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Surface Integrals and Monopole Charges in Non-Abelian Gauge Theories*

Clifford Henry Taubes†

Lyman Laboratory, Harvard University, Cambridge, MA 02138, USA

Abstract. We derive a formula which gives all the magnetic charges (topological invariants) of a monopole in the adjoint representation of a non-abelian gauge theory in terms of surface integrals at infinity.

1. Introduction

It has been known for some time that there exist topological invariants which are associated with static Yang–Mills–Higgs field configurations on Minkowski space [1–3]. In particular, suppose that the gauge group G is a simple, compact Lie group. Further, assume that the Higgs field is in the adjoint representation of the Lie algebra $\mathscr{g}$ of G. Every field configuration satisfying certain asymptotic conditions (c.f. Theorem 2.1) is known to define a gauge invariant set of integers $\{n_a\}_{a=1}^{\ell}, \ell \leqq \text{rank } G$ [3]. These integers are the aforementioned topological invariants. It is the purpose of this paper to prove that the integers $\{n_a\}_{a=1}^{\ell}$ are completely specified by surface integrals at $|x| = \infty$. For example, if $G = SU(n)$ and the representation of $\mathscr{g}$ is the defining one, then

$$Q_k = \lim_{R \to \infty} \frac{1}{4\pi} \int_{|x|=R} \operatorname{tr}(\Phi^k F_A), \qquad k \in \{1, \dots, \ell\} \tag{1.1}$$

completely determine the integers $\{n_a\}_{a=1}^{\ell}$. Here Φ is the Higgs field; the Lie algebra-valued two form F_A is the curvature of the Yang–Mills connection A; and $x = (x^1, x^2, x^3)$ are cartesian coordinates on $\mathbb{R}^3$. For example, if $G = SU(2)$ then only Q_1 is needed. In this case the right hand side of (1.1) computes the winding number of the map

$$\hat{\Phi} = \Phi/|\Phi| : S_R^2 = \{x \in \mathbb{R}^3 : |x| = R\} \to S^2 = \{\sigma \in su(2) : |\sigma| = 1\} \text{ [1]}$$

(see also [4], Proposition II.3.7.)

We remark that the right hand side of (1.1) is gauge invariant so it is not surprising that there should be some connection between the numbers $\{Q_k\}_{a=1}^{\ell}$ and the integer invariants $\{n_k\}_{k=1}^{\ell}$. This relationship is stated as Theorems 2.4–5. The proofs are contained in Sect. 3–5.

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† Junior Fellow, Harvard University Society of Fellows

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2. The Topological Invariants

We begin by establishing our notation. For convenience, we realize G as an embedded submanifold in the vector space $\mathbb{M}(m;\mathbb{C}$ or $\mathbb{R})$ of $m \times m$ matrices for some m. Let $\mathrm{tr}(\cdot)$ denote a normalized trace on $\mathbb{M}$. For $\sigma \in \mathbb{M}$,

$$|\sigma| \equiv (\mathrm{tr}(\sigma^\dagger \sigma))^{1/2}.$$

If A is a connection on the principal G bundle $\mathbb{R}^3 \times G$ then the curvature of A is the $\mathfrak{g}$ valued 2-form F_A given by the well-known formula

$$F_A = dA + A \wedge A. \tag{2.1}$$

If Φ is a section of $\mathbb{R}^3 \times \mathfrak{g}$ then A defines a covariant derivative by

$$D_A\Phi = (\nabla_A \Phi)_i dx^i = d\Phi + [A, \Phi]. \tag{2.2}$$

We identify a gauge transformation with a map $g: \mathbb{R}^3 \to G$ and (A, Φ) transforms under the action of g in the usual way.

Let $\hat{x}$ denote a point on S^2 and suppose that

$$\hat{\Phi}(\hat{x}) = \lim_{t \to \infty} \Phi(t\hat{x}) \tag{2.3}$$

defines a map of S^2 into $\mathfrak{g}$. Let $h = \hat{\Phi}(1, 0, 0)$. Then define the Lie subgroup $J \subset G$ as

$$J = \{g \in G: ghg^{-1} = h\}. \tag{2.4}$$

Theorem 2.1. (*Theorem II.3.1 of* [4].) *Let A be a C^1 connection on $\mathbb{R}^3 \times G$ and Φ a C^1 section of $\mathbb{R}^3 \times \mathfrak{g}$. Assume that*

$$\lim_{R \to \infty} \sup_{|x| = R} (1 - |\Phi|) = 0 \tag{2.5a}$$

and that for some $\delta > 0$,

$$|x|^{1+\delta} |\nabla_A \Phi| \leqq \text{const.} \tag{2.5b}$$

Then

(a) *There exists a gauge such that $\hat{\Phi}(x)$ is a continuous map from S^2 into $\mathfrak{g}$.*

(b) *The configuration (A, Φ) defines a homotopy class $[(A, \Phi)] \in \Pi_2(G/J)$.* (c.f. *Proposition 2.2*).

(c) *The class $[(A. \Phi)]$ is invariant under C^1 gauge transformations.*

(d) *Suppose that (a, ϕ) is respectively a C^1 $\mathfrak{g}$-valued 1-form and a C^1 section of $\mathbb{R}^3 \times \mathfrak{g}$ which satisfy*

$$\lim_{R \to \infty} \sup_{|x| = R} |\Phi| = 0 = \lim_{R \to \infty} \sup_{|x| = R} |x||a|. \tag{2.6}$$

Then

$$[(A + a, \Phi + \phi)] = [(A, \Phi)]. \qquad \square$$

In order to be more specific, we note that the map

$$\rho: G/J \to M = \{ghg^{-1}: g \in G\} \subset \mathfrak{g} \tag{2.7}$$

is a diffeomorphism (c.f. [5], Theorem 2.9.4).

Proposition 2.2. *The map $\hat{\Phi}(\hat{x})$ takes values in M. Further, the map*

$$\hat{h}(\hat{x}) = \rho^{-1}(\hat{\Phi}(\hat{x})) : S^2 \to G/J \tag{2.8}$$

is at least C^0. □

By definition, the class $[(A, \Phi)] = [\hat{h}]$.

We remark that Proposition 2.2 is a restatement of Lemmas II.4.3, 4 of [4]. However, there are some errata in the statement and proof of Lemma II.4.3 so we prove Proposition 2.2 in full in Sect. 4, c.f. Lemma 4.3.

The homotopy class $[(A, \Phi)] \in \Pi_2(G/J)$ is the topological invariant alluded to in the introduction. In order to categorize $[(A, \Phi)]$ we remark that the lie algebra $\mathscr{j}$ of J is necessarily the direct sum of a commuting algebra $\mathscr{h}$ of dimension ℓ and a semi-simple lie algebra $\mathscr{g}'$ of rank $s = \operatorname{rank} G - \ell$.

Theorem 2.3. *The group $\Pi_2(G/J)$ is isomorphic to $\mathbb{Z}^\ell$.* □

Thus, if (A, Φ) satisfies (2.5a, b), the topological invariant $[(A, \Phi)]$ is completely determined by a set of integers $\{n_a\}_{a=1}^{\ell}$.

Theorem 2.4. *Let $G = SU(n) \subset \mathbb{M}(n, \mathbb{C})$. Suppose that (A, Φ) satisfies the conditions of Theorem 2.1 and in addition that F_A and $D_A \Phi$ are square integrable. There exist constants b_a^k with $a, k = 1, \ldots, \ell$ which depend only on $Ad_G h$ such that*

$$n_a = \sum_{k=1}^{\ell} b_a^k \lim_{R\to\infty} \int_{|x|=R} \operatorname{tr}(\Phi^k F_A), \qquad a = 1, \ldots, \ell \tag{2.9}$$

are integers. Further, the set $\{n_a\}_{a=1}^{\ell}$ specify $[(A, \Phi)]$ in $\Pi_2(G/J)$.

Theorem 2.5. *Let $G = Sp(n) \subset \mathbb{M}(2n, \mathbb{C})$ or $G = SO(2n) \subset \mathbb{M}(2n, \mathbb{R})$ or $G = SO(2n+1) \subset \mathbb{M}(2n+1, \mathbb{R})$. Suppose that (A, Φ) satisfies the conditions of Theorem 2.1 and in addition that F_A and $D_A \Phi$ are square integrable. There exist constants b_a^k; $a, k = 1, \ldots, \ell$ which depend only on $Ad_G h$ such that*

$$n_a = \sum_{k=1}^{\ell} b_a^k \lim_{R\to\infty} \int_{|x|=R} \operatorname{tr}(\Phi^{2k-1} F_A) \tag{2.10}$$

is an integer for each $a \in \{1, \ldots, \ell\}$. Further, the set $\{n_a\}_{a=1}^{\ell}$ specifies $[(A, \Phi)]$ in $\Pi^2(G/J)$. □

There is a general formula to compute the integers $\{n_a\}_{a=1}^{\ell}$ which is valid for any Lie group and all faithful representations. This is the content of Proposition 4.5.

The proofs of Theorems 2.3 – 5 comprise the next three sections. In Sect. 3 we review the relevant topology of G/J and prove Theorem 2.3. In Sect. 4, it is proved that the integrands in (2.9, 10) are pull-backs of a linear combination of generators of H_{DR}^2 (H_{DR}^* is the DeRham cohomology complex). This fact and Hurewicz isomorphism [6] allow us to prove the general result expressed in Proposition 4.5. Theorems 2.4–2.5 follow from Proposition 4.5 and properties of the matrix representation of the classical groups. This is explained in Sect. 5.

As a parenthetical remark, we point out that in theories with no Higgs self interactions, finite energy solutions to the classical equations of motion exist for all simple groups G, subgroups J and integers $\{n_a\}_{a=1}^{\ell}$ if all the integers are the same sign (with respect to a specific choice of generators of $\Pi_2(G/J)$ [4, 7].

3. Algebraic Topology on G/J

This section is a review of the algebraic topology behind the computation of $\Pi_2(G/J)$. The important result is that the second homotopy group is isomorphic to the second cohomology group of the universal covering of G/J. The implications of this isomorphism are summarized in Proposition 3.2, and the second cohomology group of the universal cover is described in Proposition 3.3.

Let $\tilde{G}$ be the universal covering space of G. There is a natural covering projection

$$\tilde{G} \to G. \tag{3.1}$$

Further, identifying G/J with M defined by (2.7) we have

$$G/J = \{ghg^{-1} : g \in G\} = \{ghg^{-1}; g \in \tilde{G}\}. \tag{3.2}$$

Define the group $\tilde{J}$ by replacing G by $\tilde{G}$ in equation (2.4). Then

$$G/J = \tilde{G}/\tilde{J}. \tag{3.3}$$

Let $\tilde{J}_{(0)}$ be the component of $\tilde{J}$ which is path connected to the identity. The projection

$$p' : \tilde{G}/\tilde{J}_0 \to \tilde{G}/\tilde{J} = G/J \tag{3.4}$$

is a covering space map. Since $\Pi_1(\tilde{G}) = \Pi_0(\tilde{J}_{(0)}) = (0)$, the long exact homotopy sequence of fibration [6, Theorem 7.2.10] implies that $\Pi_1(\tilde{G}/\tilde{J}_{(0)}) = (0)$. Thus $\tilde{G}/\tilde{J}_{(0)}$ is the universal covering space of G/J.

Let x be the image of J in G/J and x_0 the image of $\tilde{J}_{(0)}$ in $\tilde{G}/\tilde{J}_{(0)}$. The following proposition is standard [6, Theorem 2.4.5 and 7.2.10].

Proposition 3.1. *Let M be a simply connected space and $m \in M$. Suppose that ψ maps the pair (M, m) into $(G/J, x)$ continuously. Then there exists a unique lifting*

$$\tilde{\psi} : (M, m) \to (\tilde{G}/\tilde{J}_{(0)}, x_0)$$

such that $p'\tilde{\psi} = \psi$. In addition, the projection p' induces an isomorphism between $\Pi_n(\tilde{G}/\tilde{J}_{(0)}, x_0)$ and $\Pi_n(G/J, x)$ for all $n \geqq 2$. □

Because $\Pi_1\ (\tilde{G}/\tilde{J}_{(0)}) = (0)$, the Hurewicz isomorphism is applicable, that is,

$$H_2(\tilde{G}/\tilde{J}_{(0)}; \mathbb{Z}) \approx \Pi_2(\tilde{G}/\tilde{J}_{(0)}). \tag{3.5}$$

Here $H_*(\cdot\,; \mathbb{Z})$ are the singular homology groups with integral coefficients [6, Chapt. 7.5].

The group J is a closed subgroup of G and an analytic submanifold ([5], Lemma 2.9.2.) Since $\tilde{G}$ is compact, $\tilde{J}$ has a finite number of path components. Hence $\tilde{G}/\tilde{J}_{(0)}$ is a compact manifold. By poincaré duality [6, Theorem 6.2.18],

$$H_2(\tilde{G}/\tilde{J}_{(0)}; \mathbb{Z}) \approx H^{m-2}(\tilde{G}/\tilde{J}_{(0)}); \mathbb{Z}) \tag{3.6}$$

where $m = \dim \tilde{G}/\tilde{J}_{(0)}$ and $H^*(\cdot\,; \mathbb{Z})$ are the singular cohomology groups with integral coefficients.

The Universal Coefficient Theorem [6] relates cohomology with integral coefficients to cohomology with real coefficients:

$$H^{m-2}(\tilde{G}/\tilde{J}_{(0)};\mathbb{Z})\otimes\mathbb{R}\approx H^{m-2}(\tilde{G}/\tilde{J}_{(0)};\mathbb{R}). \tag{3.7}$$

A fundamental result of R. Bott [8] is that

$$H^{m-2}(\tilde{G}/\tilde{J}_{(0)};\mathbb{Z})\approx\mathbb{Z}^{\ell}, \tag{3.8}$$

where ℓ is the dimension of the center of $\tilde{J}_{(0)}$. This gives Theorem 2.3. In particular, H^{m-2} is freely generated so

$$H^{m-2}(\tilde{G}/\tilde{J}_{(0)};\mathbb{R})\approx\mathbb{R}^{\ell}. \tag{3.9}$$

Finally, $H^{m-2}(\tilde{G}/\tilde{J}_{(0)};\mathbb{R})\approx H^{m-2}_{DR}(\tilde{G}/\tilde{J}_{(0)})\approx H^{2}_{DR}(\tilde{G}/\tilde{J}_{(0)})$. Here H^{*}_{DR} are the DeRham cohomology groups.

The above discussion is summarized in

Propositions 3.2. *There are ℓ linearly independent generators $\{q_1,\dots,q_\ell\}$ of $\Pi_2(\tilde{G}/\tilde{J})$ and ℓ linearly independent generators $\{\eta^1,\dots,\eta^\ell\}$ of $H^2_{DR}(\tilde{G}/\tilde{J}_{(0)})$. Any element $[\psi]\in\Pi_2(\tilde{G}/\tilde{J})$ has a unique expansion*

$$[\psi]=\sum_{a=1}^{\ell} n_a q_a \quad \textit{with} \quad n_a\in\mathbb{Z}. \tag{3.10}$$

There is an $\ell\times\ell$ matrix α_a^k which is independent of $[\psi]$ such that

$$n_a=\sum_{k=1}^{\ell}\alpha_a^k\int_{s^2}\psi^*(\eta^k). \tag{3.11}$$

Here $\psi\in[\psi]$ is any representative map. □

With Eq. (3.11) in mind, we examine $H^2_{DR}(\tilde{G}/\tilde{J}_{(0)})$. With $\tilde{G}\subset\mathbb{M}(m)$ as a smooth submanifold, we represent $\tilde{G}/\tilde{J}_{(0)}$ as the quotient by matrix multiplication on the right:

$$g\sim g' \text{ iff } g=g'j \quad \text{with } j\in\tilde{J}_0.$$

Let $\{x_j\}_{j=1}^{\dim\mathscr{g}}$ be an orthonormal basis for $\mathscr{g}$. That is

$$\mathrm{tr}(x_j^\dagger x_k)=\delta_{jk}. \tag{3.12}$$

The structure constants of $\mathscr{g}$, $\{C_{ij}{}^k\}$ are defined by the comutator

$$[x_i,x_j]=c_{ij}{}^k x_k. \tag{3.13}$$

We denote a basis for the left invariant 1-forms on $\mathscr{g}$ by

$$\omega^j=\mathrm{tr}(x_j^\dagger g^{-1}dg),\quad j=1,\dots,\dim\mathscr{g}. \tag{3.14}$$

Here $dg=dg_{AB}$ is the restriction to $\tilde{G}$ of the Euclidean one forms dx_{AB} on M. The one forms ω^j satisfy the equations of Maurier–Cartan, namely

$$d\omega^k=-\tfrac{1}{2}C_{ij}{}^k\omega^i\wedge\omega^j. \tag{3.15}$$

It is convenient to distinguish the elements of the basis $\{x_j\}$ which generate the subgroup $\tilde{J}_{(0)}$. Recall that the Lie algebra $\mathscr{j}=\mathscr{h}\oplus\mathscr{g}'$ where $[\mathscr{h},\mathscr{j}]=0$. Let

$\{h_a\}_{a=1}^{\ell}$ be an orthonormal basis for $\mathscr{h}$

$\{e_\alpha\}_{\alpha=1}^{\dim\mathscr{g}'}$ be an orthonormal basis for $\mathscr{g}'$

and $\{y_A\}_{A=1}^{\dim \mathscr{g} - \dim \mathscr{g}' - \ell}$ be an orthonormal basis for the remaining generators of $\mathscr{g}$. Then set

$$\begin{aligned} \omega^a &= \operatorname{tr}(h_a^\dagger g^{-1} dg), \quad & a &= 1, \dots, \ell \\ \omega^\alpha &= \operatorname{tr}(e_\alpha^\dagger g^{-1} dg), \quad & \alpha &= 1, \dots, \dim \mathscr{g}' \\ \omega^A &= \operatorname{tr}(y_A^\dagger g^{-1} dg), \quad & A &= 1, \dots, \dim \mathscr{g} - \dim \mathscr{g}' - \ell. \end{aligned} \tag{3.16}$$

Proposition 3.3. *Let $p_0 : \tilde{G} \to \tilde{G}/\tilde{J}_{(0)}$ be the canonical projection. For each $a = 1, \dots, \ell$ there exists a closed two form $\eta^a \in \Lambda^2(\tilde{G}/\tilde{J}_{(0)})$ such that*

$$p_0^* \eta^a = -\tfrac{1}{2} C^a{}_{AB} \omega^A \wedge \omega^B. \tag{3.17}$$

Further the set $\{\eta^a\}_{a=1}^{\ell}$ generates $H^2_{DR}(\tilde{G}/\tilde{J}_{(0)})$ as an $\mathbb{R}$ module.

Proof. See, for example [9] and references therein.

4. The Pull-Back of Cohomology from $\tilde{G}/\tilde{J}_{(0)}$

We begin by studying the asymptotic behavior of Φ to demonstrate that for sufficiently large r, the Higgs field $\Phi(r\hat{x})$ maps S^2 into a tubular neighborhood of the manifold M defined by (2.7). Hence we can compute the homotopy class defined by $\hat{\Phi}(\hat{x})$ from $\Phi(rx)$. The result is Proposition 4.5.

In this section it is always assumed that (A, Φ) satisfy the conditions of Theorem 2.1.

For $x \in \mathbb{R}^3$, define $\hat{x} = x/|x|$.

Lemma 4.1. *There exists a smooth gauge such that*

$$\lim_{r \to \infty} \Phi(r\hat{x}) \equiv \hat{\Phi}(\hat{x}),$$

is a continuous map of S^2 into $\mathscr{g}$. Further,

$$\lim_{r \to \infty} r^\delta |\hat{\Phi}(\hat{x}) - \Phi(r\hat{x})| \leqq \text{const.} \tag{4.1}$$

Proof. It is always possible to choose a smooth gauge in which the radial component of the connection vanishes [10]. Then (4.1) follows from (2.5.6) by integration. The continuity of $\hat{\Phi}(\hat{x})$ is proved in Lemma II.4.1 of [4].

The statements above are gauge invariant under all gauge transformations $g(r\hat{x})$ which, as a function of $y = 1/r$, are Hölder continuous with exponent $\delta \leqq 1$ as $y \to 0$.

Let $\{t_k\}_{k=1}^{\mathscr{d}}$ be an orthonormal basis for the Cartan sub-algebra $\mathscr{t} \subset \mathscr{g}$. Require that $t_a = h_a$ for $a \in \{1, \dots, \ell\}$ where $\{h_a\}_{a=1}^{\ell}$ is defined in Sect. 3. By convention $h_1 \equiv h$. It follows from (4.1) that there exists $R_0 > 0$ such that $|\Phi(x)| \geqq \frac{1}{2}$ if $|x| > R_0$. Set $v_0 = (1, 0, 0)$ and $\bar{v}_0 = (-1, 0, 0)$. For $R > R_0$ define

$$V_R = \{x \in \mathbb{R}^3 : |x| > R \text{ and } \hat{x} \in S^2 \backslash \bar{v}_0\}.$$

Lemma 4.2. *For some $R_0 < R < \infty$ there exists $g \in C^\infty(V_R, G)$ and $\{\phi_k\}_{k=1}^{\mathscr{d}} \in C^\infty(V_R)$ such that the following is true: For $x \in V_R$,*

(a) $\Phi(x) = \sum_{k=1}^{\mathfrak{a}} \phi_k g t_k g^{-1}$.

(b) $\lim_{r\to\infty} r^{\delta}|\phi_1(rx) - 1| < \text{const.}$

(c) $\lim_{r\to\infty} r^{\delta}|\phi_k(rx)| < \text{const.}, \quad \text{for } k = 2, \ldots, \mathfrak{a}.$ (4.2)

Proof. Statement (a) follows from the fact that every element in $\mathfrak{g}$ is conjugate to an element in $\mathfrak{t}$. The smoothness conditions follow because V_{R_0} is contractible and $|\Phi| > 0$. Having established (a) we compute $D_A\Phi$:

For $x \in V_{R_0}$,

$$D_A\Phi = \sum_{k=1}^{\mathfrak{a}} d\phi_k g t_k g^{-1} + \sum_{k=1}^{\mathfrak{a}} \phi_k [A + dgg^{-1}, gt_k g^{-1}]. \tag{4.3}$$

Using (4.3) and (2.5b) we obtain for $\hat{x} \in S^2 \backslash \bar{v}_0$, $r_1 > R_0$ and $k \in \{1, \ldots, \mathfrak{a}\}$ that

$$\lim_{r\to\infty} |\phi_k(r\hat{x}) - \phi_k(r_1\hat{x})| < \text{const. } r_1^{-\delta}. \tag{4.4}$$

Further, for $\hat{x}_1, \hat{x}_2 \in S^2 \backslash \bar{v}_0$, and $r > R_0$

$$\lim_{r\to\infty} r^{\delta}|\phi_k(r\hat{x}_1) - \phi_k(r\hat{x})| \leqq \text{const.} \tag{4.5}$$

The conclusion from (4.4) and (4.5) is that for each $k \in \{1, \ldots, \mathfrak{a}\}$ there exists a constant c_k such that

$$\lim_{r\to\infty} r^{\delta}|\phi_k(r\hat{x}) - c_k| \leqq \text{const.} \tag{4.6}$$

By construction, $c_k = \delta_{k1}$. Hence statements b) and c) of the Lemma.

Let

$$e_1(x) = g(x)\, h_1 g^{-1}(x) \in C^\infty(V_R; M), \tag{4.7}$$

where $g(x)$ is given in Lemma 4.2 and M is defined in (2.7).

Lemma 4.3. *The function $e_1(x)$ extends to an element in $C^\infty(\{x : |x| > R\}; M)$. In addition,*

(a) $\lim_{r\to\infty} r^{\delta}|\hat{\Phi}(\hat{x}) - e_1(r\hat{x})| \leqq \text{const.}$ (4.8)

(b) *Proposition 2.2 is true.*

(c) *For all $r > R$, $\hat{\Phi}(\hat{x})$ and $e_1(r\hat{x})$ are homotopic as maps from S^2 into M.* □

Proof. The fact that $\phi_1(x)$ never vanishes in $\{x ; |x| > R\}$ implies that $e_1(x)$ has the stated extension. Statement (a) follows from statements (b, c) of Lemma 4.2

and Eq. (4.1). Statement (a) of Lemma 4.3 implies that the sequence $\{e_1(mR\hat{x})\}_{m=1}^{\infty}$ is a Cauchy sequence in M for fixed $\hat{x}$. Hence the limit, which is $\hat{\Phi}(\hat{x})$, takes values in M. This last fact and the previous Lemma imply (b). Statement (c) is implied by statement (a).

The orbit of each h_a, $a \in \{1, \dots, \ell\}$, under $\mathrm{Ad}_{\tilde{G}}$ maps the group $\tilde{G}$ smoothly into its Lie algebra. Define $l_a : \tilde{G} \to \mathscr{g}$ by

$$\ell_a(g) = g h_a g^{-1}. \tag{4.9}$$

We note that for each $a \in \{1, \dots, \ell\}$, the map ℓ_a descends to a smooth map (which we also denote by ℓ_a) of $\tilde{G}/\tilde{J}_{(0)}$ into $\mathscr{g}$. This is because $\tilde{J}_{(0)}$ is generated by $\exp(\mathscr{j})$ and $[\mathscr{h}, \mathscr{j}] = 0$.

Let $\hat{h}$ be the map defined in Proposition 2.2. Denote by $\tilde{h}: S^2 \to \tilde{G}/\tilde{J}_{(0)}$ the unique lifting of $\hat{h}: S^2 \to \tilde{G}/\tilde{J}$ as guaranteed by Proposition 3.1.

It follows from (c) of Lemma 4.3 that for $r > R$, the maps

$$\hat{h}(r:\hat{x}) = \rho^{-1}(e(r\hat{x})) : S^2 \to \tilde{G}/\tilde{J} \tag{4.10}$$

are homotopic to $\hat{h}(\hat{x})$. In addition, the liftings of these maps,

$$\tilde{h}(r;\hat{x}) : S^2 \to \tilde{G}/\tilde{J}_{(0)} \tag{4.11}$$

are homotopic to $\tilde{h}(\hat{x})$.

For $a \in \{1, \dots, \ell\}$ define

$$e_a(r;\hat{x}) = \ell_a(\tilde{h}(r, \hat{x})). \tag{4.12}$$

Definition 4.4. *For $r > R$ and $a \in \{1, \dots, \ell\}$ define*

$$Q_a(r) = \int_{|x|=r} \mathrm{tr}(e_a(r;\hat{x}) F_A). \tag{4.13}$$

Proposition 4.5. *Let (A, Φ) be as in Theorem 2.1. Then*

$$\int_{S^2} \tilde{h}^*(\eta^a) = \lim_{r \to \infty} Q_a(r), \tag{4.14}$$

where the two cocycle $\eta^a \in H^2_{DR}(\tilde{G}/\tilde{J}_{(0)})$ is defined in Proposition 3.3 □

Proof. By assumption, A is C^1 so the integrand in (4.13) is smooth. Hence we have

$$Q_a(r) = -\int_{|x|=r} \mathrm{tr}(D_A e_a \wedge A + A \wedge A), = -\int_{\substack{|x|=r \\ \hat{x} \in S^2 \setminus \bar{v}_0}} \mathrm{tr}(D_A e_a \wedge A + A \wedge A). \tag{4.15}$$

Recall that the set V_R is contractible so that in V_R, $e_1(x)$ is given by (4.7) with $g(x) \in C^\infty(V_R; \tilde{G})$. Then up to an element of $\tilde{J}/\tilde{J}_0$ (which is a discrete group)

$$e_a(x) = g(x) h_a g^{-1}(x). \tag{4.16}$$

An explicit computation of (4.15) gives

$$Q_a(r) = - \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} \operatorname{tr}(gh_a g^{-1} gdg^{-1} \wedge gdg^{-1}) + \tfrac{1}{2} \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} \operatorname{tr}([gh_a g^{-1}, A + gdg^{-1}] \wedge (A + gdg^{-1})). \tag{4.17}$$

Use (4.3) to estimate the second term in (4.17). That is, let A_L be the component of $A + dgg^{-1}$ taking values in $g\mathscr{j}g^{-1}$ and let A_T be the orthogonal complement. Because $\phi_1 \to 1$, equations (4.3) and (2.5b) imply that

$$|[A + dgg^{-1}, gh_a g^{-1}]| \leqq \text{const.}\, r^{-1-\delta}$$

and hence that

$$|A_T| < \text{const.}\, r^{-1-\delta}. \tag{4.18}$$

Since the skew form

$$\mathscr{g} \to [gh_a g^{-1}, \mathscr{g}] \quad (a \in \{1, \dots, \ell\})$$

has a kernal which includes $\mathscr{j}$ and maps $\mathscr{j}_T$ into itself, we conclude from (4.20) that

$$|\operatorname{tr}([gh_a g^{-1}, A_T] A_T)| \leqq \text{const.}\, r^{-2-2\delta}. \tag{4.19}$$

Therefore the second term in (4.17) is order $r^{-2\delta}$. As for the first term in (4.17),

$$- \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} \operatorname{tr}(gh_a g^{-1} gdg^{-1} \wedge gdg^{-1}) = \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} g^*(-\tfrac{1}{2} C^a{}_{AB}\omega^A \wedge \omega^B),$$
$$= \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} g^* p^*_{(0)}(\eta^a) = \int_{\substack{|x|=r \\ \hat{x}\in S^2\backslash \bar{v}_0}} \tilde{h}(r;\cdot)^*(\eta^a). \tag{4.20}$$

Because $\tilde{h}(r;\cdot)$ is C^∞ we can replace the point $\bar{v}_0$ in (4.20). Since $\tilde{h}(r;\cdot)$ is homotopic to $\tilde{h}$, Eq. (4.20) implies Eq. (4.14) and the Proposition is proved.

We now expand on the remarks in the opening paragraph of this section. The manifold M is the orbit of h under Ad_G and it is a regular, compact submanifold of $\mathscr{g} \simeq \mathbb{R}^{\dim \mathscr{g}}$. As such, there exists a tubular neighborhood, $\mathcal{O}(M)$ of M in $\mathscr{g}$ and a C^∞ projection $q : \mathcal{O} \to M$. (see [11], Chap. 12 for details.) We can specify q uniquely by requiring that it be the orthogonal projection onto M defined by the Killing-form on $\mathscr{g}$. This choice has the advantage that the projection commutes with the action of Ad_G on M and $\mathcal{O}$.

Lemma's 4.1 and 4.3 imply that $\Phi(r\hat{x})$ maps S^2 into $\mathcal{O}$ for all r larger than some

r_1. It is not hard to show that

$$(q \circ \Phi)(r\hat{x}) = e_1(r\hat{x}).$$

We end this section with a summary:

Theorem 4.6. *Let (A, Φ) satisfy the conditions of Theorem 2.1 Let $\rho: G/J \to M = \mathrm{Ad}_G h$ be the canonical map. Let $q: \mathcal{O} \to M$ be a tubular neighborhood of M in $\mathscr{g}$ such that q commutes with Ad_G. Let $\ell_a: G \to \mathscr{g}$ map $g \in G$ to $\mathrm{Ad}_g h_a$. There exist constants b_a^k; $a, k \in \{1, \ldots, \ell\}$ which depend only on $\mathrm{Ad}_G h$ such that*

$$n_a = \sum_{k=1}^{\ell} b_a^k \lim_{r \to \infty} \int_{|x|=r} \mathrm{tr}(\ell_a \widetilde{(\rho^{-1} \circ q \circ \Phi)} F_A) \tag{4.21}$$

is an integer. The set $\{n_a\}_{a=1}^{\ell}$ specifies $[(A, \Phi)]$ in $\Pi^2(G/J)$. Here $\widetilde{\rho^{-1} \circ q \circ \Phi}$ is the lifting of $\rho^{-1} \circ q \circ \Phi$ to $\tilde{G}/\tilde{J}_{(0)}$. □

5. The Classical Lie Groups

Equation (4.21) can be simplified considerably if the properties of the specific matrix representation of a compact group G are taken into account. A faithful, irreducible, unitary representation of G of dimension m is equivalent to an embedding of G in $SU(m)$ or $SO(m)$ if the representation is real. Then h is an $m \times m$ matrix and we can consider the matrix powers $h^k = hh \ldots h$, k times. The set $\{h^k\}_{k=0}^{p-1}$ are linearly independent where p is the number of distinct eigenvalues of h. (By construction, h is traceless.) Let $\mathscr{p}(h)$ be the linear span of $\{h^k\}_{k=0}^{p-1}$. Since $[\mathscr{p}(h), \mathscr{j}] = 0$, the projection of $\mathscr{p}(h)$ onto $\mathscr{g}$ lies in the commuting subalgebra $\mathscr{h}$ spanned by the orthonormal basis $\{h_1 = h, h_2 \ldots h_\ell\}$. The projection of $\mathscr{p}(h)$ on $\mathscr{g}$ spans a subset $\mathscr{h}' \subseteq \mathscr{h}$, say the subspace generated by $\{h_1, \ldots, h_{\ell'}\}$ with $\ell' \leqq \ell$. Then there exist constants c_a^k where $a \in \{1, \ldots, \ell'\}$ *and* $k \in \{1, \ldots, m\}$ *such that*

$$h_a = \sum_{k=1}^{m} c_a^k h^k, \qquad a = 1, \ldots, \ell'. \tag{5.1}$$

Equation (5.1) suggests that

$$\lim_{r \to \infty} \cdot \sum_{k=1}^{m} c_a^k \int_{|x|=r} \mathrm{tr}(\Phi^k F_A) = \int_{S^2} \tilde{h}^*(\eta^a), \tag{5.2}$$

for $a \in \{1, \ldots, \ell'\}$, and where η^a is a generator of $H^2_{D\cdot R\cdot}(\tilde{G}/\tilde{J}_{(0)})$ as discussed in Sect. 3 (Proposition 3.3.) In fact this is the case with a proviso, namely F_A and $D_A\Phi$ are squared integrable.

Proposition 5.1. *Let (A, Φ) satisfy the conditions of Theorem 2.1 and in addition suppose that F_A and $D_A \Phi$ are square integrable. Let a faithful, irreducible representation of G be given by an embedding of $G \subset SU(m)$ or $SO(m)$ for some m such that (5.1) holds. Then (5.2) holds.* □

Proof. Equation (5.2) holds with Φ replaced by $e_1(r\hat{x})$ as defined in (4.7). Lemma 4.2 and Eq. (5.3) imply that there exists $r_1 > 0$ such that $D_A e_1 \in L_2(\mathbb{R}^3 \backslash \{|x| < r_1\})$. Using Kato's inequality (c.f. [4], Chap. VI.) we have $\nabla|\Phi - e_1| \in L_2(\mathbb{R}^3 \backslash \{|x| < r_1\})$ where ∇ is the ordinary derivative. Lemma's 4.1 and 4.3 imply that $|\Phi - e_1| \in L_q(\mathbb{R}^3 \backslash \{|x| < r_1\})$ for all $q > 3\delta^{-1}$. Then the Sobolev lemma (see e.g. [4], Chap. VI.) gives:

$$|\Phi - e_1| \in L_6(\mathbb{R}^3 \backslash \{|x| < r_1\}). \tag{5.3}$$

Fix once and for all a function $\beta(x) \in C_0^\infty(\mathbb{R}^3)$ with the property that

$$\begin{aligned}
&\text{(a)}\quad 0 \leqq \beta(x) \leqq 1,\\
&\text{(b)}\quad \beta(x) = 1 \quad \text{if } |x| \leqq 3/4,\\
&\text{(c)}\quad \beta(x) = 0 \quad \text{if } |x| \geqq 1.
\end{aligned} \tag{5.4}$$

Set $\beta_r(x) = \beta(x/r)$ which we define for $r > 0$. Then for $r > 2r_1$,

$$\begin{aligned}
\Delta(r) &= \left| \int_{|x|=r} \operatorname{tr}(\Phi^k F_A) - \int_{|x|=r} \operatorname{tr}(e_1^k F_A) \right| \\
&\leqq k \int_{|x| \leqq r} (1-\beta_r)(|\Phi|^{k-1}|D_A\Phi||F_A| + |D_A e||F_A|) \\
&\quad + \int_{|x|\leqq r} |d\beta_r|(1-\beta_{r/2})|\Phi - e|k(|\Phi|^{k-1}+1)|F_A|.
\end{aligned} \tag{5.5}$$

Here we have used Stoke's theorem along with the identities

$$d\beta_r = d\beta_r(1-\beta_{r/2}),$$

and

$$\Phi^k - e^k = \frac{1}{2}(\Phi - e)\left(\sum_{j=0}^{k-1} \Phi^j e^{k-1-j}\right) + \frac{1}{2}\left(\sum_{j=0}^{k-1} \Phi^j e^{k-1-j}\right)(\Phi - e). \tag{5.6}$$

Now using Hölder's inequality, we obtain

$$\begin{aligned}
\Delta(r) &\leqq \|(1-\beta_{r/2})F_A\|_{L_2}(\|D_A\Phi\|_{L_2} + \|(1-\beta_{r/2})D_A e\|_{L_2} \\
&\quad \|\nabla\beta_r\|_{L_3}\|\Phi - e\|_{L_6})k(\|\Phi\|_{L\infty}^{k-1} + 1).
\end{aligned} \tag{5.7}$$

By scaling, $\|\nabla\beta_r\|_{L_3} = \|\nabla\beta_1\|_{L_3}$ so we conclude from (5.7) that

$$\lim_{r\to\infty} \Delta(r) = 0. \tag{5.8}$$

This proves Proposition 5.1.

Proof of Theorem 2.4 Let $G = SU(n)$ acting on $\mathbb{C}^n$. The number of different eigenvalues of h is precisely the dimensions of $\mathscr{h}$. Hence the projections of the powers $\{h^k\}_{k=1}^{n-\ell}$ onto $\mathscr{h}$ span $\mathscr{h}$. Thus Theorem 2.4 follows from Proposition 5.1.

Proof of Theorem 2.5. Case 1. $G = Sp(n)$ acting on $\mathbb{C}^{2n}$. An arbitrary element in the Lie algebra $\mathfrak{sp}(n)$ has the form [12, Sect. 65]

$$A \otimes 1 + S_j \otimes \tau_j, \tag{5.9}$$

where $(1, \tau_j)$ are unit quaternious; A and S_j are real $n \times n$ matrices but $A^T = -A$ while $S_j^T = S_j$. The Cartan subalgebra may be chosen to have the form

$$t = \{D \otimes \tau_3\,; D \text{ a diagonal } n \times n \text{ real matrix}\}. \tag{5.10}$$

The eigenvalues of $h \in t$ come in $\pm$ pairs since τ_3 has eigenvalues $\pm i$. Hence we need only consider the odd powers of h (the even powers are orthogonal to $\mathfrak{sp}(n)$.) We write

$$h = H \otimes \tau_3 \qquad \text{with } H \text{ diagonal.}$$

It is not hard to see that the number of distinct eigenvalues of h (the number of distinct eigenvalues of H up to sign) is precisely the dimensions of $\mathfrak{h}$. Hence $\mathfrak{p}_{\mathrm{odd}}(h)$ spans $\mathfrak{h}$. Case 1 follows now from Proposition 5.1.

Case 2. $G = SO(2n)$ acting on $\mathbb{R}^{2n}$. The Lie algebra $\mathfrak{so}(2n)$ is the space of $2n \times 2n$ real skew-symmetric matrices. Let α, β, σ be 2×2 matrices defined by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.11}$$

Every element in $\mathfrak{so}(2n)$ has the form [12, Sect. 65]

$$A_0 \otimes 1 + A_\alpha \otimes \alpha + A_\beta \otimes \beta + S_\sigma \otimes \sigma \tag{5.12}$$

where A_0, A_α, A_β and S_σ are real $n \times n$ matrices with $A_{0,\alpha,\beta}$ antisymmetric but S_σ symmetric. An element h in the Cartan subalgebra can be put in the form

$$h = H \otimes \sigma \quad \text{with } H \text{ a diagonal } n \times n \text{ matrix} \tag{5.13}$$

Hence the eigenvalues come in $\pm$ pairs also and we need only consider the odd powers of h. Once again the number of distinct eigenvalues of h is precisely the dimension of $\mathfrak{h}$ and $\mathfrak{p}_{\mathrm{odd}}(h)$ spans $\mathfrak{h}$. Case 2 follows from Proposition 5.1. Case 3. $G = SO(2n+1)$ acting on $\mathbb{R}^{2n+1}$. The group $SO(2n+1)$ has rank n. The Lie algebra, $\mathfrak{so}(2n+1)$ is the space of $(2n \times 1) \times (2n+1)$ real skew-symmetric metrices. Every element in $\mathfrak{so}(2n+1)$ has the form

$$\left(\begin{array}{c|c} \mathfrak{so}(2n) & 0 \\ \hline 0 \ldots & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & v \\ \hline -v & 0 \end{array}\right) \tag{5.14}$$

where v is an $n \times 1$ matrix. An element h in the Cartan subalgebra may be taken to have the form

$$h = \left(\begin{array}{c|c} H \otimes \sigma & 0 \\ \hline 0 & 0 \end{array}\right) \tag{5.15}$$

where H is diagonal [12]. This case reduces to the case of $SO(2n)$.

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