforward application of the Pirogov-Sinai theory impossible. The difficulties arise from the nonpositivity of terms of the decoupling expansion, from different classical masses in different minima, and from the unboundedness of ϕ. These will be discussed in the next section, and as they arise in later chapters.

The techniques of this paper should also be useful in solving problems outside of $\mathscr{P}(\phi)_2$ theory. At no point do we use symmetries or correlation inequalities to obtain the expansion. The techniques should apply to the low temperature statistical mechanics of continuous, unbounded spins (not covered in [23, 24]). They may be applicable to the problem of proving Debye screening for arbitrary relative activities of charge species. (See [4] for a statement of this problem.)

We now state the main results. Let μ vary in some neighborhood of the origin in $\mathbb{R}^{r-1}$. Let $\mathscr{P}_{1,\mu}(\xi)$ be a polynomial with local minima at $\xi=\xi_1(\mu), \ldots, \xi_r(\mu)$. Suppose these minima are degenerate when $\mu=0$, and suppose $\mathscr{P}''_{1,\mu}(\xi_q)\equiv m_q^2(\mu)>0$ for $q=1, \ldots, r$. Assume $\mathscr{P}_{1,\mu}$ is bounded below and its minima $\xi_1, \ldots, \xi_r$ are separated by potential barriers. We give a more precise statement of the requirements on $\mathscr{P}_{1,\mu}$ in Sect. 2.1. The parameters μ^i must break the degeneracy properly, and the classical energies $\mathscr{P}_{1,\mu}(\xi_q)$ cannot be too far apart.

With

$$\mathscr{P}_{\lambda,\mu}(\xi)=\lambda^{-2}\mathscr{P}_{1,\mu}(\lambda\xi), \tag{1.1.1}$$

we construct finite volume expectations as follows. Define

$$\begin{aligned} V_q &= \int_\Lambda \left[:\mathscr{P}_{\lambda,\mu}(\phi(x)): - \mathscr{P}_{\lambda,\mu}(\xi_q) - \frac{m_q^2}{2} :(\phi(x)-\xi_q)^2: \right] dx \\ &= \int_\Lambda \left[:\mathscr{P}_{\lambda,\mu}(\psi_q(x)+\xi_q): - \mathscr{P}_{\lambda,\mu}(\xi_q) - \frac{m_q^2}{2} :\psi_q(x)^2: \right] dx, \end{aligned} \tag{1.1.2}$$

where $\psi_q(x)=\phi(x)-\xi_q$. Wick ordering is defined with respect to the covariance $(-\Delta+m_{\bar{q}}^2)^{-1}$, where $\bar{q}\in\{1, \ldots, r\}$ is fixed throughout. Let $d\mu_{m_{\bar{q}}^2}(\psi_q)$ denote the Gaussian measure in which ψ_q has mean zero and covariance $(-\Delta+m_q^2)^{-1}$. The expectation with boundary condition q is

$$\langle R\rangle_{\Lambda,q}=\frac{\int Re^{-V_q}d\mu_{m_q^2}(\psi_q)}{\int e^{-V_q}d\mu_{m_q^2}(\psi_q)}. \tag{1.1.3}$$

The measure $d\mu_{m_q^2}(\psi_q)$ supplies the missing mass term $\frac{m_q^2}{2}:\psi_q(x)^2:$ in Λ, while outside Λ it forces the field to lie near the q^{th} minimum. The scaling (1.1.1) preserves the shape of $\mathscr{P}_{1,\mu}$, while separating the minima by factors of λ^{-1} and raising potential barriers by factors of λ^{-2}. Quadratic terms are unaffected, while cubic and higher coefficients are multiplied by powers of λ.

Theorem 1.1.1. *Let λ be sufficiently small. There exists a Lipschitz continuous mapping of a neighborhood of the origin in parameter space onto a neighborhood of the origin in the boundary of the positive octant in $\mathbb{R}^r$. (This is the set of r-tuples $(a^1,\dots,a^r)$ satisfying $\inf_q a^q=0$.) The mapping inverts and the inverse is also Lipschitz continuous. Let μ be a parameter set which is the inverse image of a point with $a^q=0$. At μ, the infinite volume limit of the Schwinger functions*

$$\left\langle\prod_{i=1}^{n}\phi(x_i)\right\rangle_{\Lambda,q} \tag{1.1.4}$$

exists and the limiting functions satisfy all the Osterwalder-Schrader axioms [22] *with exponential clustering.*

The perturbation series for the q^{th}-state Schwinger functions are asymptotic as $\lambda\to 0$ whenever $a^q(\mu)=0$. As a consequence there exist $\binom{r}{k}$ hypersurfaces of codimension $k-1$ in parameter space at which k distinct phases coexist.

In the course of proving this theorem we give rather precise estimates on the positions of the phase transition hypersurfaces. In Sect. 3.7 it is shown that they deviate by $O(\lambda^{5/2})$ from the positions inferred from perturbation theory through order λ^0 for the vacuum energies.

This paper incorporates and extends the author's doctoral thesis [21], where the central ideas were worked out in a less general setting.

1.2. Physical Ideas and Outline

In this section we present some of the main ideas behind the proof of Theorem 1.1.1. An organizational overview will be given at the same time. More detailed discussions and analyses of technical problems will be found at the beginnings of Chaps. 3–5.

Chapter 2 gives a version of the mean field expansion that is suited to variable mass polynomials and to the constructions in Chap. 3. The basic objects used in the rest of the paper are defined, and estimates on them are stated in Sect. 2.5. In Sect. 2.1 we give a precise statement of requirements on polynomials, and we single out aspects that ultimately determine how small λ must be taken. A concise statement of the choice of constants used in the expansions is also given in Sect. 2.1.

The basic idea of the mean field expansion is to decompose the theory into low- and high-momentum modes. The low-momentum part resembles a lattice spin system; it is controlled with a Peierls expansion. The high-momentum part resembles a single-phase model; it is controlled with a decoupling expansion in the spirit of [17]. The low-momentum component (the "mean field") is defined by averaging the field over a square. Rigorous bounds on the deviation of the field from its average value justify the splitting and provide a basis for the vacuum energy bounds.

The mean field expansion expresses the theory in terms of a statistical ensemble of objects defined on finite regions of $\mathbb{R}^2$. (Actually, it is not a true statistical ensemble because the objects are not in general positive.) This is the starting point for the analysis of Chap. 3.

In Chap. 3 the stable phases are determined and the all-important ratio of partition function bounds are established. We use many of the ideas of Pirogov and Sinai [23, 24], though the constructions we give are quite different from analogous ones in [24]. We emphasize partition function language over the contour language of [23, 24]. We must avoid taking logarithms of the basic objects, since there is no way to keep them positive and bounded away from zero.

The constructions proceed through approximations to quantities representing the "relative energies" of the different phases. For each set of approximate energies, approximate partition functions are constructed. The physical basis of the construction is the idea that transitions out of an unstable phase should have a probability growing exponentially with the volume of the fluctuation. The coefficient should be proportional to the energy of the phase relative to the stable phases. In contrast to the true partition functions, the approximate partition functions can be estimated at this stage.

Estimating approximate partition functions is the technical crux of this work. The estimate depends on the fact that there is a form in which the expansion has positive terms. In order to be able to reduce to that form at any point, we must use a complicated inductively defined construction for the approximate partition functions. The "relative energies" acquire a dependence on the size of fluctuations.

The approximate partition functions are estimated in terms of pressures and surface energies. The pressure terms correct the approximate relative energies, and the surface terms are incorporated into the ensemble. This new information yields a better approximation to the partition function, and the procedure repeats. When the iteration converges, we have expansions for the true partition functions. The stable phases are the ones with zero relative energy. The estimate mentioned above can be applied to give good bounds on ratios of partition functions, provided a stable phase is in the denominator.

In Chap. 4 the information on partition function is used to obtain an expansion for the Schwinger functions. Ratios of constrained partition functions must be dealt with before dispensing with the machinery of Chap. 3. Constraints on the expansion for $\int R e^{-V_q} d\mu_{m_q^2}(\psi_q)$ can be handled with some techniques of Kunz and Souillard (see [1]), and the normalization $\int e^{-V_q} d\mu_{m_q^2}(\psi_q)$ can be factored out explicitly. This yields estimates on $\langle R \rangle_{\Lambda,q}$ independent of Λ, for q a stable phase. Other consequences of the convergence of the expansion are the mass gap and the asymptotic nature of perturbation theory.

Chapter 5 contains proofs of the estimates required for Chaps. 3 and 4. It is essential to take account of the effective vacuum energy arising from changes in mass. We also require some smoothness in μ for each element of the expansion. To obtain smoothness of partition functions associated with a single spin configuration, we need a new type of estimate approximating the field as a bounded spin. This estimate is what allows us to go beyond [21] to give a complete picture of the phase diagram.

2. Mean Field Expansion

2.1. Polynomials and the $\lambda \to 0$ Limit

In this section we specify precisely the polynomials and perturbations that will be treated in this paper. Our philosophy will be to specify a polynomial $\mathscr{P}_1$ and then scale $\mathscr{P}_1$ and its argument according to

$$\mathscr{P}_\lambda(\xi) = \lambda^{-2}\mathscr{P}_1(\lambda\xi), \qquad \lambda \in (0,1]. \tag{2.1.1}$$

This scaling has the effect of separating minima by λ^{-1} and increasing barriers between minima by λ^{-2}. The curvatures at the minima of $\mathscr{P}_\lambda$ are independent of λ. The coupling constant λ can be identified with $\hbar^{1/2}$, so that as λ tends to zero we approach the classical limit. Alternatively, we can think of the scaling as a continuum version of the low temperature limit for the lattice Hamiltonian $H = \frac{1}{2}|\nabla\phi|^2 + \mathscr{P}_1(\phi)$. With $\lambda = \beta^{-1/2}$, $\beta \gg 1$, the change of variable $\phi \to \lambda\phi$ sends βH to $\frac{1}{2}|\nabla\phi|^2 + \lambda^{-2}\mathscr{P}_1(\lambda\phi)$.

The scaling is particularly useful in studying $\mathscr{P}(\phi)_2$ theories that possess many phases, because it preserves whatever multiple-well structure $\mathscr{P}_1$ has. In this respect it differs from $\lambda\mathscr{P}(\phi) - \phi^2$, $\lambda \ll 1$ and $g\mathscr{P}(\phi)$, $g \gg 1$, which should yield only one- or two-phase theories [25].

We shall give a convergent r-phase expansion for the Schwinger functions when $\lambda \ll 1$ and $\mathscr{P}_\lambda$ has r important minima. (The meaning of "important" will be specified precisely below.) As parameters in $\mathscr{P}_\lambda$ are varied, a minimum may rise so far above the others that it becomes irrelevant. If this happens, we are able to give overlapping r-phase and $(r-1)$-phase expansions. Thus we can handle general polynomials except in neighborhoods of critical points.

In order to achieve coexistence of up to r-phases, we need to consider perturbations of polynomials with r degenerate minima. Quantum corrections will destroy the degeneracy, and we must be able to restore it by making small changes in the perturbing parameters. Let $\mu = (\mu^1, \ldots, \mu^{r-1})$. We take $\mathscr{P}_1 = \mathscr{P}_{1,\mu}$ and suppose that $\mathscr{P}_{1,0}$ has r degenerate minima.

We shall need some lower bounds on $\mathscr{P}_1$ to insure that the minima $\xi_1, \ldots, \xi_r$ are isolated by potential barriers and to prevent any one minimum from dominating the others too strongly. We require a weak quadratic lower bound near each minimum. Between minima, but outside a neighborhood of each minimum, we allow the lower bound to diverge quadratically towards $-\infty$ at a faster rate than the local bound. This allows $\mathscr{P}_\lambda(\xi_q) - \mathscr{P}_\lambda(\xi_{q'})$ to be $O(\lambda^{-2})$, but no larger. Thus, the lower bounds restrict the range of μ. See Fig. 2.1.1. The diverging lower bounds will be controlled by the expansion in phase configurations (Sect. 2.2). The expansion makes it unlikely for the field to lie outside of a prescribed well.

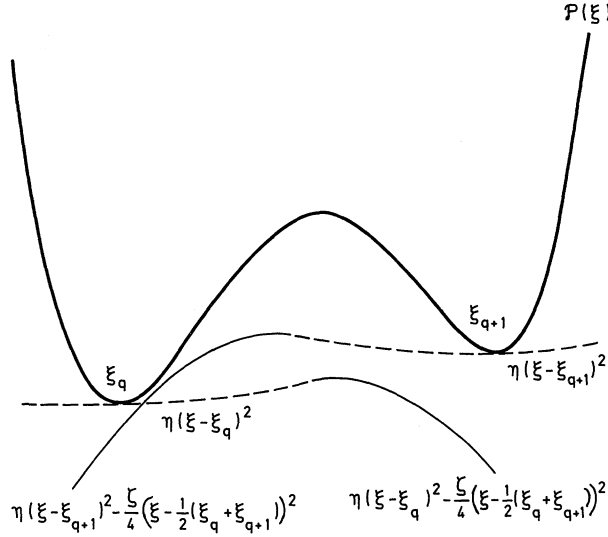

Fig. 2.1.1. Lower bounds required of $\mathscr{P}(\xi)$. ζ is fixed, while $\eta \leqq \zeta/4$ may be taken to be small. However, $\lambda \leqq \lambda_0(\eta)$

We now give a precise statement of the requirements on $\mathscr{P}_{1,\mu}$ and $\xi_1(1,\mu), \dots, \xi_r(1,\mu)$. $\{\mathscr{P}_{1,\mu}, \xi_1, \dots, \xi_r\}$ will be called $\zeta\eta C$-admissible if for all μ with $\sup_i |\mu^i| \leqq C^{-1}$

(i) $\xi_1, \dots, \xi_r$ are local minima of $\mathscr{P}_{1,\mu}$.

(ii) $\mathscr{P}_{1,0}(\xi_{q_1}) = \mathscr{P}_{1,0}(\xi_{q_2})$, $1 \leqq q_1 < q_2 \leqq r$.

(iii) Define d and $a_{j,q}(\mu)$ by the formula

$$\mathscr{P}_{1,\mu}(\xi + \xi_q) = \sum_{j=0}^{d} a_{j,q}(\mu)\xi^j .$$

Then d is even, $d \leqq C$, $|a_{d,q}(\mu)| \geqq C^{-1}$, and $|a_{j,q}(\mu)| \leqq C$, $j \geqq 2$. Of course $a_{1,q}(\mu) = 0$. Write $a_{2,q}(\mu) = \frac{1}{2}m_q^2(\mu)$, $a_{0,q}(\mu) = E_c^q(\mu)$.

(iv) $\xi_{q+1} - \xi_q \geqq C^{-1}$.

(v) There is a $\bar{q} \in \{1, \dots, r\}$ such that $e^q = E_c^q - E_c^{\bar{q}}$ satisfies the following condition. If Λ is an eigenvalue of the matrix $\left(\frac{\partial e^q}{\partial \mu^i}\right)_{\substack{q \neq \bar{q} \\ i = 1, \dots, r}}$, then $|\Lambda| \in [C^{-1}, C]$.

(vi) $\left|\frac{\partial^2 e^q}{\partial \mu^i \partial \mu^j}\right| \leqq C$, $\left|\frac{\partial}{\partial \mu^i} \xi_q\right| \leqq C$, $\left|\frac{\partial}{\partial \mu^i} a_{j,q}\right| \leqq C$, and $\left|\frac{\partial^2 m_q^2}{\partial \mu^i \partial \mu^j}\right| \leqq C$.

(vii) Put $\xi_0 = -\infty$, $\xi_{r+1} = \infty$.

For $q = 1, \dots, r$,

$$\mathscr{P}_{1,\mu}(\xi) - \mathscr{P}_{1,\mu}(\xi_q) \geqq \begin{cases} \eta(\xi - \xi_q)^2, & \xi \in (\frac{1}{2}(\xi_{q-1} + \xi_q), \frac{1}{2}(\xi_q + \xi_{q+1})) \\ \eta(\xi - \xi_q)^2 - \frac{\zeta}{4}(\xi - \frac{1}{2}(\xi_q + \xi_{q+1}))^2, & \xi \geqq \frac{1}{2}(\xi_q + \xi_{q+1}) \\ \eta(\xi - \xi_q)^2 - \frac{\zeta}{4}(\xi - \frac{1}{2}(\xi_q + \xi_{q-1}))^2, & \xi \leqq \frac{1}{2}(\xi_q + \xi_{q-1}) . \end{cases}$$

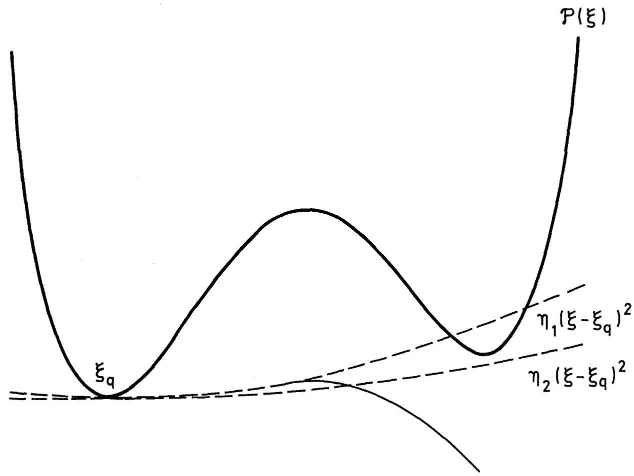

Fig. 2.1.2. If η is decreased from η_1 to η_2, then the right-hand minimum can be omitted from consideration

The constants ζ, η, and C that appear in the above definition arise as follows. ζ is a constant chosen consistently with some upper limits arising in the proof of the vacuum energy bound (Sects. 5.2 and 5.3). η may be chosen at will in the range $(0, \zeta/4]$. However, the lower bounds (vii) allow for a reasonable range of perturbations μ only if η is a small fraction of ζ. Also, the lower bounds imply that $m_q^2 \geqq 2\eta$ for all q. As η tends toward zero, we must restrict λ to smaller ranges. By adjusting η, one can exclude a minimum from consideration in a restricted range of μ (Fig. 2.1.2). Thus the domains of convergence of expansions with different values of r overlap. The constant C parametrizes various aspects of the polynomial which ultimately will affect the maximum allowable λ. Thus C and η^{-1} may be chosen arbitrarily large, but we require $\lambda < \lambda_0(\eta, C)$.

Requirement (v) insures that the parameters μ^i break the degeneracy of $\mathcal{P}_{1,0}$ and establishes a scale for μ^i. With $E_c^q(\lambda, \mu) = \mathcal{P}_{\lambda,\mu}(\xi_q)$, we have that a change $\delta\mu$ in the parameters induces a change $\delta E_c^q = O(\lambda^{-2})\delta\mu$ in the classical energies of the minima. Requirement (vi) is used to prove convergence of successive approximations to phase coexistence points. We need (v) and (vi) only in Sect. 3.7, where coexistence is established.

The μ-derivatives of the $a_{j,q}$'s can be expressed as nonsingular rational functions of μ-derivatives of the coefficients of $\mathcal{P}$. (Singularities appear only when some m_q tends to zero.)

The bound $\xi_{q+1} - \xi_q \geqq C^{-1}$ orders the minima and helps to avoid any potential critical points. Together with the lower bounds (vii), it insures that fluctuations between minima incur a certain energetic cost. The minima of $\mathcal{P}_{\lambda,\mu}$ are at $\xi_q(\lambda) = \lambda^{-1}\xi_q$ $(\lambda = 1)$. Thus they are separated by $O(\lambda^{-1})$.

Two length scales will be used in constructing the mean field expansion. One is the scale on which distant regions are decoupled in the cluster expansion. Decoupling lines are placed on a lattice with squares of length l. We choose l to be a large integer, depending on C and η. l need not diverge with λ. However, λ must be chosen sufficiently small, depending on l. The qualification $\lambda \ll 1 \ll l$ will always refer to this choice of constants. Once λ and l are chosen as described above, all our results apply to any $\zeta\eta C$-admissible family of polynomials. A second length L provides an upper bound on the scale of effects of fluctuations between minima. We take L to be the next multiple of l after $(\log\lambda)^2$.

More general polynomials can be brought into $\zeta\eta C$-admissible form by performing two operations. A dilation $\mathscr{P}\to\delta^2\mathscr{P}$, $x\to\delta^{-1}x$ can be used to bring the masses into the range $[\sqrt{2\eta}, \sqrt{2C}]$ or to increase the range of parameters consistent with the lower bounds (vii). A preliminary scaling $\mathscr{P}(\phi)\to\gamma^{-2}\mathscr{P}(\gamma\phi)$ will leave the masses fixed while bringing the leading coefficient into the range $[C^{-1}, C]$.

2.2. Expansion in Phase Boundaries

In the next three sections we present the mean field expansion [19] that we will use throughout this paper. It consists of four parts: an expansion in phase boundaries, a translation of the field, decoupling expansions, and mass shifts. The last two operations are intertwined in a way that generates an expansion with the right locality properties. Shifts of mass from the measure to the interaction or vice versa are needed so that the decoupling expansion is always controlled by factors of λ. These multiply cubic and higher order terms in the interaction, but not quadratic terms. The object of all of these operations is to represent an integral $\int \mathrm{R}e^{-V_q} d\mu_{m_q^2}(\psi_q)$ as an appropriate statistical sum of quantities defined on finite regions in $\mathbb{R}^2$. In Chaps. 3 and 4 we use the representation as a starting point for a sequence of transformations that ultimately leads to the desired estimates on the Schwinger functions.

To define the expansion, we put two lattices on $\mathbb{R}^2$. One is a unit lattice, with elementary squares Δ_i^1 having lower left corners at $i=(i_0, i_1)\in\mathbb{Z}^2\subseteqq\mathbb{R}^2$. The other is a coarser lattice, with elementary squares Δ_j having lower left corners at (lj_0, lj_1) with $l\gg 1$. We take the interaction region Λ to be a square composed of l-lattice squares.

In each unit lattice square $\Delta_i^1\subseteqq\Lambda$, we decompose the measure into r parts, one for each minimum of $\mathscr{P}_{\lambda,\mu}$. This is accomplished by inserting partitions of unity into the measure. Suppressing the dependence of ξ_q on λ and μ, we define

$$\begin{aligned} \phi(\Delta^1) &= \int_{\Delta^1} \phi(x)dx \equiv \bar{\phi}(x), \qquad x\in\Delta^1 \\ \chi_q(\xi) &= \pi^{-1/2} \int_{(\xi_{q-1}+\xi_q)/2}^{(\xi_q+\xi_{q+1})/2} e^{-(\xi-z)^2} dz, \qquad q=1,\dots,r. \end{aligned} \tag{2.2.1}$$

Then $\sum_{q=1}^{r} \chi_q(\xi)=1$. Let σ_i take the values $1,\dots,r$ at the unit square $\Delta_i^1\subseteqq\Lambda$. The expansion in phase boundaries is a result of the identity

$$\begin{aligned} 1 &= \prod_{\Delta_i^1\subseteqq\Lambda} \left(\sum_{\sigma_i=1}^{r} \chi_{\sigma_i}(\phi(\Delta_i^1))\right) \\ &= \sum_{\{\sigma_i\}} \prod_{\Delta_i^1\subseteqq\Lambda} \chi_{\sigma_i}(\phi(\Delta_i^1)). \end{aligned} \tag{2.2.2}$$

Here $\{\sigma_i\}=\Sigma$ is a spin configuration, that is, a function on the unit squares in Λ taking values $1, \ldots, r$. Define

$$\chi_\Sigma = \prod_{\Delta_i^1 \subseteq \Lambda} \chi_{\sigma_i}(\phi(\Delta_i^1)) \tag{2.2.3}$$

and insert (2.2.2) into the measure to obtain the expansion

$$\int \mathrm{Re}^{-V_q} d\mu_{m_q^2}(\psi_q) = \sum_\Sigma \int R\chi_\Sigma e^{-V_q} d\mu_{m_q^2}(\psi_q). \tag{2.2.4}$$

The formula may be interpreted as follows. Since $\chi_q(\xi)$ is close to 1 for ξ near ξ_q and close to zero elsewhere, the factor χ_Σ tends to make each $\phi(\Delta_i^1)$ lie in the vicinity of the minimum ξ_{σ_i}. By estimating the fluctuation field $\delta\phi(x)=\phi(x)-\bar{\phi}(x)$, we shall see that $\phi(x)$ also tends to lie near ξ_{σ_i}. For expectations with boundary condition q we set $\sigma_i = q$ if $\Delta_i^1 \nsubseteq \Lambda$. The boundary condition presses $\phi(x)$ to lie near ξ_q in $\sim\Lambda$ without the help of χ-factors.

Whenever $\sigma_i \neq \sigma_{i'}$ for neighboring squares Δ_i^1 and $\Delta_{i'}^1$, χ_Σ forces a fluctuation between wells of $\mathscr{P}_{\lambda,\mu}$. When this happens, the edge common to Δ_i^1 and $\Delta_{i'}^1$ will be called a phase boundary. We shall see that each configuration Σ is associated with a strong convergence factor $e^{-O(1)\lambda^{-2}|\Sigma|}$, where $|\Sigma|$ is the total length of the phase boundaries in Σ. Thus the tendency is for σ to be constant over large regions.

2.3. Translation of ψ_q

In defining the finite volume expectation with boundary condition q, we translated from ϕ to $\psi_q = \phi - \xi_q$, anticipating that $\langle\phi\rangle$ should be approximately ξ_q in the q^{th} phase. With the χ-factors present in the measure, we need a space-dependent translation in order to recover a small mean. We expect $\langle\phi(x)\rangle$ to behave roughly like

$$h(x) = \xi_{\sigma_i}, \quad x \in \Delta_i^1. \tag{2.3.1}$$

Taking into account the kinetic energy term in the action, a better approximation would be

$$g_c(x) = (\eta(-\Delta+\eta)^{-1}h)(x), \tag{2.3.2}$$

where η is the constant appearing in the lower bounds (vii), Sect. 2.1.

As in [19] we localize the definition of the translation function $g(x)$ so that it depends only on σ_i for $\mathrm{dist}(\Delta_i^1, x) \leqq L/2$. Choose a C_0^∞ function $\zeta(x)$ satisfying

$$\begin{aligned} &0 \leqq \zeta(x) \leqq 1 \\ &\zeta(x) = 0, \quad |x| > \tfrac{1}{2} \\ &\zeta(x) = 1, \quad |x| \leqq \tfrac{1}{4}. \end{aligned} \tag{2.3.3}$$

Then define

$$g(x) = \eta_\zeta \int (-\Delta+\eta)^{-1}(x-y)\zeta((x-y)/L)h(y)dy, \tag{2.3.4}$$

where

$$\eta_\zeta^{-1} = \int (-\Delta+\eta)^{-1}(y)\zeta(y/L)dy. \tag{2.3.5}$$

Due to the localization of the kernel $(-\Delta+\eta)\zeta$, we have that $g(x)$ and $(-\Delta+\eta)(g-g_c)(x)$ are independent of $h(y)$ for $\mathrm{dist}(x,y) \geqq L/2$. Thus if $\mathrm{dist}(x,\Sigma) \geqq L/2$, then $g(x)=h(x)$ and $(-\Delta+\eta)(g-g_c)(x)=0$.

Define

$$\psi(x)=\phi(x)-g(x)$$
$$=\psi_q(x)+\xi_q-g(x). \quad (2.3.6)$$

The meaning of ψ depends on Σ, though the dependence is not explicit in the notation. Let $d\mu_{m_q^2}(\psi)$ denote the measure in which $\psi(x)$ has mean zero and covariance $(-\Delta+m_q^2)^{-1}$. We claim the following formula is valid:

$$e^{-V_q}d\mu_{m_q^2}(\psi_q)=e^{-Q(\Sigma,\Lambda,q)}d\mu_{m_q^2}(\psi), \quad (2.3.7)$$

where

$$Q=\int_\Lambda [:\mathscr{P}_{\lambda,\mu}(\phi(x)):-E_c^q-\tfrac{1}{2}\eta:(\phi(x)-h(x))^2:-\tfrac{1}{2}(m_q^2-\eta):\psi(x)^2:]dx$$
$$+F_1+F_2+F_3+F_4, \quad (2.3.8)$$

$$F_1=\tfrac{1}{2}\eta\int(h-g)^2dx+\tfrac{1}{2}\int(\nabla g)^2dx, \quad (2.3.9)$$

$$F_2=\tfrac{1}{2}(m_q^2-\eta)\int_{\sim\Lambda}(g-\xi_q)^2dx, \quad (2.3.10)$$

$$F_3=\int\psi(x)(-\Delta+\eta)(g-g_c)dx, \quad (2.3.11)$$

$$F_4=(m_q^2-\eta)\int_{\sim\Lambda}\psi(x)(g-\xi_q)dx. \quad (2.3.12)$$

The formula for translation of a Gaussian measure is

$$d\mu_{m_q^2}(\psi+f)=\exp(-\tfrac{1}{2}\langle f,(-\Delta+m_q^2)f\rangle-\langle\psi,(-\Delta+m_q^2)f\rangle)d\mu_{m_q^2}(\psi). \quad (2.3.13)$$

Thus

$$d\mu_{m_q^2}(\psi_q)=d\mu_{m_q^2}(\psi+g-\xi_q)$$
$$=\exp(-\langle\psi+\tfrac{1}{2}(g-\xi_q,(-\Delta+m_q^2)(g-\xi_q)\rangle)d\mu_{m_q^2}(\psi). \quad (2.3.14)$$

Since $V_q=\int_\Lambda[:\mathscr{P}(\phi):-E_c^q-\frac{m_q^2}{2}:(\psi+g-\xi_q)^2:]dx$, the claim will follow from

$$\sum_i F_i=\int_\Lambda\left[-\frac{m_q^2}{2}:(\psi+g-\xi_q)^2:+\tfrac{1}{2}\eta:(\psi+g-h)^2:+\tfrac{1}{2}(m_q^2-\eta):\psi^2:\right]dx$$
$$+\langle\psi+\tfrac{1}{2}(g-\xi_q),(-\Delta+m_q^2)(g-\xi_q)\rangle. \quad (2.3.15)$$

The terms linear in ψ are

$$\int_\Lambda\psi[m_q^2\xi_q-\eta h-(m_q^2-\eta)g]dx+\int\psi[-m_q^2\xi_q+(-\Delta+m_q^2)g]dx. \quad (2.3.16)$$

Using $h=\xi_q$ on $\sim\Lambda$ and (2.3.2), these combine to form F_3+F_4. Of the remaining terms, $\pm\frac{m_q^2}{2}(g-\xi_q)^2$ cancel in Λ to leave

$$\int_{\sim\Lambda}\frac{m_q^2}{2}(g-\xi_q)^2dx+\int_\Lambda\tfrac{1}{2}\eta(g-h)^2dx+\tfrac{1}{2}\langle\nabla(g-\xi_q),\nabla(g-\xi_q)\rangle$$
$$=F_1+\int_{\sim\Lambda}\left[\frac{m_q^2}{2}(g-\xi_q)^2-\tfrac{1}{2}\eta(g-h)^2\right]dx. \quad (2.3.17)$$

Using again $h=\xi_q$ on $\sim\Lambda$, the proof is complète.

The grouping of terms in (2.3.7)–(2.3.12) is convenient for the vacuum energy bounds of Chap. 5. The term $-\frac{1}{2}(m_q^2-\eta):\psi(x)^2:$ will be absorbed into the Gaussian measure. The negative term $-\frac{1}{2}\eta:(\phi-h)^2:$ must be controlled by the χ-factors and by estimates on the fluctuation field $\delta\phi$. [Compare with the lower bounds (vii) of Sect. 2.1.] F_1 is the dominant effect of phase boundaries, and produces a factor $e^{-O(1)\lambda^{-2}|\Sigma|}$. F_2, F_3, and F_4 are error terms.

2.4. *Decoupling Expansion and Mass Shifts*

In this section we take each term in the expansion in phase boundaries and express it as a sum of products of terms defined on bounded regions. The localized terms are the basic objects of study in Chaps. 3 and 4.

In order to implement factorization we will use covariances with space-dependent masses and with full or partial Dirichlet data on bonds of the l-lattice. Let $\omega(x)$ take values in the set $\{m_1^2, \dots, m_r^2\}$ and be constant on every unit lattice square. Let Γ be a set of l-lattice bonds and write Δ_Γ for the Laplace operator with zero Dirichlet boundary conditions on Γ. We will use decoupling parameters $\{s_b\}$, where b indexes bonds of the l-lattice, and $s_b \in [0,1]$. With $s=\{s_b\}$, $\bar{m}=\sup_{q,\mu} m_q(\mu)$, define

$$C_{\bar{m}^2}(s)=\sum_{\Gamma\subseteqq\mathscr{B}}\prod_{b\in\Gamma^c} s_b \prod_{b\in\Gamma}(1-s_b)(-\Delta_\Gamma+\bar{m}^2)^{-1},$$
$$C_\omega(s)=(C_{\bar{m}^2}(s)^{-1}+\omega-\bar{m}^2)^{-1}, \tag{2.4.1}$$

where Γ^c is the complement of Γ in $\mathscr{B}$, the set of all l-lattice bonds. When $s_b=0$, $C_\omega(s)$ has zero Dirichlet boundary conditions on b and when $s_b=1$, $C_\omega(s)$ has no Dirichlet data on b. If $s_b=0$ for all $b\in\Gamma$, we call Γ a Dirichlet contour.

Let $d\mu_{\omega,s}(\psi)$ denote the Gaussian measure in which ψ has mean zero and covariance $C_\omega(s)$. We call $\{s_b\}$ decoupling parameters because

$$\int R_Y R_{\sim Y} d\mu_{\omega,s}(\psi)=\int R_Y d\mu_{\omega,s}(\psi)\int R_{\sim Y} d\mu_{\omega,s}(\psi) \tag{2.4.2}$$

whenever ∂Y is a Dirichlet contour. ($R_Y, R_{\sim Y}$ are supported in Y, $\sim Y$, respectively.)

We now give a decoupling expansion for each term of the expansion in spin configurations. Let $\mathscr{B}(\Sigma)\subseteqq\mathscr{B}$ be the set of l-lattice bonds that are at least a distance L from all phase boundaries of Σ. We use the sup norm $\mathrm{dist}(x,y)=\sup_{i=0,1}|x_i-y_i|$ in discussing distances between phase boundaries and l-lattice bonds. We keep $s_b=1$ for $b\notin\mathscr{B}(\Sigma)$ throughout. Rather than expand in all the s-parameters at once, we break $\mathscr{B}(\Sigma)$ into subsets. After expanding in the s-parameters corresponding to some subset, we make a mass shift. We may then proceed to the next subset and continue until all bonds of $\mathscr{B}(\Sigma)$ have been differentiated or decoupled.

In order to define the subsets of $\mathscr{B}(\Sigma)$, we construct inductively a sequence of subsets of $\mathbb{R}^2:\mathbb{R}^2=D_1\supseteqq D_2\supseteqq\dots\supseteqq D_{n(\Sigma)+1}=\emptyset$. $p(b)$, the phase of a bond in $\mathscr{B}(\Sigma)$, is defined to be the common value of nearby spins. Let D_2 be the region bounded by bonds $b\in\mathscr{B}(\Sigma)$ with $p(b)\neq p(b_0)$, where b_0 is any bond far outside of Λ. [Bonds with $p(b)=p(b_0)$ may be contained in D_2.] Then define $\mathscr{B}_1(\Sigma)=\mathscr{B}(\Sigma)\cap\{b:b\subseteqq D_1\backslash D_2\}$. In general, D_k will consist of a number of connected, simply connected components D_k^α. With $b_0\subseteqq\partial D_k^\alpha$, define D_{k+1}^α to be the region bounded by bonds $b\subseteqq D_k^\alpha$

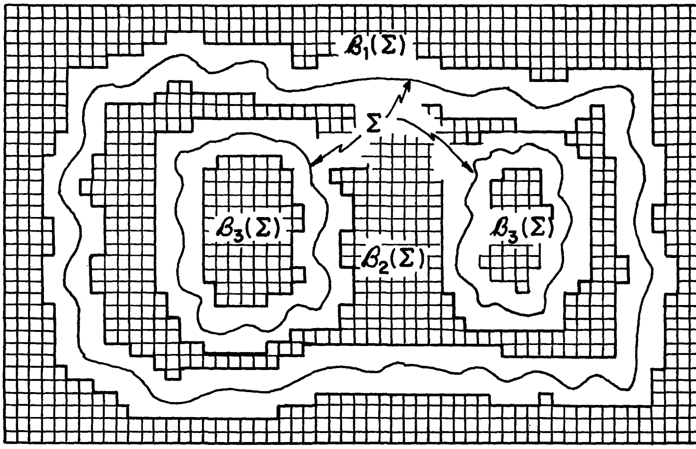

Fig. 2.4.1. The decomposition of $\mathscr{B}(\Sigma)$ into subsets $\mathscr{B}_1(\Sigma), \mathscr{B}_2(\Sigma), \ldots$ separated from each other by the phase boundary Σ

with $p(b) \neq p(b_0)$ and $b \in \mathscr{B}(\Sigma)$. Then put $D_{k+1} = \bigcup_\alpha D^\alpha_{k+1}$ and $\mathscr{B}_k(\Sigma) = \mathscr{B}(\Sigma) \cap \{b : b \subseteqq D_k \backslash D_{k+1}\}$. The construction terminates when the bonds of $\mathscr{B}(\Sigma)$ are exhausted, and we have $\mathscr{B}(\Sigma) = \mathscr{B}_1(\Sigma) \cup \ldots \cup \mathscr{B}_{n(\Sigma)}(\Sigma)$. See Fig. 2.4.1.

We associate to each k a space-dependent mass $\omega_k(x)$. Define $p(D^\alpha_k) = p(b_0)$ where $b_0 \in \partial D^\alpha_k$ if $k \geqq 2$ and b_0 is far outside of Λ if $k=1$. Then put $\omega_1(x) \equiv m^2_{p(D_1)}$. We obtain ω_{k+1} from ω_k by modifying it on each

$$\hat{D}^\alpha_{k+1} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, D^\alpha_{k+1}) \leqq L/2 + l\}$$

(using again the sup norm):

$$\omega_{k+1}(x) = m^2_{p(D^\alpha_{k+1})}, \qquad x \in \hat{D}^\alpha_{k+1}$$
$$\omega_{k+1}(x) = \omega_k(x), \qquad x \notin \bigcup_\alpha \hat{D}^\alpha_{k+1}.$$

Associated to each ω_k we have a mass-shifted interaction

$$Q_{\omega_k}(\Sigma, \Lambda, q) = Q(\Sigma, \Lambda, q) + \int_\Lambda \tfrac{1}{2}(m_q^2 - \omega_k(x)) : \psi(x)^2 : dx. \tag{2.4.3}$$

Notice that $Q_{\omega_1} = Q$.

The decoupling expansion uses the identity

$$F(s_b = 1) = F(s_b = 0) + \int_0^1 ds_b \frac{\partial}{\partial s_b} F(s_b) \tag{2.4.4}$$

for each $b \in \mathscr{B}_k(\Sigma)$ for some k. For $k=1$ we obtain

$$\begin{aligned} &\int R\chi_\Sigma e^{-Q} d\mu_{m_q^2}(\psi) \\ &= \sum_{\Gamma_1 \subseteqq \mathscr{B}_1(\Sigma), \Gamma_1 \text{ finite}} \int ds_{\Gamma_1} \partial_s^{\Gamma_1} \int R\chi_\Sigma e^{-Q} d\mu_{\omega_1, s_{\Gamma_1}}(\psi) \\ &= \sum_{\Gamma_1 \subseteqq \mathscr{B}_1(\Sigma), \Gamma_1 \text{ finite}} \int ds_{\Gamma_1} \sum_{\pi_1 \in \mathscr{P}(\Gamma_1)} \int \prod_{\alpha_1 \in \pi_1} [\tfrac{1}{2} \partial_s^\alpha C_{\omega_1}(s_{\Gamma_1}) \cdot \Delta_\psi] \\ &\quad \cdot R\chi_\Sigma e^{-Q} d\mu_{\omega_1, s_{\Gamma_1}}(\psi). \end{aligned} \tag{2.4.5}$$

We have used the following notations:

$$\partial_s^\Gamma = \prod_{b\in\Gamma} \frac{\partial}{\partial s_b},$$

$$(s_{\Gamma_1})_b = \begin{cases} s_b & \text{if } b\in\Gamma_1 \\ 0 & \text{if } b\in\mathscr{B}_1(\Sigma)\setminus\Gamma_1 \\ 1 & \text{if } b\notin\mathscr{B}_1(\Sigma), \end{cases}$$

$$\int ds_\Gamma = \int \prod_{b\in\Gamma} ds_b,$$

$\mathscr{P}(\Gamma)$ = set of partitions of Γ,

$$K\cdot\Delta_\psi = \int K(z_1,z_2)\frac{\delta}{\delta\psi(z_1)}\frac{\delta}{\delta\psi(z_2)}dz_1dz_2.$$

See [7, 17] for details.

Before proceeding to the next block of bonds, we shift mass from Q to $d\mu$:

$$e^{-Q_{\omega_1}}d\mu_{\omega_1,s_{\Gamma_1}}(\psi) = e^{-Q_{\omega_2}}d\mu_{\omega_2,s_{\Gamma_1}}(\psi)Z_{\omega_1\omega_2}(s_{\Gamma_1}), \tag{2.4.6}$$

$$Z_{\omega_1\omega_2}(s_{\Gamma_1}) = \int e^{\int 1/2(\omega_1(x)-\omega_2(x)):\psi(x)^2:dx}d\mu_{\omega_1,s_{\Gamma_1}}(\psi). \tag{2.4.7}$$

Absorbing the mass term into $d\mu$ changes the covariance to $C_{\omega_2}(s_{\Gamma_1})$, given in (2.4.1). We shall never apply s-derivatives to $Z_{\omega_1\omega_2}$ factors. As a result of the mass shift, contractions to Q_{ω_2} from the next block of s-derivatives will bring down powers of λ.

Proceed through the blocks of bonds $\mathscr{B}_2(\Sigma), \mathscr{B}_3(\Sigma), \ldots, \mathscr{B}_{n(\Sigma)}(\Sigma)$ shifting mass-squared from ω_k to ω_{k+1} after deriving bonds in $\mathscr{B}_k(\Sigma)$. We allow s-derivatives to act on covariances arising from earlier blocks. The result is

$$\begin{aligned}\int R\chi_\Sigma e^{-Q}d\mu_{m_q^2}(\psi) = &\sum_{\Gamma\subseteq\mathscr{B}(\Sigma),\,\Gamma\,\text{finite}} \int ds_\Gamma \sum_{\pi\in\mathscr{P}(\Gamma)} \prod_{j=1}^{n}\left[e^{Q_{\omega_j}-Q_{\omega_{j+1}}}\prod_{\substack{\alpha\in\pi\\ k(\alpha)=j}}\left[\tfrac{1}{2}\partial_s^\alpha C_{\omega_j}(s_\Gamma)\cdot\Delta_\psi\right]\right] \\ &\cdot R\chi_\Sigma e^{-Q_{\omega_1}}d\mu_{\omega_n,s_\Gamma}(\psi)\prod_{k=1}^{n-1} Z_{\omega_k\omega_{k+1}}(s_{\Gamma_k}).\end{aligned} \tag{2.4.8}$$

We have used the following notations:

$$\Gamma_k = \Gamma\cap\bigcup_{j\leq k}\mathscr{B}_j(\Sigma),$$

k(α) is the first integer k such that $\alpha\cap\Gamma_k\neq\emptyset$,

$$(s_{\Gamma_k})_b = \begin{cases} s_b & \text{if } b\in\Gamma_k \\ 0 & \text{if } b\in\left(\bigcup_{j\leq k}\mathscr{B}_j(\Sigma)\right)\setminus\Gamma_k \\ 1 & \text{otherwise}, \end{cases}$$

$$s_\Gamma = s_{\Gamma_n},$$

$$\omega_{n+1} = \omega_n,$$

$$\prod_{j=1}^{n}\mathcal{O}_n = \mathcal{O}_n\mathcal{O}_{n-1}\ldots\mathcal{O}_1.$$

We now write (2.4.8) in factorized form. Define $Q(Z)$ as in (2.3.8)–(2.3.12) but restricting all integrals to $Z\cap\Lambda$ or $Z\cap\sim\Lambda$. Define $Q_{\omega_k}(Z)$ similarly from (2.4.3). Let $\{Z_\kappa\}$ be the closures of the connected components of $\mathbb{R}^2\backslash\{s=0 \text{ bonds}\}$. (The $s=0$ bonds are those in $\mathscr{B}(\Sigma)\backslash\Gamma$.) Then

$$Q_{\omega_k}=\sum_\kappa Q_{\omega_k}(Z_\kappa).$$

We let $\Sigma\cap Z$ denote the restriction of the spin configuration Σ to Z, and define

$$\chi_{\Sigma\cap Z}=\prod_{\Delta_i^1\subseteq Z\cap\Lambda}\chi_{\sigma_i}(\phi(\Delta_i^1)),$$

so that

$$\chi_\Sigma=\prod_{Z_\kappa}\chi_{\Sigma\cap Z_\kappa}.$$

Using the fact that $C_{\omega(\alpha)}(s_\Gamma)$ vanishes between Z_κ and $Z_{\kappa'}$, and the fact that $C_{\omega(\alpha)}(s_\Gamma)$ restricted to Z_κ depends only on s_b with $b\subseteq Z_{\kappa'}$ we can apply (2.4.2) to obtain

$$\begin{aligned}&\int ds_\Gamma\sum_{\pi\in\mathscr{P}(\Gamma)}\int\prod_{j=1}^n\left[e^{Q_{\omega_j}-Q_{\omega_{j+1}}}\prod_{\substack{\alpha\in\pi\\k(\alpha)=j}}\left[\tfrac{1}{2}\partial_s^\alpha C_{\omega_j}(s_\Gamma)\cdot\Delta_\psi\right]\right]\\&\cdot R\chi_\Sigma e^{-Q_{\omega_1}}d\mu_{\omega_n,s_\Gamma}(\psi)\\&=\prod_\kappa\left\{\int ds_{\Gamma\cap Z_\kappa}\sum_{\pi\in\mathscr{P}(\Gamma\cap Z_\kappa)}\int\prod_{j=1}^n\left[e^{Q_{\omega_j}(Z_\kappa)-Q_{\omega_{j+1}}(Z_\kappa)}\right.\right.\\&\left.\left.\cdot\prod_{\substack{\alpha\in\pi\\k(\alpha)=j}}\left[\tfrac{1}{2}\partial_s^\alpha C_{\omega_j}(s_\Gamma)\cdot\Delta_\psi\right]\right]R_{Z_\kappa}\chi_{\cap Z_\kappa}e^{-Q_{\omega_1}(Z_\kappa)}d\mu_{\omega_n,s_\Gamma}(\psi)\right\}.\end{aligned}\tag{2.4.9}$$

We have assumed that R is a product.

The Gaussian integrals $Z_{\omega_k\omega_{k+1}}(s_{\Gamma_k})$ also factor, although not as fully as above. We associate the factors to Z_κ's as follows:

$$Z_{\omega_k\omega_{k+1}}(s_{\Gamma_k})=\prod_\kappa Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k}),\tag{2.4.10}$$

where

$$\omega_{k+1}^\kappa(x)=\begin{cases}\omega_{k+1}(x), & x \text{ in a connected component of}\\ & \text{suppt}(\omega_{k+1}-\omega_k) \text{ whose boundary lies in } Z_\kappa\\ \omega_k(x), & \text{otherwise}.\end{cases}\tag{2.4.11}$$

Notice that $\text{suppt}(\omega_k-\omega_{k+1}^\kappa)$ is always separated from $\text{suppt}(\omega_k-\omega_{k+1}^{\kappa'})$ by bonds with $s_{\Gamma_k}=0$. Hence (2.4.10) is valid. In addition, $Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k})$ depends only on $s_{\Gamma_k\cap Z_\kappa}$.

Using (2.2.4), (2.3.7), (2.4.8), and (2.4.9), the full mean field expansion becomes

$$\begin{aligned}\int Re^{-V_q}d\mu_{m_q^2}(\psi_q)=&\sum_\Sigma\sum_{\Gamma\subseteq\mathscr{B}(\Sigma),\Gamma\text{finite}}\prod_\kappa\\&\cdot\left\{\int ds_{\Gamma\cap Z_\kappa}\sum_{\pi\in\mathscr{P}(\Gamma\cap Z_\kappa)}\int\prod_{j=1}^n\left[e^{Q_{\omega_j}(Z_\kappa)-Q_{\omega_{j+1}}(Z_\kappa)}\prod_{\substack{\alpha\in\pi\\k(\alpha)=j}}\left[\tfrac{1}{2}\partial_s^\alpha C_{\omega_j}(s_\Gamma)\cdot\Delta_\psi\right]\right]\right.\\&\left.\cdot R_{Z_\kappa}\chi_{\Sigma\cap Z_\kappa}e^{-Q_{\omega_1}(Z_\kappa)}d\mu_{\omega_n,s_\Gamma}(\psi)\prod_{k=1}^{n-1}Z_{\omega_k\omega_{k+1}^\kappa}(s_{\Gamma_k})\right\}.\end{aligned}\tag{2.4.12}$$

Note that the construction insures that all spins within L of a connected Dirichlet contour have the same value. If the common value is m, we call the contour an m-contour. By the locality of the construction of $g(x)$ and $\omega_n(x)$, $Q_{\omega_n}(Z_\kappa)$ does not depend on $\Sigma \cap Z_{\kappa'}$ for $\kappa' \neq \kappa$. (It still depends on Λ and the boundary condition q.)

Before making further transformations on (2.4.12), we introduce the notion of a cluster. A cluster is a triple $(Z, \Sigma \cap Z, \Gamma \cap Z)$ satisfying certain conditions.

(i) Z is a connected union of a finite number of l-lattice squares.

(ii) $\Sigma = \Sigma \cap Z$ is a spin configuration in Z.

(iii) $\Gamma = \Gamma \cap Z$ is a set of derivative bonds in Z.

(iv) $\operatorname{dist}(\partial Z \cup \Gamma, \Sigma) \geqq L$. Here Σ denotes the phase boundaries corresponding to the spin configuration Σ.

(v) $\partial Z \cap \Gamma = \emptyset$.

(vi) Let Γ^c denote the set of all bonds $b \notin \Gamma$ such that $\operatorname{dist}(b, \Sigma) \geqq L$. $Z \backslash \Gamma^c$ is connected.

We use the symbols $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ to denote clusters. We shall often speak of clusters as if they were simply subsets of $\mathbb{R}^2$. Set theoretic operations $\cup$, $\cap$, $\backslash$, etc. should be interpreted accordingly. Let $|\mathbb{Z}|$ denote the number of l-lattice squares in $\mathbb{Z}$. The boundary of a cluster $\mathbb{Z}$ consists of a number of loops with boundary conditions determined by $\Sigma_{\mathbb{Z}}$, the spin configuration of $\mathbb{Z}$. One loop surrounds all the others and is called the external loop of $\mathbb{Z}$. Its boundary condition is called the external boundary condition of $\mathbb{Z}$. If the external boundary condition of $\mathbb{Z}$ is m, then $\mathbb{Z}$ is called an m-cluster.

We define $\varrho_{\Lambda,q}(\mathbb{Z}_\kappa)$ to be the κ-term of the product over κ in (2.4.12). The mass shifts, etc. were designed so that $\varrho_{\Lambda,q}(\mathbb{Z}_\kappa)$ is determined uniquely by $\mathbb{Z}_\kappa$, Λ, q, and R_{Z_κ}. (It does not depend on $\Sigma \cap \sim Z_\kappa$ or $\Gamma \cap \sim Z_\kappa$.)

We may write (2.4.12) in terms of clusters. Most of the compatibility conditions among Σ, Γ, and $\{Z_\kappa\}$ are handled by (i)–(vi) above.

$$\int R e^{-V_q} d\mu_{m_{\bar{q}}^2}(\varphi_q) = \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{R}^2 \\ \text{agreeing on common boundaries, } \Sigma_{\mathbb{Z}_\kappa} = q \text{ in } \mathbb{Z}_\kappa \backslash \Lambda \\ \text{only finitely many } \mathbb{Z}_\kappa \text{ have } |\mathbb{Z}_\kappa| > 1}} \prod_\kappa \varrho_{\Lambda,q}(\mathbb{Z}_\kappa) . \quad (2.4.13)$$

Adjacent clusters must have their Σ's match along common boundaries. Note that if $\mathbb{Z}$ is a single square in $\sim \Lambda$ with no R-factors, then $Q(Z) = 0$ and $\varrho_{\Lambda,q}(\mathbb{Z}) = 1$.

At several places in Chaps. 3 and 4, we shall need to resum the expansion (2.4.13) within a region of $\mathbb{R}^2$. The sum in (2.4.13) is not a true statistical sum because not all the terms are positive. Resummation restores the positivity that is needed for certain arguments. Resummation also produces the finite volume partition functions which will be the fundamental objects of study in Chap. 3. We will need to make some adjustments in classical energies to allow for the effects of mass shifts. Define Wick ground state energies

$$E_w^q = E_c^q - \frac{m_q^2}{8\pi} \log \frac{m_q^2}{m_{\bar{q}}^2} + \frac{m_q^2 - m_{\bar{q}}^2}{8\pi} . \quad (2.4.14)$$

(Recall that $m_{\bar{q}}$ is the Wick-ordering mass, and $\bar{q} \in \{1, \ldots, r\}$ is fixed throughout.)

Note that

$$E_w^q - E_c^q = \lim_{|\Lambda|\to\infty} \frac{-1}{|\Lambda|} \log \int e^{\int_\Lambda 1/2(m_{\bar{q}}^2 - m_q^2):\psi(x)^2:dx} d\mu_{m_{\bar{q}}^2}(\psi), \tag{2.4.15}$$

see Sect. 5.5.

Let $V \subseteqq \Lambda$ be a connected region composed of l-lattice squares. Let $\mathbb{V}$ consist of V together with a single choice of boundary condition $p(\mathbb{V})$ for all the boundary loops of V. Let Σ be a spin configuration in $\mathbb{V}$ compatible with the boundary condition. [Compatibility entails that $\mathrm{dist}(\Sigma, \partial V) \geqq L$.] Let $d\mu_{m_{\bar{p}(\mathbb{V})}^2, \partial\mathbb{V}}(\psi)$ be the measure with mass $m_{p(\mathbb{V})}$, and with $s=0$ on $\partial\mathbb{V}$ and $s=1$ elsewhere. Define

$$\begin{aligned} 0 \leqq Z_\Sigma(\mathbb{V}) &= e^{(E_c^{p(\mathbb{V})} - E_w^{p(\mathbb{V})})|\mathbb{V}|l^2} \\ &\cdot \int \chi_\Sigma e^{-\int_V [:\mathscr{P}(\phi(x)): - 1/2 m_{\bar{p}(\mathbb{V})}^2 :\psi(x)^2:]dx} d\mu_{m_{\bar{p}(\mathbb{V})}^2, \partial\mathbb{V}}(\psi_{p(\mathbb{V})}) \\ &= e^{-E_w^{p(\mathbb{V})}|\mathbb{V}|l^2} \int \chi_\Sigma e^{-V_{p(\mathbb{V})}(\mathbb{V})} d\mu_{m_{\bar{p}(\mathbb{V})}^2, \partial\mathbb{V}}(\psi_{p(\mathbb{V})}) \\ &= e^{(-E_c^q + E_c^{p(\mathbb{V})} - E_w^{p(\mathbb{V})})|\mathbb{V}|l^2} \int \chi_\Sigma e^{-Q_{m_{\bar{p}(\mathbb{V})}^2}(\mathbb{V})} d\mu_{m_{\bar{p}(\mathbb{V})}^2, \partial\mathbb{V}}(\psi). \end{aligned} \tag{2.4.16}$$

Since $\mathbb{V} \subseteqq \Lambda$, $Z_\Sigma(\mathbb{V})$ does not depend on Λ. Moreover, the factor $e^{-E_c^q|\mathbb{V}|l^2}$ cancels the dependence of Q on q so that $Z_\Sigma(\mathbb{V})$ is independent of q. $Z_\Sigma(\mathbb{V})$ is manifestly positive. We shall also need the "partition function"

$$Z(\mathbb{V}) = \sum_{\Sigma \text{ compatible with } \mathbb{V}} Z_\Sigma(\mathbb{V}) \geqq 0. \tag{2.4.17}$$

$Z_\Sigma(\mathbb{V})$ and $Z(\mathbb{V})$ are defined in general by taking Λ large enough to include $\mathbb{V}$. If $\mathbb{V}$ has several components, $Z_\Sigma(\mathbb{V})$ is defined by taking a product over the components.

Apply the decoupling expansions and mass shifts to (2.4.16) to obtain

$$\begin{aligned} &Z_\Sigma(\mathbb{V}) e^{(E_c^q - E_c^{p(\mathbb{V})} + E_w^{p(\mathbb{V})})|\mathbb{V}|l^2} \\ &= \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{V} \\ \Sigma_{\mathbb{Z}_\kappa} = \Sigma \cap \mathbb{Z}_\kappa}} \prod_\kappa \varrho_{\Lambda,q}(\mathbb{Z}_\kappa). \end{aligned} \tag{2.4.18}$$

Summing over Σ yields an analogous formula for $Z(\mathbb{V})$:

$$\begin{aligned} &Z(\mathbb{V}) e^{(E_c^q - E_c^{p(\mathbb{V})} + E_w^{p(\mathbb{V})})|\mathbb{V}|l^2} \\ &= \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{V} \\ \text{agreeing on common boundaries and with } \partial\mathbb{V}}} \prod_\kappa \varrho_{\Lambda,q}(\mathbb{Z}_\kappa). \end{aligned} \tag{2.4.19}$$

Equations (2.4.18) and (2.4.19), applied in reverse, resum the mean field expansion inside $\mathbb{V}$.

In the special case where $\mathbb{V} \subseteqq \Lambda$ is a single l-lattice square with boundary condition m, we define

$$Z_{\Delta m} = Z(\mathbb{V}). \tag{2.4.20}$$

Only one term contributes to the sum over Σ, because $L \geqq l$.

We put (2.4.18)–(2.4.19) into a form where the lack of dependence on Λ, q is more apparent. Define

$$\varrho(\mathbb{Z}) = \varrho_{\Lambda,q}(\mathbb{Z}) e^{-E_c^q|\mathbb{Z}|l^2}, \tag{2.4.21}$$

where Λ has been taken large enough so that $\mathbb{Z} \subseteq \Lambda$ and $\mathrm{dist}(\partial\Lambda, \partial\mathbb{Z}) \geqq L$. As with $Z_\Sigma(\mathbb{V})$, the $e^{-E_c^q|\mathbb{Z}|l^2}$ factor cancels the dependence on q, making $\varrho(\mathbb{Z})$ independent of Λ and q. Equations (2.4.18) and (2.4.19) may be rewritten as

$$\begin{aligned} Z_\Sigma(\mathbb{V}) &= e^{(E_c^{p(\mathbb{V})} - E_w^{p(\mathbb{V})})l^2|\mathbb{V}|} \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{V} \\ \Sigma_{\mathbb{Z}_\kappa} = \Sigma \cap \mathbb{Z}_\kappa}} \prod_\kappa \varrho(\mathbb{Z}_\kappa), \\ Z(\mathbb{V}) &= e^{(E_c^{p(\mathbb{V})} - E_w^{p(\mathbb{V})})l^2|\mathbb{V}|} \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ nonoverlapping, filling } \mathbb{V} \\ \text{agreeing on common boundaries and with } \partial\mathbb{V}}} \prod_\kappa \varrho(\mathbb{Z}_\kappa). \end{aligned} \tag{2.4.22}$$

2.5. *Vacuum Energy Estimates and Bounds on Terms of the Expansion*

In this section we state the main technical estimates used in Chaps. 3 and 4. The proofs will be deferred to Chap. 5.

The most important estimate is the vacuum energy bound. It provides estimates uniform in λ, despite the fact that the classical energies E_c^m are $O(\lambda^{-2})$ relative to one another.

Let $\{n(\Delta_i^1)\}$ be a set of nonnegative integers, one for each $\Delta_i^1 \subseteq Y \cap \Lambda$. Define

$$\chi_{\sigma_i}^{(n)}(\xi) = \frac{\partial^n}{\partial \xi^n} \chi_{\sigma_i}(\xi) \tag{2.5.1}$$

and put

$$\chi_{\Sigma \cap Y}^{(\prime)} = \prod_{\Delta_i^1 \subseteq Y \cap \Lambda} \chi_{\sigma_i}^{(n(\Delta_i^1))}(\phi(\Delta_i^1)). \tag{2.5.2}$$

Let $|\Sigma'|$ be the number of Δ_i^1 with $n(\Delta_i^1) > 0$.

Let $|Y|_m$ be the volume (in units of l^2) of the portion of Y with $\Sigma = m$, and let $|Y| = \sum_m |Y|_m$. Recall that $\bar{m} = \sup_{q,\mu} m_q(\mu)$.

Proposition 2.5.1. *There exists $\zeta > 0$ such that for all $C > 2$, $\eta \in (0, \zeta/4]$ there exists $\tau_2(\eta, C) > 0$, $a(\eta, C) > 0$, and $\lambda_0(\eta, C) > 0$ such that for all $\lambda \in (0, \lambda_0]$, all $\{\mathscr{P}_{\lambda,\mu}, \xi_1, \ldots, \xi_r\}$ $\zeta\eta C$-admissible and all $p \in \left[1, 1 + \frac{\eta}{30\bar{m}^2}\right]$,*

$$\begin{aligned} &\| \chi_{\Sigma \cap Y}^{(\prime)} e^{-Q_{\omega_n}(Y)} \|_{L^p(d\mu_{\omega_n, s_\Gamma}(\psi))} \\ &\leqq e^{\sum_m l^2 (E_c^q - E_c^m)|Y|_m} e^{-3\tau_2 \lambda^{-2}(|\Sigma| + |\Sigma'|)} e^{a\lambda l^2 |Y \cap \Lambda|} \prod_{\Delta_i^1} n(\Delta_i^1)! \, . \end{aligned} \tag{2.5.3}$$

The L^p estimate is needed so that we may be able to split a general integral into a Gaussian part times an integral as above via Hölder's inequality.

The next proposition provides a lower bound needed to support the interacting measure and prevent normalization factors from vanishing.

Proposition 2.5.2. *With ζ, C, η, a, λ, and $\mathscr{P}_{\lambda,\mu}$ as in Proposition 2.5.1 except that λ_0 depends also on l,*

$$(Z_{\Delta^m} e^{l^2 E_w^m})^{\pm 1} \leqq e^{a(\eta, C)\lambda l^2}. \tag{2.5.4}$$

We next state bounds on the terms of the mean field expansion. First consider the case where there are no field monomials, that is, $R=1$. Clusters with no field monomials will be denoted by the letter $\mathbb{Y}$. For the others we use the letter $\mathbb{X}$. Let $p(\mathbb{Z})$ denote the external boundary condition of $\mathbb{Z}$. If $m \neq p(\mathbb{Z})$, define $\mathrm{Int}_m \mathbb{Z}$ to be the region bounded by boundary loops of $\mathbb{Z}$ that are in the m^{th} phase. Every component of $\mathrm{Int}_m \mathbb{Z}$ is given boundary condition m.

Proposition 2.5.3. *Suppose $\lambda \ll 1 \ll l$. There exist $\tau_1(\eta, C)>0$, $\tau_2(\eta, C)>0$ such that for $R=1$, $|\mathbb{Y}| \geqq 2$,*

$$|\varrho_{\Lambda,q}(\mathbb{Y})| \leqq \lambda^{1/2} e^{-3\tau_1 l|\mathbb{Y}|} e^{-2\tau_2 \lambda^{-2}|\Sigma_{\mathbb{Y}}|}$$
$$\cdot e^{\sum_m (E_c^q - E_w^m - E_c^{p(\mathbb{Y})} + E_w^{p(\mathbb{Y})}) l^2 |\mathbb{Y}_m|} e^{\sum_{m \neq p(\mathbb{Y})} (E_c^m - E_w^m - E_c^{p(\mathbb{Y})} + E_w^{p(\mathbb{Y})}) l^2 |\mathrm{Int}_m \mathbb{Y}|} . \tag{2.5.5}$$

Take R to be of the form

$$R = \prod_{i=1}^{\nu} :\psi_m(x_i)^{k_i}: , \tag{2.5.6}$$

and let $\deg R = \sum_{i=1}^{\nu} k_i$. Let $\|\cdot\|_{L^{p'}}$ denote the norm for $L^{p'}\left(\prod_{i=1}^{\nu} \Delta_i\right)$, where Δ_i is the l-lattice square containing x_i. (The Δ_i's need not be distinct.) Assume that $\Delta_i \subseteq \mathbb{X}$ for all i. Define $N(\Delta) = \sum_{i : \Delta_i = \Delta} k_i$.

Proposition 2.5.4. *Given $p' \in [1, \infty)$, let $\lambda \ll 1 \ll l$. There exist positive constants τ_1, τ_2, K depending on p', η, C such that for* $\deg R \geqq 1$, $|\mathbb{X}| \geqq 1$,

$$\|\varrho_{\Lambda,q}(\mathbb{X})\|_{L^{p'}}$$
$$\leqq \prod_{\Delta} (N(\Delta)!)^{1/2} e^{Kl \deg R} \lambda^{-\deg R} e^{-3\tau_1 l|\mathbb{X}|} e^{-2\tau_2 \lambda^{-2}|\Sigma_{\mathbb{X}}|}$$
$$\cdot e^{\sum_m (E_c^q - E_w^m - E_c^{p(\mathbb{X})} + E_w^{p(\mathbb{X})}) l^2 |\mathbb{X}|_m} e^{\sum_{m \neq p(\mathbb{X})} (E_c^m - E_w^m - E_c^{p(\mathbb{X})} + E_w^{p(\mathbb{X})}) l^2 |\mathrm{Int}_m \mathbb{X}|} . \tag{2.5.7}$$

If $\Sigma_{\mathbb{X}} \equiv m$, then the factor $\lambda^{-\deg R}$ may be omitted.

Differences such as $E_c^m - E_w^m$ appear because of the mass shift normalization factors included in the definition of $\varrho_{\Lambda,q}(\mathbb{Z})$. These factors will extend into $\mathrm{Int}_m \mathbb{Z}$.

The next two propositions give some smoothness in μ to elements of the construction of Chap. 3. They are needed in Sect. 3.7, where the phase diagram is constructed.

Proposition 2.5.5. *Suppose $\lambda \ll 1 \ll l$. There exist $\tau_1(\eta, C)>0$, $\tau_2(\eta, C)>0$ such that for $R=1$, $|\mathbb{Y}| \geqq 1$,*

$$\left|\frac{\partial}{\partial \mu^i}\left(\varrho_{\Lambda,q}(\mathbb{Y}) e^{\sum_m (-E_c^q + E_w^m + E_c^{p(\mathbb{Y})} - E_w^{p(\mathbb{Y})}) l^2 |\mathbb{Y}|_m}\right.\right.$$
$$\left.\left.\cdot e^{\sum_{m \neq p(\mathbb{Y})} (-E_c^m + E_w^m + E_c^{p(\mathbb{Y})} - E_w^{p(\mathbb{Y})}) l^2 |\mathrm{Int}_m \mathbb{Y}|}\right)\right|$$
$$\leqq \lambda^{1/2} e^{-5/2 \tau_1 l |\mathbb{Y}|} e^{-\tau_2 \lambda^{-2}|\Sigma_{\mathbb{Y}}|} . \tag{2.5.8}$$

Proposition 2.5.6. *With ζ, C, η, λ, and ${}^{\pi}{}_{\lambda,\mu}$ as in Proposition 2.5.1, there exists $K(\eta, C)$ such that*

$$\left|\frac{\partial}{\partial\mu^i}\log(Z_\Sigma(\mathbb{V})e^{E_w^{p(\mathbb{V})}l^2|\mathbb{V}|})\right| \leqq K\lambda^{-2}l^2|\mathbb{V}|. \tag{2.5.9}$$

Without Proposition 2.5.6, only limited information on the phase diagram may be obtained – see [21].

In the next two chapters we shall use extensively the following normalized form of $\varrho_{\Lambda,q}$:

$$\begin{aligned}&\tilde{\varrho}_{\Lambda,q}(\mathbb{Z})\\ &\quad=\frac{\varrho_{\Lambda,q}(\mathbb{Z})e^{(-E_c^q+E_c^{p(\mathbb{Z})}-E_w^{p(\mathbb{Z})})l^2|\mathbb{Z}\cap\Lambda|}e^{\sum\limits_{m\neq p(\mathbb{Z})}(-E_c^m+E_w^m+E_c^{p(\mathbb{Z})}-E_w^{p(\mathbb{Z})})l^2|\mathbb{I}\mathrm{nt}_m\mathbb{Z}|}}{\prod\limits_m (Z_{\Delta^m})^{|\mathbb{Z}\cap\Lambda|_m}}.\end{aligned} \tag{2.5.10}$$

Using Propositions 2.5.2–2.5.4, we see that the energy factors cancel, yielding the bounds

$$\begin{aligned}&\tilde{\varrho}_{\Lambda,q}(\mathbb{Y})\leqq\lambda^{1/2}e^{-5/2\tau_1 l|\mathbb{Y}|-2\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|}, \qquad |\mathbb{Y}|>1\\ &\tilde{\varrho}_{\Lambda,q}(\mathbb{X})\leqq\prod_{\Delta}(N(\Delta)!)^{1/2}e^{Kl\deg R}e^{-5/2\tau_1 l|\mathbb{X}|-2\tau_2\lambda^{-2}|\Sigma_{\mathbb{X}}|}\\ &\qquad\qquad\cdot\begin{pmatrix}\lambda^{-\deg R}, & \Sigma_{\mathbb{X}}\not\equiv m\\ 1, & \Sigma_{\mathbb{X}}\equiv m\end{pmatrix}, \qquad \deg R\geqq 1\\ &\tilde{\varrho}_{\Lambda,q}(\mathbb{Y})=1, \qquad |\mathbb{Y}|=1.\end{aligned} \tag{2.5.11}$$

In case $\mathbb{Z}$ is well inside Λ, (2.4.21) implies

$$\begin{aligned}&\tilde{\varrho}_{\Lambda,q}(\mathbb{Z})\\ &\quad=\frac{\varrho(\mathbb{Z})e^{(E_c^{p(\mathbb{Z})}-E_w^{p(\mathbb{Z})})l^2|\mathbb{Z}|}e^{\sum\limits_{m\neq p(\mathbb{Z})}(-E_c^m+E_w^m+E_c^{p(\mathbb{Z})}-E_w^{p(\mathbb{Z})})l^2|\mathbb{I}\mathrm{nt}_m\mathbb{Z}|}}{\prod\limits_m (Z_{\Delta^m})^{|\mathbb{Z}|_m}}\\ &\quad\equiv\tilde{\varrho}(\mathbb{Z}),\end{aligned} \tag{2.5.12}$$

where we have made the Λ, q-independence clear by dropping the subscripts on $\tilde{\varrho}$.

3. The Search for Stable Phases: Ratios of Partition Functions

3.1. The Physics: Surface Energies and Collapsing Expansions

Chapter 3 constitutes the core of this paper. The main result to be proved is the following bound on ratios of partition functions:

$$\frac{Z(\mathbb{V})}{Z(\mathbb{V}^{q_0})}\leqq e^{2\lambda^{1/2}|\partial\mathbb{V}|}. \tag{3.1.1}$$

Here $\mathbb{V}$ and $\mathbb{V}^{q_0}$ occupy the same simply connected region in $\mathbb{R}^2$ but $\mathbb{V}^{q_0}$ has boundary condition q_0. The fact that (3.1.1) holds for arbitrary $\mathbb{V}$ for some $q_0\in\{1,\dots,r\}$ will be of the utmost importance in all subsequent developments. It is the definitive sign that q_0 is a stable phase at a given point in parameter space. One could not hope to give a convergent expansion for Schwinger functions without first attending to a bound of this type.

Pirogov and Sinai [23, 24] have developed a technique for obtaining bounds like (3.1.1) in theories lacking the special properties discussed in the introduction (symmetries, correlation inequalities, or large differences in classical energy densities). Their work applies to lattice spin systems where the spins take values in a finite set. We adapt many of the ideas in [23, 24] for the problem at hand. We encounter additional problems from the unboundedness of ϕ and from the continuum limit. In Chap. 4 we deal with a third additional complication: In contrast to [23, 24], the passage from (3.1.1) to a construction of the stable states is nontrivial.

The difficulty in proving (3.1.1) lies first of all in the fact that both numerator and denominator have exponential dependence on the volume of $\mathbb{V}$. Yet the bound states that $Z(\mathbb{V}^{q_0})$ dominates up to a small surface term. Second, there is no *a priori* way of knowing which phases q_0 should work in (3.1.1).

Heuristically speaking, the stable phases should be the ones of least energy. There is, however, no simple measure of the energy of the unstable phases. For example,

$$\lim_{|\mathbb{V}|\to\infty} \frac{-1}{|\mathbb{V}|} \log Z(\mathbb{V}^m) \tag{3.1.2}$$

is presumably the same for all m, because the theory should tunnel into a stable minimum if unstable boundary conditions are imposed.

A way of constructing a measure of the energy of unstable phases is given by Pirogov and Sinai in [24]. A theory is first expressed in terms of contours separating uniform distributions of spins. Unstable boundary conditions m should be characterized by the fact that fluctuations out of the m^{th} state should have a probability growing like $\frac{1}{\text{Norm.}} e^{a^m|\text{Volume of contour}|}$, for some $a^m > 0$. The constant a^m is a measure of the energy of the m^{th} state.

In order to achieve a situation where the partition function is given by such an ensemble of contours, one must proceed through successive approximations to a^m. In our $\mathscr{P}(\phi)_2$ model,

$$a^m = -\log Z_{\Delta^m} - \inf_m(-\log Z_{\Delta^m}) \tag{3.1.3}$$

is an excellent first approximation, as it contains the Wick ground state energy and also some fluctuations about the m^{th} minimum.

When one attempts to describe the theory using this approximation to the relative energies, one finds two types of errors. The approximate partition functions we use have an expansion

$$\log \Omega(\mathbb{V}) = s|\mathbb{V}| + \Delta(\mathbb{V}), \tag{3.1.4}$$

where the pressure s is independent of $\mathbb{V}$ and where $|\Delta(\mathbb{V})| \leqq \lambda^{1/2}|\partial\mathbb{V}|$. A difference of pressures corrects the first approximation to $\{a^m\}$. Surface energies $\Delta(\mathbb{V})$ must be incorporated into the description of the theory by including them with the "energy" of contours bounding $\mathbb{V}$.

After making the corrections, the new approximate partition function is expanded as in (3.1.4) and the procedure is repeated. As one proceeds through the iteration, higher and higher order surface effects are taken into account. When the iteration converges, one has a description of the true partition function $Z(\mathbb{V})$ in terms of an ensemble of contours with factors $e^{a^m|\text{Volume of contour}|}$. The phases q_0 for which $a^{q_0}=0$ can then be shown to dominate the others as in (3.1.1). If several phases have $a^{q_0}=0$ at some point in parameter space, then they will all coexist at that point.

Collapsing expansions are the principal tools for carrying out the procedure described above. They arise from the following construction.

Compatibility of clusters on shared boundaries is a very difficult condition to work with. Therefore, one resums the mean field expansion inside the outermost clusters (the ones not surrounded by other clusters). The result is a partition function $Z(\mathbb{V})$ for the regions surrounded by clusters. If the outer clusters are multiplied by $\frac{Z(\mathbb{V})}{Z(\mathbb{V}^q)}\cdot Z(\mathbb{V}^q)$ and if $Z(\mathbb{V}^q)$ is expanded in the numerator, then one returns to the original boundary condition q and there is no compatibility condition for the outer clusters.

To be completely relieved of compatibility conditions, one must apply the above procedure to the clusters that make up the expansion for $Z(\mathbb{V}^q)$ and continue to smaller and smaller clusters. In the end, one has an expansion purely in terms of q-clusters with no compatibility conditions. Each cluster is multiplied by a ratio of partition functions. This form of the expansion makes it clear why (3.1.1) is so important.

One can reverse the above procedure to recover the original expansion. When this is done, we say the expansion has collapsed.

Collapsing expansions were first used in quantum field theory in [20]. They were used to make the step from Z/Z bounds to a cluster expansion, rather than to prove Z/Z bounds.

The biggest problem we encounter lies in bounding partition functions that involve the factors $e^{a^m|\text{Volume}|}$. Only the crudest estimates are available for such objects. (A more familiar situation is where contours have probability $e^{-\tau|\text{Length}|}$, in which case expansion techniques give very precise bounds.) The crude bounds depend on every term in the partition function sum being positive. Unfortunately, $\varrho(\mathbb{Y})$ is not in general positive.

We discuss this problem in greater detail in Sect. 3.4. The resolution of the problem entails that each iteration step consist of an inductive construction designed so that the expansion may be collapsed at any point. Certain bounds are proven in the collapsed form of the expansion, while others are accomplished in the uncollapsed form. The constants a^m acquire a dependence on the diameter of clusters.

A second major problem arises in proving smoothness of the construction in the parameters of $\mathscr{P}$. Smoothness is needed to solve for the parameters that yield an arbitrary set of a^m's, as was done in [24]. The difficult bound is Proposition 2.5.6. One must control the bounded-spin approximation for ϕ and obtain lower bounds on $Z_\Sigma(\mathbb{V})$. These problems will be covered in Chap. 5.

3.2. Contour Models

In this section we introduce certain statistical ensembles which will be used throughout this chapter. Recall the definition of clusters given in Sect. 2.4. In most of this chapter we restrict attention to clusters with more than one l-lattice square.

Definition. *Let F be a real-valued function of clusters. F will be called a contour functional if*

$$\|F\| \equiv \sup_{\mathbb{Y}} e^{\tau_1 l|\mathbb{Y}| + \tau_2 \lambda^{-2}|\Sigma_{\mathbb{Y}}|}|F(\mathbb{Y})| < \infty . \tag{3.2.1}$$

The restriction of F to q-clusters will be denoted F^q and called a q-contour model. $\|F^q\|$ is defined by restricting the supremum in (3.2.1) *to q-clusters. Note that $F(\mathbb{Y})$ need not be positive.*

Let $\mathbb{V}$ be a finite, connected, simply connected union of l-lattice squares together with a choice of boundary condition. Such a $\mathbb{V}$ will be called a simple domain. A domain is a finite union of simple domains. The components of a domain may have different boundary conditions. $|\mathbb{V}|$ will denote the number of l-lattice squares in $\mathbb{V}$ and $|\partial\mathbb{V}|$ will denote the number of l-lattice bonds in the boundary of $\mathbb{V}$.

For any contour functional F, we define a partition function as follows. Suppose $\mathbb{V}$ is a simple domain with boundary condition q. Then

$$\Omega(F, \mathbb{V}) \equiv \sum_{\substack{\{\mathbb{Y}_s^q\} \\ \mathbb{Y}_s^q \subseteq \mathbb{V},\ \text{nonoverlapping}}} \prod_s F^q(\mathbb{Y}_s^q) . \tag{3.2.2}$$

The sum over $\{\mathbb{Y}_s^q\}$ is over collections of nonoverlapping q-clusters in $\mathbb{V}$, and $|\mathbb{Y}_s^q| > 1$. If $\{\mathbb{Y}_s^q\} = \emptyset$, the product is set equal to 1. Note that every cluster in the collection has the same external boundary condition as $\mathbb{V}$ and that neighboring clusters may disagree on boundary conditions. This is different from the expansion of Sect. 2.4, where neighboring clusters agree, and where clusters with any external boundary condition may appear.

To define $\Omega(F, \mathbb{V})$ for general domains, let $\{\mathbb{V}_i\}$ be the simple domains that make up $\mathbb{V}$. Then

$$\Omega(F, \mathbb{V}) \equiv \prod_i \Omega(F, \mathbb{V}_i) . \tag{3.2.3}$$

An example of a contour functional is $\tilde{\varrho}$, with no field monomials present ($R=1$). [See (2.5.10) and (2.5.12).] From (2.5.11) we see that $\tilde{\varrho}(\mathbb{Y})$ has the requisite decay. In fact, there is room to spare in the exponential bound on $|\tilde{\varrho}(\mathbb{Y})|$ so that

$$F_L(\mathbb{Y}) \equiv \tilde{\varrho}(\mathbb{Y}) e^{L(\mathbb{Y})} \tag{3.2.4}$$

is also a contour functional if $L(\mathbb{Y}) \leqq \tau_1 l|\mathbb{Y}|$.

Much of this chapter is devoted to finding an $L(\mathbb{Y})$ such that the true partition function is given by

$$Z(\mathbb{V}) = \Omega(F_L, \mathbb{V}) . \tag{3.2.5}$$

This will only be possible if $\mathbb{V}$ has thermodynamically stable boundary conditions. (For the other boundary conditions, we will need the generalized contour models introduced in Sect. 3.4.) $L(\mathbb{Y})$ will be a sum of several terms, one of which is $\log \Omega(F, \mathbb{V})$ for some F and for $\mathbb{V}$ a domain surrounded by $\mathbb{Y}$. In order to preserve the bound $L(\mathbb{Y}) \leqq \tau_1 l |\mathbb{Y}|$ we will have to cancel the volume growth of $\log \Omega$ and leave just a surface term. In the next section we will show how to separate the surface effect from the volume effect and obtain bounds on each.

3.3. *Expansions for the Pressure and Surface Energy*

We make use of some algebraic properties of partition function sums in order to write $\Omega(F, \mathbb{V})$ as an exponential of some quantity. Then taking the logarithm and separating surface effects from volume effects will be easy. The main barrier to taking the logarithm of the expansion (3.2.2) is the constraint that no two clusters can overlap. Following [1], we implement the constraint using functions of clusters $U(\mathbb{Y}_1, \mathbb{Y}_2)$. The technique can be traced back to similar ideas in statistical mechanics [26] and in quantum field theory [8]. The U's defined here are special cases of ones in Chap. 4.

Define

$$U(\mathbb{Y}_1, \mathbb{Y}_2) = \begin{cases} 0 & \text{if } \mathbb{Y}_1 \text{ and } \mathbb{Y}_2 \text{ overlap (i.e., } |\mathbb{Y}_1 \cap \mathbb{Y}_2| \geqq 1) \\ 1 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Assume $\mathbb{V}$ is a simple domain with boundary condition q. Then the expansion for $\Omega(F, \mathbb{V})$ (3.2.2) becomes

$$\Omega(F, \mathbb{V}) = \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1^q, \ldots, \mathbb{Y}_k^q)} \prod_{s_1 < s_2} U(\mathbb{Y}_{s_1}^q, \mathbb{Y}_{s_2}^q) \prod_s F^q(\mathbb{Y}_s^q). \tag{3.3.2}$$

We have changed from summing over sets of $\mathbb{Y}_s^q$'s to summing over ordered families of $\mathbb{Y}_s^q$'s, hence the $1/k!$. The terms in this sum involving only nonoverlapping $\mathbb{Y}_s^q$'s are unchanged, while the others vanish. In the remainder of this section it will be assumed that all clusters have external boundary condition q and the superscript q will be omitted.

Define some more operations $A = U - 1$, and expand the product of U's using the formula

$$\prod_{\mathscr{L} \in \mathscr{S}} (1 + A(\mathscr{L})) = \sum_{G \subseteqq \mathscr{S}} \prod_{\mathscr{L} \in G} A(\mathscr{L}). \tag{3.3.3}$$

Here $\mathscr{S}$ is a set of unordered pairs of clusters (called lines $\mathscr{L}$). G is a subset of $\mathscr{S}$ and can be visualized as a graph connecting clusters with the lines $\mathscr{L}$. Equation (3.3.2) becomes

$$\Omega(F, \mathbb{V}) = \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)} \sum_G \prod_{\mathscr{L} \in G} A(\mathscr{L}) \prod_s F(\mathbb{Y}_s). \tag{3.3.4}$$

A graph G is called connected if the lines in G form a connected graph joining all the clusters in G. Every graph breaks into its connected components (which may consist of single clusters with no lines). Let $G_1, \ldots, G_n$ be the connected components of G. We sum separately over the $\mathbb{Y}_s$'s and G_j's in each component, using the identity

$$\sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)} \sum_G = \sum_n \frac{1}{n!} \prod_{j=1}^n \left(\sum_{k_j} \frac{1}{k_j!} \sum_{(\mathbb{Y}_1^{(j)}, \ldots, \mathbb{Y}_{k_j}^{(j)})} \sum_{G_j} \right). \tag{3.3.5}$$

To understand where the factorials come from, fix k, n, and $\{k_j\}$. The sums on the right do not allow for the $\mathbb{Y}$'s in the connected parts of G to be chosen arbitrarily from $(\mathbb{Y}_1, \dots, \mathbb{Y}_k)$, but always consecutively. Therefore the right-hand side should have a factor of the number of choices of n subsets with sizes $k_1, \dots, k_n$ from k objects, or $\frac{k!}{k_1! \dots k_n!}$. Having made this correction, the right-hand side counts permutations of the components of G as separate terms. Hence a factor $\frac{1}{n!}$ must be included to cancel this overcounting. Dividing both sides by $k!$ yields (3.3.5).

The expansion for $\Omega(F, \mathbb{V})$ takes the form

$$\Omega(F,\mathbb{V}) = \sum_n \frac{1}{n!} \prod_{j=1}^{n} \left(\sum_{k_j} \frac{1}{k_j!} \sum_{(\mathbb{Y}_1^{(j)}, \dots, \mathbb{Y}_{k_j}^{(j)})} \sum_{G_j} \prod_{\mathscr{L} \in G_j} A(\mathscr{L}) \prod_{s=1}^{k_j} F(\mathbb{Y}_s^{(j)}) \right). \tag{3.3.6}$$

This formula is valid because the A's in each term of $\prod_{j=1}^{n}$ do not depend on clusters in other terms. We observe that all the terms of $\prod_{j=1}^{n}$ are identical. Therefore we may take the logarithm to obtain

$$\log \Omega(F,\mathbb{V}) = \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1, \dots, \mathbb{Y}_k)} \sum_{G_c} \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F(\mathbb{Y}_s), \tag{3.3.7}$$

where the subscript c on G serves to remind us that the graph is connected. There are no connected graphs on zero clusters, so the $k=0$ term vanishes.

The number of clusters in $\mathbb{V}$ is finite, so the sums $\sum_{(\mathbb{Y}_1, \dots, \mathbb{Y}_k)}$, $\sum_G$ in the above discussion are all finite. The sums over k_j are infinite in (3.3.6), but since they are absolutely convergent sums (see below), there is no problem in deriving (3.3.6) from (3.3.4).

To see where the volume growth in (3.3.7) comes from, observe that translating all the $\mathbb{Y}$'s in some term produces another legitimate term, provided the $\mathbb{Y}$'s do not run across the boundary of $\mathbb{V}$. Therefore, summing only over families of $\mathbb{Y}$'s such that $\bigcup_s \mathbb{Y}_s$ contains some l-lattice square Δ should, up to surface effects, yield the coefficient of volume growth of $\log \Omega(F, \mathbb{V})$. Actually, we should divide each term by $\left|\bigcup_s \mathbb{Y}_s\right|$ because that is the number of times the term is counted when we sum over $\Delta \subseteq \mathbb{V}$.

In the remainder of this section, we suppose that $F(\mathbb{Y})$ is invariant under translations of $\mathbb{Y}$. This property holds for $\tilde{\varrho}(\mathbb{Y})$ because of its construction via a sufficiently large Λ. The pressure associated to a translation invariant contour model F^q is defined as follows:

$$s(F^q) \equiv \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1^q, \dots, \mathbb{Y}_k^q) : \bigcup_s \mathbb{Y}_s^q \supseteq \Delta} \left| \bigcup_s \mathbb{Y}_s^q \right|^{-1} \sum_{G_c} \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F^q(\mathbb{Y}_s^q). \tag{3.3.8}$$

Here, the clusters are not restricted to lie within a particular domain, as was the case with (3.3.7). The pressure is independent of the square used to define it, by the translation invariance of F. Note that $s(F^{q_1})$ will in general be different from $s(F^{q_2})$.

Subtracting $s(F^q)|\mathbb{V}|$ from $\log\Omega(F,\mathbb{V})$ should leave a remainder growing as $|\partial\mathbb{V}|$. Define

$$\Delta(F,\mathbb{V}) \equiv \log\Omega(F,\mathbb{V}) - s(F^q)|\mathbb{V}|$$

$$= \sum_k \frac{1}{k!} \sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k) \\ \mathbb{V}\cap\bigcup_s \mathbb{Y}_s \neq \emptyset \neq \sim\mathbb{V}\cap\bigcup_s \mathbb{Y}_s}} \frac{-\left|\mathbb{V}\cap\bigcup_s \mathbb{Y}_s\right|}{\left|\bigcup_s \mathbb{Y}_s\right|} \sum_{G_c} \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F(\mathbb{Y}_s). \quad (3.3.9)$$

The second line follows from (3.3.7) and (3.3.8). All the terms of (3.3.7) are cancelled, leaving graphs containing squares of both $\mathbb{V}$ and $\sim\mathbb{V}$. These do not appear in $\log\Omega(F,\mathbb{V})$, but appear exactly $\left|\mathbb{V}\cap\bigcup_s \mathbb{Y}_s\right|$ times in $s(F^q)|\mathbb{V}|$.

If $\mathbb{V}$ has several components $\mathbb{V}_i$, each with the same boundary condition, define

$$\Delta(F,\mathbb{V}) = \sum_i \Delta(F,\mathbb{V}_i). \quad (3.3.10)$$

We now estimate the pressure and surface energy, using a lemma proven in Sect. 4.4.

Lemma 3.3.1. *Suppose $\lambda \ll 1 \ll l$. If F^q is any q-contour model with $\|F^q\| \leqq 1$, then*

$$\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k):\ \bigcup_s \mathbb{Y}_s \supseteq \Delta \\ \sum_s |\mathbb{Y}_s| = N}} \left| \sum_{G_c} \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_{s=1}^{k} F^q(\mathbb{Y}_s) \right| \leqq k!\,\|F^q\| e^{-\tau_1 l(k+N)/4}. \quad (3.3.11)$$

There are two sources of convergence. The first is that $A(\mathbb{Y}_1,\mathbb{Y}_2)=0$ unless $\mathbb{Y}_1$ overlaps $\mathbb{Y}_2$. The second is the exponential decay of $F(\mathbb{Y})$ with $|\mathbb{Y}|$ and with $|\Sigma_{\mathbb{Y}}|$ [see Eq. (3.2.1)].

Proposition 3.3.2. *Suppose $\lambda \ll 1 \ll l$. Let F^q be a translation invariant q-contour model with $\|F^q\| \leqq 1$. Then*

$$|s(F^q)| \leqq \|F^q\|, \quad (3.3.12)$$

where $s(F^q)$ is defined in (3.3.8). *Moreover, if $\mathbb{V}$ is a domain with only q-boundary conditions, then*

$$\log\Omega(F,\mathbb{V}) = s(F^q)|\mathbb{V}| + \Delta(F,\mathbb{V}) \quad (3.3.13)$$

with

$$|\Delta(F,\mathbb{V})| \leqq \|F^q\|\,|\partial\mathbb{V}| \quad (3.3.14)$$

and $\Delta(F,\mathbb{V})$ given by (3.3.10) *and* (3.3.9). *Finally, if F_1^q and F_2^q are two q-contour models as above, then*

$$|s(F_1^q) - s(F_2^q)| \leqq \|F_1^q - F_2^q\| \quad (3.3.15)$$

and

$$|\Delta(F_1^q,\mathbb{V}) - \Delta(F_2^q,\mathbb{V})| \leqq \|F_1^q - F_2^q\|\,|\partial\mathbb{V}|. \quad (3.3.16)$$

Proof. By summing over $k \geqq 1$ and $N \geqq 2$ in (3.3.8) and applying the lemma, we obtain (3.3.12). Every graph in (3.3.9) must contain a square at the boundary of $\mathbb{V}$.

Applying Lemma 3.3.1 to each such square, and using $\frac{\left|\mathbb{V}\cap\bigcup_s \mathbb{Y}_s\right|}{\left|\bigcup_s \mathbb{Y}_s\right|}\leqq 1$, we obtain the bound $|\Delta(F,\mathbb{V})|\leqq \|F^q\|\,|\partial\mathbb{V}|$. Equation (3.3.14) follows by summing over the components of $\mathbb{V}$. To prove (3.3.15), interpolate between F_1 and F_2 in the formula for $s(F^q)$. Let $F_t^q(\mathbb{Y})=tF_2^q(\mathbb{Y})+(1-t)F_1^q(\mathbb{Y})$. Then

$$|s(F_1^q)-s(F_2^q)|=\left|\int_0^1 dt\frac{d}{dt}s(F_t^q)\right|$$

$$=\left|\int_0^1 dt\sum_k\frac{1}{k!}\sum_{(\mathbb{Y}_1^q,\dots,\mathbb{Y}_k^q):\bigcup_s\mathbb{Y}_s^q\supseteq\Delta}\left|\bigcup_s\mathbb{Y}_s^q\right|^{-1}\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\sum_{\hat{s}=1}^{k}(F_2^q(\mathbb{Y}_{\hat{s}}^q)-F_1^q(\mathbb{Y}_{\hat{s}}^q))\prod_{\substack{s=1\\ s\neq\hat{s}}}^{k}F_t^q(\mathbb{Y}_s^q)\right|$$

$$\leqq\|F_2^q-F_1^q\|\sum_k\frac{1}{k!}\sum_{(\mathbb{Y}_1^q,\dots,\mathbb{Y}_k^q):\bigcup_s\mathbb{Y}_s^q\supseteq\Delta}\left|\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{s=1}^{k}\hat{F}^q(\mathbb{Y}_s^q)\right|,\tag{3.3.17}$$

where $\hat{F}^q(\mathbb{Y}_s^q)=e^{-\tau_1 l|\mathbb{Y}_s^q|-\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}_s^q}|}$. The sum over $\hat{s}$ is controlled by $\left|\bigcup_s\mathbb{Y}_s^q\right|^{-1}$, and we have used $\|F_t^q\|\leqq 1$. Applying Lemma 3.3.1 as before, we obtain (3.3.15).

For (3.3.16), interpolate as above. The sum over squares Δ lining $\partial\mathbb{V}$ yields the factor $|\partial\mathbb{V}|$, as in (3.3.14). The factor k from the sum over $\hat{s}$ is controlled by $e^{-\tau_1 lk/8}$, taken out of (3.3.11). □

3.4. Contour Models with Parameters: An Inductive Construction

In this section we introduce ensembles whose purpose is to describe the unstable phases as well as the stable ones. The basic idea is as in [24]. Contour functionals are multiplied by factors that grow exponentially with the volume of interiors of clusters. The coefficient of volume growth can be thought of as the loss in energy density (gain in probability) achieved by fluctuating into a stable phase. The result is not a contour functional, but the new partition function can be crudely dominated by the old one times an exponential factor. When we have managed to express the partition function of our $\mathscr{P}(\phi)_2$ model as a partition function for a contour model with parameter, the crude bound will yield the ratio of partition function estimate that this chapter is devoted to.

Carrying out the above plan in the context of quantum field theory produces some problems. The most serious is the fact that $\tilde{\varrho}(\mathbb{Y})$ is not in general positive. Therefore, multiplying terms of associated partition function sums by factors greater than 1 does not necessarily increase the sums. There is, however, a form in which the mean field expansion has positive terms. This is after the expansion in spin configurations, but before the introduction of decoupling lines. Our strategy will be to multiply terms of the expansion by volume-divergent factors when it is the undecoupled form, and then use the decoupled form to make estimates that depend on decoupling. In order to work things so that one can go back and forth between the two forms at any time, we are forced to use a rather complicated procedure. We must exploit at each stage the collapsing expansions that result when terms are multiplied by ratios of partition functions. (In [24] a collapsing expansion was obtained only after the iterations had converged.)

We will work with contour functionals F_L, where

$$F_L(\mathbb{Y}) \equiv \tilde{\varrho}(\mathbb{Y}) e^{L(\mathbb{Y})} \tag{3.4.1}$$

and where certain growth conditions are imposed on L.

Definition. *Let $\delta(\mathbb{Y})$ denote the largest dimension of $\mathbb{Y}$, measuring distances in units of l along the directions of the l-lattice. We call $\delta(\mathbb{Y})$ the diameter of $\mathbb{Y}$. L, a real-valued translation invariant function of clusters, is called a vacuum functional if*

$$|||L||| \equiv \sup_{\mathbb{Y}} \frac{|L(\mathbb{Y})|}{\delta(\mathbb{Y})^3} < \infty \tag{3.4.2}$$

and

$$L(\mathbb{Y}) \leqq \tau_1 l |\mathbb{Y}|. \tag{3.4.3}$$

In d dimensions, we would use $\delta(\mathbb{Y})^{d+1}$ in place of $\delta(\mathbb{Y})^3$. We could also use the norm of [24] which allows $|L(\mathbb{Y})|$ to grow exponentially with $|\mathbb{Y}|$. Our stronger growth condition is possible because in the iteration of Sect. 3.6, errors accumulate more slowly with $\delta(\mathbb{Y})$ than they do in [24].

The splitting of F_L into $\tilde{\varrho}$ and e^L avoids the problem of taking the logarithm of $\tilde{\varrho}$. Notice that $e^{L(\mathbb{Y})}$ can be very small, but it cannot be very large, by (3.4.3). From (2.5.11) we have that F_L is a contour functional if L is a vacuum functional. Moreover,

$$\|F_L\| \leqq \lambda^{1/2}. \tag{3.4.4}$$

Given a contour functional F_L, define for each q

$$a^q(F_L) = -s(F_L^q) - \log Z_{\Delta^q} + \text{const}, \tag{3.4.5}$$

where the constant is adjusted so that

$$\inf_q a^q(F_L) = 0. \tag{3.4.6}$$

From (3.4.4), Proposition 3.3.2, and Proposition 2.5.2, we see that up to $O(\lambda^{1/2})$ corrections, $a^q(F_L)/l^2$ is the Wick ground state energy of the q^{th} well of $\mathscr{P}$ relative to the smallest such energy. Thus $a^q(F_L)$ may be as large as $O(\lambda^{-2} l^2)$.

Definition. *The interior domain of $\mathbb{Y}$, denoted $\mathrm{Int}\,\mathbb{Y}$, is defined as follows. Suppose $\mathbb{Y}$ has external boundary condition q. Consider the closure of the union of all the q-squares of $\mathbb{Y}$ (the ones with $p(\Delta^1) = q$). Of its connected components, let $\mathbb{Y}^{\text{ext}}$ be the one bordering on the outer boundary of $\mathbb{Y}$. Call a boundary loop of $\mathbb{Y}$ inner if it is not contained in $\mathbb{Y}^{\text{ext}}$. $\mathrm{Int}\,\mathbb{Y}$ is defined as the region bounded by the inner loops of $\mathbb{Y}$, with each component given the boundary condition associated to the loop bounding that component. $|\mathrm{Int}\,\mathbb{Y}|$ will denote the number of l-lattice squares in $\mathrm{Int}\,\mathbb{Y}$.*

Define $\mathbb{Y}^{\text{int}} = (\mathbb{Y} \cup \mathrm{Int}\,\mathbb{Y}) \backslash \mathbb{Y}^{\text{ext}}$. $\Sigma_{\mathbb{Y}^{\text{int}}}$, the spin configuration associated with $\mathbb{Y}^{\text{int}}$, is defined by extending $\Sigma_{\mathbb{Y}} \cap \mathbb{Y}^{\text{int}}$ to $\mathrm{Int}\,\mathbb{Y}$ so that the phase that exists on the boundary of a component of $\mathrm{Int}\,\mathbb{Y}$ persists in all of that component. See Fig. 3.4.1.

We define the partition function for a contour model with parameter in a manner closely related to Eq. (3.2.2). If $\mathbb{V}$ is a simple domain with boundary condition q, then

$$\Omega^a(F_L, \mathbb{V}) = \sum_{\substack{\{\mathbb{Y}_s^q\} \\ \mathbb{Y}_s^q \subseteq \mathbb{V},\ \text{nonoverlapping}}} \prod_s F_L(\mathbb{Y}_s^q) e^{a^q(F_L) \left|\bigcup_s \mathbb{Y}_s^{q\,\text{int}}\right|}. \tag{3.4.7}$$

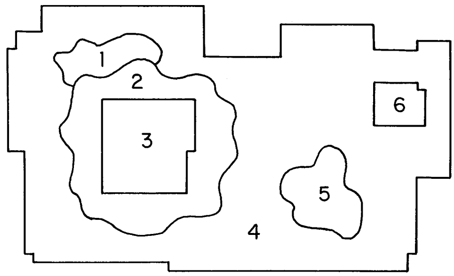

Fig. 3.4.1. The construction of the regions $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$, $\mathbb{Y}^{\mathrm{int}}$, $\mathbb{Y}^{\mathrm{ext}}$ associated to a cluster $\mathbb{Y}$. Let $\mathbb{Y}$ consist of regions 1, 2, 4, 5 with curved lines indicating the phase boundaries of $\mathbb{Y}$. Then $\mathbb{Y}^{\mathrm{int}}$ is the union of regions 1, 2, 3, 5; $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ is region 3; and $\mathbb{Y}^{\mathrm{ext}}$ is region 4. $\mathbb{Y}^{\mathrm{int}}$ is decomposed into subregions $\{R_i\}_{i=1,2,3}$. One labeling is as follows: $R_1=$ region 1; $R_2=$ regions 2, 3; $R_3=$ region 5. With this labeling, $\mathbb{I}\mathrm{nt}_{R_i}\mathbb{Y}=\emptyset$ for $i=1,3$ and $\mathbb{I}\mathrm{nt}_{R_2}\mathbb{Y}=$ region 3

Except for the factors $e^{a^q(F_L)\left|\bigcup_s \mathbb{Y}_s^{q\,\mathrm{int}}\right|}$, this is the same as $\Omega(F_L, \mathbb{V})$. More general domains, with possibly different boundary conditions on different components $\mathbb{V}_i$, are handled by taking a product:

$$\Omega^a(F_L, \mathbb{V}) = \prod_i \Omega^a(F_L, \mathbb{V}_i). \tag{3.4.8}$$

$a^q(F_L)$, the coefficient of interior volume, will vary from component to component, because q varies from component to component.

Starting from a vacuum functional L, we construct a new vacuum functional $\mathcal{N}(L)$ by successively making changes in $L(\mathbb{Y})$ for $\mathbb{Y}$'s with larger and larger diameters. The vacuum functional that results when changes have been made through $\delta(\mathbb{Y})=n$ will be denoted L_n. We then define

$$\mathcal{N}(L)(\mathbb{Y}) = L_n(\mathbb{Y}), \tag{3.4.9}$$

where n is any integer greater than or equal to $\delta(\mathbb{Y})$.

$L_n(\mathbb{Y})$ is defined to be zero if $\mathbb{Y}$ is a cluster without phase boundaries with $\delta(\mathbb{Y}) \leqq n$. If $n \leqq 2L/l$, the only other possibility is $\delta(\mathbb{Y}) > n$, in which case $L_n(\mathbb{Y}) = L(\mathbb{Y})$. This begins an inductive construction.

Given $L_1, \ldots, L_n$, we construct L_{n+1}. Define $F_k = F_{L_k}$ for $k \leqq n$. Suppose $\mathbb{Y}$ has external boundary condition q. If $\delta(\mathbb{Y}) \neq n+1$, $L_{n+1}(\mathbb{Y}) = L_n(\mathbb{Y})$. For $\delta(\mathbb{Y}) = n+1$, we proceed as follows. Let $\{R_i\}$ be the connected regions of constant phase in $\mathbb{Y}^{\mathrm{int}}$. Squares are considered connected if they abut on an edge or on a corner. See Fig. 3.4.1. Let $\mathbb{I}\mathrm{nt}_{R_i}\mathbb{Y}$ denote the components of $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ that are contained in R_i. Let $\mathbb{I}\mathrm{nt}^q_{R_i}\mathbb{Y}$ be the domain obtained by placing q-boundary conditions on every component of $\mathbb{I}\mathrm{nt}_{R_i}\mathbb{Y}$. Let $n_i = [\delta(R_i)]$ be the integer part of the diameter of R_i. Then we define

$$L_{n+1}(\mathbb{Y}) = \sum_i \left[A(R_i) + \log \frac{\Omega^a(F_{n_i}, \mathbb{I}\mathrm{nt}_{R_i}\mathbb{Y})}{\Omega(F_{n_i}, \mathbb{I}\mathrm{nt}^q_{R_i}\mathbb{Y})} \right], \tag{3.4.10}$$

where $A(R_i)$ is defined as follows.

R_2 is said to surround R_1 if every smooth path from R_1 to infinity intersects R_2 in a curve of positive length. R_2 is called the closest region surrounding R_1 if every other region from $\{R_i\}$ that surrounds R_1 surrounds R_2 also. Denote by $V(R_i)$ the volume (in units of l^2) of the union of R_i with all the regions surrounded by R_i. Let $p(R_i)$ denote the phase of R_i and let $m(R_i)$ denote the phase of the closest region surrounding R_i. If no regions surround R_i, then $m(R_i)=q$, the phase of $\mathbb{Y}^{\text{ext}}$ which surrounds all the regions.

If R_i is not surrounded by any other R, then

$$A(R_i) \equiv \left[-a^{m(R_i)}(F_{n_i}) + \log \frac{Z_{\Delta^{p(R_i)}}}{Z_{\Delta^{m(R_i)}}} \right] V(R_i). \tag{3.4.11}$$

Otherwise, let $R_{i_2(R_i)}$ be the closest region surrounding R_i. Then

$$A(R_i) \equiv \left[-a^{m(R_i)}(F_{n_i}) + a^{m(R_i)}(F_{n_{i_2(R_i)}}) + \log \frac{Z_{\Delta^{p(R_i)}}}{Z_{\Delta^{m(R_i)}}} \right] V(R_i). \tag{3.4.12}$$

This completes the inductive construction.

Notice that the construction of L_{n+1} used $F_1, \dots, F_n$, not just F_n. The reason is so that when we assemble a set of clusters with their factors e^L, the $a^{m(R_i)}(F_{n_i})$ coefficients associated with various regions will depend only on Σ, and not on Γ. This will permit us to resum the decoupling expansion, leaving only an expansion in phase boundaries where all terms are positive.

To prove that the construction does not produce L_n's that are not vacuum functionals, and to obtain useful properties of the Ω's, we state a proposition and carry it along as an inductive hypothesis.

Proposition 3.4.1. *Suppose $\lambda \ll 1 \ll l$. Let L be any vacuum functional, and for any n let $L_1, L_2, \dots, L_n$ be constructed from L as described above. Then $L_1, \dots, L_n$ are vacuum functionals, and $|||L_k||| \leqq |||L||| + O(\lambda^{-2} l^2)$ for $1 \leqq k \leqq n$.*

Suppose $\mathbb{V}$ is a simple domain with diameter $\delta(\mathbb{V})$ and boundary condition q. A representation for $\Omega^q(F_j, \mathbb{V})$ may be obtained for $j \leqq n$ as follows. For any spin configuration in $\mathbb{V}$ define regions R_i as in the construction given above. Define n_i, $p(R_i)$, $m(R_i)$, and $i_2(R_i)$ as above. If $R_{i_2(R_i)}$ does not exist, then set $n_{i_2(R_i)}=j$. If $\delta(\mathbb{V}) \leqq j \leqq n$ then

$$\Omega^q(F_j, \mathbb{V}) = \sum_{\substack{\Sigma \cap \mathbb{V} \\ \text{compatible with } \mathbb{V}}} \frac{Z_\Sigma(\mathbb{V})}{\prod\limits_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}} \prod_i e^{[-a^{m(R_i)}(F_{n_i}) + a^{m(R_i)}(F_{n_{i_2(R_i)}})] V(R_i)}. \tag{3.4.13}$$

The ordinary contour model partition function $\Omega(F_j, \mathbb{V})$ is given by the same formula except that for i such that $R_{i_2(R_i)}$ does not exist, the corresponding term in $\prod\limits_i$ has $a^{m(R_i)}(F_{n_{i_2(R_i)}})$ replaced by zero. By the nonnegativity of Z_Σ, we have

$$\frac{Z_{\Sigma \equiv q}(\mathbb{V})}{\prod\limits_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}} \leqq \Omega^q(F_j, \mathbb{V}) \leqq \Omega(F_j, \mathbb{V}) e^{a^q(F_j)|\mathbb{V}|}. \tag{3.4.14}$$

Proof. To avoid confusion, we suggest setting $E_c^m = E_w^m$ and $Z_{\Delta^m}=1$ for a first reading. We proceed by induction on n. For small n, $L_n(\mathbb{Y}) = L(\mathbb{Y})$ or zero, and the bound on $|||L_k|||$ is trivial. There can be no phase boundaries in $\mathbb{V}$, so $\operatorname{Int}\mathbb{Y} = \mathbb{Y}^{\text{int}} = R_i = \emptyset$ for all $\mathbb{Y} \subseteqq \mathbb{V}$. Therefore, $\Omega^q(F_j, \mathbb{V}) = \Omega(F_j, \mathbb{V}) = \Omega(\tilde{\varrho}, \mathbb{V})$. The decoupling expansion for $\frac{Z_{\Sigma \equiv q}(\mathbb{V})}{\prod\limits_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}}$ is exactly $\Omega(\tilde{\varrho}, \mathbb{V})$, so (3.4.13) follows.

To prove the proposition at stage n, suppose it holds for $\bar{n}<n$. We need the following bounds on $L_n(\mathbb{Y})$, for $\delta(\mathbb{Y})=n$:

$$-O(\lambda^{-2}l^2)\delta(\mathbb{Y})^3 \leqq L_n(\mathbb{Y}) \leqq \tau_1 l|\mathbb{Y}|. \tag{3.4.15}$$

Suppose $\mathbb{Y}$ has external boundary condition q. Visualize the accumulation of terms in $\sum_i A(R_i)$ as follows: For each i, $A(R_i)$ associates terms $\pm a^{m(R_i)}(F)/l^2$ and $\frac{1}{l^2}\log\frac{Z_{\Delta^{p(R_i)}}}{Z_{\Delta^{m(R_i)}}}$ to every unit square contained in, or surrounded by, R_i. Many of the $\pm\frac{1}{l^2}\log Z_\Delta$ terms cancel. If R_1 surrounds or contains a unit square and R_2 is the closest region surrounding R_1, then $\frac{1}{l^2}\log Z_{\Delta^{p(R_2)}}$ cancels with $-\frac{1}{l^2}\log Z_{\Delta^{m(R_1)}}$. The only uncancelled terms are $\frac{1}{l^2}\log Z_{\Delta^{p(R \supseteq \Delta^1)}}$ and $-\frac{1}{l^2}\log Z_{\Delta^q}$, the latter coming from the region with no R_i surrounding it.

The terms $\pm a^{m(R_i)}(F)/l^2$ in (3.4.12) would cancel if F_{n_i} were the same as $F_{n_{i_2(R_i)}}$. Keep track of them as follows. For each unit square Δ^1 of $\mathbb{Y}^{\text{int}}$, let $R_{i_1(\Delta^1)}$ be the region containing Δ^1, and let $R_{i_2(\Delta^1)}, R_{i_3(\Delta^1)}, \ldots, R_{i_\omega(\Delta^1)}$ be the successive regions surrounding Δ^1 such that each one is the closest region surrounding the last. Since $i_\alpha(\Delta^1)$ is independent of Δ^1 as long as Δ^1 is chosen from a fixed region, we can define more generally $i_\alpha(X)=i_\alpha(\Delta^1 \subseteqq X)$ for X a subset of some R_i. This agrees with our definition of $i_2(R_i)$ above. $R_{i_1(\Delta^1)}, \ldots, R_{i_{\omega-1}(\Delta^1)}$ contribute differences of a's and $R_{i_\omega(\Delta^1)}$ contributes only $-a^q(F_{n_{i_\omega(\Delta^1)}})/l^2$. We have derived the following equation:

$$\sum_i A(R_i)=\frac{1}{l^2}\sum_{\Delta^1 \subseteqq \mathbb{Y}^{\text{int}}}\left[\sum_{\alpha=1}^{\omega(\Delta^1)-1}\left(a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_{\alpha+1}(\Delta^1)}}) - a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_\alpha(\Delta^1)}})\right) - a^q(F_{n_{i_\omega(\Delta^1)}}) + \log\frac{Z_{\Delta^{p(R_{i_1(\Delta^1)})}}}{Z_{\Delta^q}}\right]. \tag{3.4.16}$$

Consider first the terms $\Delta^1 \subseteqq \mathbb{Y}^{\text{int}} \cap \mathbb{Y}$. The difference of a's is small because $F_{n_{i_{\alpha+1}}}$ differs from $F_{n_{i_\alpha}}$ only on clusters with diameter larger than n_{i_α}. Thus for every phase p,

$$\begin{aligned}|s(F^p_{n_{i_{\alpha+1}}}) - s(F^p_{n_{i_\alpha}})| &\leqq \|F^p_{n_{i_{\alpha+1}}} - F^p_{n_{i_\alpha}}\| \\ &\leqq e^{-\tau_1 l(n_{i_\alpha}+1)}\lambda^{1/2}.\end{aligned} \tag{3.4.17}$$

We have used Proposition 3.3.2 and the fact that $\tilde{\varrho}$ has extra exponential decay [see (2.5.11)] so that the possible growth from e^L and the large factor in the norm can be dominated and a factor $e^{-\tau_1 l(n_{i_\alpha}+1)}$ remains. Using the definition of a^m (3.4.5)–(3.4.6), we obtain

$$|a^m(F_{n_{i_{\alpha+1}(\Delta^1)}}) - a^m(F_{n_{i_\alpha(\Delta^1)}})| \leqq 2\lambda^{1/2}e^{-\tau_1 l(n_{i_\alpha(\Delta^1)})}. \tag{3.4.18}$$

Therefore,

$$\begin{aligned}&\left|\sum_{\alpha=1}^{\omega(\Delta^1)-1}\left(a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_{\alpha+1}(\Delta^1)}}) - a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_\alpha(\Delta^1)}})\right)\right| \\ &\leqq 2\lambda^{1/2}\sum_{\alpha=1}^{\infty} e^{-\tau_1 \alpha} \leqq O(1)\lambda^{1/2}.\end{aligned} \tag{3.4.19}$$

We are using the fact that $F_{n_{i_\alpha}}$ and $F_{n_{i_{\alpha+1}}}$ differ only on large clusters for large α. Hence the corresponding change in a is small.

The terms $-a^q+\log Z_{\Delta^p}-\log Z_{\Delta^q}$ are handled by noting that

$$a^q-a^p=s(F^p)-s(F^q)+\log Z_{\Delta^p}-\log Z_{\Delta^q}, \tag{3.4.20}$$

so that

$$\begin{aligned}-a^q+\log Z_{\Delta^p}-\log Z_{\Delta^q}&=-a^p+s(F^q)-s(F^p)\\ &\leqq s(F^q)-s(F^p)\\ &\leqq O(1)\lambda^{1/2}.\end{aligned} \tag{3.4.21}$$

Hence the contribution to L from $\Delta^1\subseteqq\mathbb{Y}$ is bounded by $O(\lambda^{1/2})|\mathbb{Y}|$.

Next consider $\Delta^1\subseteqq\mathbb{V}$, where $\mathbb{V}$ is a simple domain of $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$. We bound

$$\left|\sum_{\alpha=1}^{\omega(\Delta^1)-1}(a^m(F_{n_{i_{\alpha+1}}})-a^m(F_{n_{i_\alpha}}))\right|$$

as before, but noting that $n_{i_\alpha(\Delta^1)}\geqq\delta(\mathbb{V})$, which greatly reduces (3.4.17). Equation (3.4.19) becomes

$$\begin{aligned}&\left|\sum_{\alpha=1}^{\omega(\Delta^1)-1}(a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_{\alpha+1}(\Delta^1)}})-a^{m(R_{i_\alpha(\Delta^1)})}(F_{n_{i_{\alpha+1}(\Delta^1)}}))\right|\\ &\leqq 2\lambda^{1/2}\sum_{\alpha=1}^{\infty}e^{-\tau_1(\alpha+l\delta(\mathbb{V}))}\leqq O(1)\lambda^{1/2}e^{-\tau_1 l\delta(\mathbb{V})}.\end{aligned} \tag{3.4.22}$$

Summing over $\Delta^1\subseteqq\mathbb{V}$ and over $\mathbb{V}\subseteqq\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ yields a bound $O(\lambda^{1/2})|\mathbb{Y}|$ for the contribution of these terms, since the number of components of $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ is less than $|\mathbb{Y}|$.

Up till now we have been bounding the effects due to making the a-factors depend only on spin configurations. We next attend to the contribution from the ratio of partition functions, which is physically the interesting term. Consider each simple domain $\mathbb{V}$ of $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ separately. Note that $\bar{n}\equiv n_{i_1(\mathbb{V})}<n$ because $\delta(R_{i_1(\mathbb{V})})\leqq\delta(\mathbb{V})-2L$. Thus we can apply an induction hypothesis from Proposition 3.4.1:

$$\Omega^a(F_{\bar{n}},\mathbb{V})\leqq\Omega(F_{\bar{n}},\mathbb{V})e^{a^{p(\mathbb{V})}(F_{\bar{n}})|\mathbb{V}|}\leqq e^{(s(F_{\bar{n}}^{p(\mathbb{V})})+a^{p(\mathbb{V})}(F_{\bar{n}}))|\mathbb{V}|}e^{\Delta(F_{\bar{n}},\mathbb{V})}. \tag{3.4.23}$$

Here $p(\mathbb{V})$ is the phase of $\mathbb{V}$, and we have used Proposition 3.3.2. In addition,

$$\Omega(F_{\bar{n}},\mathbb{V}^q)=e^{s(F_{\bar{n}}^q)|\mathbb{V}|}e^{\Delta(F_{\bar{n}},\mathbb{V}^q)}, \tag{3.4.24}$$

where $\mathbb{V}^q$ is $\mathbb{V}$ with boundary condition changed to q. We can now bound the remaining contributions to L_n from $\mathbb{V}$ as follows:

$$\begin{aligned}&\log\frac{\Omega^a(F_{\bar{n}},\mathbb{V})}{\Omega(F_{\bar{n}},\mathbb{V}^q)}-a^q(F_{n_{i_\omega(\mathbb{V})}})+\log\frac{Z_{\Delta^{p(\mathbb{V})}}}{Z_{\Delta^q}}\\ &\leqq\Delta(F_{\bar{n}},\mathbb{V})-\Delta(F_{\bar{n}},\mathbb{V}^q)+|\mathbb{V}|\\ &\quad\cdot(s(F_{\bar{n}}^{p(\mathbb{V})})+a^{p(\mathbb{V})}(F_{\bar{n}})-s(F_{\bar{n}}^q)-a^q(F_{n_{i_\omega(\mathbb{V})}})+\log Z_{\Delta^{p(\mathbb{V})}}-\log Z_{\Delta^q})\\ &=\Delta(F_{\bar{n}},\mathbb{V})-\Delta(F_{\bar{n}},\mathbb{V}^q)+|\mathbb{V}|(a^q(F_{\bar{n}})-a^q(F_{n_{i_\omega(\mathbb{V})}}))\\ &\leqq 2\lambda^{1/2}|\partial\mathbb{V}|+2\lambda^{1/2}|\mathbb{V}|e^{-\tau_1 l\delta(\mathbb{V})}\\ &\leqq O(\lambda^{1/2})|\partial\mathbb{V}|.\end{aligned} \tag{3.4.25}$$

Using the definition of the a's, we see that all of the volume factors would have cancelled exactly, except for the difference between $a^q(F_{\bar{n}})$ and $a^q(F_{n_{i_\omega(\mathbb{V})}})$. This difference is bounded as in (3.4.18). What remains is a surface effect. This near-cancellation of volume effects is simply the result of a judicious choice of volume terms in L. Summing over $\mathbb{V}\subseteq\mathbb{I}\mathrm{nt}\,\mathbb{Y}$, and using the other results above, we obtain the bound $L_n(\mathbb{Y})\leqq O(\lambda^{1/2})|\mathbb{Y}|$.

We proceed to the lower bound on $L_n(\mathbb{Y})$. Except for (3.4.25) and (3.4.21), all of the above bounds are bounds on absolute values and can be used for the lower bound. From (3.4.14), we have

$$\log\frac{\Omega^q(F_{\bar{n}},\mathbb{V})}{\Omega(F_{\bar{n}},\mathbb{V}^q)}\geqq\log\frac{Z_{\Sigma\equiv q}(\mathbb{V})}{\prod\limits_{\Delta\subseteq\mathbb{V}}Z_{\Delta^q}}-\Delta(F_{\bar{n}},\mathbb{V}^q)-s(F_{\bar{n}}^q)|\mathbb{V}|$$
$$\geqq -O(\lambda^{1/2})|\mathbb{V}|. \tag{3.4.26}$$

In the last step we have used the decoupling expansion to show that

$$\frac{Z_{\Sigma\equiv q}(\mathbb{V})}{\prod\limits_{\Delta\subseteq\mathbb{V}}Z_{\Delta^q}}=\sum_{\{\mathbb{Y}_s^q\}:\Sigma_{\mathbb{Y}_s^q}=\emptyset,\ \text{nonoverlapping}}\prod_s\tilde{\varrho}(\mathbb{Y}_s^q)$$
$$=\Omega(F_0,\mathbb{V}), \tag{3.4.27}$$

where $F_0(\mathbb{Y})=\tilde{\varrho}(\mathbb{Y})$ if $\Sigma_{\mathbb{Y}}=\emptyset$ and $F(\mathbb{Y})=0$ otherwise. Since $\|F_0\|\leqq\lambda^{1/2}$, Proposition 3.3.2 yields (3.4.26).

The other terms $-a^q(F_{n_{i_\omega(\mathbb{V})}})+\log Z_{\Delta^p}(\mathbb{V})-\log Z_{\Delta^p}$ occurring in (3.4.25) and (3.4.21) are bounded below by $O(\lambda^{-2}l^2)$. Altogether we have

$$L_n(\mathbb{Y})\geqq -O(\lambda^{-2}l^2)|\mathbb{Y}\cup\mathbb{I}\mathrm{nt}\,\mathbb{Y}|$$
$$\geqq -O(\lambda^{-2}l^2)\delta(\mathbb{Y})^2. \tag{3.4.28}$$

Thus $|||L_n(\mathbb{Y})|||\leqq O(\lambda^{-2}l^2)+|||L|||$. We include $|||L|||$ in this estimate because of contributions to L_n from $\mathbb{Y}$'s with $\delta(\mathbb{Y})>n$. The need for the $\frac{1}{\delta(\mathbb{Y})^3}$ factor in the definition of $|||L|||$ will not appear until later in this chapter.

We now prove (3.4.13) and the analogous statement for $\Omega(F_j,\mathbb{V})$. The proof is the core of our construction. Four basic steps are involved. The expansion for $\Omega^q(F_j,\mathbb{V})$ is collapsed, and the induction hypothesis is applied for $\mathbb{I}\mathrm{nt}\,\mathbb{Y}$ with $\mathbb{Y}\subseteq\mathbb{V}$. Then after a complicated matching and cancellation of volume factors, the decoupling expansion is resummed to complete the proof. The first step relies on our technique of modifying L one diameter at a time. The matching and cancellations are possible because we have anticipated in L and in (3.4.13) the structure of the terms that arise. The resummation is possible because the volume terms depend only on Σ (not on Γ).

We can assume $\delta(\mathbb{V})\leqq j=n$ because the other cases are covered by the inductive hypothesis. Given $\{\mathbb{Y}_s^q\}$, a set of nonoverlapping clusters in $\mathbb{V}$, define the outer clusters to be the ones not contained in $\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s^q$ for any $\mathbb{Y}_s^q$ in the set. In (3.4.7) we separate the outer clusters from the rest and sum over the rest first. This sum factors into separate sums over each component of $\bigcup_{\mathbb{Y}_s^q\ \text{outer}}\mathbb{I}\mathrm{nt}\,\mathbb{Y}_s^q$. In each component, the sum is an unconstrained partition function sum as in (3.2.2).

$\left|\bigcup_s \mathbb{Y}_s^{q\,\mathrm{int}}\right|$ can be determined from the outer clusters only. Thus we have

$$\Omega^a(F_n, \mathbb{V}) = \sum_{\substack{\{\mathbb{Y}_s^q\}\ \text{outer} \\ \mathbb{Y}_s^q \subseteqq \mathbb{V},\ \text{nonoverlapping}}} \prod_s [F_n(\mathbb{Y}_s^q)\Omega(F_n, \mathrm{Int}^q \mathbb{Y}_s^q)] e^{a^q(F_n)\left|\bigcup_s \mathbb{Y}_s^{q\,\mathrm{int}}\right|}. \tag{3.4.29}$$

We have changed the boundary conditions of $\mathrm{Int}\,\mathbb{Y}_s^q$ to q in $\Omega(F_n, \mathrm{Int}^q \mathbb{Y}_s^q)$ because all clusters must have external boundary condition q.

Divide $L_n(\mathbb{Y})$ into three parts [for $\delta(\mathbb{Y}) \leqq n$]:

$$L_n(\mathbb{Y}) = L_n^a(\mathbb{Y}) + L_n^\Delta(\mathbb{Y}) + L_n^\Omega(\mathbb{Y}). \tag{3.4.30}$$

L_n^a contains all the a-terms in (3.4.11) and (3.4.12); L_n^Δ contains the $\log Z_\Delta$ terms; and L_n^Ω is the $\log \Omega^a/\Omega$ term in (3.4.10). The partition functions in (3.4.29) cancel with the ones in $e^{L_n^\Omega(\mathbb{Y})}$ because F_n differs from $F_{n_{i_1(\mathbb{V})}}$ only on clusters that do not contribute to $\Omega(F_n, \mathbb{W}^q)$. This cancellation is the signature of a collapsing expansion. Here $\mathbb{W}$ is a component of $\mathrm{Int}\,\mathbb{Y}_s^q$. We obtain

$$\begin{aligned}
&F_n(\mathbb{Y}_s^q)\Omega(F_n, \mathrm{Int}^q \mathbb{Y}_s^q) \\
&\quad = \tilde{\varrho}(\mathbb{Y}_s^q) e^{L_n^a(\mathbb{Y}_s^q)} e^{L_n^\Delta(\mathbb{Y}_s^q)} \prod_{\mathbb{W} \subseteqq \mathrm{Int}\,\mathbb{Y}_s^q} \Omega^a(F_{n_{i_1(\mathbb{W})}}, \mathbb{W}) \\
&\quad = \tilde{\varrho}(\mathbb{Y}_s^q) e^{L_n^a(\mathbb{Y}_s^q)} e^{L_n^\Delta(\mathbb{Y}_s^q)} \prod_{\mathbb{W} \subseteqq \mathrm{Int}\,\mathbb{Y}_s^q} \\
&\qquad \cdot \left[\sum_{\Sigma \cap \mathbb{W}} \frac{Z_\Sigma(\mathbb{W})}{\prod_{\Delta \subseteqq \mathbb{W}} Z_{\Delta^{p(\mathbb{W})}}} \prod_i e^{[-a^{m(R_i)}(F_{n_i}) + a^{m(R_i)}(F_{n_{i_2(R_i)}})]V(R_i)} \right] \\
&\quad = \frac{\varrho(\mathbb{Y}_s^q) e^{(E_c^q - E_{\mathbb{W}}^q)l^2|\mathbb{Y}_s^q|} e^{\sum_{n \neq q} (-E_c^n + E_{\mathbb{W}}^n + E_c^q - E_{\mathbb{W}}^q)l^2 |\mathrm{Int}_n \mathbb{Y}_s^q|}}{\prod_{\Delta \subseteqq \mathbb{Y}_s^q \cup \mathrm{Int}\,\mathbb{Y}_s^q} Z_{\Delta^q}} e^{L_n^a(\mathbb{Y}_s^q)} \\
&\qquad \cdot \prod_{\mathbb{W} \subseteqq \mathrm{Int}\,\mathbb{Y}_s^q} \left[\sum_{\Sigma \cap \mathbb{W}} Z_\Sigma(\mathbb{W}) \prod_i e^{[-a^{m(R_i)}(F_{n_i}) + a^{m(R\ )}(F_{n_{i_2(R_i)}})]V(R_i)} \right].
\end{aligned} \tag{3.4.31}$$

We have applied the inductive hypothesis (3.4.13) to $\mathbb{V}$, which must have diameter less than n. In the last step, we have used our previous analysis of $L_n^\Delta(\mathbb{Y}_s^q)$: It is a sum over $\Delta^1 \subseteqq \mathbb{Y}_s^{q\,\mathrm{int}}$ of $\frac{1}{l^2}\log\frac{Z_{\Delta^{p(\Delta^1)}}}{Z_{\Delta^q}}$. For $\Delta^1 \subseteqq \mathbb{Y}_s^q$, $Z_{\Delta^{p(\Delta^1)}}$ cancels the $1/Z_{\Delta^{p(\Delta^1)}}$ factor in $\tilde{\varrho}(\mathbb{Y}_s^q)$. For $\Delta^1 \subseteqq \mathrm{Int}\,\mathbb{Y}_s^q$, it cancels the $1/Z_{\Delta^{p(\mathbb{V})}}$ in (3.4.31). The end result is that $\tilde{\varrho}(\mathbb{Y}_s^q)$ is changed to $\varrho(\mathbb{Y}_s^q)$ with the inclusion of some energy factors and the Z_Δ's are as in the last line of (3.4.31). Recall from Proposition 3.4.1 that $n_{i_2(R_i)} = n_{i_1(\mathbb{W})}$ if R_i has no region surrounding it.

Combine (3.4.29) with (3.4.31). Each term in the sum

$$\sum_{\{\mathbb{Y}_s^q\}\ \text{outer}} \prod_{\mathbb{W}} \left[\sum_{\Sigma \cap \mathbb{W}} \cdots \right]$$

determines a spin configuration $\Sigma_{\mathbb{V}}$ in all of $\mathbb{V}$. We fix $\Sigma_{\mathbb{V}}$ and sum over all $\{\mathbb{Y}_s^q\}$ compatible with $\Sigma_{\mathbb{V}}$. Compatibility here means that the $\mathbb{Y}_s^q$'s do not overlap, that they agree with $\Sigma_{\mathbb{V}}$, and that no $\mathbb{Y}_s^q$ is contained in any $\mathrm{Int}\,\mathbb{Y}_{s'}^q$. After some shuffling of energy factors, the result is

$$\Omega^a(F_n, \mathbb{V}) = \sum_{\Sigma_{\mathbb{V}}} \sum_{\substack{\{\mathbb{Y}_s^q\}\text{ outer} \\ \text{compatible with } \Sigma_{\mathbb{V}}}} e^{a^q(F_n)\left|\bigcup_s \mathbb{Y}_s^{q\,\text{int}}\right|}$$

$$\cdot \prod_s \left[\frac{\varrho(\mathbb{Y}_s^q) e^{(E_c^q - E_w^q)l^2|\mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q|}}{\prod_{\Delta \subseteqq \mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q} Z_{\Delta^q}} e^{L_n^a(\mathbb{Y}_s^q)} \prod_{\mathbb{W} \subseteqq \mathbb{I}\text{nt}\,\mathbb{Y}_s^q} \right.$$

$$\left. \cdot \left[e^{(-E_c^{p(\mathbb{W})} + E_w^{p(\mathbb{W})})l^2|\mathbb{W}|} Z_{\Sigma_{\mathbb{V}} \cap \mathbb{W}}(\mathbb{W}) \prod_i e^{[-a^{m(R_i)}(F_{n_i}) + a^{m(R_i)}(F_{n_{i_2(R_i)}})]V(R_i)} \right] \right]$$

$$= \sum_{\Sigma_{\mathbb{V}}} \prod_i e^{[-a^{m(R_i)}(F_{n_i}) + a^{m(R_i)}(F_{n_{i_2(R_i)}})]V(R_i)} \sum_{\substack{\{\mathbb{Y}_s^q\}\text{ outer} \\ \text{compatible with } \Sigma_{\mathbb{V}}}} \prod_s$$

$$\cdot \left[\frac{\varrho(\mathbb{Y}_s^q) e^{(E_c^q - E_w^q)l^2|\mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q|}}{\prod_{\Delta \subseteqq \mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q} Z_{\Delta^q}} \prod_{\mathbb{W} \subseteqq \mathbb{I}\text{nt}\,\mathbb{Y}_s^q} [e^{(-E_c^{p(\mathbb{W})} + E_w^{p(\mathbb{W})})l^2|\mathbb{W}|} Z_{\Sigma_{\mathbb{V}} \cap \mathbb{W}}(\mathbb{W})] \right]. \quad (3.4.32)$$

We have combined all the e^a factors into a simple form that depends only on the regions R_i associated with $\Sigma_{\mathbb{V}}$. For a region R_i in some $\mathbb{W}$ such that $R_{i_2(R_i)}$ does not exist in $\mathbb{W}$, we have

$$a^{m(R_i)}(F_{n_{i_2(R_i)}}) \equiv a^{p(\mathbb{W})}(F_{n_{i_1(\mathbb{W})}}) = a^{m(R_i)}(F_{n_{i_2(R_i)}}),$$

where for the second equality $i_2(R_i)$ and $m(R_i)$ are defined in terms of the regions associated with $\Sigma_{\mathbb{V}}$. For regions R_i such that $R_{i_2(R_i)}$ does not exist in $\mathbb{V}$, the $a^{m(R_i)}(F_{n_{i_2(R_i)}})$ is defined to be $a^q(F_n)$ and such terms arise from the overall $e^{a^q(F_n)\left|\bigcup_s \mathbb{Y}_s^{q\,\text{int}}\right|}$ factor. [This would be absent if we were dealing with $\Omega(F_n, \mathbb{V})$ instead of $\Omega^a(F_n, \mathbb{V})$.] For all the other regions, the last step is a straightforward transcription of terms from the previous step [from $L_n^a(\mathbb{Y}_s^q)$ or from $\mathbb{W}$-terms].

Equation (3.4.13) now follows from the following string of identities:

$$\sum_{\{\mathbb{Y}_s^q\}} \prod_s [\ldots] = \frac{e^{(E_c^q - E_w^q)l^2|\mathbb{V}|}}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}} \sum_{\substack{\{\mathbb{Y}_s^q\}\text{ outer} \\ \text{compatible with } \Sigma_{\mathbb{V}}}} \prod_{\Delta \subseteqq \mathbb{V} \setminus \bigcup_s (\mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q)} \varrho(\Delta^q)$$

$$\cdot \prod_s \left[\varrho(\mathbb{Y}_s^q) \prod_{\mathbb{W} \subseteqq \mathbb{I}\text{nt}\,\mathbb{Y}_s^q} [e^{(-E_c^{p(\mathbb{W})} + E_w^{p(\mathbb{W})})l^2|\mathbb{W}|} Z_{\Sigma_{\mathbb{V}} \cap \mathbb{W}}(\mathbb{W})] \right]$$

$$= \frac{e^{(E_c^q - E_w^q)l^2|\mathbb{V}|}}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}} \sum_{\substack{\{\mathbb{Y}_s^q\}\text{ outer} \\ \text{compatible with } \Sigma_{\mathbb{V}}}} \prod_{\Delta \subseteqq \mathbb{V} \setminus \bigcup_s (\mathbb{Y}_s^q \cup \mathbb{I}\text{nt}\,\mathbb{Y}_s^q)} \varrho(\Delta^q)$$

$$\cdot \prod_s \left[\varrho(\mathbb{Y}_s^q) \sum_{\substack{\{\mathbb{Y}_{r_s}\}\text{ nonoverlapping, } |\mathbb{Y}_{r_s}| \geqq 1 \\ \Sigma_{\mathbb{Y}_{r_s}} = \Sigma_{\mathbb{V}} \cap \mathbb{Y}_{r_s},\text{ filling } \mathbb{I}\text{nt}\,\mathbb{Y}_s^q}} \prod_{r_s} \varrho(\mathbb{Y}_{r_s}) \right]$$

$$= \frac{e^{(E_c^q - E_w^q)(^2|\mathbb{V}|}}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}} \sum_{\substack{\{\mathbb{Y}_s\}\text{ nonoverlapping, } |\mathbb{Y}_s| \geqq 1 \\ \Sigma_{\mathbb{Y}_s} = \Sigma_{\mathbb{V}} \cap \mathbb{Y}_s,\text{ filling } \mathbb{V}}} \prod_s \varrho(\mathbb{Y}_s)$$

$$= \frac{Z_{\Sigma_{\mathbb{V}}}(\mathbb{V})}{\prod_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}}. \quad (3.4.33)$$

The first step uses the identity $Z_{\Delta^q}=e^{(E_e^q-E_w^q)l^2}\varrho(\Delta^q)$ to rearrange energy factors. The second and last steps are simply the cluster expansion performed at fixed Σ [see (2.4.18) and (2.4.21)]. Note that $\mathbb{Y}_{r_s}$ and $\mathbb{Y}_s$ need not have q-boundary conditions, and they may be single l-squares. The third step follows because the inner sums and the presence of the $\varrho(\Delta^q)$'s outside relieve the constraint that $\{\mathbb{Y}_s^q\}$ be a set of outer q-clusters, yielding an overall summation over $\{\mathbb{Y}_s\}$ as above.

The analog of (3.4.13) for $\Omega(F_n, \mathbb{V})$ follows exactly as above, with the only change being the absence of the factor $e^{a^q(F_n)\left|\bigcup_s \mathbb{Y}_s^{q\,\mathrm{int}}\right|}$. This leads to the modification in (3.4.13) stated in Proposition 3.4.1. Equation (3.4.14) follows immediately. This completes the proof of Proposition 3.4.1. □

3.5. An Equation That Yields the Ratio of Partition Function Estimate

In the last section we defined a transformation $L\mapsto\mathcal{N}(L)$ of vacuum functionals. We now explore the consequences of having a fixed point for this transformation, that is, a vacuum functional L that solves the equation

$$L=\mathcal{N}(L). \tag{3.5.1}$$

If L solves (3.5.1), then $L_1, L_2, \ldots$ must all be equal to L, because of the way $\mathcal{N}(L)$ was constructed. Thus $F_{L_n}=F_L$ for all n and the terms $-a^{m(R_i)}(F_{n_i})$ $+a^{m(R_i)}(F_{n_{i_2(R_i)}})$ in (3.4.13) exactly cancel. Therefore, Proposition 3.4.1 implies that

$$\Omega^a(F_L, \mathbb{V})=\sum_{\Sigma\cap\mathbb{V}}\frac{Z_\Sigma(\mathbb{V})}{\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^q}}=\frac{Z(\mathbb{V})}{\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^q}}, \tag{3.5.2}$$

where $\mathbb{V}$ has boundary condition q. Place a subscript 0 on letters denoting phases q such that $a^q(F_L)=0$, for example q_0. There must be at least one such phase because of the condition $\inf\limits_q a^q(F_L)=0$. These phases will turn out to be the thermodynamically stable ones. If $\mathbb{V}$ has boundary condition q_0, then there is no difference between $\Omega^a(F_L, \mathbb{V})$ and $\Omega(F_L, \mathbb{V})$ because the elimination of the term $a^{m(R_i)}(F_{n_{i_2(R_i)}})$ in (3.4.13) has no effect. Therefore,

$$\Omega(F_L, \mathbb{V}^{q_0})=\frac{Z(\mathbb{V}^{q_0})}{\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^{q_0}}}. \tag{3.5.3}$$

Theorem 3.5.1. *Suppose $L=\mathcal{N}(L)$ and $\lambda\ll 1\ll l$. Let $\mathbb{V}$ be a simple domain with boundary condition q, and let $\mathbb{V}^{q_0}$ be the same domain but with some stable boundary condition ($a^{q_0}(F_L)=0$) replacing q. Then*

$$\frac{Z(\mathbb{V})}{Z(\mathbb{V}^{q_0})}\leqq e^{2\lambda^{1/2}|\partial\mathbb{V}|}. \tag{3.5.4}$$

Proof. From (3.5.2) and (3.5.3) we have

$$\begin{aligned}\frac{Z(\mathbb{V})}{Z(\mathbb{V}^{q_0})}&=\frac{\Omega^a(F_L, \mathbb{V})\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^q}}{\Omega(F_L, \mathbb{V}^{q_0})\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^{q_0}}}\\&\leqq\frac{\Omega(F_L, \mathbb{V})e^{a^q(F_L)|\mathbb{V}|}\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^q}}{\Omega(F_L, \mathbb{V}^{q_0})\prod\limits_{\Delta\subseteq\mathbb{V}} Z_{\Delta^{q_0}}},\end{aligned} \tag{3.5.5}$$

where (3.4.14) has been applied from Proposition 3.4.1. Using the estimates of Proposition 3.3.2, we obtain

$$\log\frac{Z(\mathbb{V})}{Z(\mathbb{V}^{q_0})} \leqq (a^q(F_L)+s(F_L^q)-s(F_L^{q_0})+\log Z_{\Delta^q}-\log Z_{\Delta^{q_0}})|\mathbb{V}|$$
$$+\Delta(F_L,\mathbb{V})-\Delta(F_L,\mathbb{V}^{q_0})$$
$$\leqq 2\lambda^{1/2}|\partial\mathbb{V}|. \tag{3.5.6}$$

The terms proportional to $|\mathbb{V}|$ cancel exactly, by the definition of the a^q's. This completes the proof. □

Theorem 3.5.1 will be of crucial importance in Chap. 4, where an expansion is given for the Schwinger functions in the stable phases.

3.6. Solving $L=\mathcal{N}(L)$ by Iteration

Starting with $L=0$, we produce a sequence $0, \mathcal{N}(0), \mathcal{N}(\mathcal{N}(0)), \ldots$ and show that the sequence converges in the norm $|||\cdot|||$. The limiting vacuum functional will satisfy $L=\mathcal{N}(L)$. Convergence is very rapid, with each iteration making a change a factor $e^{-\tau_2\lambda^{-2}}$ smaller than the previous change. This is due to the fact that $L\equiv 0$ except for $\mathbb{Y}$'s with some phase boundaries. Proving convergence involves obtaining Lipschitz conditions on various aspects of the construction of $\mathcal{N}(L)$.

Lemma 3.6.1. *Suppose $\lambda \ll 1 \ll l$. Let L and L' be two vacuum functionals such that $L(\mathbb{Y})=L'(\mathbb{Y})$ on $\mathbb{Y}$'s with no phase boundaries. For each q, the following bound holds:*

$$\|F_L-F_{L'}\| \leqq e^{-4\tau_2\lambda^{-2}}|||L-L'|||. \tag{3.6.1}$$

Proof. By definition, $F_L^q(\mathbb{Y})=\tilde{\varrho}(\mathbb{Y})e^{L(\mathbb{Y})}$. By Proposition 3.3.2, the lemma follows from the bound

$$\sup_{\mathbb{Y}} e^{\tau_1 l|\mathbb{Y}|+\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|}|\tilde{\varrho}(\mathbb{Y})e^{L(\mathbb{Y})}-\tilde{\varrho}(\mathbb{Y})e^{L'(\mathbb{Y})}| \leqq e^{-4\tau_2\lambda^{-2}}|||L-L'|||. \tag{3.6.2}$$

Let $L^t(\mathbb{Y})=(1-t)L(\mathbb{Y})+tL'(\mathbb{Y})$. Then

$$|e^{L(\mathbb{Y})}-e^{L'(\mathbb{Y})}| \leqq \left|\int_0^1 dt\frac{d}{dt}e^{L^t(\mathbb{Y})}\right|$$
$$\leqq |L(\mathbb{Y})-L'(\mathbb{Y})|\sup_t e^{L^t(\mathbb{Y})}$$
$$\leqq |||L-L'|||\delta(\mathbb{Y})^3 e^{\tau_1 l|\mathbb{Y}|}. \tag{3.6.3}$$

The bound (3.6.2) follows because $e^{-\tau_1 l|\mathbb{Y}|}\delta(\mathbb{Y})^3 \leqq 1$ so that the decay of $\tilde{\varrho}(\mathbb{Y})$ (2.5.11) dominates both the diverging factors of (3.6.3) and the growth from the definition of $\|\cdot\|$. The fact that $L \neq L'$ only on clusters with phase boundaries insures the presence of the factor $e^{-4\tau_2\lambda^{-2}}$ from $|\tilde{\varrho}(\mathbb{Y})|$, since the minimum length of phase boundary is 4. □

Let $|\mu_1-\mu_2|$ denote $\sup_i |\mu_1^i-\mu_2^i|$.

Lemma 3.6.2. *Suppose $\lambda \ll 1 \ll l$. Then for L a vacuum functional,*

$$\|\tilde{\varrho}_{\mu_1}e^L-\tilde{\varrho}_{\mu_2}e^L\| \leqq \lambda^{1/2}|\mu_1-\mu_2|. \tag{3.6.4}$$

Proof. We need the bound

$$|\tilde{\varrho}_{\mu_1}(\mathbb{Y})e^L - \tilde{\varrho}_{\mu_2}(\mathbb{Y})e^L| \leqq \lambda^{1/2} e^{-\tau_1 l|\mathbb{Y}| - \tau_2 \lambda^{-2}|\Sigma_{\mathbb{Y}}|}|\mu_1 - \mu_2|. \tag{3.6.5}$$

Apply Proposition 2.5.5 to $\varrho_{\Lambda,q}(\mathbb{Y})$ and to

$$\varrho_{\Lambda,q}(\Delta^m)e^{(-E_c^q + E_c^m)l^2} = Z_{\Delta^m} e^{E_{\mathrm{w}}^m l^2}$$

to obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial\mu^i}\tilde{\varrho}(\mathbb{Y})\right| \\
&\leqq \frac{\left|\frac{\partial}{\partial\mu^i}\left(\varrho_{\Lambda,q}(\mathbb{Y})e^{\sum\limits_m(-E_c^q + E_{\mathrm{w}}^m + E_c^{p(\mathbb{Y})} - E_{\mathrm{w}}^{p(\mathbb{Y})})l^2|\mathbb{Y}|_m} e^{\sum\limits_{m \neq p(\mathbb{Y})}(-E_c^m + E_{\mathrm{w}}^m + E_c^{p(\mathbb{Y})} - E_{\mathrm{w}}^{p(\mathbb{Y})})l^2|\mathrm{Int}_m \mathbb{Y}|}\right)\right|}{\prod\limits_m (Z_{\Delta^m} e^{l^2 E_{\mathrm{w}}^m})^{|\mathbb{Y}|_m}} \\
&\quad + |\tilde{\varrho}(\mathbb{Y})| \sum_m |\mathbb{Y}|_m \frac{\left|\frac{\partial}{\partial\mu^i}(Z_{\Delta^m} e^{l^2 E_{\mathrm{w}}^m})\right|}{Z_{\Delta^m} e^{l^2 E_{\mathrm{w}}^m}} \\
&\leqq \lambda^{1/2} e^{-(5/2\tau_1 l - a\lambda l^2)|\mathbb{Y}|} e^{-\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|} + \lambda^{1/2} e^{-5/2\tau_1 l|\mathbb{Y}|} e^{-2\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|}|\mathbb{Y}|\,\lambda^{1/2} e^{a\lambda l^2} \\
&\leqq \lambda^{1/2} e^{-2\tau_1 l|\mathbb{Y}|} e^{-\tau_2\lambda^{-2}|\Sigma_{\mathbb{Y}}|}.
\end{aligned} \tag{3.6.6}$$

Since $e^L \leqq e^{\tau_1 l|\mathbb{Y}|}$, (3.6.5) follows. □

Lemmas 3.6.1 and 3.6.2 imply that

$$\begin{aligned}
\|F_{\mu_1 L_1} - F_{\mu_2 L_2}\| &\leqq \|F_{\mu_1 L_1} - F_{\mu_1 L_2}\| + \|F_{\mu_1 L_2} - F_{\mu_2 L_2}\| \\
&\leqq e^{-4\tau_2\lambda^{-2}}|||L_1 - L_2||| + \lambda^{1/2}|\mu_1 - \mu_2|.
\end{aligned} \tag{3.6.7}$$

Proposition 3.3.2 then yields the bound

$$|s(F^q_{\mu_1 L_1}) - s(F^q_{\mu_2 L_2})| \leqq e^{-4\tau_2\lambda^{-2}}|||L_1 - L_2||| + \lambda^{1/2}|\mu_1 - \mu_2|. \tag{3.6.8}$$

Proposition 3.6.3. *Suppose $\lambda \ll 1 \ll l$. Let L_n be constructed from $L, L_1, \ldots, L_{n-1}$ at the parameter set μ_1 as in an elementary step of the construction of $\mathcal{N}(L)$. Let L_n' be constructed from $L', L_1', \ldots, L_{n-1}'$ at parameter set μ_2. Write $|||\delta L|||$ for*

$$\max\{|||L - L'|||, |||L_1 - L_1'|||, \ldots, |||L_{n-1} - L_{n-1}'|||\},$$

and assume all vacuum functionals vanish on clusters with no phase boundaries. Then for $\delta(\mathbb{Y}) = n$,

$$\frac{|L_n(\mathbb{Y}) - L_n'(\mathbb{Y})|}{\delta(\mathbb{Y})^3} \leqq e^{-\tau_2\lambda^{-2}}|||\delta L||| + \lambda^{-2} l^5 |\mu_1 - \mu_2|. \tag{3.6.9}$$

Proof. We have from (3.4.10) that

$$L_n(\mathbb{Y}) = \sum_i [A(R_i) + \log\Omega^a(F_{n_i}, \mathrm{Int}_{R_i}(\mathbb{Y})) - \log\Omega(F_{n_i}, \mathrm{Int}_{R_i}(\mathbb{Y}))]. \tag{3.6.10}$$

Consider the $\log\Omega^{(a)}$ terms first, and use (3.4.13):

$$\log\Omega^a_{\mu_2}(F_{n_i}', \mathbb{V}) - \log\Omega^a_{\mu_1}(F_{n_i}, \mathbb{V}) = \int_0^1 dt \frac{d}{dt}\log\left[\sum_\Sigma \frac{Z_\Sigma(\mathbb{V})_{\mu_t}}{\prod\limits_{\Delta \subseteqq \mathbb{V}} Z_{\Delta^q}(\mu_t)} \prod_i e^{[-a_t^* + a_t^{**}]V(R_i)}\right]. \tag{3.6.11}$$

Here $\mu_t=(1-t)\mu_1+t\mu_2$ and

$$a_t^*=(1-t)a^{m(R_i)}(F_{n_i})+ta^{m(R_i)}(F'_{n_i})$$
$$a_t^{**}=(1-t)a^{m(R_i)}(F_{n_{i_2(R_i)}})+ta^{m(R_i)}(F'_{n_{i_2(R_i)}}). \tag{3.6.12}$$

This may be estimated as

$$|\log\Omega^a_{\mu_2}(F'_{n_i},\mathbb{V})-\log\Omega^a_{\mu_1}(F_{n_i},\mathbb{V})|$$
$$\leqq \sup_t\left\langle \frac{\left|\frac{d}{dt}(Z_\Sigma(\mathbb{V})_{\mu_t}e^{E^q_w l^2|\mathbb{V}|})\right|}{Z_\Sigma(\mathbb{V})_{\mu_t}e^{E^q_w l^2|\mathbb{V}|}}\right.$$
$$\left.+|\mathbb{V}|\frac{\left|\frac{d}{dt}(Z_{\Delta^q}(\mu_t)e^{E^q_w l^2})\right|}{Z_{\Delta^q}(\mu_t)e^{E^q_w l^2}}+\sum_i|a_0^*-a_1^*-a_1^{**}+a_0^{**}|V(R_i)\right\rangle_t, \tag{3.6.13}$$

where

$$\langle\cdot\rangle_t=\frac{\sum_\Sigma\cdot Z_\Sigma(\mathbb{V})_{\mu_t}\prod_i e^{[-a_t^*+a_t^{**}]V(R_i)}}{\sum_\Sigma Z_\Sigma(\mathbb{V})_{\mu_t}\prod_i e^{[-a_t^*+a_t^{**}]V(R_i)}}. \tag{3.6.14}$$

The first two terms are bounded uniformly in Σ by $2rk\lambda^{-2}l^2|\mathbb{V}|\,|\mu_1-\mu_2|$, by Proposition 2.5.6. Equation (3.6.8) produces the bound

$$|a_1^*-a_0^*|+|a_1^{**}-a_0^{**}|$$
$$\leqq 8e^{-4\tau_2\lambda^{-2}}|||\delta L|||+8\lambda^{1/2}|\mu_1-\mu_2|+8r\sup_{q,t,j}\left|\frac{\partial}{\partial\mu^j}(\log Z_{\Delta^q}(\mu_t)-\log Z_{\Delta^{\bar q}}(\mu_t))\right|$$
$$\leqq 8e^{-4\tau_2\lambda^{-2}}|||\delta L|||+O(\lambda^{-2}l^2)|\mu_1-\mu_2|. \tag{3.6.15}$$

Here, Propositions 2.5.5 and 2.5.2 bound the derivative of

$$\log(Z_{\Delta^q}e^{E^q_w l^2})=\log\varrho_{\Lambda,q}(\Delta^q)$$

by $\lambda^{1/2}e^{a\lambda l^2}$, and similarly for $\bar q$. The derivative of $l^2(E_c^{\bar q}-E_c^q)$ is $O(\lambda^{-2}l^2)$, by condition (v), Sect. 2.1. The derivative of

$$E_c^q-E_w^q=\frac{m_q^2}{8\pi}\log\frac{m_q^2}{m_{\bar q}^2}-\frac{m_q^2-m_{\bar q}^2}{8\pi}$$

is $O(1)$, by conditions (vi), (vii), and (iii).

Hence,

$$\sum_i|\log\Omega^a_{\mu_2}(F'_{n_i},\mathrm{Int}_{R_i}(\mathbb{Y}))-\log\Omega^a_{\mu_1}(F_{n_i},\mathrm{Int}_{R_i}(\mathbb{Y}))|$$
$$\leqq 2rk\lambda^{-2}l^2|\mathrm{Int}\,\mathbb{Y}|\,|\mu_1-\mu_2|$$
$$+\left[\sup_{\Sigma_{\mathrm{Int}\,\mathbb{Y}}}\sum_{R_i\subseteq\mathrm{Int}\,\mathbb{Y}}V(R_i)\right](8e^{-4\tau_2\lambda^{-2}}|||\delta L|||+O(\lambda^{-2}l^2)|\mu_1-\mu_2|). \tag{3.6.16}$$

The supremum is bounded by $l^3\delta(\mathbb{Y})^3$ because each unit square is counted no more than $l\delta(\mathbb{Y})$ times in $\sum_{R_i\subseteq\mathrm{Int}\,\mathbb{Y}}$ and there are no more than $l^2\delta(\mathbb{Y})^2$ squares to consider.

Thus (3.6.16) is bounded by

$$l^3\delta(\mathbb{Y})^3(8e^{-4\tau_2\lambda^{-2}}|||\delta L|||+O(\lambda^{-2}l^2)|\mu_1-\mu_2|).$$

The same bound applies to the terms $\log\Omega(F_{n_i},\mathrm{Int}_{R_i}(\mathbb{Y}))$.

Consider next the terms

$$A(R_i)=[-a^*+(a^{**})+\log Z_{\Delta^{p(R_i)}}-\log Z_{\Delta^{m(R_i)}}]V(R_i). \qquad (3.6.17)$$

(The a^{**}-term may be absent.) The change in a^*, a^{**}, or $\log(Z_{\Delta^p}/Z_{\Delta^m})$ going from μ_1, L to μ_2, L' is bounded as before. Thus the change in $\sum_i A(R_i)$ is bounded by

$$l^3\delta(\mathbb{Y})^3(8e^{-4\tau_2\lambda^{-2}}|||\delta L|||+O(\lambda^{-2}l^2)|\mu_1-\mu_2|.$$

This completes the proof of Proposition 3.6.3. □

Theorem 3.6.4. *Suppose $\lambda \ll 1 \ll l$. The equation $L=\mathcal{N}(L)$ has a solution in the space of vacuum functionals that vanish on clusters with no phase boundaries. Let L_μ denote the solution obtained by successive iteration of $L=0$ at the parameter set μ. Then*

$$|||L_{\mu_1}-L_{\mu_2}|||\leqq O(\lambda^{-2}l^5)|\mu_1-\mu_2|. \qquad (3.6.18)$$

Proof. We verify the bound

$$|||\mathcal{N}_{\mu_1}(L)-\mathcal{N}_{\mu_2}(L')|||\leqq e^{-\tau_2\lambda^{-2}}|||L-L'|||+O(\lambda^{-2}l^5)|\mu_1-\mu_2|, \qquad (3.6.19)$$

where $\mathcal{N}_{\mu_1}, \mathcal{N}_{\mu_2}$ are the mappings corresponding to parameter sets μ_1, μ_2. We assume L, L' vanish on clusters with no phase boundaries – this property is preserved by the mappings. Proposition 3.6.3 contains the required bound on the difference between $\mathcal{N}_{\mu_1}(L)(\mathbb{Y}_n)\equiv L_n(\mathbb{Y}_n)$ and $\mathcal{N}_{\mu_2}(L')(\mathbb{Y}_n)\equiv L_n'(\mathbb{Y}_n)$ as long as $|||L_k-L_k'|||\leqq|||L-L'|||$ for all $k\leqq n-1$. Assume by induction that $|||L_k-L_k'|||\leqq|||L-L'|||$ for all $k\leqq n-2$. (For small n this is trivial because then $L_k=L$ and $L_k'=L'$.) Then

$$\begin{aligned}&|||L_{n-1}-L_{n-1}'|||\\&\leqq \sup_{\mathbb{Y}_{n-1}:\delta(\mathbb{Y}_{n-1})=n-1}\left\{|||L-L'|||,\frac{|L_{n-1}(\mathbb{Y}_{n-1})-L_{n-1}'(\mathbb{Y}_{n-1})|}{(n-1)^3},|||L_{n-2}-L_{n-2}'|||\right\}\\&\leqq|||L-L'|||. \end{aligned} \qquad (3.6.20)$$

The second term in the supremum is bounded by Proposition 3.6.3, using the induction hypothesis. This proves (3.6.19).

Proposition 3.4.1 implies that $|||\mathcal{N}(0)|||\leqq O(\lambda^{-2}l^2)$. Therefore, the Lipschitz condition (3.6.19) implies that the mappings $\mathcal{N}_{\mu_\alpha}$ have fixed points L_{μ_α} satisfying the bound (3.6.18). This completes the proof. □

3.7. The Phase Diagram

This section is devoted to analyzing the phase diagram of our model. We expect to find $\binom{r}{k}$ hypersurfaces of codimension $k-1$ in parameter space on which $a^q=0$ for kq's. On such hypersurfaces, we will be able to construct k distinct states. The diagram is constructed by showing that the mapping $\mu \to a^1(F_{L_\mu}), \ldots, a^r(F_{L_\mu})$ can be inverted and is in fact Lipschitz continuous in both directions. (The image of the map is a neighborhood of the origin in the boundary of the positive octant in $\mathbb{R}^r$.)

To invert the map $\mu \to \{a^q\}$, choose $a_0^1, \ldots, a_0^r$ with $\inf_q a_0^q = 0$ and such that $\{(a_0^q - a_0^{\bar{q}})/l^2\}_{q \neq \bar{q}}$ is in the neighborhood of the origin spanned by $\{e^q(\mu)\}_{q \neq \bar{q}}$ for $|\mu| \leqq C^{-1}/2$. We wish to find a parameter set μ such that $a^q(F_{L_\mu}) - a^{\bar{q}}(F_{L_\mu}) = a_0^q - a_0^{\bar{q}}$. Rewrite this equation as

$$z^q(\mu) + e_w^q(\mu) = (a_0^q - a_0^{\bar{q}})/l^2, \tag{3.7.1}$$

where

$$\begin{aligned} z^q(\mu) &= l^{-2}(-\log Z_{\Delta^q}(\mu) + \log Z_{\Delta^{\bar{q}}}(\mu) - s(F_{L_\mu}^q) + s(F_{L_\mu}^{\bar{q}})) - E_w^q + E_w^{\bar{q}} \\ e_w^q &= E_w^q - E_w^{\bar{q}} = e^q - \frac{m_q^2}{8\pi} \log \frac{m_q^2}{m_{\bar{q}}^2} + \frac{m_q^2 - m_{\bar{q}}^2}{8\pi}. \end{aligned} \tag{3.7.2}$$

Lemma 3.7.1. *Suppose $\lambda \ll 1 \ll l$. Then*

$$|z^q(\mu_1) - z^q(\mu_2)| \leqq 2\lambda^{1/2}|\mu_1 - \mu_2|. \tag{3.7.3}$$

Proof. Equation (3.6.8) and Theorem 3.6.4 imply that

$$\begin{aligned} |s(F_{L_{\mu_1}}^q) - s(F_{L_{\mu_2}}^q)| &\leqq e^{-4\tau_2 \lambda^{-2}} O(\lambda^{-2} l^5)|\mu_1 - \mu_2| + \lambda^{1/2}|\mu_1 - \mu_2| \\ &\leqq \tfrac{3}{2}\lambda^{1/2}|\mu_1 - \mu_2|. \end{aligned} \tag{3.7.4}$$

Propositions 2.5.5 and 2.5.2 bound the change in

$$l^{-2} \log Z_{\Delta^q} + E_w^q = l^{-2} \log \varrho_{\Lambda, q}(\Delta^q)$$

by $l^{-2}\lambda^{1/2} e^{a\lambda l^2} \leqq \frac{1}{4}\lambda^{1/2}$, and similarly for $\bar{q}$. This completes the proof. □

Theorem 3.7.2. *Suppose $\lambda \ll 1 \ll l$. Let a_0^q satisfy $\inf_q a_0^q = 0$ and suppose $(a_0^q - a_0^{\bar{q}})/l^2$ lies in the neighborhood of the origin spanned by $e^q(\mu)$ for $|\mu| \leqq C^{-1}/2$. Suppose further that $E_w^q(\mu_0) - E_w^{\bar{q}}(\mu_0) = (a_0^q - a_0^{\bar{q}})/l^2$. Then there exists a parameter set μ_a such that $a^q(F_{L_{\mu_a}}) = a_0^q$. Moreover, $|\mu_a - \mu_0| \leqq O(\lambda^{5/2})$ and*

$$|\mu_{a_1} - \mu_{a_2}| \leqq O(\lambda^2 l^{-2}) \sup_q |a_1^q - a_2^q|, \tag{3.7.5}$$

$$|a^q(F_{L_{\mu_1}}) - a^q(F_{L_{\mu_2}})| \leqq O(\lambda^{-2} l^2)|\mu_1 - \mu_2|. \tag{3.7.6}$$

Proof. Define

$$D_{qi}(\mu) = \frac{\partial}{\partial \mu^i}(E_w^q - E_w^{\bar{q}}) = \frac{\partial}{\partial \mu^i}\left(e^q - \frac{m_q^2}{8\pi}\log\frac{m_q^2}{m_{\bar{q}}^2} + \frac{m_q^2 - m_{\bar{q}}^2}{8\pi}\right). \tag{3.7.7}$$

The matrix index q does not take the value $\bar{q}$. By condition (v), Sect. 2.1, the eigenvalues of $\lambda^2 \frac{\partial e^q}{\partial \mu^i}$ are in some interval $[C^{-1}, C]$. The μ-derivatives of the other terms in (3.7.7) are $O(1)$, so $\lambda^2 D_{qi}(\mu)$ has the same property, for λ small enough. Let $D_{iq}^{-1} D_{qj} = \delta_{ij}$. Then the matrix D is $O(\lambda^{-2})$ and D^{-1} is $O(\lambda^2)$.

Given some μ, define $T\mu$, the next approximation to the coexistence point, by

$$(T\mu)^i = \mu^i - D_{iq}^{-1}(\mu)(z^q(\mu) + e_w^q(\mu) - (a_0^q - a_0^{\bar{q}})/l^2). \tag{3.7.8}$$

Convergence of this iteration depends on a Lipschitz condition. Let $\mu_t^i=(1-t)\mu_1^i+t\mu_2^i$. Then

$$\begin{aligned}|(T\mu_2)^i-(T\mu_1)^i| = &\left|\int_0^1 dt\frac{d}{dt}[\mu_t^i-D_{iq}^{-1}(\mu_t)(e_w^q(\mu_t)-(a_0^q-a_0^{\bar q})/l^2)]\right.\\ &\left.-D_{iq}^{-1}(\mu_1)(z^q(\mu_2)-z^q(\mu_1))-\left[\int_0^1 dt\frac{d}{dt}D_{iq}^{-1}(\mu_t)\right]z^q(\mu_2)\right|\\ \leqq &\sup_t\left|D_{im}^{-1}\left(\frac{\partial}{\partial\mu^k}D_{mj}\right)(\mu_2^k-\mu_1^k)D_{jq}^{-1}(e_w^q-(a_0^q-a_0^{\bar q})/l^2)\right|\\ &+\sum_q D_{iq}^{-1}(\mu_1)+2\lambda^{1/2}|\mu_1-\mu_2|\\ &+\sup_t\left|D_{im}^{-1}\left(\frac{\partial}{\partial\mu^k}D_{mj}\right)(\mu_2^k-\mu_1^k)D_{jq}^{-1}\right||z^q(\mu_2)|\,.\end{aligned} \tag{3.7.9}$$

The term $\frac{d}{dt}\mu_t^i$ canceled with $D_{iq}^{-1}(\mu_t)\frac{\partial}{\partial t}e_w^q(\mu_t)$. All three terms in the last line have one more D^{-1} than D, yielding a factor $O(\lambda^2)$. $\Big($Condition (vi) bounds second derivatives of $e^q(\mu,\lambda=1)$ and $m_q^2(\mu)$, so that $\frac{\partial}{\partial\mu^k}D_{mj}\leqq O(1)\lambda^{-2}$.$\Big)$

Suppose that $|\mu_1-\mu_0|$, $|\mu_2-\mu_0|$ are $O(\lambda^{5/2})$. Then $|e_w^q(\mu_t)-(a_0^q-a_0^{\bar q})/l^2|\leqq O(\lambda^{1/2})$ and the first term is less than $O(\lambda^{5/2})|\mu_1-\mu_2|$, as is the second. Since $s(F_{L_{\mu_2}}^q)$ and $l^{-2}\log(Z_{\Delta^q}(\mu_2)e^{l^2E_w^q})$ are $O(\lambda^{1/2})$, we have $|z^q(\mu_2)|\leqq O(\lambda^{1/2})$ and thus

$$|(T\mu_1)^i-(T\mu_2)^i|\leqq O(\lambda^{5/2})|\mu_1-\mu_2|\,. \tag{3.7.10}$$

Start the iteration at $\mu=\mu_0$. (We can always find μ_0 by the implicit function theorem.) Then

$$(T\mu)^i=\mu_0^i-D_{iq}^{-1}(\mu_0)z^q(\mu_0)=\mu_0^i+O(\lambda^{5/2})\,.$$

Applying (3.7.10), we see that successive iterations never leave the range $\mu=\mu_0+O(\lambda^{5/2})$. The Lipschitz condition now implies the existence of a fixed point μ_a such that $T\mu_a=\mu_a$ and $|\mu_a-\mu_0|\leqq O(\lambda^{5/2})$. This in turn implies that $z^q(\mu)+e_w^q(\mu)=(a_0^q-a_0^{\bar q})/l^2$, and therefore $a^q(F_{L_{\mu_a}})=a_0^q$.

It is clear from (3.7.8) that

$$|(T(a_1)\mu)^i-(T(a_2)\mu)^i|\leqq O(\lambda^2l^{-2})\sup_q|a_1^q-a_2^q|\,, \tag{3.7.11}$$

which implies the corresponding bound on the solutions, (3.7.5). The bound (3.7.6) follows from (3.6.8) and Theorem 3.6.4, as in the proof of Lemma 3.7.1. This completes the proof. □

Theorem 3.5.1 now yields the bound

$$\frac{Z(\mathbb{V}^q)}{Z(\mathbb{V}^{q_0})}\leqq e^{2\lambda^{1/2}|\partial\mathbb{V}|} \tag{3.7.12}$$

whenever $\mu=\mu_a$ with $a^{q_0}=0$.

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