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Cancellations of Infrared Divergences in the Two-Dimensional Non-Linear σ Models

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Abstract. In the two-dimensional $O(N)$ nonlinear σ models, the expectation value of any $O(N)$ invariant observable is shown to have an infrared finite weak coupling perturbative expansion, although it is computed in the "wrong" spontaneously broken symmetry phase. This result is proved by extracting all infrared divergences of any bare Feynman amplitude at $D = 2 - \varepsilon$ dimension. The divergences cancel at any order only for invariant observables. The renormalization at $D = 2$ preserves the infrared finiteness of the theory.

1. Introduction

Two-dimensional σ-models have raised an increasing interest during the last years, owing to their similarity with four-dimensional gauge theories, their simpler structure and the development of powerful nonperturbative methods. In this paper we deal with the weak coupling perturbative approach. This approach suffers from the fact that the naive vacuum state is not the true one, as presumably is the case for four-dimensional gauge theories. Indeed, the perturbative expansion has to be made around a peculiar classical solution, i.e. in the spontaneously broken symmetry phase, although such a phase cannot exist in two-dimensional space [1, 2]. So the symmetry has to be dynamically restored for any positive coupling constant [3, 4, 5]. A drastic consequence of the fact that the perturbative expansion is made in the wrong phase is that this expansion has very important infrared divergences, since even the free propagator of a massless Goldstone boson does not exist at two dimensions. For this reason the first perturbative calculations have been performed by introducing a symmetry breaking term which makes the theory infrared (I.R.) finite (and then by setting this term to zero) [4, 5].

However, S. Elitzur, following a remark of A. Jevicki about the effective potential of the $O(N)$ σ model [6], conjectured that any $O(N)$ invariant observable has an infrared finite expectation value to any order in perturbation expansion [7], and checked the fact up to third order of the two-point function. Various computations have been made by some authors [8, 9] for the $O(N)$ and $G \otimes G$ chiral models which have verified the conjecture in many cases and used it to study these models. Moreover, this result is very similar to what is expected for four-dimensional gauge theories, a namely that some gauge invariant quantities should be

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I.R. finite, although the nature of the divergences, as of the invariant states, is much more complicated for gauge theories. Nevertheless, Elitzur's argument is very insufficient and may not be considered as a complete proof.

In this paper we present a general proof of Elitzur's conjecture for the $O(N)$ nonlinear σ model. The infrared problem is shown to be disconnected with the ultraviolet (U.V.) problem, as assumed in [7, 8]. We use dimensional regularization (which has many advantages for this purpose), and prove that the regularized vacuum expectation value of any $O(N)$ invariant observable is I.R. finite at any order of perturbation at dimension $D = 2 - \varepsilon$. The U.V. renormalization at $D = 2$ may then be performed.

The key to the proof is the explicit extraction of the I.R. divergent part of any integral present in the perturbative expansion of any observable of the fields. For this purpose we use and adapt the general method developed by M. C. Bergère and Y. M. P. Lam in [10, 11] for studying the asymptotic expansion of Feynman amplitudes, and we also use results of [12] about dimensional regularization. This analysis is very technical and uses general technics of Mellin transform and subtraction operators. Only its final result is needed to show the mechanism of cancellations of I.R. divergences, so for clarity we shall first present this last point.

This paper is organized as follows:

In Sect. 2 we briefly recall the perturbative expansion of the $O(N)$ σ model. We then present in Lemma 2.1 the result of Sect. 3, namely the I.R. behaviour of any Feynman integral at $2 - \varepsilon$ dimensions. Then this result is used to exhibit the mechanism of cancellation of I.R. divergences for invariant observables. We finally deal with the problem of renormalization.

Section 3 is devoted to the analysis of I.R. divergences of the $O(N)$ σ model. We first present (in part A) the method of [10, 11] of analysis of asymptotic behavior, and introduce the main tools which shall be used. This method allows the extraction of I.R. divergences of any graph at generic (nonexceptional) momenta. This is performed for the graphs of the σ model in part B.

Divergences remain at exceptional momenta (this is related to the distribution-like character of the amplitudes in two-dimensional momentum space). This problem is discussed in part C. One has to look at the limit of nonexceptional momenta tending toward exceptional ones. This may not be studied by the former methods. We do not give a complete solution of this problem but present arguments for the general decomposition. This decomposition is proved in particular for the 2-point function, and a complete proof shall be given elsewhere.

2. I. R. Divergences of the $O(N)$ σ Model and Their Cancellations

We consider the Euclidean $O(N)$ nonlinear σ model. The $O(N)$ invariant action is

$$A_0 = \frac{1}{2g} \int d^D x (\partial_\mu \vec{S})(\partial^\mu \vec{S}) \tag{2.1}$$

where $\vec{S}(x)$ is a N-component real field with the usual constraint

$$\| \vec{S} \| = \sum_i (S^i(x))^2 = 1. \tag{2.2}$$

To obtain in I.R. finite perturbative expansion around $g = 0$, let us note

$$S^i(x) = \sqrt{g}\pi^i(x) \qquad i = 1, N-1$$
$$S^N(x) = \sigma(x) = \sqrt{1 - g\vec{\pi}^2(x)} \tag{2.3}$$

and let us introduce a magnetic field H in the N direction, so that the action becomes:

$$A_H = \frac{1}{g}\int d^Dx\left[\frac{\partial_\mu\vec{S}\cdot\partial^\mu\vec{S}}{2} - H(\sigma(x) - 1)\right] \tag{2.4}$$

$$= \frac{1}{2}\int d^Dx\left[(\partial_\mu\vec{\pi})^2 + H\vec{\pi}^2\right] + \frac{1}{g}\left[(\partial_\mu(\sigma - 1))^2 + H(\sigma - 1)^2\right]. \tag{2.5}$$

The expectation value of any observable $\mathcal{O}(\vec{\pi})$ is given by the functional integral

$$\langle \mathcal{O} \rangle_H = \frac{\int \mathcal{D}(\vec{\pi})\mathcal{O}(\vec{\pi})e^{-A_H(\pi)}}{\int \mathcal{D}(\vec{\pi})e^{-A_H(\pi)}} \tag{2.6}$$

where $\mathcal{D}(\vec{\pi})$ is the invariant measure

$$\mathcal{D}(\vec{\pi}) = \prod_x \frac{d\vec{\pi}(x)}{\sqrt{1 - g\vec{\pi}^2(x)}} \tag{2.7}$$

Since the interaction term may be written $\frac{1}{g}[\sigma - 1](-\Delta_x + H)(\sigma - 1)$, we associate to each vertex an "interaction line" which gives the factor $(-\Delta_x + H)$ of the vertex (that is a factor $p^2 + H$ in impulsion representation, where p is the total impulsion incoming to the $(\sigma - 1)$: (see Fig. 1)). The graphs of the model are considered as a set of usual propagator lines and of "interaction lines".

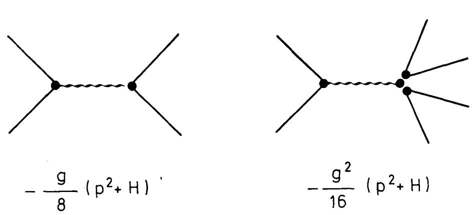

Fig. 1. Interaction vertices of first and second order. The wavy "interaction line" represents the term $p^2 + H$ where p is the impulsion carried through the line

We shall argue that the I.R. and the U.V. divergences of the model are completely disconnected. So, in order not to deal with U.V. divergences, we shall use dimensional regularization throughout this paper and study I.R. divergences at a dimension $D = 2 - \varepsilon$. This symmetry preserving regularization is much simpler than others (for instance lattice regularization used by Elitzur [7]) and was already used to study the I.R. problem of σ model in [8, 9]. In particular, it is well-known that with this procedure, the measure term in (2.7), which is proportional to

$\delta(0) = \int d^D k$, gives zero. Another advantage is that (as we shall see in the following), at $D = 2 - \varepsilon$ dimension, amplitudes diverge only as power of $H^{-\varepsilon}$ as $H \to 0$, and there is no more $\log H$ divergences (which make the analysis of I.R. divergences much simpler.

We now present the results of the analysis of the I.R. behaviour of any Feynman amplitude of the $O(N)$ σ model which is performed in Sect. 3. We shall show that those I.R. divergences may be cancelled by the introduction of I.R. counterterms (but at the price of strong modification of the functional integral). Then we shall exhibit identities relative to invariant operators that prove that these I.R. counterterms reduce to zero for these $O(N)$ invariant functions, so that the I.R. finiteness will be proved.

From Sect. 3, given a graph G, the subgraphs which give I.R. divergences are the "dominant subgraphs" E of G defined as:

Definition 2.1. A subgraph E of G is said "dominant" if:

(a) E contains all external vertices of G (that is vertices where external impulsions (or positions) are attached.

(b) E contains no disconnected part, that is has no connected part which does not contain any external vertex.

(c) There is no "interaction line" of $G - E$ attached to E.

A dominant subgraph may be disconnected (see Fig. 2).

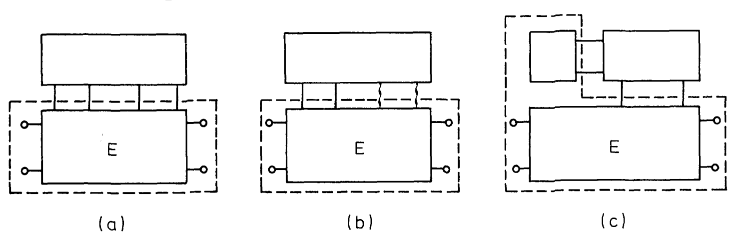

Fig. 2a a dominant subgraph E of a graph G. **b** an essential E of G which does not satisfy condition (**b**) of Definition 2.1. **c** an essential E of G which does not satisfy condition (**c**) of Definition 2.1

The main result of Sect. 3 is the following Lemma, which gives the I.R. behaviour of any amplitude.

Lemma 2.1. For any graph G, the I.R. behaviour of the amplitude $I_G(x, H)$ at $D = 2 - \varepsilon$ is given by a sum of contributions relative to dominant subgraphs of G:

$$I_G(x, H) = \sum_{\substack{E \subseteq G \\ \text{dominant}}} F_E(x) I_{[\widetilde{G/E}]}(H) + 0(H^{1 - \varepsilon L(G)/2}). \tag{2.8}$$

The F_E's are finite functions of the external positions x (or well-defined distribution of the external momenta p).

$I_{[\widetilde{G/E}]}(H)$ is the regularized amplitude of the graph $[\widetilde{G/E}]$ obtained by shrinking into one vertex the whole dominant E. It diverges like a pure power of $H^{-\varepsilon/2}$, namely

$$I_{[\widetilde{G/E}]}(H) = \text{const}\, H^{-(\varepsilon/2) L([\widetilde{G/E}])} \tag{2.9}$$

where $L([\widetilde{G/E}])$ is the number of loops of the graph $[\widetilde{G/E}]$.

The asymptotic expansion (2.8) is valid for $|\mathrm{Re}\,\varepsilon| < \frac{2}{L(G)}$.

The finite term of this expansion is the term relative to the dominant G itself (which is the only dominant such that $L(E) = L(G)$), that is $F_G(x)$. It follows immediately from this lemma that for $D > 2(\varepsilon < 0)$, the amplitude is I.R. finite, that is

$$I_G(x, 0) = F_G(x) \qquad \text{if } \varepsilon < 0 \tag{2.10}$$

and that for $\varepsilon > 0$, $F_G(x)$ is the analytic continuation of $I_G(x, 0)$ in the half plane $\varepsilon > 0$. Of course, the F_E's and the $I_{[\widetilde{G/E}]}$'s always have ultraviolet poles at $\varepsilon = 0$, since we deal with regularized amplitudes.

The result of Lemma 2.1 is quite similar to the Wilson Operator Product Expansion [13], and indeed is obtained by the same methods, since in Sect. 3 we transform by homogeneity the problem of small masses ($H \to 0$) into a problem of large momenta (see Fig. 3).

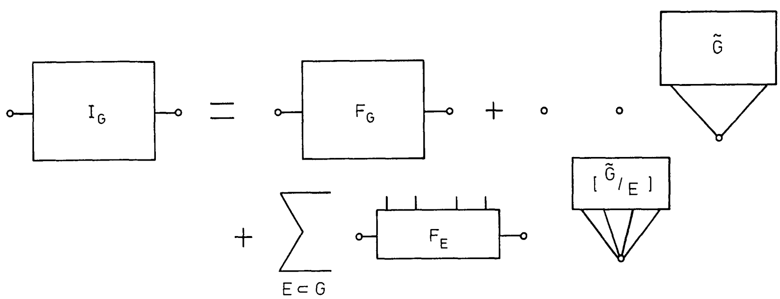

Fig. 3. A graphical interpretation of Lemma 2.1 (and of Eq. 3.43) giving the I.R. expansion of a graph of the 2-point function

We may invert (2.8) to express F_G as a function of the I_E's. A single recursive argument leads to:

Lemma 2.2.

$$F_G(x) = \sum_{\substack{E \subseteq G \\ \text{dominant}}} I_E(x, H) A_{[\widetilde{G/E}]}(H) + 0(H^{1-(\varepsilon/2)L(G)}). \tag{2.11}$$

The coefficients $A_{[\widetilde{G/E}]}(H)$ diverge as $H^{-(\varepsilon/2)L(\widetilde{G/E})}$, like $I_{[\widetilde{G/E}]}(H)$, and are given by a sum over all nests of dominant subgraphs of $[\widetilde{G/E}]$, considered as a graph which appears in the perturbative expansion of $\langle (\vec{\pi}^2(x)^p \rangle$ (where $2p$ is the number of lines of $G-E$ attached to E).

The functions $A_{[\widetilde{G/E}]}(H)$ appears as I.R. counterterms to be added to the I.R. divergent amplitude according to (2.11) to obtain a finite amplitude F. The introduction of those counterterms corresponds to a strong (nonlocal) modification of the functional integral. We shall show that this modification reduces to zero for $O(N)$ invariant functions, so that the bare amplitude I_G may be replaced by the I.R. finite amplitude F_G in the perturbative expansion of these functions. So, having isolated I.R. singularities for any amplitude in Lemma 2.1. and 2.2., we now deal with the explicit mechanism of cancellation.

Let us consider a $O(N)$ invariant function $\mathcal{O}(\vec{\pi})$. In presence of a symmetry breaking term $-\frac{H}{g}(\sigma-1)$ in the action, the vacuum expectation value of $\mathcal{O}$ is given by the functional integral (2.6).

Following an idea of Elitzur [7], let us perform an arbitrary rotation R of angle θ in a direction $\vec{u}$ of the tangent plane of the $\vec{\pi}$ ($|\vec{u}|=1$), so that the fields are changed into

$$\vec{\pi} \longrightarrow {}^R\vec{\pi} = \vec{\pi} + \left[(\cos\theta - 1)(\vec{\pi}\cdot\vec{u}) + \frac{1}{\sqrt{g}}\sin\theta\sigma\right]\vec{u} \tag{2.12}$$

$$\sigma \longrightarrow {}^R\sigma = \cos\theta\cdot\sigma - \sqrt{g}\sin\theta(\vec{\pi}\cdot\vec{u}). \tag{2.13}$$

Since the measure $\mathcal{D}$ the function $\mathcal{O}$ and the invariant action A_0 are invariant under this rotation, the only term that is changed in the functional integral (2.6) is the symmetry breaking term which becomes

$$\frac{H}{g}(1 - {}^R\sigma) = \frac{H}{g}[1 - \cos\theta\cdot\sigma + \sqrt{g}\sin\theta(\vec{\pi}\cdot\vec{u})]. \tag{2.14}$$

Defining a parameter a by

$$a = \frac{1}{\sqrt{g}}\tan\theta \tag{2.15}$$

and rescaling H into

$$H \to H/\cos\theta = H\sqrt{1+ga^2} \tag{2.16}$$

we obtain

$$\langle\mathcal{O}(\vec{\pi})\rangle_{H\sqrt{1+ga^2}} = \frac{\int\mathcal{D}(\vec{\pi})\mathcal{O}(\vec{\pi})\exp\left[-A_H(\vec{\pi}) - aH\int d^Dx(\vec{\pi}\cdot\vec{u})\right]}{\int\mathcal{D}(\vec{\pi})\exp\left[-A_H(\vec{\pi}) - aH\int d^Dx(\vec{\pi}\cdot\vec{u})\right]}. \tag{2.17}$$

Eq. (2.17) corresponds to the following identity between vacuum expectation values for any $O(N)$ invariant observable $\mathcal{O}$ and any a.

$$\langle\mathcal{O}(\vec{\pi})\rangle_{H\sqrt{1+ga^2}} = \frac{\langle\mathcal{O}(\vec{\pi})\exp(-aH\int d^Dx(\vec{\pi}\cdot\vec{u}))\rangle_H}{\langle\exp(-aH\int d^Dx(\vec{\pi}\cdot\vec{u}))\rangle_H}. \tag{2.18}$$

To see the consequence of this identity at the perturbative level, let us develop in g and a both sides of (2.18). We shall note perturbative expansion of $\langle\mathcal{O}(\vec{\pi})\rangle_H$

$$\langle\mathcal{O}(\vec{\pi})\rangle_H = \sum_{N=0}^{\infty} g^N\mathcal{O}_N(H) \tag{2.19}$$

where $\mathcal{O}_N$ is the sum of the amplitudes of all graphs of order N which appear in the perturbative expansion of $\langle\mathcal{O}\rangle$. Developing $\sqrt{1+ga^2}$ we get the following expansion of the l.h.s. of (2.18)

$$\langle\mathcal{O}(\vec{\pi})\rangle_{H\sqrt{1+ga^2}} = \sum_{N=0}^{\infty} g^N\left[\mathcal{O}_N(H) + \sum_{P=1}^{N} a^{2P}\mathcal{P}_P\left(H\frac{\partial}{\partial H}\right)\mathcal{O}_{N-P}(H)\right] \tag{2.20}$$

where the $\mathscr{P}_P$ are polynomials of degree P and of valuation 1 (they have no term of order zero).

We now look at the r.h.s. of (2.18). $\mathcal{O}(\vec{\pi})$ being invariant is an even function of $\vec{\pi}$, so that, by parity, only even powers of a occur in the expansion. We obtain

$$\text{r.h.s. (2.18)} = \sum_{N=0}^{\infty} g^N \left[\mathcal{O}_N(H) + \sum_{P=1}^{N} a^{2P} \mathcal{O}_N^P(H) \right] \tag{2.21}$$

where the $\mathcal{O}_N^P(H)$ is sum of the graphs which occur in the perturbative expansion of $\langle \mathcal{O}(\vec{\pi})([H\int d^D x(\vec{\pi}\cdot\vec{u})]^{2P}/2P!\rangle$, that is graphs of $\mathcal{O}(\vec{\pi})$ with $2P$ insertions of $H(\vec{\pi}\cdot\vec{u})$ at zero momenta. Moreover, the presence of these insertions at the denominator of (2.18) ensures that not only vacuum diagrams disappear, but also diagrams with disconnected part where there are only insertions of $H(\vec{\pi}\cdot\vec{u})$ (see Fig. 4). (The factor $1/(2P)!$ is cancelled by a factor $(2P)!$ which comes from the contractions in Wick's theorem, since these $2P$ insertions are undiscernable). The insertions being made at zero momenta, the contribution of each line joining an insertion to the graph is $\frac{H}{H} = 1$, so that the graphs may be seen as graphs of $\mathcal{O}$ with "truncated insertions" of $(\vec{\pi}\cdot\vec{u})$ at zero momenta (see Fig. 4).

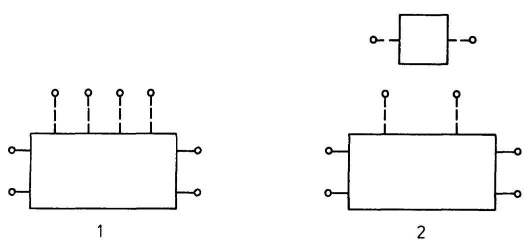

Fig. 4. Example of graphs with four insertions of $H(\vec{\pi}.\vec{u})$. Graph 1 contributes in (2.22), and graph 2 does not

Let us note by $\mathscr{G}_N^P(\mathcal{O})$ the set of all graphs that contribute to $\mathcal{O}_N^P$, and $C(E)$ the counting factor of a graph E of $\mathscr{G}_N^P(\mathcal{O})$ in $\mathcal{O}_N^P$. We have

$$\mathcal{O}_N^P(x, H) = \sum_{E \in \mathscr{G}_N^P(\mathcal{O})} C(E) I_G(x, H) \tag{2.22}$$

and we identify trivially $\mathcal{O}_N^0$ with $\mathcal{O}_N$.

Identifying term by term (2.20) and (2.21), we get the following identity, which is the perturbative version of identity (2.18):

$$\mathcal{O}_N^P(H) = \mathscr{P}_P\left(H \frac{\partial}{\partial H} \right) \mathcal{O}_{N-P}(H). \tag{2.23}$$

The crucial point of the mechanism of cancellation is that the graphs of the $\mathscr{G}_N^P(\mathcal{O})$ are exactly the "dominant subgraphs", as defined previously, that appear Lemma 2.1 and 2.2 in the I.R. expansion of the graphs of the perturbative expansion of $\langle \mathcal{O} \rangle$.

So let us come back to the result of Lemma 2.2.

Given a graph G of $\mathscr{G}_N^0(\mathcal{O})$, a dominant subgraph E of G appears to be a graph of some $\mathscr{G}_M^P(\mathcal{O})(M \leq N, P \leq M)$ where M is the order of the graph E and $2P$ is the number of lines of $G-E$ attached to E. Similarly, the reduced graph $[\widetilde{G/E}]$ appears to be a graph of order $N-M$ of the perturbative expansion of $(\vec{\pi}^2(0))^P$, that is a graph of $\mathscr{G}^0_{N-M}(\vec{\pi}^2(0)^P)$. It is shown in Appendix A that the counting factors are such that the sum over the graphs G of the decomposition (2.11) may be factorized into a sum over the dominant graphs E times a sum over the reduced graphs, so we have the following decomposition:

$$\sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) = \mathscr{O}_N(x,H) + \sum_{\substack{0\leqq M\leqq N\\ 1\leqq P\leqq M}} \mathscr{O}^P_M(x,H)\cdot\mathscr{A}_{P_{N-M}}(H) + O(H^{1-\varepsilon N/2}) \quad (2.24)$$

where the $\mathscr{O}^P_M(x,H)$ are given by (2.22) and where the $\mathscr{A}_{P_{N-M}}(H)$ are given by

$$\mathscr{A}_{P_{N-M}}(H) = \sum_{S\in\mathscr{G}^0_{N-M}(\pi^2(0)^P)} C(S).A_S(H). \quad (2.25)$$

The $\mathscr{A}_{P_Q}(H)$'s diverge as a power of $H^{-(\varepsilon/2)(P+Q)}$, except for the term $P=0, Q=0$, which corresponds to the graph reduced to a point and which is set equal to 1.

We now have all the elements needed to prove the I.R. cancellation. We have the theorem. :

Theorem 2.1. *The bare vacuum expectation value of any $\mathscr{O}(N)$ invariant function $\mathscr{O}$ is infrared finite at any order of perturbative expansion at dimension $D=2-\varepsilon$. The term of order N of the development of $\langle\mathscr{O}\rangle$, $\mathscr{O}_N$, is given by the sum of the finite part F_G of the amplitudes of the graphs G which appear in $\mathscr{O}_N$, namely*

$$\mathscr{O}_N(x,H) = \sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) + O(H^{1-\varepsilon N/2}). \quad (2.26)$$

Proof. Let us assume that the theorem is true at any order $M<N$. (This recursive hypothesis is trivially satisfied at order $N=0$). Then, any derivative versus $\ln H$ of $\mathscr{O}_M(M<N)$ has a zero limit, namely

$$\left(H\frac{\partial}{\partial H}\right)^q \mathscr{O}_M = O(H^{1-\varepsilon M/2}) \quad \forall q\geqq 1, M<N. \quad (2.27)$$

From (2.23), we deduce immediately that the $\mathscr{O}^P_N$ have a zero limit.

$$\mathscr{O}^P_N = O(H^{1-\varepsilon(N-P)/2}) \quad \forall\, p\geqq 1. \quad (2.28)$$

This is also true for the $\mathscr{O}^P_M(M<N)$ by the recursive hypothesis. So, for ε small enough, we may add to $\mathscr{O}_N$ any linear combination of the $\mathscr{O}_{M,P}$, provided that their coefficients diverge as power of $H^{-\varepsilon/2}$. In particular, we may take the I.R. counter-terms $\mathscr{A}_{P_{N-M}}$. So (2.28) and (2.24) lead immediately to:

$$\sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) = \mathscr{O}_N(x,H) + O(H^{1-\varepsilon N/2}). \quad (2.29)$$

The l.h.s. of (2.29) being I.R. finite, the theorem is proved at order N.

We finally deal with the problem of renormalization. As claimed in the introduction, we have seen that regularized vacuum expectation values of any invariant function are I.R. finite, so that the I.R. problem is disconnected from the U.V. one. The renormalization of the σ-model in dimension two is performed in references [4, 5]. It is proved that, for soft invariant operators (that is for local functions of the fields without derivatives of the fields $\vec{\pi}$), only two counterterms are needed, corresponding to the renormalization of the fields and of the coupling constant [5]. The bare field $\vec{S}_0$ and the bare coupling constant g_0 are related to the renormalized ones $\vec{S}_R$ and g_R by

$$\vec{S}_0 = \sqrt{Z}\,\vec{S}_R \tag{2.30}$$

$$g_0 = Z_1 g_R. \tag{2.31}$$

The symmetry breaking term $\frac{1}{g}H\sigma$ has the dimension of the field $\vec{S}$ and so needs no additional counterterm. The "bare magnetic field" H_0 is related to the renormalized one by

$$H_0 = \frac{Z_1}{\sqrt{Z}} H_R. \tag{2.32}$$

The counterterms Z and Z_1 being independent of H, their perturbative expansions (in g_R or in g_0) are obviously I.R. finite (but of course U.V. divergent, and have poles at $\varepsilon = 0$). So we let the reader convince himself that the renormalization of any soft operator does not introduce any additional I.R. divergences. So the perturbative expansion of any renormalized $O(N)$ invariant soft operator is I.R. finite at any order of perturbation.

The renormalization of invariant operators of higher dimension (that is with derivatives of the fields) is more subtle. Because the $O(N)$ transformation laws of the fields are modified by renormalization, the invariant operators are mixed with what seems to be non-invariant ones [5]. The problem of the I.R. finiteness of those objects is discussed in [14] and it may be shown that they are also I.R. finite.

3. Extraction of I.R. Divergences of Regularized Amplitudes at $D = 2 - \varepsilon$

A. Introduction

We now present the analysis of the I.R. divergences of the regularized amplitudes at $D \leqq 2$. The general method of analysis of the asymptotic expansion of Bergère and Lam exposed in [10, 11] is adapted to study the I.R. limit of graphs at generic (nonexceptional) momenta. In our case, some simplifications occur, since we deal with regularized (instead of renormalized) amplitudes. However, we adapt their procedure of extraction of singularities of the Mellin transform to extract not only the dominant part of the asymptotic expansion, but subdominant ones. Then we give arguments, but not a complete proof, for the extraction of singularities at exceptional momenta. Let us first introduce the main tools which are used.

Schwinger representation of dimensionally regularized amplitude

The Feynman amplitude are written in the α-Schwinger representation.

Each propagator of a line a is written:

$$\frac{1}{p^2 + H} = \int_0^\infty d\alpha\, e^{-\alpha(p^2 + H)} \tag{3.1}$$

and the contribution of the "interaction lines" introduced in Sect. 2 is written

$$(p^2 + H) \int_0^\infty d\alpha \left(\frac{\partial}{\partial \alpha} \right)^2 e^{-\alpha(p^2+H)}. \tag{3.2}$$

Performing the integration over internal momenta, we obtain the Schwinger representation for the amplitude of a graph G.

$$I_G(p, H) = \int_0^{+\infty} \prod_{a \in G} d\alpha_a \mathscr{D}_G \left[\exp\left(- \sum_{a \in G} \alpha_a H - p_i d_{G_{ij}}(\alpha) p_j \right) P_G(\alpha)^{-D/2} \right]. \tag{3.3}$$

$\mathscr{D}_G$ is the differential operator

$$\mathscr{D}_G = \prod_{a \in \mathscr{I}_G} \left(\frac{\partial}{\partial \alpha_\alpha} \right)^2 \tag{3.4}$$

where $\mathscr{I}_G$ is the set of "interaction lines" of the graph $G.P_G$ and $d_{G_{ij}}$ are the Symanzik functions, characteristic of the topology of the graph.

Dimensional regularization is performed by taking D complex in (3.3) [12]. This integral is U.V. convergent for Re D small enough, and has an analytic continuation in D meromorphic with poles at dimension D such that the superficial degree of divergence ω_S of a connected one-particle irreducible (1.P.I.) subgraph S is a positive or null integer. ω_S is defined in our case as:

$$\omega_S(D) = \frac{DL(S)}{2} - p(S) + i(S) \tag{3.5}$$

where $L(S)$ is the number of independent loops of S and $p(S)$ (respectively $i(S)$) is the number of propagator lines (respectively interaction lines) in S.

According to [12] when (2.10) is divergent, the regularized integral is given, for Re D away from the U.V. poles, by the convergent integral.:

$$I_G = \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R} \left[\mathscr{D}_G \exp\left(- \left[\sum_a \alpha_a H + p d_G p \right] \right) P_G^{-D/2} \right] \tag{3.6}$$

where $\mathscr{R}$ is the subtraction operator defined in [15] as a sum over nests of divergent subgraphs at the dimension D of products of Taylor operators:

$$\mathscr{R} = 1 + \sum_{\mathscr{N}} \prod_{S \in \mathscr{N}} (-\tau_S^{-\ell(S)}). \tag{3.7}$$

Each Taylor operator acts as follows:

The α'_a variables ($a \in S$) of the subgraph S are scaled by ρ. One writes the generalized Laurent expansion in ρ of a function f on which $\tau_S^- \ell^{(S)}$ is applied as

$$f(\alpha, \rho) = \sum_{p=0}^{\infty} \rho^{p_0 + p} f_p(\alpha). \tag{3.8}$$

(In our case p_0 is complex, and depends on D.) The Taylor operator τ only retains the terms with Re $(p_0 + p) \leqq -\ell(S)$, where

$$\ell(S) = p(S) + i(S) \tag{3.9}$$

is the number of lines of S, and then takes $\rho = 1$.

We recall that a nest $\mathcal{N}$ is a set of subgraphs S such that, given two subgraphs S_1 and S_2 of $\mathcal{N}$, either S_1 is included in S_2, or S_2 is included in S_1.

Applied on the integrand of (3.6), R may be rewritten as a sum over Zimmermann forests of connected 1.P.I. divergent subgraphs [15]. The action of $\mathcal{R}$ in (3.6) depends on the values of $\operatorname{Re} D$ and $\mathcal{R}$ performs subtractions at zero external momenta and internal mass H, so that it is not a renormalization. Indeed the singularities of I_G are always present, and appear where D tends towards a pole where the integral (3.6) is no more convergent.

Mellin transform and the I.R. *asymptotic expansion*

We now give the principle of the study of the I.R. behaviour, as exposed in [10, 11]. We want to study the limit as $H \to 0$ of the regularized amplitude $I_G(p, H)$ for a dimension D less than two (namely $D = 2 - \varepsilon$ with $0 < \varepsilon < \frac{2}{L(G)}$. Scaling α into α/H in (3.6) we get the homogeneity relation:

$$I_G(p, H) = H^{\omega_G} I_G\left(\frac{1}{\sqrt{H}} p, 1\right). \tag{3.10}$$

The Mellin transform of (3.10) is defined in [11] as

$$M_G(p, x) = \int_0^{+\infty} d\lambda (1 - \tau_\lambda^{-1}) \lambda^{-x-1} I_G\left(p, \frac{1}{\lambda}\right). \tag{3.11}$$

The integral (3.10) is convergent at infinity for x great enough (the τ_λ^{-1} ensures the convergence at zero). It is shown in [16] that using the integral representations (3.6) and (3.11), we may intervert integration in λ and α, to get the integral representation of M_G:

$$M_G(p, x) = \Gamma(-x - \omega_G) \int_0^{+\infty} \prod_G d\alpha\, \mathcal{R}\left\{ \mathcal{D}_G \exp\left(-\sum_G \alpha \right) P_G^{-D/2} (p d_G p)^{x + \omega_G} \right\}. \tag{3.12}$$

The integral in α is convergent for x great enough, $\mathcal{R}$ being given by (3.7). The singularities of the Γ function are related to the behaviour of I_G as $H \to +\infty$ and need not be considered. The integral in α which defines the function

$$F_G(x) = M_G(x)/\Gamma(-x - \omega_G) \tag{3.13}$$

may be analytically continued into a meromorphic function of x, with real poles in decreasing order $x_0 > x_1 > \ldots > x_i > \ldots$. If we know the Laurent expansion of $M_G(x)$ around these poles, namely

$$M_G(x) = \sum_{p=1}^{p_{\max}(N)} \frac{a_{N,p}}{(x - x_N)^p} + \text{regular part at } x = x_N, \tag{3.14}$$

we obtain by inverse Mellin transform the asymptotic expansion of $I_G(p, H)$, that is:

$$I_G(H) \simeq \sum_{N=0}^{\infty} H^{-x_N} \sum_{p=1}^{p_{\max}(N)} \frac{a_{N,p}}{\Gamma(p)} \ln^{p-1}(1/H). \tag{3.15}$$

So, we have to analyse the singularities of the function $F_G(x)$ which arise, as for

U.V. singularities in (3.3), from the divergent behavior of the integrand of (3.12) when some subset of α tends towards zero. We now recall the notions of [11] which are used to analyse those divergences.

Definition 3.1. Given a graph G and a set of external momenta $\{p_i\}$, a subgraph S of G is *essential* if, setting all α's relative to S equal to zero, the function $pd_G p$ is set to zero. This is equivalent to the fact that all external vertices belong to S, and that the sum of the momenta attached to any connected part of S is zero.

The notion of essential subgraph depends on the external momenta $\{p_i\}$. At non-exceptional momenta, that is, in the euclidean case, if any partial sum of p_i's is different from zero, a subgraph is essential if and only if all external vertices of G belong to the same connected part of S.

Definition 3.2. A set of subgraphs of G is *misjoint* if they have no line in common, and if the number of loops of their union is equal to the number of loops of the individual subdiagrams. (The subdiagrams may have vertices in common).

Definition 3.3. A set ψ of subgraphs of G is called a *Q-extended forest* if it satisfies:

(a) Any subset of mutually noninclusive elements is misjoint.

(b) The union of nonessential elements of ψ is nonessential.

(c) Every essential element E of ψ has no disconnected part, that it has no subgraph E' such that E' and $E - E'$ are disconnected and such that $E - E'$ is still essential.

(In [11] Bergère and Lam do not consider the condition *c*) but always consider Q-extended forests with this condition.)

Definition 3.4. An essential E has an inactive part E' if E' and $E - E'$ are misjoint and if $E - E'$ is still essential.

We now come back to the function $F_G(x)$. It is shown in [16] that (D being fixed away from an U.V. pole) the poles of the function $F_G(x)$ are given by essential subgraphs of G, and are characterized by

$$x + \omega(G) - \omega(E) = -n \qquad n \text{ positive or null integer,} \tag{3.16}$$

E being an essential of G.

However, from [12] and [16], as for the dimensionally regularized integral (3.6), for Re x away from such poles, the integral (3.12) is absolutely convergent and defines the function $F_G(x)$, provided that the $\mathscr{R}$ operator is given by the nest formula (3.7), and consequently subtracts not only U.V. divergent subgraphs, but also divergent essential subgraphs.

We now extend the results of Bergère and Lam [10, 11]. It is proved that the nest formula (3.7) for $\mathscr{R}$ may be replaced, when acting upon a function like in (3.12), by a formula over Q-extended forests, namely:

$$\mathscr{R} = 1 + \sum_{\psi Q\text{ext.forests}} \prod_{S\in\psi} (-\tau_S^{-\ell_S}) \tag{3.17}$$

where the sum runs over all Q-extended forests of divergent nonessential connected 1.P.I. subgraphs and of essential subgraphs (with no disconnected part from the Definition 3.3) of G.

If we scale by ρ the α's of a subgraph S in the integrand of (3.12) the first term of the Laurent expansion in ρ is proved to be [11]

$$\rho^{-\omega_S-\ell_S}\left\{\mathscr{D}_S P_S^{-D/2}\times\mathscr{D}_{[G/S]}\exp\left(-\sum_{[G/S]}\alpha\right)P_{[G/S]}^{D/2}pd_{[G/S]}p^{x+\omega_G}\right\} \tag{3.18}$$

if S is nonessential, and

$$\rho^{x+\omega_G-\omega_S-\ell_S}\left\{\mathscr{D}_S P_S^{-D/2}pd_S p^{x+\omega_G}\times\mathscr{D}_{[G/S]}\exp\left(-\sum_{[G/S]}\alpha\right)P_{[G/S]}^{-D/2}\right\} \tag{3.19}$$

if S is essential.

Since $F_G(x)$ has singularities only at values of x such that the action of some Taylor τ_S has a discontinuity, we deduce immediately that:

1) $F_G(x)$ has singularities given by (3.16), only for the essential subgraphs with no disconnected part. (Other essentials give no singularities)

2) x being fixed away from these poles, the only forests which contribute in (3.17) are forests of nonessential connected 1.P.I. subgraphs S such that $\omega_S > 0$ (to deal with U.V. singularities) and of essential subgraphs with no disconnected part such that

$$\operatorname{Re} x < \omega_E - \omega_G. \tag{3.20}$$

So, (3.12) gives an integral representation of $M_G(x)$ around any poles x_i. To extract the Laurent expansion of M_G at x_i, we may perform explicitly a Cauchy integral around x_i to compute residues. The general method presented here is now applied to the Green's functions of the $O(N)$ σ model.

B. The Case of Nonexceptional Momenta

The I.R. behaviour of an amplitude is in general different at exceptional and at nonexceptional momenta (since the essential subgraphs are different). So we first deal with the I.R. divergences of the $O(N)$ σ model at nonexceptional momenta (that is when any partial sum of external momenta is nonzero).

Let us consider a connected graph G that appears in the perturbative expansion of a N point function at $D = 2 - \varepsilon$ (with $N \geqq 2$). At nonexceptional momenta, any essential subgraph E without disconnected part in connected and contains all external vertices. Then we have

$$\omega_E - \omega_G = \frac{\varepsilon}{2}[L(G) - L(E)] - n_i \tag{3.21}$$

where n_i is the number of "interaction lines" in $G - E$ attached to E. (See Fig. 2). From (3.22), only essentials with $n_i = 0$ give poles at $x \geqq 0$. Let us call such essentials "leading essentials." (The leading essentials are the connected dominant subgraphs of G).

The function $F_G(x)$ has singularities at

$$x_L = \frac{\varepsilon L}{2} \qquad L = 0, \ldots, L(G) \tag{3.22}$$

since the superficial degree of convergence of G is

$$\omega_G = (1-N) - \frac{\varepsilon}{2} L(G). \tag{3.23}$$

The singularities of F_G do not interfere with those of the function $\Gamma(-x-\omega_G)$ in the definition of $M_G(x)$. Moreover, from the general analysis of (**A**), the following poles of F_G are smaller than $-1+\frac{\varepsilon}{2}L(G)$. (See Fig. 5.)

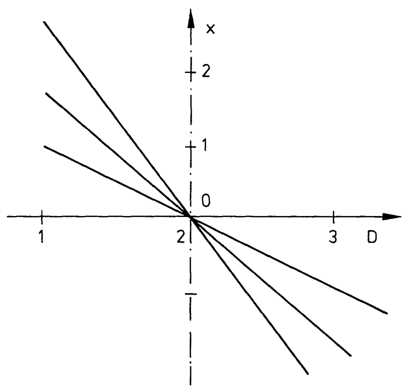

Fig. 5. Singularities in x of the Mellin transform $M_G(x)$ around $D=2$

To extract the Laurent expansion around x_L

$$M_G(x) = \sum_{p=1}^{p_{\max}} a_p \frac{1}{\left[x - \frac{\varepsilon}{2}L\right]^p}, \tag{3.24}$$

we have only to write the Cauchy integral

$$a_p = \frac{1}{2\pi}\left[\int\limits_{\substack{-\infty \\ \varepsilon L/2 < \sigma_+ < ((\varepsilon/2)(L+1))}}^{+\infty} dy M_G(\sigma_+ + iy)(\sigma_+ + iy - x_L)^{p-1} - \int\limits_{\substack{-\infty \\ (\varepsilon/2)(L-1) < \sigma_- < \varepsilon L/2}}^{+\infty} dy M_G(\sigma_- + iy)(\sigma_- + iy - x_L)^{p-1}\right] \tag{3.25}$$

and to use the integral representation (3.12) of $M_G(x)$ in the bands

$$B_L = \left\{\frac{\varepsilon L}{2} < \operatorname{Re} x < \frac{\varepsilon}{2}(L+1)\right\} \tag{3.26a}$$

and

$$B_{L-1} = \left\{\frac{\varepsilon}{2}(L-1) < \operatorname{Re} x < \frac{\varepsilon}{2}L\right\}. \tag{3.26b}$$

From (3.17), in B_L, (respectively B_{L-1}) in the expression of $\mathscr{R}$, only Q-extended forests which contain leading essential subgraphs E such that $L(G)-L(E)>L$ (respectively $L(G)-L(E)\geqq L$) contribute. The Γ function in (3.12) ensures convergence at infinity of the integral (3.25). So, we invert integrations in y and α in (3.25), to obtain for a_p a convergent integral representation where only the Q-extended forests are present which contribute in B_{L-1} and do not in B_L, namely those which contain one leading essential such that $L(G)-L(E)=L$.

We get

$$a_p = -\int_0^{+\infty}\prod_G d\alpha \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy(\sigma_- + iy - x_L)^{p-1}\Gamma(-\sigma_- - iy - \omega_G)$$
$$\times\left(\sum_{\substack{\psi \text{ containing}\\ \text{a lead. ess. } E\\ \text{such that } L(G/E)=L}}\prod_{S\in\psi}(-\tau_S^{-\ell_S})\left\{\mathscr{D}_G \exp\left(-\sum_G \alpha\right)P_G^{-D/2}(pd_G p)^{\sigma_- + iy+\omega_G}\right\}\right). \tag{3.27}$$

Let us consider such a Q-extended forest ψ. It is obvious that it contains only one leading essential E such that $L(G/E)=L$, and that it is the greatest essential of ψ. The action of $\tau_E^{-\ell_E}$ onto the function $\{\ \ \}$ in (3.27) is given by (3.19); we get a factorization into a part relative to E and a part relative to $(G/E]$:

$$\tau_E^{-\ell_E}\{\ \ \} = [\mathscr{D}_E\cdot P_E^{-D/2}(pd_E p)^{\sigma_- + iy+\omega_G}]\times\left[\mathscr{D}_{[G/E]}\exp\left(-\sum_{[G/E]}\alpha\right)P_{[G/E]}^{D/2}\right] \tag{3.28}$$

where $[G/E]$ is the reduced graph where E is shrunk to a point in G. Reorganizing the sum order Q-extended forests as a sum over leading essentials E such that $L(G/E)=L$ and a sum over Q-extended forests containing E, and factorizing over forests of $[G/E]$ and of E we get finally for a_p, up to the problems of convergence of the integrals, which shall be discussed later:

$$a_p = \sum_{\substack{E \text{ leading ess.}\\ \text{such that}\\ L(G/E)=L}}\int_0^{+\infty}\prod_{[G/E]} d\alpha\left[1+\sum_{\substack{Q \text{ ext}\\ \text{forests}\\ \text{in } [G/E]}}\prod_S(-\tau_S^{-\ell_S})\right]\left[\mathscr{D}_{[G/E]}\exp\left(-\sum_{[G/E]}\alpha\right)P_{[G/E]}^{-D/2}\right]$$
$$\times\int_0^{+\infty}\prod_E d\alpha\frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\,\Gamma(-\sigma_- - iy - \omega_G)(\sigma_- + iy - \omega(G/E))^{p-1}$$
$$\times\left\{1+\sum_{\substack{Q \text{ ext. forests}\\ \text{in } E \text{ which do}\\ \text{not contain } E}}\prod_S(-\tau_S^{-\ell_S})\right\}\left\{\mathscr{D}_E P_E^{-D/2}(pd_E p)^{\sigma_- + iy+\omega_G}\right\}. \tag{3.29}$$

The term relative to $[G/E]$ is simply the regularized amplitude $I_{[G/E]}(H=1)$. In the integral relative to E the sum runs over all Q-extended forests in E which do not contain E itself. Let us note this term R_E^p. To perform the integration over y, we have to make precise the dependence of the integrand in (3.29) on y. From the expansion properties (3.18) and (3.19), we have for a given Q-extended forest ψ:

$$\prod_{S\in\psi}(-\tau_S^{-\ell_S})\mathscr{D}_E P_E^{-D/2}(pd_E p)^{\sigma_- + iy+\omega_G} = Z(\alpha,D)(pd_{[E_{\min}]_\psi}p)^{\sigma_- + iy+\omega_G} \tag{3.30}$$

where $E_{\min}$ is the smallest leading essential in ψ (if ψ does not contain a leading essential, we take $E_{\min} = E$), and where $[E_{\min}]_{\psi}$ is the reduced graph obtained by shrinking to a point every subgraph of ψ in $E_{\min}$.

We now perform the integration over y. Denoting $z = -\sigma_{-} - iy - \omega(G/E)$, we compute

$$\frac{1}{2i\pi}\int_{\mathrm{Im}\, z=-\infty}^{\mathrm{Im}\, z=+\infty} dz\Gamma(z-\omega_E)[pd_{[E_{\min}]_{\psi}^{P}}]^{-z+\omega_E} z^{p-1}$$
$$= (pd_{[E_{\min}]_{\psi}^{P}})^{\omega_E} \quad G_p(pd_{[E_{\min}]_{\psi}^{P}}, \omega_E). \tag{3.31}$$

$G_p(x, \omega)$ is defined as

$$G_p(x,\omega) = \left(-x\frac{\partial}{\partial x}\right)^{p-1}(x^{-\omega}e^{-x}). \tag{3.32}$$

So finally for R_E^p we get the following integral representation

$$R_E^p = \int_0^\infty \prod_E d\alpha \sum_{\substack{\psi\ Q\text{-ext. forest} \\ \psi \not\ni E \\ (\psi \text{ eventually empty})}} G_p(pd_{[E_{\min}]_{\psi}^{P}}, \omega_E)$$
$$\left[\prod_{S\in\psi}(-\tau_S^{-\ell_S})\right]\left[\mathscr{D}_E P_E^{-D/2}(pd_E\, p)_E^{\omega}\right]. \tag{3.33}$$

In (3.33) we sum over all Q-extended forests which do not contain E, including the empty forest. In fact, the R_E^p are null if $p > 1$. Indeed, let us scale α into $\lambda\alpha$ in the integral (3.33). We get the same integral, except that $G_p(pd_{[E_{\min}]_{\psi}^{P}}, \omega_E)$ is changed into $G_p(\lambda pd_{[E_{\min}]_{\psi}^{P}}, \omega_E)$. Differentiating with respect to λ and using (3.32) we get

$$\lambda\frac{\lambda}{\partial\lambda}\mathscr{R}_E^P = \mathscr{R}_E^{P+1} = 0, \tag{3.34}$$

so that only R_E^1 is nonzero and shall be noted R_E in the following. The conclusion of this study is that $M_G(x)$ has a single pole at $x_L = \frac{\varepsilon}{2}L$. Its residue is given by

$$\mathrm{Res}_{\varepsilon L/2}\{M_G(x)\} = \sum_{\substack{\text{leading essential} \\ E \text{ such that } L(G/E)=L}} I_{[G/E]}(H=1). \quad R_E(p). \tag{3.35}$$

This may be done at every pole $\frac{\varepsilon L}{2}$, so that we recover all leading essentials. By the inverse Mellin transform, we deduce immediately the exact asymptotic behaviour of $I_G(p, H)$ as $H \to 0$.

Theorem. 3.1. *Any regularized amplitude of the $O(N)$ σ model at nonexceptional momenta has the following I.R. asymptotic behaviour at $D = 2 - \varepsilon$:*

$$I_G(p,H) = \sum_{n=L(G)}^{0} H^{-\varepsilon n/2} \sum_{\substack{\text{lead ess. } E \\ \text{such that } L(G/E)=n}} I_{[G/E]}(H=1)R_E(p) + 0(H^{1-\varepsilon L(G)/2}).$$

The divergent terms $H^{-\varepsilon L(G/E)/2} I_{(G/E)}$ (1) are relative only to reduced subgraphs $[G/E]$ and are, by homogeneity, equal to $I_{(G/E)}(H)$. The part $R_E(p)$ associated to a leading essential E is I.R. finite and is given by (3.33) with $p = 1$.

The finite part of the expansion (3.36) is the term at $n = 0$. The only leading essential such that $L(G/E) = 0$ is the graph G itself, so that the finite part of (3.36) is $R_G(p)$.

To complete the proof, we have yet to prove that the integral (3.33) which defines $R_E(p)$ is convergent. This is now obvious, because we know that

$$R_E(p) = \text{Residue at } x = 0 \quad \text{of } M_E(p, x).$$

$R_E(p)$ is directly given by the convergent integral representation (3.33) with $E = G, L = 0, p = 1$.

This suggests another representation of $R_G(p)$. We may extract the pole at $x = 0$ simply by scaling $\alpha \to \lambda\alpha$ in the integral representation (3.12) of $M_G(x)$. We thus get

$$\lambda^{-x} M_G(p, x) = \Gamma(-x - \omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\{\mathscr{D}_G e^{-\lambda\Sigma\alpha} p_G^{-D/2} (pd_G p)^{x+\omega_G}\}. \quad (3.37)$$

Differentiating with respect to λ and setting λ equal to one, we get

$$M_G(p, x) = \frac{1}{x}\Gamma(-x - \omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\left\{\mathscr{D}_G e^{-\sum_G \alpha}\left(\sum_G \alpha\right) p_G^{-D/2} (pd_G p)^{x+\omega_G}\right\}. \quad (3.38)$$

So we have

$$R_G(p) = \Gamma(-\omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\left\{\mathscr{D}_G e^{-\sum_G \alpha}\left(\sum_G \alpha\right) p_G^{-D/2} (pd_G p)^{\omega_G}\right\} \quad (3.39)$$

where $\mathscr{R}$ is given by (3.17)

The result of theorem 3.1 is clearly close to Lemma 2.1. But only connected dominants are present in the decomposition. We shall indicate in the next section how nonconnected dominants have to the introduced to take into account the problem at exceptional momenta.

We finally mention that in (3.36), some leading essentials give a zero contribution, that is to say $R_E = 0$. It is the case of the "essentials with inactive parts" (Definition 3.4) which may be proved not to contribute in the decomposition (3.36).

C. The Case of Exceptional momenta

However, we have not yet extracted all I.R. divergences. Indeed, the $R_G(p)$'s are well-defined functions of p at nonexceptional momenta, but they diverge at exceptional momenta, and so are not well-defined distributions of the p's. This difficulty occurs with the bare propagator:

The propagator $1/p^2$ is not a distribution at $D \leq 2$, since the integral $\int d^D p \frac{1}{p^2}$ is not convergent at zero. As already mentioned in [8, 9], to obtain the well defined distribution $Fp\frac{1}{p^2}$ (Finite part of $\frac{1}{p^2}$, see [17]) we have to subtract a divergent term at $p^2 = 0$, since $Fp\left(\frac{1}{p^2}\right)$ is defined as

$$Fp\frac{1}{p^2} = \lim_{H\to 0}\frac{1}{p^2+H} - \delta(p)\int d^Dk\frac{1}{k^2+H}. \tag{3.40}$$

This problem is not an academic one, since we have to define I.R. finite distributions in impulsion space if we want, for instance, to obtain a finite function in position space by the Fourier transform. So we have to analyze the behaviour of the functions $R_G(p)$ as the impulsions p tend toward some exceptional impulsions. This analysis needs different methods, since the methods used in 3. A only allow us to analyze the I.R. limit at fixed momenta. We shall not give a complete proof in the general case but shall discuss here two points:

—First the extraction of the singular part may be performed very easily for graphs of the 2-point function.

—Second we shall give arguments for the general form of decomposition which leads to Lemma 2.1.

Let us first look at the two-point function. By homogeneity, the function $R_E(p)$ relative to any leading essential E depends on the external impulsion p by a power

$$R_E(p) = \text{const}\,[p^2]^{-1-\varepsilon L(E)/2} \tag{3.41}$$

and is not a distribution. The finite part of $R_E(p)$ is defined as for the propagator by

$$F_E(p) = R_E(p) - \delta(p)\int d^DkR_E(k). \tag{3.42}$$

We incorporate this equation in the asymptotic expansion (3.36) of $I_G(p, H)$ (G being a graph of the two point function) and get

$$I_G(p, H) = \delta(p)\int d^DkI_G(k, H) + \sum_{\substack{\text{lead ess.}\\ E\subseteq G}} F_E(p)I_{[\widetilde{G/E}]}(H) + \ldots \tag{3.43}$$

The expansion is now made in terms of well defined distributions, the new term may be rewritten

$$\delta(p)\int d^DkI_G(k, H) = F_{S_0}(p)I_{[\widetilde{G/S_0}]}(H) \tag{3.44}$$

where $\delta(p) = F_{S_0}(p)$ is the contribution (I.R. finite) of the graph S_0 which contains only the two external vertices and where $[\widetilde{G/S_0}]$ is the graph obtained from G by reducing to a point S_0 (Fig. 3).

We now generalize this result. It follows from the general arguments of (A) that, at exceptional momenta, the essentials which give the leading I. R. singularities are not the "essential leadings" described in (B). More precisely, any set of exceptional momenta is characterized by a (unique) partition of the external vertices,

$$\{1 \ldots N\} = I_1 \cup \ldots \cup I_p \qquad (p \geqq 1) \tag{3.45}$$

such that the subfamily of external momenta attached to any element of the partition I_q has its sum equal to zero and any partial sum nonzero. Then the analysis of (B) may be performed.

The leading subgraphs are then subgraphs E of G with p disconnected parts $E_1 \dots E_p$, each E_q being a "leading essential" for the set of momenta relative to I_q, that is to say (see Fig. 6):

—E_q is connected and contains the vertices of I_q.
—No interaction line of $[G/E]$ is attached to E.

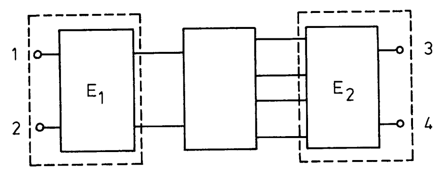

Fig. 6. Example of a leading essential $E = E_1 \cup E_2$ of a 4-point graph at exceptional moment $p_1 + p_2 = 0$

One may obtain an asymptotic expansion of the infrared divergences at exceptional momenta analog to (3.36), where the leading subgraphs previously defined play the role of the leading essential in (3.36). Those infrared divergences are of course more important than at nonexceptional momenta, since the amplitudes diverge at least as rapidly as a negative integer power of H. We expect that a quite similar expansion is obtained for $R_G(p)$ when the momenta p (nonexceptional) tend toward exceptional ones, and that, taking into account all cases of exceptional momenta, the I.R. distribution "Finite part" of $R_G(p)$ may be expressed in terms of all possible leading essentials of G that is precisely all "dominant subgraphs" of G (see Definition 2.1), and of I.R. divergent parts relative to the corresponding reduced graphs, as done for the two-point function. As explained before, a complete discussion of this problem should be much more cumbersome than the analysis of (B) and shall not be presented here. We present only the expected result, which appears to be the natural generalization of Theorem 3.1, where all possible leading essentials (in all cases of external momenta) are present.

The regularized amplitude of a graph G has the following I.R. asymptotic expansion in terms of finite distributions (in momentum space) relative to the dominant subgraphs of G:

$$I_G(p,H) = \sum_{\substack{E \subseteq G \\ \text{Dominant} \\ \text{Subgraphs}}} F_E(p) I_{[\widetilde{G/E}]}(H) + O(H^{1-\varepsilon L(G)/2}). \tag{3.46}$$

In (3.46) the sum runs over all dominant subgraphs E of G. $F_E(p)$ is a finite distribution of the external momenta, which is in fact the "finite part" of the function $R_E(p)$. $I_{[\widetilde{G/E}]}(H)$ is the amplitude of the graph obtained by shrinking the graph E to one point. As in Theorem 3.1, this term diverges as a pure power of $H^{-\varepsilon/2}$, namely $H^{-(\varepsilon/2)L(\widetilde{G/E})}$ (except for the graph G itself).

Equation (3.46) coincides with the Theorem 2.1 at nonexceptional momenta. Indeed, in that case, only connected leading essentials give a contribution and the distribution $F_E(p)$ coincides with the function $R_E(p)$.

This result is also in agreement with the decomposition of the two-point function given in (3.43). Indeed, at $p=0$, the only dominant subgraph which has to be taken into account is the graph S_0 (composed only of the two external vertices). The other dominant subgraphs necessarily have an inactive part (which is the subgraph itself) and so give a zero contribution, from the remark that ends part (B).

Of course, we have checked Eq. (3.46) on simple four point graphs. This equation is precisely the result of Lemma 2.1 (expressed in momentum space instead of position space).

4. Conclusion

In this paper we have presented a general proof of the perturbative I.R. finiteness of the vacuum expectation value of any $O(N)$ invariant function of the $O(N)$ nonlinear σ-model. The proof was performed by using general methods of extracting the I.R. divergences of *any* amplitude of the perturbative expansion. We recall that the extraction is made in Sect. 3 in the case of nonexceptional momenta and extended (but not completely proved) at exceptional momenta. The general I.R. behaviour allows us to exhibit in Sect. 2 the mechanism of cancellations.

For explicit perturbative computations, we have shown that we may replace the "bare" I.R. divergent amplitudes I_G by I.R. subtracted ones, the R_G given by Eq. (3.39), provided that we deal with invariant quantities. It may be shown that, for non-invariant quantities, this operation is equivalent to an average of the orientation of the symmetry breaking magnetic field H in (2.4). The amplitude of the magnetic field also has to be modified, and the average has to be performed with some weight over the sphere; this weight is related perturbatively to the I.R. counterterms of (2.11) in some complicated way, and so diverges as the symmetry breaking term tends towards zero.

The result of this paper is that, as expected, the perturbative expansion of any invariant function of the $O(N)$ two dimensional σ model is free of I.R. divergences although computed in the wrong phase where the symmetry is spontaneously broken. However, as argued in [8], the fact that the symmetry is dynamically restored so that there are no more long distance correlations, may not be seen at any order of the perturbative expansion (where there are always such correlations), but only by dealing with the full Green's functions.

Let us finally mention that the analysis of I.R. divergences at two dimensions presented here may be extended to other two-dimensional models, for instance the chiral models or the generalized σ-models [18], where such cancellations of I.R. divergences are also expected to occur.

Appendix A

Given a graph G belonging to $\mathcal{G}_N^0(\mathcal{O})$ and a dominant subgraph E in G, E belongs to some $\mathcal{G}_M^P(\mathcal{O})$ ($M \leqq N, P \leqq M$) where M is the order of E and $2P$ is the number of lines of $G-E$ attached to E. Similarly, let us consider the graph $G-E$ as a graph of order $(N-M)$ of the perturbative expansion of the operator:

$$\frac{1}{2P!}\left[\int d^D x \vec{\pi}.\right]^{2P}.$$

The counting factor $C(G)$ is the number of contractions which leads to G by applying Wick theorem in the expansion of $\langle \mathcal{O} \rangle$. Separating those contractions into contractions which lead to E plus contractions which lead to $G-E$ plus contractions between E and $G-E$ which lead to G we get the following relation between factors of G, E and $G-E$.

$$n_1 C(G) = n_2 n_3 C(E) C(G-E), \tag{A.1}$$

where n_1, n_2 and n_3 are defined as:

n_1 = number of ways to decompose G into E and $G-E$.

n_2 = number of ways to link the $2p$ lines of $G-E$ to E in order to reobtain G.

$n_3 = (N-1)^q$ where q is the number of lines carrying the $(N-1)$ internal indices of the field $\vec{\pi}$ closed by the former operation which get G from $G-E$ and E.

n_1 and n_2 are factors coming from Wick theorem, n_3 is a symmetry factor depending on the group $O(N)$.

Now, from (A.1) we may reorganize the sum over the G in (2.11) as a sum over the dominant subgraphs E times a sum over the graphs $G-E$, where we link all "free" lines of $G-E$ to a point in all the possible ways. We then obtain the reduced graphs $[\widetilde{G/E}]$ as graphs of the perturbative expansion of $(\vec{\pi}^2(0))^p$, with the corresponding counting factor. We thus obtain

$$\sum_{\mathcal{G}_N^0(\mathcal{O})} C(G) F_G = \sum_{\mathcal{G}_N^0(\mathcal{O})} C(G) I_G + \sum_{\substack{0 \leq M \leq N \\ 0 \leq P \leq M}} \left(\sum_{\mathcal{G}_M^P(\mathcal{O})} C(E) I_E \right) \left(\sum_{\mathcal{G}_{M-N}^0(\vec{\pi}^{2P})} C(S) A_S \right) \tag{A.2}$$

which gives (2.24).

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