Commun. Math. Phys. 81, 39–88 (1981)

Communications in
Mathematical
Physics

Absolutely Continuous Invariant Measures for One-Parameter Families of One-Dimensional Maps

M. V. Jakobson

Central Scientific-Research Economic Institute, Smolenskii Boulevard, Moscow G-117, USSR

Abstract. Given a one-parameter family $f_\lambda(x)$ of maps of the interval [0, 1], we consider the set of parameter values λ for which f_λ has an invariant measure absolutely continuous with respect to Lebesgue measure. We show that this set has positive measure, for two classes of maps: i) $f_\lambda(x)=\lambda f(x)$ where $0<\lambda\leqq 4$ and $f(x)$ is a function C^3-near the quadratic map $x(1-x)$, and ii) $f_\lambda(x)=\lambda f(x)$ (mod 1) where f is C^3, $f(0)=f(1)=0$ and f has a unique nondegenerate critical point in [0, 1].

0. Introduction

Dynamical systems generated by noninvertible maps of an interval into itself have been intensely studied recently. The most widely considered was the family $f_\lambda: x\to\lambda x(1-x)$, $x\in[0,1]$, $0\leqq\lambda\leqq 4$.

It is well-known that if f_λ has an attracting periodic orbit $\bar\alpha=(\alpha_1,\ldots,\alpha_n)$ then all probabilisitic f_λ-invariant measures are singular with respect to a Lebesgue measure dx, and the iterations $f^n_{\lambda *}dx$ converge in the weak $*$-topology to the discrete invariant measure supported by $\bar\alpha$.

It is probable (but not proved) that this situation is typical from the topological point of view, i.e. for a general one-parameter family of smooth mappings $f_\lambda: I\to I$, $\lambda\in\Lambda$, there is an open and dense subset Λ_0 of Λ such that for $\lambda\in\Lambda_0$, the set of limit points for $f^n_{\lambda *}dx$ consists of a finite number of measures supported by periodic attracting orbits.

We show in the present paper that this is not so from the metric point of view. Namely we prove for a certain class of one-parameter families f_λ that the set

$$\Lambda_1=\{\lambda: f_\lambda \text{ has an invariant finite measure } \mu_\lambda \text{ absolutely continuous with respect to } dx\ (\mu_\lambda < dx)\}$$

has a positive measure in Λ.

In the classical case $x\to 4x(1-x)$ considered by Ulam and von Neumann in [1], the invariant measure $\mu(dx)$ has density $\varrho(x)=\dfrac{1}{\pi\sqrt{x(1-x)}}$. In [2] Bunimovič

0010-3616/81/0081/0039/$10.00

constructed absolutely continuous measures for the piecewise smooth mappings $x \to n \sin \pi x \pmod 1$, $n \in \mathbb{Z}$. Ruelle in [3] considered $f_\lambda : x \to \lambda x(1-x)$ and proved that an invariant measure $\mu_\lambda < dx$ exists for $\lambda = 3{,}678\ldots$ – chosen in such a way that the third iterate of the critical point, $f_\lambda^3(\frac{1}{2})$, falls into the unstable fixed point $x = 1 - \frac{1}{\lambda}$. Bowen in [4] found sufficient conditions for the existence of an invariant measure $\mu_\lambda < dx$ for $f_\lambda(x) = \lambda x(1-x)$, when $\frac{1}{2}$ is a preimage of a periodic unstable point. In [5] it was shown that the cardinality of $\{\lambda : f_\lambda$ has an invariant measure $\mu_\lambda < dx\}$ is that of the continuum for the family $x \to \lambda x(1-x)$ and any C^2-family f_λ sufficiently close to $\lambda x(1-x)$. Similar results were obtained by Misiurewicz [6] and Szlenk [7] for a class of mappings with negative Schwarzian derivative. Ognev in [8] proved for $x \to \lambda x(1-x)$ that if $\frac{1}{2}$ is a preimage of a periodic unstable point, then the density of the invariant measure is analytic. Ito, Tanaka, Nakada in [9] studied the space of parameters of unimodal linear transformations and found explicitly the densities of the invariant measures.

Collet and Eckmann in [10] proved for a particular family $f_\delta(x)$ that f_δ has sensitive dependence with respect to initial conditions in the sense of Guckenheimer [11] for a set of δ of positive measure. The mappings f_γ obtained with our construction are also sensitive dependent. It is unknown whether sensitive dependence implies existence of absolutely continuous invariant measure.

We shall consider two kinds of one-parameter families $f_\lambda(x)$.

1. Piecewise smooth families $x \mapsto \lambda f(x) \pmod 1$, where $f(x) : [0,1] \to [0,1]$ is a C^3-map with a single nondegenerate critical point, $f(0) = f(1) = 0$, and λ is a big parameter.

2. Smooth families $x \mapsto \lambda x(1-x) 0 \leqq \lambda \leqq 4$, and $\lambda \cdot f(x)$ with $f(x)$ sufficiently close to $x(1-x)$ in $C^3([0,1],[0,1])$.

We formulate now our main results.

Theorem A. *Let $f_\lambda : x \to \lambda f(x) \pmod 1$ be a piecewise smooth family. There exists $T_0 > 0$, such that for any $\varepsilon > 0$ there is an $L(\varepsilon)$, so that if $L \geqq L(\varepsilon)$ then the interval $[L, L+T_0]$ on the λ-axis contains a set $\mathcal{M}$ satisfying*

i) $\operatorname{mes} \mathcal{M} > T_0 - \varepsilon$;

ii) $\forall \lambda \in \mathcal{M}$ *f_λ admits an invariant measure $\mu_\lambda < dx$.*

Theorem B. *Let $f_\lambda(x)$ be one of the smooth families mentioned above. Then there is a set of positive measure A_1 so that for $\lambda \in A_1$ f_λ admits an invariant measure $\mu_\lambda < dx$.*

Remark. The parameter values λ_1 such that the critical point of f_{λ_1} is contained in the preimage of an unstable periodic orbit (e.g. $\lambda_1 = 4$ for $\lambda \cdot x(1-x)$, or in the preimage of a certain invariant unstable Cantor set (see [5])) turn out to be one-sided Lebesgue points of A_1, i.e. $\forall \varepsilon > 0 \exists \delta > 0$, such that

$$\operatorname{mes}\{\lambda \in A_1 : \lambda_1 \geqq \lambda \geqq \lambda_1 - \delta\} > \delta(1-\varepsilon).$$

In Sects. 1–12 we prove Theorem A for the family $x \to \lambda x(1-x) \pmod 1$. In Sect. 13 we point out modifications concerning the case of an arbitrary family $x \to \lambda \cdot f(x) \pmod 1$ and show how to reduce the proof of Theorem B to the proof of Theorem A.

1. Idea of Proof

The number T_0 for the family $f_\lambda: x \to \lambda x(1-x)$ (mod 1) equals 4: as λ varies from L to $L+4$, the image of the critical point $f_\lambda(\frac{1}{2}) = \frac{\lambda}{4}$ (mod 1) passes over the entire interval [0, 1]). In order to prove Theorem A we must find for a given $\varepsilon > 0$ an $L(\varepsilon)$ such that, if $L \geqq L(\varepsilon)$ then the interval $[L, L+4]$ contains a set $\mathcal{M}$ so that $\operatorname{mes}\mathcal{M} > 4 - \varepsilon$ and for any $\lambda \in \mathcal{M} f_\lambda$ has an invariant measure $\mu_\lambda < dx$. Without loss of generality we can assume that λ varies from $N_0 = 4k_0$ to $N_0 + 4$, $k_0 \in \mathbb{Z}_+$. For a smooth map $g(\lambda, x)$ we shall use the notation Dg, $D^2 g$ for $\frac{\partial g(\lambda, x)}{\partial x}, \frac{\partial^2 g(\lambda, x)}{\partial x^2}$.

The central part of the proof of Theorem A is the construction for $\lambda \in \mathcal{M}$ of a special partition ξ_λ of [0, 1]. The elements of ξ_λ are intervals $\Delta_i(\lambda)$, $i \in \mathbb{Z}_+$, which satisfy the following conditions:

i) $\operatorname{int}\Delta_i(\lambda) \cap \operatorname{int}\Delta_j(\lambda)) = \emptyset$.

ii) $\forall i \exists n_i \in \mathbb{Z}_+$ such that $f_\lambda^{n_i}$ maps $\Delta_i(\lambda)$ diffeomorphically onto [0, 1].

iii) $\inf_{\Delta_i \in \xi_\lambda} \min_{x \in \Delta_i} |Df_\lambda^{n_i}(x)| > \lambda^{c_0}$ for some $c_0 > 0$ (λ is a big parameter here, so $\lambda \gg 0$).

iv) $\sup_{\Delta_i \in \xi_\lambda} \max_{x \in \Delta_i} \left|\frac{D^2 f_\lambda^{n_i}(x)}{Df_\lambda^{n_i}(x)}\right| \cdot |\Delta_i(\lambda)| < 1 + \lambda^{-t_1}$, for some $t_1 > 0$.

Let $\mathcal{X}(\lambda)$ be the union of all elements $\Delta_i(\lambda)$ of ξ_λ. Then $\mathcal{X}(\lambda) = [0, 1]$ (mod 0).

The set $\mathcal{M}$ and the sets $\mathcal{X}(\lambda)$ for $\lambda \in \mathcal{M}$ are constructed by induction. $\mathcal{M}$ is obtained as an intersection $\mathcal{M} = \bigcap_{n=0}^{\infty} \mathcal{M}_n$, where

$$\mathcal{M}_0 = [N_0, N_0 + 4], \qquad \mathcal{M}_{n+1} \subset \mathcal{M}_n,$$

$$\operatorname{mes}\mathcal{M}_{n+1} > (1 - \varepsilon_{n+1}) \operatorname{mes}\mathcal{M}_n, \qquad \sum_{n=1}^{\infty} \varepsilon_n = O(\lambda^{-t_2}), \qquad t_2 > 0.$$

At the nth induction step, we define for any $\lambda \in \mathcal{M}_{n-1}$ a set $\mathcal{X}_n(\lambda) \subset [0, 1]$ which is the union of a countable number of intervals $\Delta_i^{(k)}(\lambda)$, $k = 1, \ldots, n$. The intervals constructed at step k do not change at the next steps. The sets $\mathcal{X}_n(\lambda)$ satisfy the following properties:

$$\mathcal{X}_n(\lambda) \subset \mathcal{X}_{n+1}(\lambda); \qquad \operatorname{mes}\mathcal{X}_n(\lambda) > 1 - \lambda^{-t_3 n}, \qquad t_3 > 0.$$

Finally we set $\mathcal{X}(\lambda) = \bigcup_{n=1}^{\infty} \mathcal{X}_n(\lambda)$. Any element $\Delta_i(\lambda)$ of ξ_λ coincides with one of $\Delta_i^{(n)}(\lambda)$.

Let us define the map $T_\lambda: \mathcal{X}(\lambda) \to [0, 1]$ by $T_\lambda | \Delta_i(\lambda) = f_\lambda^{n_i}$. The results of Adler [12] and Walters [13] imply the existence and the uniqueness of a T_λ-invariant measure $\nu_\lambda < dx$. The endomorphism ([0, 1], T_λ, ν_λ) is exact, and its natural extension is a Bernoulli shift. The f_λ-invariant measure μ_λ is constructed from ν_λ.

2. First Steps of the Inductive Construction

The graph of the map f_λ consists of a lot of monotone branches which we denote by $f(\lambda, x)$ and the middle parabola denoted by $h(\lambda, x)$. The domains of $f(\lambda, x)$ and

$h(\lambda, x)$ depend continuously on λ. When $\lambda = 4k_0$, a new middle branch is born, which exists for $\lambda \leqq 4(k_0+1)$ and then breaks up into two monotone branches.

We shall denote by $\Delta f(\lambda, x)$ the domain of $f(\lambda, x)$, by $x_{\min}(\lambda)$ the endpoint nearest to $\frac{1}{2}$ of the interval $\Delta f(\lambda, x)$, and by $x_{\max}(\lambda)$ the other endpoint of $\Delta f(\lambda, x)$. We shall distinguish $[a, b]$ from $[b, a]$ according to its position relative to $\frac{1}{2}$ and not according to its orientation.

We fix a positive number $s < \frac{1}{13}$.

Step 1

Pick the branch $f'(\lambda, x)$ of f_λ whose domain $\Delta f'(\lambda, x) = \Delta'(\lambda) = [x'_{\min}(\lambda), x'_{\max}(\lambda)]$ is contained in $[0, \frac{1}{2}]$ and is closest to $\frac{1}{2}$, subject to the condition

$$|x'_{\min}(\lambda) - \tfrac{1}{2}| > \lambda^{-s} \quad \text{for all } \lambda \in \mathscr{M}_0 .$$

Denote by $\Delta''(\lambda) = \Delta f''(\lambda, x)$ the analogous interval in $[\frac{1}{2}, 1]$. Define $\delta_1(\lambda) = [x'_{\min}(\lambda), x''_{\min}(\lambda)]$, noting that $\delta_1(\lambda)$ has the form

$$\delta_1(\lambda) = [\tfrac{1}{2} - r_1(\lambda), \tfrac{1}{2} + r_1(\lambda)], \qquad r_1(\lambda) > \lambda^{-s} \tag{2.1}$$

and let $\mathscr{X}_1(\lambda) = [0, 1] \backslash \delta_1(\lambda)$. Thus,

$$[0, 1] = \mathscr{X}_1(\lambda) \bigcup \delta_1(\lambda) .$$

Both $\mathscr{X}_1(\lambda)$ and $\delta_1(\lambda)$ are the union of several domains of branches, $\Delta f(\lambda, x)$, varying continuously with λ.

Since

$$|Df(\lambda, x)| = 2\lambda |x - \tfrac{1}{2}| ,$$

we have

$$|\Delta'(\lambda)| < \tfrac{1}{2}\lambda^{-1+s}$$

and

$$\left| \frac{dx'_{\min}(\lambda)}{d\lambda} \right| = \left| \frac{\partial f(\lambda, x)/\partial \lambda}{\partial f(\lambda, x)/\partial x} \right|_{x = x'_{\min}(\lambda)} < \frac{1}{8\lambda^{1-s}} .$$

This implies

$$\frac{1}{\lambda^s} < r_1(\lambda) < \frac{1}{\lambda^s} + \frac{1}{\lambda^{1-s}} = \frac{1}{\lambda^s}\left[1 + \frac{1}{\lambda^{1-2s}}\right] . \tag{2.1a}$$

In order to construct the set $\mathscr{M}_1$ we consider the domains $\Delta f(\lambda, x) = [x_{\min}(\lambda), x_{\max}(\lambda)]$ satisfying

$$|x_{\min}(\lambda) - \tfrac{1}{2}| > 1/\lambda^{s/2} .$$

We obtain as above that for any such domain

$$|\Delta f(\lambda, x)| < \tfrac{1}{2}\lambda^{-1+s/2}$$

$$\left| \frac{dx_{\min}(\lambda)}{d\lambda} \right| < \tfrac{1}{8}\lambda^{-1+s/2} .$$

The top of the graph, $h(\lambda, \frac{1}{2})$, moves with velocity

$$\frac{dh(\lambda,\frac{1}{2})}{d\lambda}=\frac{1}{4}. \tag{2.1b}$$

A comparison of velocities shows that to each branch $f_i(\lambda, x)$ with domain $\Delta_i(\lambda)$ there corresponds a uniquely defined interval $\mathscr{I}_i=\mathscr{I}(\Delta_i)$ of λ-values such that, as λ ranges over $\mathscr{I}_i$, the top $h(\lambda,\frac{1}{2})$ ranges over $\Delta_i(\lambda)$ and its image $f_i(\lambda, h(\lambda,\frac{1}{2}))$ ranges over $[0,1]$.

So we define $\mathscr{M}_1$ as the union of these $\mathscr{I}_i$:

$$\mathscr{M}_1=\bigcup\{\mathscr{I}_i=\mathscr{I}(\Delta_i)|(\forall\lambda\in\mathscr{M}_0)|x_{\min}(\lambda)-\tfrac{1}{2}|>1/\lambda^{s/2}\}.$$

It follows from the estimates (2.1), (2.1a) and (2.1b) that

$$\operatorname{mes}\mathscr{M}_1>4\left[1-\max_{N_0\leqq\lambda\leqq N_0+4}\operatorname{mes}\mathscr{X}_1(\lambda)\right]>4\left[1-\frac{2(1+\gamma_1)}{N_0^{s/2}}\right], \tag{2.2}$$

where

$$\gamma_1<1/N_0^{1-s}.$$

Step 2. Construction of $\mathscr{X}_2(\lambda)$

Let us denote by f_1 the branches $f(\lambda,x)$ such that $\Delta f\subset\mathscr{X}_1(\lambda)$ and by g the branches with $\Delta g\subset\delta_1(\lambda)$. Let us consider compositions $f_1\circ g$. Any domain Δg can be represented in the form

$$\Delta g=\bigcup\Delta(f_1\circ g)\cup\bigcup g^{-1}(\delta_1). \tag{2.3}$$

Choose an interval

$$\delta_2(\lambda)=\left[\frac{1}{2}-\frac{c_{21}}{\lambda^{2s}},\frac{1}{2}+\frac{c_{22}}{\lambda^{2s}}\right],\qquad 1<c_{21},c_{22}<1+O(1/\lambda^{1-3s})$$

which is a union of domains $\Delta(f_1\circ g)$ and $g^{-1}\delta_1$. We shall use g_1 to denote $g|\delta_1\backslash\delta_2$ and f_{21} to denote $f_1\circ g_1$. Then (2.3) implies

$$\delta_1=\bigcup\Delta f_{21}\cup\bigcup g_1^{-1}\delta_1\cup\delta_2. \tag{2.4}$$

For any particular branch $\tilde{g}_1$ we have

$$\tilde{g}_1^{-1}(\delta_1)=\bigcup\tilde{g}_1^{-1}(\Delta f_{21})\cup\bigcup\tilde{g}_1^{-1}\circ g_1^{-1}(\delta_1)\cup\tilde{g}_1^{-1}\delta_2,$$

where the large unions are over all f_{21} and g_1 respectively. Denote the branches $f_{21}\circ g_1$ by f_{22}. Since $\Delta(f_{21}\circ g_1)=g_1^{-1}(\Delta f_{21})$, we can rewrite (2.4) as

$$\delta_1=\bigcup\Delta f_{21}\cup\bigcup\Delta f_{22}\cup\bigcup g_1^{-2}(\delta_1)\cup\bigcup g_1^{-1}(\delta_2)\cup\delta_2, \tag{2.5}$$

where g_1^{-2} denotes any composition of the form $\tilde{g}_1^{-1}\circ\tilde{\tilde{g}}_1^{-1}$. Proceeding in the same way we obtain the representation

$$\begin{aligned}\delta_1=&\bigcup\Delta f_{21}\cup\bigcup\Delta f_{22}\cup\ldots\cup\bigcup\Delta f_{2k}\cup\bigcup g_1^{-(k-1)}(\delta_2)\\&\cdot\cup\ldots\cup\bigcup g_1^{-1}(\delta_2)\cup\delta_2\cup\bigcup g_1^{-k}\delta_1,\end{aligned} \tag{2.6}$$

where

$$f_{2\ell}=f_{21}\circ g_{1i_1}\circ\dots\circ g_{1i_{\ell-1}}$$
$$g_1^{-r}=g_{1i_r}^{-1}\circ\dots\circ g_{1i_1}^{-1}.$$

Any branch g_1 satisfies

$$|Dg_1|>2\lambda^{1-2s}$$
$$|D^2g_1|=2\lambda \tag{2.7}$$

from which it follows (see for example [11]) that

$$\lim_{k\to\infty}\operatorname{mes}[\bigcup g_1^{-k}(\delta_1)]=0.$$

Therefore, we can write

$$\delta_1=\bigcup_{k=1}^{\infty}(\Delta f_{2k})\cup\bigcup_{k=1}^{\infty}g_1^{-k}(\delta_2)\cup\delta_2(\operatorname{mod}0), \tag{2.8}$$

where mod 0 means we neglect sets with zero Lebesgue measure. (Hereafter, in analogous equalities, "mod 0" will be understood.) Using the notation f_2 for all the f_{2k}, $k=1,2,\dots$, we obtain

$$[0,1]=\bigcup\Delta f_1\cup\bigcup\Delta f_2\cup\bigcup_{k=1}^{\infty}g_1^{-k}(\delta_2)\cup\delta_2 \tag{2.9}$$

or

$$[0,1]=\mathscr{X}_2(\lambda)\cup\bigcup_{k=1}^{\infty}g_1^{-k}(\delta_2)\cup\delta_2, \tag{2.10}$$

where by construction $\mathscr{X}_2(\lambda)$ is partitioned by the various domains Δf_1 and Δf_2 constructed in steps 1 and 2. These domains will be elements of the partition ξ_λ.

Now (2.3) and (2.8) induce an analogous structure inside δ_2:

$$\delta_2=\bigcup\Delta(f_1\circ g)\cup\bigcup_{k=1}^{\infty}\Delta(f_{2k}\circ g)\cup\bigcup_{n=0}^{\infty}g^{-1}\circ g_1^{-n}(\delta_2). \tag{2.11}$$

Notice that one of the g's in (2.11) stands for h. Suppose $h(\frac{1}{2})\in\Delta\tilde{f}_1$. Then for any other branch $f_1\neq\tilde{f}_1$ either $f_1\circ h$ has two monotone branches or none; similarly h^{-1} on δ_2 has two or no monotone branches. The only branch of parabolic type in (2.11) is $\tilde{f}_1\circ h$.

We see from (2.10) that $\mathscr{X}_2(\lambda)$ is the complement (mod 0) of the preimages of δ_2 under the various branches $g_1^k(k\geqq 0)$. At the end of the next section, we will see that $\mathscr{M}_2$ is the set of those $\lambda\in\mathscr{M}_1$ for which the appropriate branch f_1 takes the critical value $h(\lambda,\frac{1}{2})$ into the complement of the g_1^k-preimages of an interval $\hat{\delta}_2$ which is also small but much larger than δ_2.

3. Step $n+1$. Geometrical Part

We assume after step n that the set $\mathscr{M}_n$ has been defined and for every $\lambda\in\mathscr{M}_n$ the set $\mathscr{X}_n(\lambda)$ has been constructed. Every $\mathscr{X}_n(\lambda)$ is a countable union of domains $\Delta f_k(\lambda,x)$,

$k=1,2,\ldots,n$, where we use f_k to denote a branch constructed at step k. The interval $[0,1]$ can be represented (mod 0) in the following form:

$$[0,1]=\left[\bigcup_{k=1}^{n}\left(\bigcup \Delta f_k\right)\right]\cup\left[\bigcup_{m=1}^{\infty}\left(\bigcup \delta_n^{-m}\right)\right]\cup\delta_n\,. \tag{3.1}$$

Here the interval

$$\delta_n=\delta_n(\lambda)=\left[\tfrac{1}{2}-\frac{c_{n1}}{\lambda^{sn}},\tfrac{1}{2}+\frac{c_{n2}}{\lambda^{sn}}\right],\qquad 1\leqq c_{n1},c_{n2}\leqq 1+0\left(\frac{1}{\lambda^{tn}}\right),\qquad t=\frac{\alpha}{10},$$

and δ_n^{-m} are various diffeomorphic preimages of δ_n. We shall denote by $G_n:\delta_n^{-m}\to\delta_n$ the corresponding diffeomorphisms without pointing out their dependence on m; if $m=0$, $G_n=\mathrm{Id}$.

In order to describe the representation of δ_n analogous to (3.1) we need some additional notation. Let F_{n-1} be a composition of maps f_k constructed at the previous steps:

$$F_{n-1}=f_{i_{n-1}}\circ f_{i_{n-2}}\circ\ldots\circ f_{i_2}\circ f_{i_1},\qquad i_1=1,\qquad i_2\in[1,2],\ldots,i_{n-1}\in[1,n-1]\,.$$

We shall distinguish two kinds of branches for various powers of f with domains inside δ_n: the first have the form $F_{n-1}\circ g(\lambda,x)$ ($F_{n-1}\circ h(\lambda,x)$ for the central branch) where g denotes the initial map $x\to\lambda x(1-x)$: and the second kind are all the remaining branches, mapping their domains diffeomorphically onto $[0,1]$, and denoted by $\hat{f}_n(\lambda,x)$. So we assume δ_n has the following representation after Step n:

$$\delta_n=\left(\bigcup \Delta F_{n-1}\circ g\right)\cup\left(\bigcup \Delta\hat{f}_n\right)\cup\left[\bigcup_{m=m_n}^{\infty}\left(\bigcup\delta_n^{-m}\right)\right]. \tag{3.2}$$

Now for any $\lambda\in\mathcal{M}_n$ we describe the construction of $\mathcal{X}_{n+1}(\lambda)$. The estimates which allow us to realize this construction are adduced in subsequent sections.

a) We consider the compositions $f_k\circ F_{n-1}\circ g$ and $f_k\circ\hat{f}_n$ for all $f_k(k\in[1,n])$, $F_{n-1}\circ g$, and $\hat{f}_n$. Then the domains $\Delta F_{n-1}\circ g$ and $\Delta\hat{f}_n$ have the following representations

$$\left.\begin{aligned}\Delta F_{n-1}\circ g&=\left[\bigcup_{k=1}^{n}\left(\bigcup\Delta f_k\circ F_{n-1}\circ g\right)\right]\cup\left[\bigcup_{m=0}^{\infty}\left(\bigcup(F_{n-1}\circ g)^{-1}(\delta_n^{-m})\right)\right]\\ \Delta\hat{f}_n&=\left[\bigcup_{k=1}^{n}\left(\bigcup\Delta f_k\circ\hat{f}_n\right)\right]\cup\left[\bigcup_{m=0}^{\infty}\left(\bigcup\hat{f}_n^{-1}(\delta_n^{-m})\right)\right]\end{aligned}\right\}. \tag{3.3}$$

Notice that the representation (3.3) for $\Delta F_{n-1}\circ h$ contains only the members corresponding to Δf_k and δ_n^{-m} which lie in the image of $F_{n-1}\circ h$.

b) In (3.3) some new preimages of δ_n arose, namely $(F_{n-1}\circ g)^{-1}\delta_n^{-m}$ and $\hat{f}_n^{-1}\delta_n^{-m}$. We still denote them δ_n^{-m}, but the corresponding diffeomorphisms $G_n\circ F_{n-1}\circ g$ and $G_n\circ\hat{f}_n$ will be denoted by G_n'. Let us rewrite (3.3) in the form

$$\left.\begin{aligned}\Delta F_{n-1}\circ g&=\left(\bigcup\Delta f_k\circ F_{n-1}\circ g\right)\cup\left(\bigcup\delta_n^{-m}\right)\\ \Delta\hat{f}_n&=\left(\bigcup\Delta f_k\circ\hat{f}_n\right)\cup\left(\bigcup\delta_n^{-m}\right)\end{aligned}\right\}. \tag{3.4}$$

Now we choose an interval $\delta_{n+1}(\lambda)$ composed of whole elements of the partition generated in (3.2) and (3.4):

$$\delta_{n+1}(\lambda)=\left[\frac{1}{2}-\frac{c_{n+1,1}(\lambda)}{\lambda^{s(n+1)}},\frac{1}{2}+\frac{c_{n+1,2}(\lambda)}{\lambda^{s(n+1)}}\right],\qquad 1\leqq c_{n+1,i}\leqq 1+0\left(\frac{1}{\lambda^{t(n+1)}}\right) \tag{3.5}$$

c) We shall distinguish the maps with domains in $\delta_n\backslash\delta_{n+1}$, thus we use some additional notation.

Let $g_n=\lambda x(1-x) \pmod 1)|\delta_n\backslash\delta_{n+1}$. We shall use $f_{n+1\,1}$ to denote the branches $f_k\circ F_{n-1}\circ g_n$ and $f_k\circ\hat{f}_n|\delta_n\backslash\delta_{n+1}$. Finally, we shall use $\tilde{G}_n$ to denote the G_n or G'_n with domain inside $\delta_n\backslash\delta_{n+1}$. Using (3.2) and (3.4) we obtain the following representation of δ_n:

$$\delta_n=(\bigcup\Delta f_{n+1\,1})\cup\left(\bigcup_{m=m_n}^{\infty}\delta_n^{-m}\right)\cup\delta_{n+1}\,. \tag{3.6}$$

Let us define recurrently the branches $f_{n+1\,k}$, $k=2,3,\ldots$. If $\Delta f_{n+1\,k-1}\subset\delta_n\backslash\delta_{n+1}$ and $\tilde{G}_n:\delta_n^{-m}\to\delta_n$, then $f_{n+1\,k}=f_{n+1\,k-1}\circ\tilde{G}_n$. Any branch $f_{n+1\,k}$ maps $\tilde{G}_n^{-1}(\Delta f_{n+1\,k-1})$ onto [0, 1]. For any given $N\in\mathbb{Z}_+$ we can rewrite (3.6) proceeding as in Sect. 2:

$$\delta_n=\left[\bigcup_{k=1}^{N}(\bigcup\Delta f_{n+1\,k})\right]\cup\left(\bigcup_{m=m_n}^{\infty}\delta_{n+1}^{-m}\right)\cup\left(\bigcup_{m=N\cdot m_n}^{\infty}\delta_n^{-m}\right)\cup\delta_{n+1} \tag{3.7}$$

The preimages δ_{n+1}^{-m} and δ_n^{-m} in (3.7) have the form $(\tilde{G}_{n_1}\circ\tilde{G}_{n_2}\circ\ldots\circ\tilde{G}_{n_p})^{-1}\delta_{n+1}$ (respectively δ_n) and the branches $f_{n+1\,k}$ have the form

$$f_{n+1\,k}=f_{n+1\,1}\circ\tilde{G}_{n_1}\circ\tilde{G}_{n_2}\circ\ldots\circ\tilde{G}_{n_p}\,.$$

If $n>I$, there is an infinite number of $\tilde{G}_n$, and there is no uniform estimate $|D^2\tilde{G}_n|<\text{const}$. However using a generalization of one result of [14] (see Lemma 1 below) we obtain

$$\lim_{N\to\infty}\operatorname{mes}\left(\bigcup_{m=N\cdot m_n}^{\infty}\delta_n^{-m}\right)=0\,. \tag{3.8}$$

This implies

$$\delta_n=\left[\bigcup_{k=1}^{\infty}(\bigcup\Delta f_{n+1\,k})\right]\cup\left[\bigcup_{m=m_n}^{\infty}(\bigcup\delta_{n+1}^{-m})\right]\cup\delta_{n+1}\,. \tag{3.9}$$

Apart from $\delta_n^{-m}\subset\delta_n\backslash\delta_{n+1}$ we have $\delta_n^{-m}\subset[0,1]\backslash\delta_n$ and $\delta_n^{-m}\subset\delta_{n+1}$ (domains of G_n and G'_n from (3.1), (3.2), (3.4)). Then (3.9) induces in any such domain $\delta_n^{-m}=G_n^{-1}\delta_n$ the corresponding decomposition

$$\delta_n^{-m}=(\bigcup\Delta f_{n+1\,k}\circ G_n)\cup(\bigcup\delta_{n+1}^{-m})\,, \tag{3.10}$$

where $\delta_{n+1}^{-m}=G_n^{-1}\circ\tilde{G}_{n_p}^{-1}\circ\ldots\circ\tilde{G}_{n_1}^{-1}\delta_{n+1}$.

We shall use $f_{n+1\,k}$ to denote $f_{n+1\,k-1}\circ G_n$ for any G_n with domain $\delta_n^{-m}\subset[0,1]\backslash\delta_n$; f_{n+1} to denote $f_{n+1\,k}$ for any k; F_n to denote $f_k\circ F_{n-1}$; $\hat{f}_{n+1}$ to denote $f_k\circ\hat{f}_n$ for $\hat{f}_n$ such that $\Delta\hat{f}_n\subset\delta_{n+1}$, and also $\hat{f}_{n+1}$ to denote $f_{n+1}\circ G_n$ and

$f_{n+1}\circ G'_n$ with ΔG_n (respectively $\Delta G'_n)\subset\delta_{n+1}$; G_{n+1} to denote any composition of the form $\tilde{G}_{n_1}\circ\tilde{G}_{n_2}\circ\ldots\circ\tilde{G}_{n_p}|\delta^{-m}_{n+1}$, or $\tilde{G}_{n_1}\circ\ldots\circ\tilde{G}_{n_p}\circ G_n|\delta^{-m}_{n+1}$ or $\tilde{G}_{n_1}\circ\ldots\circ\tilde{G}_{n_p}\circ G'_n|\delta^{-m}_{n+1}$.

With these notations we have:

$$[0,1]=\left[\bigcup_{k=1}^{n+1}\left(\bigcup\Delta f_k\right)\right]\cup\left[\bigcup_{m=1}^{\infty}\left(\bigcup\delta^{-m}_{n+1}\right)\right]\cup\delta_{n+1} \tag{3.11}$$

and

$$\delta_{n+1}=\left(\bigcup\Delta F_n\circ g\right)\cup\left(\bigcup\Delta\hat{f}_{n+1}\right)\cup\left[\bigcup_{m=m_{n+1}}^{\infty}\left(\bigcup\delta^{-m}_{n+1}\right)\right]. \tag{3.12}$$

(3.11) and (3.12) correspond to (3.1) and (3.2) with n replaced by $n+1$. So we have described Step $n+1$ on the x-axis for any $\lambda\in\mathcal{M}_n$.

d) According to the induction hypothesis $\mathcal{M}_n$ is the union of a countable set of closed intervals with disjoint interiors and some set $\mathcal{F}_n$ consisting of limit points of such intervals.

$$\mathcal{M}_n=\left(\bigcup\mathcal{I}_n\right)\bigcup\mathcal{F}_n.$$

We assume inductively that $\mathcal{F}_n\subset\mathcal{M}$, and define $\mathcal{M}_{n+1}\cap\mathcal{I}_n$ for all $\mathcal{I}_n$. We fix some positive $\alpha\leqq s/4$. As λ varies over $\mathcal{I}_n$, the top of the central branch $F_{n-1}\circ h(\lambda,\frac{1}{2})$ varies over some Δf_{k_0} and $f_{k_0}\circ F_{n-1}\circ h(\lambda,\frac{1}{2})$ varies over $[0,1]$. Moreover when λ varies in $\mathcal{I}_n$ all the maps $F, G, f, \hat{f}$ constructed at previous steps vary continuously. Let $\mathcal{I}'_n$ be one of these components of $\mathcal{M}_n$. In order to construct the set $\mathcal{M}_{n+1}\cap\mathcal{I}'_n$ we shall point out the admissible positions for the top $f_{k_0}\circ F_{n-1}\circ h(\lambda,\frac{1}{2})$. Let $\mathcal{I}'_n=[a_n,b_n]$. When constructing $\delta_{n+1}(\lambda)$, we shall choose it varying continuously when $\lambda\in\mathcal{I}'_n$ and still satisfying (3.5). Then we shall expand $\delta_{n+1}(\lambda)$ almost homothetically and obtain an interval $\hat{\delta}_{n+1}(\lambda)$ varying continuously with $\lambda\in\mathcal{I}'_n$, composed of whole domains Δf_k and δ^{-m}_{n+1} and satisfying for $\lambda\in\mathcal{I}'_n$ the following

$$\lambda^{\alpha(n+1)}|\delta_{n+1}(\lambda)|\leqq|\hat{\delta}_{n+2}(\lambda)|\leqq\lambda^{\alpha(n+1)}\left(1+0\left(\frac{1}{\lambda^{t(n+1)}}\right)\right)|\delta_{n+1}(\lambda)| \tag{3.13}$$

For any preimage $\delta^{-m}_{n+1}=\mathrm{G}^{-1}_{n+1}\,\delta_{n+1}\subset[0,1]\backslash\delta_{n+1}$ the corresponding domain $\hat{\delta}^{-m}_{n+1}=\mathrm{G}^{-1}_{n+1}\hat{\delta}_{n+1}$ turns out to be defined and the lengths of δ^{-m}_{n+1} and $\hat{\delta}^{-m}_{n+1}$ are still related by (3.13). Then we define

$$\mathcal{M}_{n+1}\cap\mathcal{I}_n=\left\{\lambda: f_{k_0}\circ F_{n-1}\circ h(\lambda,\tfrac{1}{2})\in[0,1]\Big\backslash\bigcup_m\bigcup\hat{\delta}^{-m}_{n+1}(\lambda)\right\}.$$

The condition $f_{k_0}\circ F_{n-1}\circ h(\lambda,\frac{1}{2})\in\hat{\delta}^{-m}_{n+1}$ defines an interval in $\mathcal{I}_n$. Thus $\mathcal{M}_{n+1}\cap\mathcal{I}_n$ is the complement of the union of these intervals. $\mathcal{M}_{n+1}\cap\mathcal{I}_n$ consists of intervals $\mathcal{I}'_{nk}=\{\lambda: f_{k_0}\circ F_{n-1}\circ h(\lambda,\frac{1}{2})\in\Delta f_k(\lambda)\}$ and of a limit set $\mathcal{F}_{n+1}(\mathcal{I}_n)$. As λ varies over $\mathcal{I}'_{nk}$, $f_k\circ f_{k_0}\circ F_{n-1}\circ h(\lambda,\frac{1}{2})$ varies over $[0,1]$.

So we have

$$\mathcal{M}_{n+1}\cap\mathcal{I}_n=\left(\bigcup_k\mathcal{I}'_{nk}\right)\cup\mathcal{F}_{n+1}(\mathcal{I}_n) \tag{3.14}$$

and finally

$$\mathcal{M}_{n+1}=\left(\bigcup_{\mathcal{I}_n}(\mathcal{M}_{n+1}\cap\mathcal{I}_n)\right)\cup\mathcal{F}_n. \tag{3.15}$$

4. Estimates for Fluctuation of Derivative

Let $f: \Delta \to I$ be a C^2-diffeomorphism of some closed interval. Then by differentiating $\log|Df(z)|$, we see that

$$\max_{x,y\in\Delta}\left|\frac{Df(x)}{Df(y)}\right| \leqq \exp\left(\max_{z\in\Delta}\left|\frac{D^2f(z)}{Df(z)}\right|\cdot|\Delta|\right). \tag{4.1}$$

We shall use the notation $\mu(f,\Delta)=\max_{x\in\Delta}\left|\frac{D^2f(x)}{Df(x)}\right|\cdot|\Delta|$ and when there is no doubt about the domain of f, we shall often write $\mu(f)$. Let $f_1:\Delta_1 \xrightarrow{\text{onto}} I$, $f_2:\Delta_2 \xrightarrow{\text{onto}} J \supset \Delta_1$ be as above, $\Delta_{12}=f_2^{-1}\Delta_1 \subset \Delta_2$. Then $f_1\circ f_2(\Delta_{12})=I$. Using the mean value theorem and (4.1) we obtain

$$\begin{aligned}
\mu(f_1\circ f_2,\Delta_{12}) &= \max_{x\in\Delta_{12}}\left|\frac{D^2(f_1\circ f_2)(x)}{D(f_1\circ f_2)(x)}\right||\Delta_{12}| \\
&= \max_{x\in\Delta_{12}}\left|\frac{D^2f_1(f_2(x))\cdot[Df_2(x)]^2+Df_1(f_2(x))\cdot D^2f_2(x)}{Df_1(f_2(x))\cdot Df_2(x)}\right||\Delta_{12}| \\
&\leqq \left[\max_{y\in\Delta_1}\left|\frac{D^2f_1(y)}{Df_1(y)}\right||\Delta_1|\right]\cdot\left[\max_{x\in\Delta_{12}}|Df_2(x)|\cdot\frac{|\Delta_{12}|}{|\Delta_1|}\right] \\
&\quad+\max_{x\in\Delta_{12}}\left|\frac{D^2f_2(x)}{Df_2(x)}\right|\cdot|\Delta_2|\cdot\frac{|\Delta_{12}|}{|\Delta_2|} \\
&\leqq \mu(f_1)\cdot\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right|+\mu(f_2)\cdot\frac{|\Delta_{12}|}{|\Delta_2|}.
\end{aligned} \tag{4.2}$$

Since by (4.1)

$$\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right| \leqq \exp\left[\mu(f_2)\cdot\frac{|\Delta_{12}|}{|\Delta_2|}\right]$$

and

$$\frac{|\Delta_{12}|}{|\Delta_2|}=\left|\frac{Df_2(\eta_2)}{Df_2(\eta_{12})}\right|\frac{|\Delta_1|}{|J|}\leqq[\exp\mu(f_2)]\frac{|\Delta_1|}{|J|}$$

we obtain

$$\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right| \leqq \exp\left[\mu(f_2)\{\exp\mu(f_2)\}\frac{|\Delta_1|}{|J|}\right]. \tag{4.3}$$

Consequently

$$\begin{aligned}
\mu(f_1\circ f_2\cdot\Delta_{12}) &\leqq \mu(f_1)\exp\left[\{\mu(f_2)\exp\mu(f_2)\}\cdot\frac{|\Delta_1|}{|J|}\right] \\
&\quad+\{\mu(f_2)\exp\mu(f_2)\}\cdot\frac{|\Delta_1|}{|J|}.
\end{aligned} \tag{4.4}$$

Using the notation $\nu(f,\Delta)=\mu(f,\Delta)\exp\mu(f,\Delta)$, (4.4) is equivalent to

$$\mu(f_1\circ f_2,\Delta_{12})\leqq\mu(f_1)\exp\left[\nu(f_2)\cdot\frac{|\Delta_1|}{|J|}\right]+\nu(f_2)\cdot\frac{|\Delta_1|}{|J|}. \tag{4.5}$$

Let $h(x)=ax^2$, and let Δ denote an interval in $\mathbb{R}_+$; let H denote the distance from Δ to 0, so that $\Delta=(H,H+|\Delta|)$, and suppose $f:\Delta\to I$ is a C^2 diffeomorphism. Let $\delta=[x_{\min},x_{\max}]\subset\mathbb{R}_+$, be one of the two diffeomorphic preimages of $\Delta:d=h^{-1}(\Delta)\cap\mathbb{R}_+$. We obtain as above

$$\mu(f\circ h,\delta)\leqq\mu(f)\cdot\max_{x,y\in\delta}\left|\frac{Dh(x)}{Dh(y)}\right|+|\Delta|\max_{x,y\in\delta}\left|\frac{D^2h(x)}{[Dh(y)]^2}\right|$$

and thus

$$\mu(f\circ h,\delta)\leqq\mu(f)\frac{x_{\max}}{x_{\min}}+\frac{|\Delta|}{2ax_{\min}^2}.$$

Since $ax_{\max}^2=H+|\Delta|$, and $ax_{\min}^2=H$, we have

$$\frac{x_{\max}}{x_{\min}}=\sqrt{1+\frac{|\Delta|}{H}}<1+\frac{|\Delta|}{2H}.$$

This implies

$$\mu(f\circ h,\delta)<\mu(f)\left(1+\frac{|\Delta|}{2H}\right)+\frac{|\Delta|}{2H} \tag{4.6}$$

or

$$\mu(f\circ h,\delta)<\mu(f)\left(1+\frac{|\Delta|}{2ax_{\min}^2}\right)+\frac{|\Delta|}{2ax_{\min}^2} \tag{4.7}$$

5. Preliminary Lemma

We shall use the following several times

Lemma 1. *Let $I\cup J=N$ be an interval, $I=\bigcup_{i=1}^{\infty}\Delta\varphi_i$, where*

1) *φ_i are C^2-diffeomorphisms from their domains onto N;*
2) $\operatorname{int}\Delta\varphi_i\cap\operatorname{int}\Delta\varphi_j=\emptyset$, $i\neq j$;
3) $|D\varphi_i|>\bar{c}_1<1$;
4) $\mu(\varphi_i)<\bar{c}_2$;
5) $\operatorname{mes}J>0$; $\operatorname{mes}I\cap J=0$.

Then $I=\bigcup_{k=1}^{\infty}\varphi^{-k}J$ (mod 0), where $\varphi^{-k}J=\bigcup_{i_1\ldots i_k}\varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}J$.

Proof. Since φ_i is onto, $\Delta\varphi_i=\varphi_i^{-1}J\cup\varphi_i^{-1}I$. Thus

$$I=\bigcup_i\Delta\varphi_i=\bigcup_i(\varphi_i^{-1}J\cup\varphi_i^{-1}I)=\varphi^{-1}J\cup\left[\bigcup_{i_1}\varphi_{i_1}^{-1}\left(\bigcup_{i_2}\varphi_{i_2}^{-1}J\cup\varphi_{i_2}^{-1}I\right)\right]$$
$$=\varphi^{-1}J\cup\varphi^{-2}J\cup\varphi^{-2}I.$$

In a similar way we obtain for any N

$$I=\bigcup_{k\leqq N}\varphi^{-k}J\cup\varphi^{-N}I. \tag{5.1}$$

For any $i_1, \ldots, i_k$,

$$\varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}I=\left[\bigcup_i \varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}\circ\varphi_i^{-1}I\right]\cup\left[\bigcup_i \varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}\circ\varphi_i^{-1}J\right]. \tag{5.2}$$

Suppose there were a constant $\theta>0$ independent of k such that for any $i_1, \ldots, i_k$

$$\frac{\operatorname{mes}\varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}J}{\operatorname{mes}\varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}I}>\theta. \tag{5.3}$$

Then it would follow from (5.2) that

$$\operatorname{mes}\varphi^{-(k+1)}I<(1+\theta)^{-1}\operatorname{mes}\varphi^{-k}I,$$

thus $\lim_{k\to\infty}\operatorname{mes}\varphi^{-k}I=0$, and in view of (5.1) this would prove Lemma 1. Note that for $k=1$, (5.3) follows from hypothesis (5).

Consider a C^2 diffeomorphism $\varphi^k=\varphi_{i_k}\circ\ldots\circ\varphi_{i_1}:\varphi_{i_1}^{-1}\circ\ldots\circ\varphi_{i_k}^{-1}N\to N$. By the mean value theorem and by (4.1), a proof of (5.3) would follow from a uniform upper bound on the quantities $\mu(\varphi^n)$ independent of n. We will show

$$\mu(\varphi^n)<\left(\sum_{i=0}^{\infty}\frac{\bar{c}_2\exp\bar{c}_2}{\bar{c}_1^i}\right)\exp\left(\sum_{i=1}^{\infty}\frac{\bar{x}_2\exp\bar{c}_2}{\bar{c}_1^i}\right). \tag{5.4}$$

We prove (5.4) by induction. From (4.5),

$$\mu(\varphi^n)=\mu(\varphi^{n-1}\circ\varphi)\leqq\mu(\varphi^{n-1})\exp\left[v(\varphi)\frac{|\Delta\varphi^{n-1}|}{|N|}\right]+v(\varphi)\frac{|\Delta\varphi^{n-1}|}{|N|}. \tag{5.5}$$

According to hypotheses 3 and 4

$$v(\varphi)<\bar{c}_2\exp\bar{c}_2$$

and

$$|\Delta\varphi^{n-1}|<|N|/\bar{c}_1^{n-1}.$$

Thus

$$\mu(\varphi^n)\leqq\mu(\varphi^{n-1})\exp[v(\varphi)/\bar{c}_1^{n-1}]+v(\varphi)/\bar{c}_1^{n-1}. \tag{5.6}$$

Suppose for $k\leqq n-1$ that

$$\mu(\varphi^k)\leqq\left(\sum_{i=0}^{k-1}v(\varphi)/\bar{c}_1^i\right)\exp\left(\sum_{i=1}^{k-1}v(\varphi)/\bar{c}_1^i\right).$$

(Note that for $k=1$, the second factor above equals 1 and this becomes the obvious inequality $\mu(\varphi)<v(\varphi)$.) Then, using (5.6),

$$\begin{aligned}\mu(\varphi^n)&\leqq\left(\sum_{i=0}^{n-2}\frac{v(\varphi)}{\bar{c}_1^i}\right)\exp\left(\sum_{i=1}^{n-2}\frac{v(\varphi)}{\bar{c}_1^i}\right)\exp\left(\frac{v(\varphi)}{\bar{c}_1^{n-1}}\right)+\frac{v(\varphi)}{\bar{c}_1^{n-1}}\\&\leqq\left(\sum_{i=0}^{n-2}\frac{v(\varphi)}{\bar{c}_1^i}\right)\exp\left(\sum_{i=1}^{n-1}\frac{v(\varphi)}{\bar{c}_1^i}\right)+\frac{v(\varphi)}{\bar{c}_1^{n-1}}\exp\left(\sum_{i=1}^{n-1}\frac{v(\varphi)}{\bar{c}_1^i}\right)\\&\leqq\left(\sum_{i=0}^{n-1}\frac{v(\varphi)}{\bar{c}_1^i}\right)\exp\left(\sum_{i=1}^{n-1}\frac{v(\varphi)}{\bar{c}_1^i}\right)\end{aligned}$$

and (5.4) is proved.

6. Transition from n to $n+1$, I. Hypotheses of Induction. Estimates of Derivatives

(3.1) and (3.2) give us the following representation of $[0,1]$ after Step n:

$$[0,1]=\left[\bigcup_{k=1}^{n}(\bigcup \Delta f_k)\right]\cup\left[\bigcup_{m=1}^{\infty}(\bigcup \delta_n^{-m})\right]\cup(\bigcup \Delta F_{n-1}\circ g)\cup(\bigcup \Delta \hat{f}_n) \tag{6.1}$$

All domains in (6.1) depend on λ which varies in $\mathscr{J}_n$, but throughout Sects. 6 and 7. λ will be fixed. Any δ_n^{-m} in (6.1) is a preimage of δ_n under some diffeomorphism denoted by G_n. For given δ_n^{-m} let $p=\max\left\{k:\delta_n^{-m}\subset\left[\frac{1}{2}-\frac{1}{\lambda^{sk}},\frac{1}{2}+\frac{1}{\lambda^{sk}}\right]\right\}$. Then we shall use the notation $G_{n,p}$ for G_n.

Let $0<s\leqq\frac{1}{13}$, $1<\alpha\leqq s/4$ be constants defined in Sects. 2 and 3, $c_0=1-s$, $c_1=1-2s$, $c_2=1-s+\alpha$, $\gamma=1-3s$, $t=\alpha/10$, $v=\frac{2(s-\alpha)}{c_0}$. Now we formulate the hypotheses of induction.

a) *Hypotheses on derivatives:*

$a_{1n}^1)$ $|Df_k|>2^k\cdot\lambda^{c_1 k}$ } $k=1,\dots,n$
$a_{1n}^2)$ $|Df_k|>2\lambda^{c_0}$ }

$a_{2n})$ $|DF_{n-1}|>2^{n-1}\lambda^{c_0(n-1)}$

$a_{3n})$ $|D\hat{f}_n|>2^n\cdot\lambda^{c_1 n}$

$a_{4n}^1)$ $|DG_{n,p}|>\lambda^{s(1-v)p}$

$a_{4n}^2)$ $|DG_n|>2\lambda^{c_2/2}$.

b) *Hypotheses on μ:*

$b_{1n})$ $\mu(f_k)<\left(\sum_{i=1}^{k}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\prod_{i=1}^{k}\left(1+\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\exp\left(\sum_{i=1}^{k}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)$, $k=1,2,\dots,n$.

$b_{2n})$ $\mu(F_{n-1})<\left(\sum_{i=1}^{n-1}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\prod_{i=1}^{n-1}\left(1+\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\exp\left(\sum_{i=1}^{n-1}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)$

$b_{3n})$ $\mu(\hat{f}_n)<\left(\sum_{i=1}^{n}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\prod_{i=1}^{n}\left(1+\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)\cdot\exp\left(\sum_{i=1}^{n}\frac{1}{2^i\cdot\lambda^{\gamma i}}\right)$

$b_{4n})$ $\mu(G_n)<\frac{1}{\lambda^{\alpha n}}$.

We suppose a_{in}, b_{in} to be true and we have to prove a_{in+1}, b_{in+1}.

Remark VI/1. At the beginning of Step $n+1$ we constructed some new preimages δ_n^{-m} with corresponding maps denoted by G_n' ($G_n':\delta_n^{-m}\to\delta_n$, see Sect. 3). We have to prove that G_n' also satisfy the conditions a_{4n}, b_{4n} which we denote in this case a_{4n}', b_{4n}'.

Remark VI/2. Some additional induction hypotheses related to the variation of λ will be formulated below. In particular the possibility of choice of intervals δ_n, $\hat{\delta}_n$ will be proved, and estimates of sizes of these intervals and their preimages will be given in Sect. 10. Now we shall use (3.5) and (3.13) with n instead of $n+1$ (this is assumed inductively) and with $n+1$ (this will be proved in Sect. 10). One easily checks there is no vicious circle here.

a_{1n+1}) According to the construction of Sect. 3, $\{f_{n+1}\}=\bigcup_{i=1}^{\infty}\{f_{n+1i}\}$ where $f_{n+11}=f_k\circ F_{n-1}\circ g_n$ $\left(\text{with } g_n=\lambda\cdot x^2\Big|\left\{|x|>\frac{1}{\lambda^{s(n+1)}}\right\}\text{ in local coordinates near }\frac{1}{2}\right)$, or $f_{n+11}=f_k\circ\hat{f}_n$. In the first case a_{1n}, a_{2n} and the form of g_n above imply

$$|Df_{n+11}|\geqq 2\lambda^{c_0}\cdot 2^{n-1}\cdot\lambda^{c_0(n-1)}\cdot 2\lambda\cdot\frac{1}{\lambda^{s(n+1)}}>2^{n+1}\cdot\lambda^{c_1(n+1)}.$$

In the second case a_{1n}, a_{3n} imply

$$|Df_{n+11}|\geqq 2\lambda^{c_0}\cdot 2^n\cdot\lambda^{c_1 n}>2^{n+1}\cdot\lambda^{c_1(n+1)}.$$

Thus a^1_{1n+1} is true for f_{n+11}. The choice of s implies $2c_1>c_0$, hence a^1_{1n} implies a^2_{1n} for $n\geqq 2$. All f_{n+1k}, $k\geqq 2$ are compositions of the form $f_{n+1k}=f_{n+1k-1}\circ G_n$ or $f_{n+1k}=f_{n+1k-1}\circ G'_n$ with $\Delta G'_n\subset\delta_n\backslash\delta_{n+1}$. According to a^2_{4n}, $|DG_n|>1$. $(a^2_{4n})'$ proved below is much stronger than $|DG'_n|>1$, and G'_n under consideration satisfies $|DG'_n|>2^n\cdot\lambda^{c_1 n}$. Indeed, $G'_n=G_n\circ F_{n-1}\circ g_n$ or $G'_n=G_n\circ\hat{f}_n$. In both cases a_{2n} and a_{3n} imply as above $|DG'_n|>2^n\cdot\lambda^{c_1 n}$. □

a_{2n+1}) $F_n=f_k\circ F_{n-1}$. Hence a_{1n} and a_{2n} imply a_{2n+1}. □

$a^{2\prime}_{4n}$) We consider $G_n:\delta_n^{-m}\to\delta_n$, $G'_n=G_n\circ F_{n-1}\circ g$ or $G'_n=G_n\circ\hat{f}_n$ and their domains $\delta_n^{-N}=(F_{n-1}\circ g)^{-1}(\delta_n^{-m})$ or $\delta_n^{-M}=\hat{f}_n^{-1}(\delta_n^{-m})$. The most complicated is the case of central branch $F_{n-1}\circ h$. We omit indices and use δ to denote δ_n^{-m} (if $m=0$, $\delta=\delta_n$), G to denote G_n (if $m=0$, $G=\text{id}$), ℓ to denote $(F_{n-1}\circ h)^{-1}\delta$. We estimate $|D(G\circ F_{n-1}\circ h)|$. Let $H=\text{dist}(\delta, F_{n-1}\circ h(\frac{1}{2}))$. The induction construction of Step n implies that the top $F_{n-1}\circ h(\frac{1}{2})$ lies outside an interval $\hat{\delta}$ corresponding to δ. Thus (see (3.13) with n instead of $n+1$)

$$H\geqq(\lambda^{\alpha n}-1)\cdot|\delta|/2.$$

Let $H_1=\text{dist}(F^{-1}_{n-1}\delta, h(\frac{1}{2}))$. It follows from (4.1) and b_{2n} that

$$H_1>(\lambda^{\alpha n}-1)\cdot|F^{-1}_{n-1}\delta|\cdot 2^{-1}\cdot\exp\left(-\frac{1+\varepsilon_{6.2}}{2\lambda^{\gamma}}\right),\quad\text{where } \varepsilon_{6.2}=O(\lambda^{-\gamma})\tag{6.2}$$

Remark VI/3. Several constants $0\leqq\varepsilon_{i,k}<\lambda^{-t}$ are indexed according to the numbers of inequalities in which they occur.

Let $\ell=[x_{\min}, x_{\max}]$. We have, using the local coordinate,

$$h(x_{\min})=\lambda\cdot x^2_{\min}=H_1,\qquad x_{\min}\sqrt{H_1\lambda^{-1}},\qquad |Dh|_\ell|\geqq 2\lambda|x_{\min}|=2\sqrt{\lambda\cdot H_1}.$$

In consequence of $|\delta|=|F^{-1}_{n-1}\delta|\cdot|DF_{n-1}(\theta)|$, for some $\theta\in F^{-1}_{n-1}\delta$, we obtain

$$|Dh|_\ell|\geqq\sqrt{\frac{2\cdot\lambda^{\alpha n+1}|\delta|}{|DF_{n-1}(\theta)|}(1-\varepsilon_{6.3})}\tag{6.3}$$

Since $|D(F_{n-1}\circ h)|=|DF_{n-1}|\cdot|Dh|$, we have, using (4.1) and b_{2n}, for any $x\in\Delta F_{n-1}$

$$|D(F_{n-1}\circ h)|_\ell|\geqq\sqrt{2\cdot\lambda^{\alpha n+1}\cdot|DF_{n-1}(x)|\cdot|\delta|}(1-\varepsilon_{6.4})\tag{6.4}$$

If $\delta=\delta_n$, then (6.4), $|\delta_n|>2\cdot\lambda^{-sn}$ and a_{2n} imply

$$|D(F_{n-1}\circ h)|_\ell|\geqq(\sqrt{2}\cdot\lambda^{\frac{1}{2}(c_1+\alpha)})^n\cdot\sqrt{2\lambda^s}\cdot(1-\varepsilon_{6.5}). \tag{6.5}$$

If $\delta=\delta_n^{-m}=G^{-1}\delta_n$ we obtain, using a_{4n}

$$|D(G\circ F_{n-1}\circ h)|_\ell|\geqq(\sqrt{2}\cdot\lambda^{\frac{1}{2}(c_1+\alpha)})^n\sqrt{\lambda^{s+c_2/2}}\cdot 2(1-\varepsilon_{6.6}). \tag{6.6}$$

(6.5) and (6.6) imply $(a_{4n}^2)'$ for $G_n'=G_n\circ F_{n-1}\circ n$. In the case $G_n'=G_n\circ F_{n-1}\circ g$ we have in (6.2) $H_1>\frac{1}{2}\cdot|\Delta F_{n-1}|\exp\left(-\frac{1+\varepsilon_{6.2}}{2\lambda^\gamma}\right)$, which leads to better estimates. In the case $G_n'=G_n\circ\hat{f}_n$ $(a_{4n}^2)'$ is obvious because of a_{3n} and a_{4n}^2. □

a_{4n+1}^2) Follows from a_{4n}^2, $(a_{4n}^2)'$ and the definition:

$G_{n+1}=G_{n_1}\circ\ldots\circ G_{n_p}$. □

a_{3n+1}) If $\hat{f}_{n+1}=f_{n+1}\circ G_{n_1}\circ\ldots\circ G_{n_p}$, a_{3n+1} follows from a_{1n+1}, a_{4n}^2 and $(a_{4n}^2)'$. If $\hat{f}_{n+1}=f_k\circ\hat{f}_n$, a_{3n+1} follows from a_{1n} and a_{3n}. □.

Remark VI/4. The inequalities (6.5), (6.6) show that the derivatives of G_n' grow exponentially with n, but this is not sufficient to prove $(a_{4n}^1)'$. Indeed, let n_1 be so that $F_{n_1-1}\circ h(\lambda,\frac{1}{2})$ may lie in the domain δ_1. As δ_1 contains Δf_2 of arbitrary small diameter, the interval $\Delta(f_2\circ F_{n_1-1}\circ h)=\Delta(F_{n_1}\circ h)$ may also be arbitrarily small and the corresponding $\delta_{n_1}^{-M}=(F_{n_1}\circ h)^{-1}\delta_{n_1}$ is contained in δ_N with arbitrarily large N. However $|DF_{n_1}|$ turns out to be very large in this case, which implies $(a_{4n}^1)'$.

a_{4n}^1) We use the notation introduced in the proof of $(a_{4n}^2)'$. According to the definition, the domain $\ell=(F_{n-1}\circ h)^{-1}\delta$ of $G_{n,p}'$ is so that $\ell\subset(\frac{1}{2}-\lambda^{-sp},\frac{1}{2}+\lambda^{-sp})$, but $\ell\not\subset(\frac{1}{2}-\lambda^{-s(p+1)},\frac{1}{2}+\lambda^{-s(p+1)})$. Let $\ell\subset(\frac{1}{2},\frac{1}{2}+\lambda^{-sp})$. Then $H_1=\lambda x_{\min}^2<\lambda^{1-2sp}$. It follows from (6.2)

$$\frac{\lambda}{\lambda^{2sp}}>\frac{\lambda^{\alpha n}}{2}\cdot\frac{|\delta|(1-\varepsilon_{6.7})}{|DF_{n-1}(\theta_1)|}. \tag{6.7}$$

(6.7) together with b_{2n} imply for any $x\in\Delta F_{n-1}$

$$|DF_{n-1}(x)|>\frac{\lambda^{\alpha n-1}\cdot\lambda^{2sp}}{2}\cdot|\delta|(1-\varepsilon_{6.8}). \tag{6.8}$$

Thus we can rewrite (6.4) as

$$|D(F_{n-1}\circ h)|_\ell|\geqq\lambda^{\alpha n}\cdot\lambda^{sp}\cdot|\delta|\cdot(1-\varepsilon_{6.9}). \tag{6.9}$$

From $|\delta|=\frac{|\delta_n|}{|DG_n(\theta_2)|}$, and $|\delta_n|>2\cdot\lambda^{-sn}$ we obtain using b_{4n}

$$|DG_{n,p}|=|D(G_n\circ F_{n-1}\circ h)|_\ell|\geqq 2\cdot\lambda^{\alpha n}\cdot\lambda^{s(p-n)}\cdot(1-\varepsilon_{6.10}). \tag{6.10}$$

Let us compare p and n. Let $\mathscr{D}_{n-1}=(\frac{1}{2}-u_{n-1}^1,\frac{1}{2}+u_{n-1}^2)$ be the domain of $F_{n-1}\circ h$, and $p_1=\max\{q:u_{n-2}^2<\lambda^{-sq}\}$. Then $p\geqq p_1$. We have in the local coordinate system, using a_{2n},

$$2^{n-1}\cdot\lambda^{c_0(n-1)+1}\cdot(u_{n-1}^2)^2<F_{n-1}\circ h(\tfrac{1}{2})\leqq 1.$$

Thus

$$\lambda^{-s(p_1+1)} < u_{n-1}^2 < [(\sqrt{2})^{n-1}\lambda^{(c_0 n+s)1/2}]^{-1}. \tag{6.11}$$

(6.11) implies $n < 2s(p_1+\frac{1}{2})/c_0$, which gives for v a somewhat worse estimate than $2(s-\alpha)/c_0$. We prefer to improve it instead of taking a different v. It suffices to make $F_{n-1}\circ h(\lambda,\frac{1}{2})$ lie outside $(\frac{1}{2}-\lambda^{-s/2},\frac{1}{2}+\lambda^{-s/2})$ for the first two steps. This gives an extra factor $\lambda^{-s/2}$ on the right side of (6.11). Hence

$$n < \frac{2s}{c_0} p_1. \tag{6.12}$$

Remark VI/5. For a given n_0 we may introduce the additional condition

$$F_{n-1}\circ h(\lambda,\tfrac{1}{2}) \in [0,1]\setminus[\tfrac{1}{2}-\lambda^{-s/2},\tfrac{1}{2}+\lambda^{-s/2}] \quad n \leqq n_0$$

(above $n_0=2$). This simplifies the estimates, but as follows from Sect. 11, it gives an extra factor of $(1-2N_0^{-s/2})^{n_0}$ in the estimate of $\mathcal{M}$. However, this factor can be made arbitrarily close to 1 by taking N_0 sufficiently large.

As $p \geqq p_1$, (6.10) and (6.12) imply

$$|DG'_{n,p}| > \lambda^{ps\left(1-\frac{2(s-\alpha)}{c_0}\right)} = \lambda^{ps(1-v)} \tag{6.13}$$

which finishes the proof of $(a_{4n}^1)'$ for $\ell = (F_{n-1}\circ h)^{-1}\delta$.

If $G'_{n,p} = G_n \circ F_{n-1}\circ g$, the estimate of H_1 (see the proof of $(a_{4n}^2)'$) implies that (6.10) turns out into $|DG'_{n,p}| > \lambda^{sp}(1-\varepsilon_{6.10})$. Finally when $G'_{n,p} = G_n\circ \hat{f}_n$, notice that any $\hat{f}_n$ is a composition of the form $\varphi\circ G_k$, $k \leqq n-1$, where G_k satisfies a_{4k}^1, and $|D\varphi| > 1$. □

a_{4n+1}^1) Follows from a_{4n}^1, a_{4n}^2, $(a_{4n}^1)'$, $(a_{4n}^2)'$ and the definition of G_{n+1}. □

7. Transition from n to $n+1$, II. Estimates of μ

b_{4n+1}) Let $G'_n = G_n\circ F_{n-1}\circ h : I \to \delta_n$, where

$$G_n : \Delta G_n = \delta_n^{-m} \to \delta_n, \quad \text{and} \quad \ell = (F_{n-1}\circ h)^{-1}\delta_n^{-m}.$$

We estimate $\mu(G'_n,\ell)$ first.

According to (4.5)

$$\mu(G_n\circ F_{n-1}, F_{n-1}^{-1}\Delta G_n) \leqq \mu(G_n)\exp(v(F_{n-1})\cdot|\Delta G_n|) + v(F_{n-1})\cdot|\Delta G_n|.$$

We have

$$|\delta_n| = |\Delta G'_n|\cdot|DG_n(\theta_0)|, \quad |\delta_n| < 2\cdot\lambda^{-sn}(1+O(\lambda^{-tn})), \quad |DG_n| > 2\lambda^{c_2/2}.$$

In consequence of b_{2n}, $v(F_{n-1}) = 1+O(\lambda^{-\gamma})$. Thus we obtain, using a_{2n}, b_{4n}:

$$\mu(G_n\circ F_{n-1}, F_{n-1}^{-1}\Delta G_n) < \lambda^{-\alpha n}\cdot\exp\left[\frac{2(1+O(\lambda^{-\gamma}))}{|DG_n(\theta_0)|\lambda^{c_0(n-1)+sn}\cdot 2^{n-1}}\right] + \frac{2(1+O(\lambda^{-\gamma}))}{|DG_n(\theta_0)|\lambda^{c_0(n-1)+sn}\cdot 2^{n-1}} = \frac{1+\varepsilon_{7.1}}{\lambda^{\alpha n}}. \tag{7.1}$$

Proceeding along the line of the proof of (6.2), and using (7.1) and (4.6) with $\Delta = F_{n-1}^{-1}\Delta G_n$, $H > \frac{1}{2}(1-O(\lambda^{-\gamma}))\cdot\lambda^{\alpha n}\cdot|\Delta|$ we obtain

$$\mu(G_n \circ F_{n-1} \circ h, \ell) < \frac{1+\varepsilon_{7.1}}{\lambda^{\alpha n}}\left(1 + \frac{1+O(\lambda^{-\gamma})}{\lambda^{\alpha n}}\right) + \frac{1+O(\lambda^{-\gamma})}{\lambda^{\alpha n}} < \frac{2(1+\varepsilon_{7.2})}{\lambda^{\alpha n}}. \quad (7.2)$$

The proof for $G_n' = G_n \circ F_{n-1} \circ g$ and $G_n' = G_n \circ \hat{f}_n$ is similar and gives a better estimate

$$\mu(G_n') < (1+\varepsilon_{7.3})\lambda^{-\alpha n}. \quad (7.3)$$

Then we consider $G_{n+1} = \tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_p} : \delta_n^{-M} \to \delta_n$, $\delta_n^{-M} \subset \delta_n \backslash \delta_{n+1}$. When estimating $\mu(G_{n+1}, \delta_n^{-M})$ we use the proof of Lemma 1 with $\varphi_i = \tilde{G}_{n_i}$, $\bar{c}_2 = (1+\varepsilon_{7.3})\lambda^{-\alpha n}$, according to (7.3) and b_{4n}, and $\bar{c}_1 = \max(\lambda^{c_2/2}, \lambda^{s[n(1-\nu)-1]})$, according to a_{4n}. Then (5.4) gives

$$\mu(G_{n+1}, \delta_n^{-M}) < (1+\varepsilon_{7.4})\lambda^{-\alpha n}. \quad (7.4)$$

The estimates (3.5) of $|\delta_n|$ and $|\delta_{n+1}|$ imply

$$\frac{|\delta_{n+1}|}{|\delta_n|} < \frac{1+\varepsilon_{7.5}}{\lambda^s}. \quad (7.5)$$

Considering $\delta_{n+1}^{-M} = G_{n+1}^{-1}\delta_{n+1}$ and applying (7.4), (7.5) we obtain

$$\mu(G_{n+1}, \delta_{n+1}^{-M}) \leqq \mu(G_{n+1}, \delta_n^{-M}) \cdot \frac{|\delta_{n+1}^{-M}|}{|\delta_n^{-M}|} \leqq \mu(G_{n+1}, \delta_n^{-M}) \cdot \frac{|\delta_{n+1}|}{|\delta_n|}$$

$$\cdot \exp \mu(G_{n+1}, \delta_n^{-M}) < \frac{(1+\varepsilon_{7.4})(1+\varepsilon_{7.5})(1+\varepsilon_{7.6})}{\lambda^{\alpha n + s}} < \frac{1}{\lambda^{\alpha(n+1)}} \quad (7.6)$$

which proves b_{4n+1} for $\delta_{n+1}^{-M} \subset \delta_n \backslash \delta_{n+1}$.

Any $G_{n+1} : \delta_{n+1}^{-N} \to \delta_{n+1}$ for $\delta_{n+1}^{-N} \subset [0,1] \backslash (\delta_n \backslash \delta_{n+1})$ is either a restriction of $G_n : \delta_n^{-N} \to \delta_n$ on $\delta_{n+1}^{-N} \subset \delta_n^{-N}$, or a composition of the form $\tilde{G}_{n+1} \circ G_n$ or $\tilde{G}_{n+1} \circ G_n'$, where $\mu(\tilde{G}_{n+1})$ satisfies (7.6), $\mu(G_n)$ satisfies b_{4n} and $\mu(G_n')$ satisfies (7.2). The case of restriction is treated along the lines of (7.5), (7.6). In the other cases, (4.5) together with a_{4n}^2 imply

$$\mu(G_{n+1}, \delta_{n+1}^{-N}) \leqq \prod_{i=4}^{6} (1+\varepsilon_{7.1}) \cdot \frac{1}{\lambda^{\alpha n + s}} \exp\left(\frac{\exp(3\cdot\lambda^{-\alpha n})}{2\lambda^{\alpha n + c_2/2}}\right) + \frac{\exp(3\cdot\lambda^{-\alpha n})}{2\lambda^{\alpha n + c_2/2}} \quad (7.7)$$

which proves b_{4n+1}. □

b_{1n+1}) $\{f_{n+1}\} = \bigcup_{k=1}^{\infty} \{f_{n+1\,k}\}$, where $f_{n+1\,1} = f_k \circ F_{n-1} \circ g_n$, $k \in [1, n]$, or $f_{n+1\,1} = f_k \circ \hat{f}_n$, and $f_{n+1\,k}$ are obtained from $f_{n+1\,1}$ using consecutive compositions with different sorts of $\tilde{G}_n$ and $\dot{G}_n$. Let us begin with $f_{n+1\,1} = f_k \circ F_{n-1} \circ g_n$. (4.5) implies:

$$\mu(f_k \circ F_{n-1}) \leqq \mu(f_k)\exp(\nu(F_{n-1})\cdot|\Delta f_k|) + \nu(F_{n-1})\cdot|\Delta f_k|.$$

We have $\nu(F_{n-1}) = \frac{1+\varepsilon_{7.8}}{2\lambda^{\gamma}}$ (in consequence of b_{2n}), $|\Delta f_k| < \frac{1}{2^k \lambda^{c_1 k}}$ (in consequence of a_{1n}), thus

$$\mu(f_k \circ F_{n-1}) \leqq \mu(f_k)\exp\left(\frac{1+\varepsilon_{7.8}}{2^{k+1}\cdot\lambda^{c_1 k + \gamma}}\right) + \frac{1+\varepsilon_{7.8}}{2^{k+1}\cdot\lambda^{c_1 k + \gamma}}. \quad (7.8)$$

Let Δ be the domain of $f_k \circ F_{n-1}$. Then (3.5) and (4.7) used with $a=\lambda$, imply

$$\mu(f_k \circ F_{n-1} \circ g_n) \leqq \mu(f_k \circ F_{n-1})(1+|\Delta| \cdot \lambda^{2s(n+1)-1}) + |\Delta| \cdot \lambda^{2s(n+1)-1}.$$

We have $|\Delta| < 2^{-n} \cdot \lambda^{-c_0 n}$, because of a_{1n}^2 and a_{2n}, and thus

$$\mu(f_k \circ F_{n-1} \circ g_n) \leqq \mu(f_k \circ F_{n-1}) \left(1 + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0-2s)(n+1)}}\right) + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0-2s)(n+1)}}. \quad (7.9)$$

In a similar way one verifies using a_{3n} and b_{3n} that $\mu(f_{n+1\,1} = f_k \circ \hat{f}_n)$ also satisfies (7.8).

Using b_{1n}, (7.8) and (7.9), we obtain

$$\begin{aligned} \mu(f_{n+1\,1}) \leqq & \left[\left(\sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \prod_{i=1}^{k} \left(1 + \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\exp \sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right)\right. \\ & \left. \cdot \exp\left(\frac{1+\varepsilon_{7.8}}{2^{k+1}\lambda^{c_1 k+\gamma}}\right) + \frac{1+\varepsilon_{7.8}}{2^{k+1}\lambda^{c_1 k+\gamma}}\right] \\ & \cdot \left(1 + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0-2s)(n+1)}}\right) + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0-2s)(n+1)}}. \end{aligned} \quad (7.10)$$

Since $c_1 - s = c_0 - 2s = \gamma$, we have

$$(1+\varepsilon_{7.8}) \cdot 2^{-(k+1)} \cdot \lambda^{-c_1 k - \gamma} = (1+\varepsilon_{7.8}) \cdot 2^{-(k+1)} \cdot \lambda^{-\gamma(k+1)} \cdot \lambda^{-sk} \ll 2^{-(k+1)} \cdot \lambda^{-\gamma(k+1)}$$

and

$$\lambda^{-s} \cdot 2^{-n} \cdot \lambda^{-(c_0-2s)(n+1)} = (2\lambda^{-s}) \cdot 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)} < 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)}.$$

Therefore

$$\begin{aligned} \mu(f_{n+1\,1}) < & \left(\sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\exp \sum_{i=1}^{k+1} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\prod_{i=1}^{k+1} \left(1 + \frac{1}{2^i \lambda^{\gamma i}}\right)\right) \\ & + \frac{1}{2^{k+1} \cdot \lambda^{\gamma(k+1)}} \left[\frac{(1+\varepsilon_{7.8})[1 + (2\lambda^{-s}) \cdot 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)}]}{\lambda^{sk}} + \frac{2}{\lambda^s \cdot 2^{n-k} \cdot \lambda^{\gamma(n-k)}}\right]. \end{aligned} \quad (7.11)$$

Since $k \leqq n$, the factor in square brackets is less than 1, which implies $b_{1\,n+1}$ for $f_{n+1\,1}$.

If $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n+1} = f_{n+1\,1} \circ \tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_p}$ we have using (4.5), (3.5), (7.4) and $a_{1\,n+1}^1$

$$\mu(f_{n+1\,k}) \leqq \mu(f_{n+1\,1}) \exp\left(\frac{1+\varepsilon_{7.12}}{2^{n+1} \cdot \lambda^{s+\alpha n+\gamma(n+1)}}\right) + \frac{1+\varepsilon_{7.12}}{2^{n+1} \cdot \lambda^{s+\alpha n+\gamma(n+1)}}. \quad (7.12)$$

Substituting (7.10) in (7.12) we obtain $b_{1\,n+1}$ as above. The same reasoning proves $b_{1\,n+1}$ for $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n+1} \circ G_n$. □

$b_{2\,n+1}$) The proof is similar to the above proof of $b_{1\,n+1}$. □

$b_{3\,n+1}$) For $\hat{f}_{n+1} = f_k \circ \hat{f}_n$ with $\Delta \hat{f}_n \subset \delta_{n+1}$ and for $\hat{f}_{n+1} = f_{n+1} \circ G_n$ the proof is similar. For $\hat{f}_{n+1} = f_{n+1} \circ G'_n$ (7.2) is applied. □

8. Measure of Holes After Step $n+1$

For any $\lambda \in \mathcal{M}_n$ we estimate the measure of the union $\delta_n(\lambda) \cup \bigcup_{m=1}^{\infty} (\bigcup \delta_n^{-m}(\lambda))$, where $\delta_n^{-m}(\lambda) \subset [0,1] \backslash \delta_n(\lambda)$.

Lemma 2. *There exists an $\varepsilon < \lambda^{-t}$ so that for any $k \in \mathbb{Z}_+ \backslash \{0\}$*

$$\operatorname{mes}\left[\delta_k \cup \bigcup_{m=1}^{\infty} (\bigcup \delta_k^{-m})\right] < \frac{(1+\varepsilon)^k}{\lambda^{sk}}.$$

Proof. We proceed by induction and assume that after Step n:

i) The estimate of Lemma 2 holds for $k=n$;

ii) To any hole δ_n^{-m} there corresponds a unique hole $\delta_{n-1}^{-m} \supset \delta_n^{-m}$ and a set $K_{n,m} = K_{n,m}(\delta_n^{-m}) \subset \delta_{n-1}^{-m}$, such that $K_{n,m} \subset \mathcal{X}_n$ and for some $\varepsilon_{8.1} = O(\lambda^{-t})$

$$\frac{\operatorname{mes} \delta_n^{-m}}{\operatorname{mes} K_{n,m}} < \frac{1+\varepsilon_{8.1}}{\lambda^s}. \tag{8.1}$$

Remark VIII/1. The proof of Lemma 4 in Sect. 10 implies ii above. However we prove ii here in order to separate the proof of Lemma 2.

Remark VIII/2. We shall use here that the intervals δ_n, δ_{n+1} constructed in Sect. 10 are chosen so as to have $\delta_{n-1}^{-m} \subset \delta_n \backslash \delta_{n+1}$ for the holes δ_{n-1}^{-m} corresponding to holes $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$.

We began Step $n+1$ by taking compositions $f_k \circ (F_{n-1} \circ g)$ or $f_k \circ \hat{f}_n$ and creating new holes of the form $(F_{n-1} \circ g)^{-1} \delta_n^{-m}$, $\hat{f}_n^{-1} \delta_n^{-m}$. Let $\delta_{nn+1} = \delta_n \backslash \delta_{n+1}$. There are holes δ_n^{-m} of two kinds inside δ_{nn+1}: the old ones $\delta_n^{-m} \subset \delta_{n-1}^{-m}$, and the new ones $\tilde{\delta}_n^{-M} = (F_{n-1} \circ g_n)^{-1} \delta_n^{-m}$, or $\tilde{\delta}_n^{-M} = \hat{f}_n^{-1} \delta_n^{-m}$ for $\Delta \hat{f}_n \subset \delta_{nn+1}$, $m = 0, 1, \ldots$. Let

$$p_{nn+1} = \operatorname{mes}[(\bigcup \delta_n^{-m}) \cap \delta_{nn+1}], \quad \tilde{p}_{nn+1} = \operatorname{mes}[(\bigcup \tilde{\delta}_n^{-m}) \cap \delta_{nn+1}].$$

Then (8.1) implies

$$p_{nn+1} < |\delta_{nn+1}| \cdot (1+\varepsilon_{8.1}) \cdot \lambda^{-s}. \tag{8.2}$$

One obtains similarly to (7.9) $\mu(F_{n-1} \circ g_n) < 1 + \lambda^{-\gamma}$. Then i) implies

$$\begin{aligned} \tilde{p}_{nn+1} &< |\delta_{nn+1}| \cdot (\exp \mu(F_{n-1} \circ g_n)) \cdot [(1+\varepsilon_{8.1}) \cdot \lambda^{-s}]^n \\ &< |\delta_{nn+1}| \cdot (1+\lambda^{-\gamma}) \cdot [(1+\varepsilon_{8.1}) \cdot \lambda^{-s}]^n. \end{aligned} \tag{8.3}$$

The construction of Sect. 3 implies the one-to-one correspondence between δ_{n+1}^{-m} and corresponding $\delta_n^{-m} (\delta_{n+1}^{-m} \subset \delta_n^{-m} \subset \delta_{nn+1})$. We have, according to the construction, $\operatorname{mes}(\bigcup \Delta f_{n+1\,1}) \geq (\operatorname{mes} \delta_{nn+1}) - p_{nn+1} - \tilde{p}_{nn+1}$. Now, we let $K_{n+1} = K_{n+1,0} = \bigcup \Delta f_{n+1\,1}$ correspond to δ_{n+1}. In consequence of (8.2) and (8.3) we have

$$\begin{aligned} \frac{\operatorname{mes} \delta_{n+1}}{\operatorname{mes} K_{n+1}} &< \frac{2(1+o(\lambda^{-t(n+1)}))}{\lambda^{s(n+1)}} \\ &: \left[\frac{2}{\lambda^{sn}}\left(1 - \frac{1+o(\lambda^{-t(n+1)})}{\lambda^s}\right) \cdot \left(1 - \frac{1+\varepsilon_{8.1}}{\lambda^s} - (1+\lambda^{-\gamma})\left(\frac{1+\varepsilon_{8.1}}{\lambda^s}\right)^n\right)\right]. \end{aligned} \tag{8.4}$$

The right part of (8.4) is less than $(1+\varepsilon_{8.1})\lambda^{-s}$ for a suitable $\varepsilon_{8.1} = O(\lambda^{-r})$, $s > r > t$.

We let $K_{n+1,m}=G_{n+1}^{-1}(K_{n+1})$ correspond to $\delta_{n+1}^{-m}=G_{n+1}^{+1}(\delta_{n+1})$. We have

$$\frac{\operatorname{mes}\delta_{n+1}^{-m}}{\operatorname{mes}K_{n+1,m}}<\frac{\operatorname{mes}\delta_{n+1}}{\operatorname{mes}K_{n+1}}\cdot\exp\mu(G_{n+1}). \tag{8.5}$$

Because of b_{4n+1}, the right side of (8.4) with the additional factor $\exp\mu(G_{n+1})$ is still less than $(1+\varepsilon_{8.1})\lambda^{-s}$ and (8.1) is proved for $k=n+1$. Lemma 2 with $\varepsilon=\varepsilon_{8.1}$ follows now from

$$\operatorname{mes}(\bigcup\delta_{n+1}^{-m})<(1+\varepsilon)\lambda^{-s}\cdot\operatorname{mes}(\bigcup K_{n,m})<(1+\varepsilon)\lambda^{-s}\cdot\operatorname{mes}(\bigcup\delta_n^{-m})<\left(\frac{1+\varepsilon}{\lambda^s}\right)^{n+1}.$$

The estimates of Sects. 6–8 prove the following

Proposition 1. *Let $\lambda\in[N_0, N_0+4]$ be so that for any $n=1,2,\ldots,F_{n+1}\circ h(\lambda,\frac{1}{2})\in[0,1]\setminus\bigcup_{m=0}^{\infty}(\bigcup\hat{\delta}_n^{-m})$. Then a partition ξ_λ as in Sect. 1 exists.*

Remark VIII/3. Notice that if λ is such that at step n $F_{n-1}\circ h(\lambda,\frac{1}{2})$ falls into a limit set $\mathscr{F}_n$ defined in Sect. 3 the condition of Proposition 1 will be satisfied. It is certainly so at Step n, and at subsequent steps the holes $\hat{\delta}_p^{-m}$ lie either in $\hat{\delta}_n^{-m}$, or in the intervals $F_{n-1}\circ g$, $\hat{f}_n$ constructed at Step n (there is no middle branch $F_p\circ h$ for $p\geqq n$). The estimates of Sects. 6–8 are even better in this case.

Remark VIII/4. If we suppose $F_{n-1}\circ h(\lambda,\frac{1}{2})$ is outside $\hat{\delta}_1(\lambda)=(\frac{1}{2}-\lambda^{-(s-\alpha)}, \frac{1}{2}+\lambda^{-(s-\alpha)})$ for all n, the above condition of Proposition 1 will be satisfied. In particular, if $h(\lambda,\frac{1}{2})$ falls into some f_λ-invariant set (e.g. periodic orbit or invariant Cantor set of [5]) lying outside $\hat{\delta}_1(\lambda)$, λ satisfies this condition. Thus card$\{\lambda$ satisfying Proposition 1$\}$ equals the continuum. One can check however, using estimates of Sect. 11, that $\operatorname{mes}\{\lambda: F_{n-1}\circ h(\lambda,\frac{1}{2})\in[0,1]\setminus\hat{\delta}_1(\lambda)\}=0$.

9. Velocities of Endpoints of Domains $\Delta f_n(\lambda)$

Let f_n be one of the maps constructed at step n, with domain $\Delta f_n=[x_{1n}, x_{2n}]$. In this section we prove the following

Lemma 3. *There is an $\varepsilon=O(\lambda^{-s(1-\nu)})$ such that for $i=1,2$*

$$\left|\frac{dx_{in}(\lambda)}{d\lambda}\right|<\frac{\lambda^{sn}(1+\varepsilon)}{8\lambda}.$$

Proof. Any $x_{ik}(\lambda)$ satisfies $f_k(\lambda, x_{ik}(\lambda))=0$ or 1. Thus

$$\left|\frac{dx_{ik}(\lambda)}{d\lambda}\right|=\left|\frac{\partial f_k(\lambda,x_{ik}(\lambda))/\partial\lambda}{\partial f_k(\lambda,x_{ik}(\lambda))/\partial x}\right|.$$

We proceed by induction as in the main construction. Consider the maps $f_k(2\leqq k\leqq n)$, $G_n:\delta_n^{-m}\to\delta_n$, and $\hat{f}_n$. Assume inductively that the following estimates hold:

$$c_{1k})\ \left|\frac{\partial f_k(\lambda,x)/\partial_\lambda}{\partial f_k(\lambda,x)/\partial x}\right|<\frac{\lambda^{sk}}{8\lambda}\left[1+\sum_{i=1}^{k-1}\frac{1+\varepsilon}{\lambda^{s(1-\nu)i}}\right].$$

c_{2n}) Let H_n denote either G_n or $\hat{f}_n$, and pick p so that, if $p \leqq n$, then $\Delta H_n \subset [0,1] \backslash \delta_p$, while if $p > n$, then $\operatorname{dist}(\Delta H_n, \frac{1}{2}) > \lambda^{-sp}$. Let $N = \max(n, p)$. Then

$$\left|\frac{\partial H_n(\lambda, x)/\partial \lambda}{\partial H_n(\lambda, x)/\partial x}\right| < \frac{\lambda^{sN}}{8\lambda}\left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-\nu)i}}\right].$$

For $k=1$, these estimates are proven in Sect. 2. We will prove $c_{1\,n+1}$ in the various cases that arise from the construction. $c_{2\,n+1}$ is similar. In particular, c_{1n} implies Lemma 3.

Suppose $\varphi_i(\lambda, x)$, $i = 1, \ldots, n$ are C^1 functions, and define

$$F(\lambda, x) = \varphi_n(\lambda, \varphi_{n-1}(\lambda, \ldots, \varphi_1(\lambda, x) \ldots)).$$

One sees that

$$\frac{\partial F}{\partial \lambda} = \sum_{k=1}^{n} \left[\left(\frac{\partial \varphi_k}{\partial \lambda}\right) \prod_{i=k+1}^{n} \frac{\partial \varphi_i}{\partial x}\right]$$

so that, at any point (λ_0, x_0) in the domain of F,

$$\frac{\partial F/\partial \lambda}{\partial F/\partial x} = \sum_{k=1}^{n} \left[\left(\frac{\partial \varphi_k/\partial \lambda}{\partial \varphi_k/\partial x}\right) \cdot \left(\prod_{i=1}^{k-1} \frac{\partial \varphi_i}{\partial x}\right)^{-1}\right], \tag{9.1}$$

where the partials of φ_i are evaluated at $(\lambda_0, \varphi_{i-1}(\lambda_0, \ldots, \varphi_1(\lambda_0, x_0) \ldots))$.

To prove $c_{1\,n+1}$, we first consider the case

$$f_{n+1\,1} = f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g_n(\lambda, x).$$

Since

$$g_n(\lambda, x) = \lambda x(1-x) \quad |x - \tfrac{1}{2}| > \lambda^{-s(n+1)},$$

we have

$$|\partial g_n/\partial \lambda| < \tfrac{1}{4}, \; |\partial g_n/\partial x| > 2\lambda/\lambda^{s(n+1)}.$$

Using (9.1), a_{1n}^2 and $c_{1k}(k = i_1, \ldots, i_n)$, we obtain

$$\begin{aligned}\left|\frac{\partial f_{n+1\,1}/\partial \lambda}{\partial f_{n+1\,1}/\partial x}\right| &< \frac{\lambda^{s(n+1)}}{2\lambda}\left[\frac{1}{4} + \frac{\lambda^s(1+\varepsilon)}{8\lambda}\left(1 + \frac{1}{\lambda^{s(1-\nu)} - 1}\right)\right. \\ &\quad \left.\cdot \left(1 + \frac{\lambda^s}{2\lambda^{c_0}} + \ldots + \left(\frac{\lambda^s}{2\lambda^{c_0}}\right)^{n-1}\right)\right] \\ &< \frac{\lambda^{s(n+1)}}{8\lambda}\left[1 + \frac{1+\varepsilon_{9.2}}{2\lambda^{c_0}}\right].\end{aligned} \tag{9.2}$$

This proves $c_{1\,n+1}$ in case $f_{n+1\,1} = f_k \circ F_{n-1} \circ g_n$.

In case $f_{n+1\,1} = f_k \circ \hat{f}_n$, (9.1), c_{1k} and c_{2n}, and a_{3n} imply

$$\left|\frac{\partial f_{n+1\,1}/\partial \lambda}{\partial f_{n+1\,1}/\partial x}\right| < \frac{\lambda^{s(n+1)}}{8\lambda}\left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-\nu)i}} + \frac{1}{\lambda^{\gamma n}}\right]. \tag{9.3}$$

Similarly, if $F_{n-1} = f_{i_{n-1}} \circ \ldots \circ f_{i_1}$ and $|x - \frac{1}{2}| > \lambda^{-sp}$, then

$$\left|\frac{\partial (F_{n-1} \circ g)/\partial \lambda}{\partial (F_{n-1} \circ g)/\partial x}\right| < \frac{\lambda^{sp}}{8\lambda}\left(1 + \frac{1+\varepsilon_{9.4}}{2\lambda^{c_0}}\right). \tag{9.4}$$

Now let $G'_n = G_n \circ F_{n-1} \circ g$, where $\Delta(F_{n-1} \circ g) \subset [0,1] \setminus \left(\frac{1}{2} - \frac{1}{\lambda^{sp}}, \frac{1}{2} + \frac{1}{\lambda^{sp}}\right)$, and $\Delta G_n \subset [0,1] \setminus \delta_n$. Using (9.1), c_{2n}, a_{2n} and a^2_{4n}, we see that

$$\left|\frac{\partial G'_n/\partial\lambda}{\partial G'_n/\partial x}\right| < \frac{\lambda^{sp}}{8\lambda}\left(1 + \frac{1+\varepsilon_{9.4}}{2\lambda^{co}}\right) + \frac{\lambda^{sn}(1+\varepsilon)}{(8\lambda)(2\lambda)^{co(n-1)}(2\lambda^{1-sp})} \sum_{i=0}^{n-1} \lambda^{-s(1-v)i}$$

$$= \frac{\lambda^{sp}}{8\lambda}\left(1 + \frac{1+\varepsilon_{9.5}}{2\lambda^{co}}\right). \tag{9.5}$$

On the other hand, for $G'_n = G_n \circ \hat{f}_n$ where $\Delta \hat{f}_n \subset [0,1] \setminus \left(\frac{1}{2} - \frac{1}{\lambda^{sp}}, \frac{1}{2} + \frac{1}{\lambda^{sp}}\right)$, we obtain an estimate similar to (9.3):

$$\left|\frac{\partial G'_n/\partial\lambda}{\partial G'_n/\partial x}\right| < \frac{\lambda^{sp}}{8\lambda}\left(1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{\gamma n}}\right). \tag{9.6}$$

Finally, let $\tilde{G}_n = G_n$ or G'_n, $\Delta \tilde{G}_n \subset \delta_n \setminus \delta_{n+1}$. Then in c_{2n}, (9.4) and (9.6), $p = n+1$. Now a^1_{4n} implies $|D\tilde{G}_n| > \lambda^{ns(1-v)}$. Hence using (9.1), (9.2) or (9.3) and c_{2n}, (9.5) and (9.6), we obtain for $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_{k-1}}$

$$\left|\frac{\partial f_{n+1\,k}/\partial\lambda}{\partial f_{n+1\,k}/\partial x}\right| < \frac{\lambda^{s(n+1)}}{8\lambda}\left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{\gamma n}}\right]\left[1 + \frac{1}{\lambda^{s(1-v)n}} + \ldots + \frac{1}{\lambda^{s(1-v)(k-1)n}}\right]$$

$$< \frac{\lambda^{s(n+1)}}{8\lambda}\left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{\gamma n}}\right]\left[1 + \frac{1}{\lambda^{s(1-v)n} - 1}\right]$$

$$< \frac{\lambda^{s(n+1)}}{8\lambda}\left[1 + \sum_{i=1}^{n} \frac{1+\varepsilon}{\lambda^{s(1-v)i}}\right] \tag{9.7}$$

for a suitable $\varepsilon = O(\lambda^{-s(1-v)})$.

This proves $c_{1\,n+1}$; the proof of $c_{2\,n+1}$ is similar.

10. Construction of $\delta_{n+1}(\lambda)$ and $\hat{\delta}_{n+1}(\lambda)$. Structure of $\mathscr{X}_{n+1}$ in a $\lambda^{(-s+2\alpha)(n+1)}$-Neighborhood of δ_{n+1}

a) Recall that at step $n+1$ of the induction construction, we consider λ contained in an interval $\mathscr{J}_n = [\lambda_{0n}, \lambda_{1n}]$. As λ varies in $\mathscr{J}_n$, all the maps under consideration together with their domains vary continuously with λ.

The induction hypotheses a_{in} in Sect. 6 imply the following estimates on the diameters of the domains appearing at step $n+1$:

$$\left.\begin{aligned} &|\Delta f_k \circ \hat{f}_n| < \lambda^{-s} \cdot (2\lambda^{c_1})^{-(n+1)} \\ &|\delta^{-m}_{np_0}| < |\delta_n| \cdot \lambda^{-s(1-v)p_0} \\ &|\Delta f_k \circ F_{n-1} \circ g| < (2\lambda^{c_1})^{-(n+1)}. \end{aligned}\right\} \tag{10.1}$$

In the second estimate of (10.1), we write δ_n^{-m} *as* $\delta_{np_0}^{-m}$, where p_0 denotes the minimum integer p such that

$$\delta_n^{-m} \subset [\tfrac{1}{2} - \lambda^{-sp}, \tfrac{1}{2} + \lambda^{-sp}].$$

In the third estimate, recall that $\Delta F_{n-1} \circ g = [x_{\min}, x_{\max}]$ with

$$|x_{\min} - \tfrac{1}{2}| > \lambda^{-s(n+1)}.$$

For any $G_n : \delta_n^{-k} \to \delta_n$ we have, according to b_{4n} of Sect. 6, that $\mu(G_n) < \lambda^{-\alpha n}$. But actually for $\delta_n^{-k} \subset [0,1] \backslash \delta_{n+1}$, we can strongly enlarge δ_n^{-k} and still have the maps G_n defined with $\mu(G_n)$ small. Let us consider the homothetic transformation

$$\psi_n(\lambda) : x \mapsto \tfrac{1}{2} + (x - \tfrac{1}{2})\lambda^{2\alpha n}.$$

It follows from the condition $\alpha \leqq s/4$ that for $n \geqq 3$ one can define

$$q(n) = \max\{q : \psi_n(\lambda)\delta_n(\lambda) \subset \delta_q(\lambda)\} \geqq 1.$$

Remark X/1. For $n=1$, the endpoints of $\psi_1(\lambda)\delta_1(\lambda)$ belong to $\bigcup \Delta f_1$. We define $\delta_{q(1)}(\lambda)$ for all $\lambda \in \mathscr{J}_n$ as the minimal interval containing $\psi_1(\lambda)\delta_1(\lambda)$ of the form $[x_{1\max}(\lambda), x_{2\max}(\lambda)]$, where $x_{1\max}(\lambda) \in [0, \tfrac{1}{2}]$ and $x_{2\max}(\lambda) \in [\tfrac{1}{2}, 1]$ are endpoints of domains Δf_1. We define $\delta_{q(2)}(\lambda)$ in an analogous way whenever $\psi_2(\lambda)\delta_2(\lambda) \not\subset \delta_1(\lambda)$.

It follows from the construction of Sect. 3 that for every interval $G_n^{-1}\delta_n$ (or $(G_n')^{-1}\delta_n$) which lies outside the domain $\Delta F_{q(n)} \circ h$, the corresponding preimage $G_n^{-1}\delta_{q(n)}$ is defined. Indeed, the maps G_n under consideration are those compositions of $G_{q(n)}$ and $F_k \circ g$ or $\hat{f}_k$, $q(n) \leqq k \leqq n$, which map their domains onto $[0,1]$. Using Lemma 1 and following the proof of b_{4n}, we get for some $\varepsilon_{10.2} < \lambda^{-t}$

$$\mu(G_n, \delta_{q(n)}^{-k}) < (1 + \varepsilon_{10.2})\lambda^{-\alpha q(n)}. \tag{10.2}$$

From the definition of $q(n)$ for $n \geqq 3$ it follows that

$$q(n) \geqq \max\left\{q : \left(1 - 2\frac{\alpha}{s}\right)n > 1\right\}.$$

Since $2\dfrac{\alpha}{s} \leqq \tfrac{1}{2}$, we get

$$q(n) \geqq \max\left\{q : \frac{n}{2} > q\right\} = \begin{cases} \dfrac{n}{2} - 1 & \text{for} \quad n \text{ even} \\[2ex] \dfrac{n}{2} - \tfrac{1}{2} & \text{for} \quad n \text{ odd}. \end{cases}$$

In particular, we always have

$$q(n) \geqq \frac{n}{2} - 1. \tag{10.3}$$

We shall show that for $n \geqq 3$

$$\Delta F_{q(n)} \circ h \subset \left(\tfrac{1}{2} - \frac{1}{\lambda^{s(n+1)}}, \tfrac{1}{2} + \frac{1}{\lambda^{s(n+1)}}\right). \tag{10.4}$$

Let $\Delta F_{q(n)} \circ h = [\frac{1}{2} - v_{1n}, \frac{1}{2} + v_{2n}]$, $v_{in} > 0$. In a way analogous to (6.11) we get

$$v_{in} < 1/[(\sqrt{2})^{q(n)} \lambda^{(c_0 q(n)+1)/2}]. \tag{10.5}$$

From (10.3) and (10.5) we obtain that for (10.4) it is enough to have $\frac{c_0}{2}\left(\frac{n}{2} - 1\right) + \frac{1}{2} > s(n+1)$, or taking into account that $c_0 = 1 - s$, $\left(\frac{1-s}{4} - s\right) n > \frac{s}{2}$. This holds for $s \leqq \frac{1}{13}$, $n \geqq 1$.

From the fact that for $s \leqq \frac{1}{13}$ the domain of the central branch $\Delta h \subset \delta_6$, it follows that for $n \leqq 5$ if $G_n^{-1} \delta_n \subset \delta_n \backslash \delta_{n+1}$, then $G_n^{-1}[0,1]$ is defined.

In such a way, for all $n \geqq 1$ and for all domains

$$G_n^{-1} \delta_n \subset [0,1] \Big\backslash \left(\frac{1}{2} - \frac{1}{\lambda^{s(n+1)}}, \frac{1}{2} + \frac{1}{\lambda^{s(n+1)}}\right)$$

the preimage

$$G_n^{-1} \delta_{q(n)} \supset G_n^{-1}(\psi_n(\lambda) \delta_n(\lambda))$$

is defined.

b) Let us estimate the length of $\mathscr{J}_n$. When λ varies in $\mathscr{J}_n$

$$f_{i_{n-1}} \circ f_{i_{n-2}} \circ \ldots \circ f_{i_1} \circ h(\lambda, \tfrac{1}{2})$$

varies in one of Δf_{i_n} and $f_{i_n} \circ f_{i_{n-1}} \ldots f_{i_1} \circ h(\lambda, \frac{1}{2})$ varies in $[0,1]$. We have

$$\frac{\partial(f_{i_n} \circ \ldots \circ f_{i_1} \circ h(\lambda, \frac{1}{2}))}{\partial \lambda} = \prod_{k=1}^{n} \frac{\partial f_{i_k}}{\partial x} \left[\frac{1}{4} + \sum_{j=1}^{n} \frac{\partial f_{i_j}/\partial \lambda}{\partial f_{i_j}/\partial x} \left(\prod_{\ell=1}^{j-1} \frac{\partial f_{i_\ell}}{\partial x}\right)^{-1}\right], \tag{10.6}$$

where the arguments of $f_{i_p}(\lambda, x)$ are $x = f_{i_{p-1}} \circ f_{i_{p-2}} \circ \ldots \circ f_{i_1} \circ h(\lambda, \frac{1}{2})$. In consequence of c_{1k} and a_{1n}^2, the sum in brackets is larger than $\frac{1}{4} - \frac{(1+\varepsilon_{10.6})\lambda^s}{8\lambda} > \frac{1}{4}(1 - \lambda^{-c_0})$. We shall use $v_n(\lambda)$ to denote the velocity of the top. We have

$$v_n(\lambda) = \left|\frac{\partial(f^n \circ \ldots \circ f^1 \circ h(\lambda, \frac{1}{2}))}{\partial \lambda}\right| > (2\lambda^{c_0})^n \cdot \frac{1}{4}\left(1 - \frac{1}{\lambda^{c_0}}\right). \tag{10.7}$$

Thus

$$|\mathscr{J}_n| < 4 \cdot (1 - \lambda_{0n}^{-c_0})^{-1} \cdot (2\lambda_{0n}^{c_0})^{-n}. \tag{10.8}$$

We formulate the induction conditions on the choice of $\delta_n(\lambda)$.

i) The interval $\delta_n(\lambda)$ is of the form:

$$\delta_n(\lambda) = (\tfrac{1}{2} - c_{n1}(\lambda) \cdot \lambda^{-sn}, \tfrac{1}{2} + c_{n2}(\lambda) \cdot \lambda^{-sn}), \qquad 1 \leqq c_{ni}(\lambda) < 1 + o(\lambda^{-tn}). \tag{10.9}$$

ii) If for some δ_n^{-k} $\delta_{q(n)}^{-k} \cap \delta_n \neq \emptyset$, then $\delta_{q(n)}^{-k} \subset \delta_n$.

iii) If a_n is an endpoint of δ_n, then a_n coincides with a common endpoint of two intervals: some Δf_n exterior to δ_n and some $\Delta F_{n-1} \circ g$ or $\Delta \hat{f}_n$ interior to δ_n.

According to the construction of Sect. 3 we consider intervals $\Delta f_k \circ F_{n-1} \circ g$, $\Delta f_k \circ \hat{f}_n$, $(G_n')^{-1} \delta_n$, and have to choose an interval $\delta_{n+1}(\lambda)$ satisfying the above conditions and varying continuously with $\lambda \in [\lambda_{0n}, \lambda_{1n}]$.

Consider the point $\xi_{0n}=\frac{1}{2}-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\lambda_{0n}^{-s(n+1)}$. For $k=q(n)$, (ii) implies that if two intervals $\delta_{q(n)}^{-m}$ intersect, then one of them contains the other. Let $\bar{\delta}_{q(n)}^{-m}$ be the maximal interval containing ξ_{0n}. Then we replace ξ_{0n} by ξ_{1n} which is the endpoint of $\bar{\delta}_{q(n)}^{-m}$. If ξ_{0n} is not contained in any $\delta_{q(n)}^{-m}$, but is inside some interval $\Delta f_k \circ F_{n-1} \circ g$ or $\Delta f_k \circ \hat{f}_n$, we let ξ_{1n} be the right endpoint of this interval. If ξ_{0n} is a limit point of δ_n^{-m} we obtain any of the previous cases with an arbitrary small perturbation of ξ_{0n}. The estimates (10.1), c_{2n}, (10.8), (10.9) show that when $\lambda \geqq \lambda_{0n}$ varies in $\mathscr{I}_n$

$$\begin{aligned}\tfrac{1}{2} &-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\cdot\lambda_{0n}^{-s(n+1)}-(1+\varepsilon_{10.10})\lambda_{0n}^{-s}(2\lambda_{0n}^{c_1})^{-(n+1)}<\xi_{1n}(\lambda)\\ &<\tfrac{1}{2}-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\cdot\lambda_{0n}^{-s(n+1)}+2(1+\varepsilon_{10.10})\lambda_{0n}^{-[sq(n)+s(1-\nu)]n}\\ &\quad+(1+\varepsilon_{10.10})\lambda_{0n}^{-s}(2\lambda_{0n}^{c_1})^{-(n+1)}. \end{aligned}\tag{10.10}$$

We shall show that for $n\geqq 7$ $\lambda_{0n}^{-[sq(n)+s(1-\nu)n]} \ll \lambda_{0n}^{-\left(\frac{\alpha}{8}+s\right)(n+1)}$. For this it is enough to have $sq(n)+s(1-\nu)n>\left(s+\frac{\alpha}{8}\right)(n+1)$. Since

$$q(n)\geqq\frac{n}{2}-1,\qquad \alpha\leqq\frac{s}{4},\qquad \nu=\frac{2(s-\alpha)}{1-s}<\frac{2s}{1-s},$$

we get the inequality $n\left(\frac{1}{2}-\frac{2s}{1-s}-\frac{1}{32}\right)>2+\frac{1}{32}$, which holds for $n\geqq 7$, $s\leqq\frac{1}{13}$.

For $n\leqq 6$ the check that for $\delta_{q(n)}^{-1}\subset\delta_n\backslash\delta_{n+1}$, $|\delta_{q(n)}^{-1}|\ll\frac{1}{\lambda^{s(n+1)}}$ is straightforward. The worst estimates correspond to $n=6$. Since $q(6)\geqq 2$ and $Dh|_{\delta_6\backslash\delta_7}>2\lambda^{1-7s}$, we get

$$|h^{-1}\delta_2|<\frac{1+\varepsilon}{\lambda^{1-5s}}\ll\frac{1}{\lambda^{7s}}$$

for $s\leqq\frac{1}{13}$.

Taking into account (10.8) and the formula $c_1=1-2s\geqq 11s$, we obtain from (10.10)

$$\tfrac{1}{2}-(1+2\cdot\lambda^{-\frac{\alpha}{8}(n+1)})\lambda^{-s(n+1)}<\xi_{1n}(\lambda)<\tfrac{1}{2}-\lambda^{-s(n+1)}$$

and we can make ξ_{1n} the left endpoint of $\delta_{n+1}(\lambda)$. The analogous choice of the right endpoint gives us

$$\delta_{n+1}(\lambda)=(\tfrac{1}{2}-(1+c_{n+1\,1})\lambda^{-s(n+1)},\tfrac{1}{2}+(1+c_{n+1\,2})\lambda^{-s(n+1)})c_{n+1\,i}=o(\lambda^{-t(n+1)}).\tag{10.11}$$

One easily checks that $\delta_{n+1}(\lambda)$ also satisfies (ii) and (iii).

c) We then construct an enlarged interval $\hat{\delta}_{n+1}(\lambda)$. We begin by expanding $\delta_{n+1}(\lambda_{0n})$ with a homothetic transformation

$$\varphi_{n+1}: x\to\tfrac{1}{2}+(x-\tfrac{1}{2})\lambda_{0n}^{\alpha(n+1)}(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)}).$$

Then we proceed with the endpoints of $\varphi_{n+1}\delta_{n+1}(\lambda_{0n})$ as above, i.e. using a small perturbation we make the endpoints of $\varphi_{n+1}\delta_{n+1}(\lambda_{0n})$ coincide with endpoints of some interval Δf_k, $k \leqq n$. One checks as above, that this can be done so that the interval $\hat{\delta}_{n+1}(\lambda)$ satisfies for all $\lambda \in \mathscr{J}_n = [\lambda_{0n}, \lambda_{1n}]$ the inequalities:

$$\lambda^{\alpha(n+1)} |\delta_{n+1}(\lambda)| < |\hat{\delta}_{n+1}(\lambda)| < \lambda^{\alpha(n+1)} |\delta_{n+1}(\lambda)| \left(1 + O(\lambda^{-\frac{\alpha}{8}(n+1)})\right). \tag{10.12}$$

As $\hat{\delta}_{n+1}(\lambda) \subset \delta_{q(n+1)}(\lambda)$, for any $\delta_{n+1}^{-k}(\lambda) = G_{n+1}^{-1} \delta_{n+1}(\lambda)$ the corresponding interval $\hat{\delta}_{n+1}^{-k} = G_{n+1}^{-1} \hat{\delta}_{n+1}(\lambda)$ is defined. Taking into account an additional factor $\exp \mu(G_{n+1}, \hat{\delta}_{n+1}^{-k}) < (1+\varepsilon)\lambda^{-(s-\alpha)(n+1)}$ we still have

$$\lambda^{\alpha(n+1)} |\delta_{n+1}^{-k}(\lambda)| < |\hat{\delta}_{n+1}^{-k}(\lambda)| < \lambda^{\alpha(n+1)} \cdot (1 + o(\lambda^{-t(n+1)})) \cdot |\delta_{n+1}^{-k}(\lambda)| \tag{10.13}$$

d) When estimating $\operatorname{mes} \mathscr{M}_{n+1}$ we shall use the following

Lemma 4. *For any n there is a set $L_n \subset \mathscr{X}_n$ corresponding to δ_n, and for any $\delta_n^{-k} \subset [0,1] \backslash \delta_n$ there is a corresponding set $L_n^{-k} \subset \mathscr{X}_n$, such that*

(a) *if $\delta_n^{-k_1} \neq \delta_n^{-k_2}$ then $L_n^{-k_1} \cap L_n^{-k_2} = \emptyset$*

and

(b) $\operatorname{mes}(L_n^{-k}) > (1-\varepsilon_0)\lambda^{2\alpha n} \operatorname{mes}(\delta_n^{-k})$, *with* $\varepsilon_0 = O(\lambda^{-2\alpha})$.

Proof. In addition to the estimate $\alpha \leqq s/4$, we will suppose that α has the form

$$\alpha = s/2k_0,$$

where k_0 is an integer $\geqq 2$. This assumption is not really necessary, but it simplifies the notation.

If an interval δ with center x_0 and a number $c > 0$ are given, we shall denote by $c \cdot \delta$ the image of δ under the homothetic transformation $x \to x_0 + (x - x_0) \cdot c$. Further, we shall use $\delta_n^{(r)}$ to denote the set $\lambda^{2\alpha r} \cdot \delta_n \backslash \lambda^{2\alpha(r-1)} \cdot \delta_n$.

Let $\varepsilon_1 = 3 \cdot \lambda^{-2\alpha}$, $\psi_0 = 0$, $\psi_i = (2 \cdot \lambda^{-(s-2\alpha)})^i$, $i \geqq 1$, $c_n = \prod_{i=0}^{n-1} (1 + \psi_i)$, $c = \lim_{n \to \infty} c_n$.

We prove Lemma 4 by induction. We assume that L_n, L_n^{-k} are constructed and consist of Δf_r, $r \leqq n$, and that the following property holds: For any δ_n^{-k}, $k = 0, 1, \ldots$ there exists an increasing sequence of intervals $(\lambda^{2\alpha r} \cdot \delta_n)^{-k}$, $r = 0, 1, \ldots, R = R(\delta_n^{-k}) \geqq n$, such that

$$\frac{\operatorname{mes}(\lambda^{2\alpha(r-1)} \cdot \delta_n)^{-k}}{\operatorname{mes}(L_n^{-k} \cap \delta_n^{(r)})} < \frac{c_n(1+\varepsilon_1)}{\lambda^{2\alpha}}. \tag{10.14}$$

We define L_{n+1}^{-m} corresponding to δ_{n+1}^{-m} and prove (10.14) for $n+1$. Then Lemma 4 follows with $1 - \varepsilon_0 = c^{-1} \cdot (1+\varepsilon_1)^{-1}$.

Consider $\lambda^{2\alpha n} \cdot \delta_n \subset \delta_{q(n)}$. Condition ii and the construction of δ_{n+1} imply $\delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}$ for $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$. Considering maximal elements $\bar{\delta}_{q(n)}^{-m}$ among $\{\delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}\}$ and the corresponding diffeomorphisms $\bar{G}_n^{-m}$, we transmit the structure from $\lambda^{2\alpha n} \cdot \delta_n$ into each $\bar{\delta}_{q(n)}^{-m}$ and obtain that corresponding to any $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$ one can pick $L_n^{-m} \subset \delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}$ so that $L_n^{-m} \cap L_n^{-r} = \emptyset$, if $\delta_n^{-m} \neq \delta_n^{-r}$ and (10.14) multiplied by an additional factor $\exp(\lambda^{-\alpha q(n)})$ holds for L_n^{-m}.

Let us consider the domain $V_{n+1} = \lambda^{2\alpha n} \cdot \delta_n \backslash \lambda^{2\alpha(n+1)} \cdot \delta_{n+1}$. Taking into account (10.9), (10.11), and $s = 2k_0\alpha$, $k_0 \geqq 2$, we obtain

$$\operatorname{mes}(V_{n+1} \triangle (\lambda^{2\alpha n} \cdot \delta_n \backslash \lambda^{2\alpha[n-(k_0-1)]} \cdot \delta_n)) = o(\lambda^{-(s-2\alpha)(n+1)}).$$

Together with (10.14) this implies

$$c_n(1+\varepsilon_1)(1+\chi_{1\,n+1})\cdot \operatorname{mes}(L_n \cap V_{n+1}) > \lambda^{2\alpha}\cdot \operatorname{mes}(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1}) \tag{10.15}$$

(here and below $\chi_{i\,n+1} = o(\lambda^{-t(n+1)})$).

For $\delta_{n+1}^{-m} \subset \delta_n^{-m} \subset \delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}$ the corresponding set V_{n+1}^{-m} is defined and

$$c_n(1+\varepsilon_1)(1+\chi_{2\,n+1})\cdot \operatorname{mes}(L_n^{-m} \cap V_{n+1}^{-m}) > \lambda^{2\alpha}\cdot \operatorname{mes}(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}. \tag{10.16}$$

We define $\bar{L}_{n+1} = \bigcup (L_n^{-m} \cap V_{n+1}^{-m})$ where the sum is taken over all $L_n^{-m} \subset \delta_n \backslash \delta_{n+1}$. For any $r \geqq 1$ such that $\delta_n \supset \lambda^{2\alpha r}\cdot \delta_{n+1}$ consider $\delta_{q(n)}^{-m} \subset \delta_{n+1}^{(r)}$ and corresponding $(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}$, $V_{n+1}^{-m} \subset \delta_{q(n)}^{-m}$. Since the dimensions of $\delta_{q(n)}^{-m}$ are small compared to $\delta_{n+1}^{(r)}$ (see the proof of 10.10 above) we obtain from (10.16) that

$$\frac{\operatorname{mes}(\lambda^{2\alpha(r-1)}\cdot \delta_{n+1})}{\operatorname{mes}(\bar{L}_{n+1} \cap \delta_{n+1}^{(r)})} < \frac{1}{\lambda^{2\alpha}} \cdot \frac{(1-\lambda^{-2\alpha})^{-1}}{1 - \dfrac{c_n\cdot(1+\varepsilon_1)(1+\chi_{2n})}{\lambda^{2\alpha}}}. \tag{10.17}$$

Besides, for any $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$,

$$(L_n^{-m} \cap [(\lambda^{2\alpha n}\cdot \delta_n)^{-m} \backslash V_{n+1}^{-m}]) = (L_n^{-m} \cap (\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}).$$

All L_n^{-m} consist of domains $\Delta \hat{f}_n$ and $\Delta(F_{n-1} \circ g)$. At Step $n+1$ when constructing $\Delta f_{n+1\,1}$ we reproduce the structure from $[0,1] \backslash \delta_n$ on each $\Delta \hat{f}_n$ or $\Delta(F_{n-1} \circ g_n)$ using respectively $\hat{f}_n^{-1}$ or $(F_{n-1} \circ g_n)^{-1}$. We denote by $(\delta_n^{-m})'$, $(\delta_{q(n)}^{-m})'$ the new preimages of δ_n, $\delta_{q(n)}$ under $\hat{f}_n^{-1}$, $(F_{n-1} \circ g_n)^{-1}$. The estimate of $\operatorname{mes}\left[\bigcup_{m=0}^{\infty} \bigcup \delta_n^{-m}\right]$ from Sect. 8 together with the estimate of $\mu(G_{q(n)}, [0,1] \backslash \delta_{n+1})$ show that after excluding the set $\bigcup_{m=0}^{\infty} \bigcup (\delta_{q(n)}^{-m})'$ from each $\Delta \hat{f}_n$ or $\Delta(F_{n-1} \circ g_n)$ the measures of $\bar{L}_{n+1}$ and of any $L_n^{-m} \cap (\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}$ are multiplied by a factor larger than $1-(2\cdot \lambda^{-(s-2\alpha)})^n$. This factor implies the passage from ε_n to ε_{n+1} in estimates (10.14) for $(\lambda^{2\alpha r} \cdot \delta_{n+1})^{-m}$. We let $(L_n^{-m})' \cap [(\lambda^{2\alpha(n+1)} \delta_{n+1})^{-m}]$ correspond to $(\delta_{n+1}^{-m})'$. Thus to each δ_{n+1}^{-m}, $(\delta_{n+1}^{-m})' \subset \delta_n \backslash \delta_{n+1}$ uniquely corresponds its $\lambda^{2\alpha(n+1)}$–enlargement which does not intersect $\bar{L}_{n+1} \backslash \bigcup (\delta_{q(n)}^{-m})'$. We now set

$$L_{n+1} = \begin{cases} L_n & \text{outside } \delta_n \\ \bar{L}_{n+1} \backslash \bigcup (\delta_{q(n)}^{-m})' & \text{inside } \delta_n \backslash \delta_{n+1}. \end{cases}$$

Notice that $L_{n+1} \cap \delta_n \backslash \delta_{n+1}$ consists of $\Delta f_{n+1\,1}$. (10.17) together with the estimate of $\bigcup (\delta_{q(n)}^{-m})'$ gives

$$\frac{\operatorname{mes}(\lambda^{2\alpha(\mu-1)}\cdot \delta_{n+1})}{\operatorname{mes}(L_{n+1} \cap \delta_{n+1}^{(\mu)})} < \lambda^{-2\alpha}(1+2.5\cdot \lambda^{-2\alpha}) \tag{10.18}$$

and (10.14) follows for δ_{n+1}. The maps G_n^{-1}, $G_n'^{-1}$ and their compositions transmit (10.18) on $(\delta_{n+1}^{(r)})^{-m} \subset \delta_n^{-m} \backslash \delta_{n+1}^{-m}$ with an additional factor $\exp \lambda^{-\alpha q(n)}$. Joining it to the above estimate of

$$\operatorname{mes}[(L_n^{-m} \cap (\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}) \backslash \bigcup (\delta_{q(n)}^{-m})']$$

finishes the proof of (10.14) and of Lemma 4.

Remark X/2. The above construction is similar to one used in Sect. 8 in order to estimate the measure of holes at Step $n+1$.

Remark X/3. R which bounds r in (10.14), may be much larger than n. For example, the construction implies that the consecutive $\lambda^{2\alpha r}$-enlargements of δ_n are taken until we obtain the whole interval [0, 1].

11. The Positivity of Measure

Remember that at step $n+1$ we consider $\lambda \in \mathscr{J}_n = [\lambda_{0n}, \lambda_{1n}]$. As λ varies in $\mathscr{J}_n$, $F_{n-1} \circ h(\lambda, \frac{1}{2})$ traverses some Δf_n and $f_n \circ F_{n-1} \circ h(\lambda, \frac{1}{2}) = F_n \circ h(\lambda, \frac{1}{2})$ traverses [0, 1]. The set $\mathscr{X}_{n+1}(\lambda) = \bigcup_{k=1}^{n+1} \Delta f_k$ is defined for all $\lambda \in \mathscr{J}_n$, and all the domains $\Delta f_k = (\Delta f_k)_\lambda$ as well as the holes $\delta_{n+1}^{-m}(\lambda)$ and their enlargements $\hat{\delta}_{n+1}^{-m}(\lambda)$ vary continuously with $\lambda \in \mathscr{J}_n$. We then define $\mathscr{M}_{n+1} \cap \mathscr{J}_n$ as the set consisting of those $\lambda \in \mathscr{J}_n$ for which

$$F_n \circ h(\lambda, \tfrac{1}{2}) \in \mathscr{X}_{n+1}(\lambda) \Big\backslash \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda).$$

We saw in Sect. 10 that the velocity of the top satisfies

$$v_n(\lambda) = \left| \frac{\partial}{\partial \lambda} F_n \circ h(\lambda, \tfrac{1}{2}) \right| > (2\lambda^{c_0})^n [4(1+\varepsilon_{11.1})]^{-1}. \tag{11.1}$$

At the same time the endpoints $x_k(\lambda)$ of $\Delta f_k(\lambda)$, $k \leqq n+1$, move with velocities

$$\left| \frac{dx_k}{d\lambda} \right| < \frac{(1+\varepsilon_{11.2})}{8\lambda} \cdot \lambda^{s(n+1)}. \tag{11.2}$$

(11.1) and (11.2) imply that for any Δf_k, $k \leqq n+1$, the condition $F_n \circ h(\lambda, \frac{1}{2}) \in \Delta f_k(\lambda)$ defines an interval $\mathscr{J}(\Delta f_k) \subset \mathscr{J}_n$, as does the condition $F_n \circ h(\lambda, \frac{1}{2}) \in \hat{\delta}_{n+1}^{-m}(\lambda)$.

A priori the condition

$$\operatorname{mes} \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda) < [(1+\varepsilon)\lambda^{-(s-\alpha)}]^{n+1}$$

does not imply the predominance of

$$\left\{ \lambda \in \mathscr{J}_n : F_n \circ h(\lambda, \tfrac{1}{2}) \in [0, 1] \Big\backslash \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda) \right\}$$

in $\mathscr{J}_n$, and we have to do some additional estimates. In consequence of Lemma 4 for $k = n+1$, to any $\delta_{n+1}^{-k} = G_{n+1}^{-1} \delta_{n+1}$ there corresponds uniquely a set $L_{n+1}^{-k} = G_{n+1}^{-1} L_{n+1} \subset \mathscr{X}_{n+1} \bigcap \delta_{q(n+1)}^{-k}$ such that

$$\operatorname{mes} L_{n+1}^{-k} > (1-\varepsilon)\lambda^{2\alpha(n+1)} |\delta_{n+1}^{-k}|.$$

We define $\hat{L}_{n+1}^{-k} = (L_{n+1}^{-k} \backslash \hat{\delta}_{n+1}^{-k})$. Thus for any $\lambda \in \mathscr{J}_n$ the following estimate holds:

$$\operatorname{mes} \hat{L}_{n+1}^{-k} > (1-\varepsilon_{11.3})\lambda^{\alpha(n+1)} |\hat{\delta}_{n+1}^{-k}|, \quad k = 0, 1, \ldots, \varepsilon_{11.3} = O(\lambda^{-2\alpha}) \tag{11.3}$$

Let $\mathcal{J}=\mathcal{J}(\delta_{q(n+1)}^{-k})=[\lambda_0,\lambda_1]$ be an interval on the λ-axis such that $F_n\circ h(\lambda,\frac{1}{2})\in\delta_{q(n+1)}^{-k}$ when $\lambda\in\mathcal{J}$. Because of the definition of $q(n)$, $|\delta_{q(n+1)}^{-k}|<(1+o(\lambda^{-t(n+2)}))\cdot\lambda^{2\alpha(n+1)}\cdot\lambda^{2s}\cdot|\delta_{n+1}^{-k}|$. Then the comparison of velocities (11.1) and (11.2) implies

$$|\mathcal{J}|<\frac{4\cdot\lambda_0^{2\alpha(n+1)+2s}}{(2\lambda_0^{\text{co}})^n}(1+\varepsilon_{11.4})\cdot|\delta_{n+1}^{-k}(\lambda_0)|\,. \tag{11.4}$$

When λ passes $\mathcal{J}$, the measures of $\hat{\delta}_{n+1}^{-k}$ and $\hat{L}_{n+1}^{-k}$ vary in particular because of the variation of $\partial/\partial x(G_{n+1}^{-1})$. We shall show this variation is small.

a)

Lemma 5. *Let Γ_p denote δ_p if $p\leqq n$, and $(\frac{1}{2}-\lambda^{-sp},\frac{1}{2}+\lambda^{-sp})$ if $p>n$. Let $F_\lambda(x)$ be one of the diffeomorphisms $G_n(\lambda,x)$, $\hat{f}_n(\lambda,x)$, or $f_n(\lambda,x)$, and suppose $\Delta F_\lambda(x)\subset[0,1]\backslash\Gamma_p$. Let $F_\lambda^{-1}(z)$ be the inverse diffeomorphism, and let $N=N(F)$ be the number of iterations of the initial map $g_\lambda:x\mapsto\lambda x(1-x)$ mod 1 corresponding to F_λ (i.e., $F_\lambda=g_\lambda^N$). Then*

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial F_\lambda^{-1}}{\partial z}\right|<\frac{\lambda^{2sp}}{8\lambda}N\left|\frac{\partial F_\lambda^{-1}}{\partial z}\right|\sum_{i=0}^{n}\lambda^{-si}. \tag{11.5}$$

Proof. We proceed by induction. Assuming Lemma 5 holds for $k\leqq n$, we need to prove the corresponding estimates for $n+1$. We begin by estimating $\frac{\partial}{\partial\lambda}\frac{\partial\phi^{-1}}{\partial z}$ for a composition of maps. Let

$$\phi(\lambda,x)=\varphi_n\circ\varphi_{n-1}\circ\ldots\circ\varphi_1(\lambda,x),$$

where our notation is similar to that in the calculations for (9.1). Several applications of the chain rule give

$$\frac{\partial}{\partial\lambda}(\partial\phi^{-1}/\partial z)=[\partial\phi^{-1}/\partial z]\sum_{i=1}^{n}\left[\frac{\partial/\partial\lambda(\partial\varphi_i^{-1}/\partial z)}{\partial\varphi_i^{-1}/\partial z}-\frac{\partial^2\varphi_i/\partial x^2}{(\partial\varphi_i/\partial x)^2}\frac{\partial(\varphi_{i+1}^{-1}\circ\ldots\circ\varphi_n^{-1})}{\partial\lambda}\right], \tag{11.6}$$

where as before the arguments of φ_i and its derivatives are λ and $\varphi_{i-1}\circ\ldots\circ\varphi_1(x)$ while those of φ_i^{-1} are λ and $\varphi_{i+1}^{-1}\circ\ldots\circ\varphi_n^{-1}(z)$, $z=\phi(\lambda,x)$ (for $i=n$, $\varphi_n^{-1}=\varphi_n^{-1}(\lambda,z)$, and there is no second term in the brackets).

Let $F_1=f_{n-1}\circ\ldots\circ f_1\circ g$, $\Delta F_1\subset[0,1]\backslash\Gamma_p$, and let N_i denote the number of iterations corresponding to f_i. The expression $\partial/\partial\lambda(\varphi_{i+1}^{-1}\circ\ldots\circ\varphi_n^{-1})$ in (11.6) equals $\partial x_{i+1}/\partial\lambda$, where $x_{i+1}(\lambda)$ satisfies

$$\varphi_n\circ\ldots\circ\varphi_{i+1}(\lambda,x_{i+1}(\lambda))=z\,.$$

In our case $\varphi_1=g$, $\varphi_{i+1}=f_i$ $1\leqq i\leqq n-1$ and the estimates of Sect. 9 give

$$|\partial/\partial\lambda(f_i^{-1}\circ\ldots\circ f_{n-1}^{-1})(\lambda,z)|<\frac{1+\varepsilon_1}{8\lambda}\lambda^{si}.$$

For $g=\lambda x(1-x)$ we have

$$\frac{\partial}{\partial\lambda}\frac{\partial g^{-1}}{\partial z}=\frac{1}{2\lambda^2(x-\frac{1}{2})},\qquad\frac{-\partial^2 g/\partial x^2}{(\partial g/\partial x)^2}=\frac{1}{2\lambda(x-\frac{1}{2})^2}.$$

For f_i we have by estimates b_{2n} of Sect. 6 that

$$\frac{\partial^2 f_i/\partial x^2}{(\partial f_i/\partial x)^2} < 1+\varepsilon_2 .$$

Thus, (11.5) and (11.6) give

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial F_1^{-1}}{\partial z}\right| \leqq \left|\frac{\partial F_1^{-1}}{\partial z}\right| \cdot \left[\left(\frac{\lambda^{sp}}{2\lambda^2}+\frac{\lambda^{2sp}}{2\lambda}(1+\varepsilon_1)\frac{\lambda^s}{8\lambda}\right)+\left(\frac{\lambda^{2s}}{8\lambda}N_1+\frac{\lambda^{2s}}{8\lambda}(1+\varepsilon_1)(1+\varepsilon_2)\right)\right.$$
$$\left.+\ldots+\left(\frac{\lambda^{2s(n-2)}}{8\lambda}N_{n-2}+\frac{\lambda^{s(n-1)}}{8\lambda}(1+\varepsilon_1)(1+\varepsilon_2)\right)+\frac{\lambda^{2s(n-1)}}{8\lambda}N_{n-1}\right] \quad (11.7)$$

Let $F_2 = G\circ F_1$, where $G=G_n : \Delta G \to \delta_n$, $\Delta G \subset [0,1]\setminus\delta_n$, and N_G is the number of iterates for G. The estimates of Sect. 9 imply $\left|\frac{\partial G^{-1}(z,\lambda)}{\partial\lambda}\right| < \frac{\lambda^{sn}(1+\varepsilon_1)}{8\lambda}$.

Because $\frac{|\partial^2 F_1/\partial x^2|}{(\partial F_1/\partial x)^2} < \nu(F_1, \Delta(F_1))$ for $x\in\Delta F_1$, we obtain using (4.7), a_{2n} and b_{2n} that

$$\frac{|\partial^2 F_1/\partial x^2|}{(\partial F_1/\partial x)^2} < (1+O(\lambda^{-\gamma}))\left(1+\frac{\lambda^{2sp}}{\lambda^{s/2}(2\lambda^{\mathrm{co}})^{n+1}}\right)$$
$$+\frac{\lambda^{2sp}}{\lambda^{s/2}(2\lambda^{\mathrm{co}})^{n+1}} = \frac{\lambda^{2sp}(1+\varepsilon_3)}{\lambda^{s/2}(2\lambda^{\mathrm{co}})^{n+1}} + (1+\varepsilon_3).$$

Using (11.6) for $F_2 = G\circ F_1$ we have

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial F_2^{-1}}{\partial z}\right| < \left[\left|\frac{\partial}{\partial\lambda}\frac{\partial F_1^{-1}}{\partial z}\right| \Big/ \left|\frac{\partial F_1^{-1}}{\partial z}\right| + \left(\frac{\lambda^{2sp}(1+\varepsilon_3)}{\lambda^{s/2}\cdot(2\lambda^{\mathrm{co}})^{n+1}}+(1+\varepsilon_3)\right)\right.$$
$$\left.\cdot\frac{\lambda^{sn}(1+\varepsilon_1)}{8\lambda}+\frac{\lambda^{2sn}}{8\lambda}\cdot N_G\right]\cdot\left|\frac{\partial F_2^{-1}}{\partial z}\right|. \quad (11.8)$$

Substituting (11.7) in (11.8) we obtain (11.5) for $G_n' = G_n\circ F_{n-1}\circ g$ constructed at the beginning of step $n+1$ (we have besides an additional factor less than $\lambda^{-\mathrm{co}}$ in the right part of (11.5)). The proof for $G_n' = G_n\circ \hat f_n$, $\Delta \hat f_n \subset [0,1]\setminus \Gamma_p$, is analogous. Considering $p=n+1$ in (11.7) we obtain the assertion of Lemma 5 for $f_{n+1\,1}$. Then we consider the compositions $f_{n+1\,k} = f_{n+1\,1}\circ \tilde G_{n_k}\circ\ldots\circ\tilde G_{n_1}$. The induction hypotheses and the previous estimates give

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial G_{n_i}'^{-1}}{\partial z}\right| < \frac{\lambda^{2s(n+1)}}{8\lambda}\cdot N_i\left|\frac{\partial G_{n_i}^{-1}}{\partial z}\right|\cdot\sum_{i=0}^{n}\lambda^{-si}.$$

The estimates of Sect. 9 give

$$\left|\frac{\partial(\tilde G_{n_1}^{-1}\circ\ldots\circ\tilde G_{n_k}^{-1}\circ f_{n+1\,1}^{-1})}{\partial\lambda}\right| < \frac{\lambda^{s(n+1)}}{8\lambda}(1+\varepsilon_1).$$

Taking into account

$$\left|\frac{\partial^2 G_{n_i}}{\partial x^2}\Big/\left(\frac{\partial G_{n_i}}{\partial x}\right)^2\right| < (1+\varepsilon_2)\lambda^{-\alpha n},$$

(11.6) implies

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial f_{n+1\,k}^{-1}}{\partial z}\right| < \left[\sum_{i=1}^{k}\left(\frac{\lambda^{2s(n+1)}}{8\lambda}\cdot N_i\cdot\sum_{i=0}^{n}\lambda^{-si}+\frac{(1+\varepsilon_2)(1+\varepsilon_1)\cdot\lambda^{s(n+1)}}{\lambda^{\alpha n}\cdot 8\lambda}\right)\right.$$
$$\left.+\frac{\lambda^{2s(n+1)}}{8\lambda}\cdot N(f_{n+1\,1})\right]\left|\frac{\partial f_{n+1\,k}^{-1}}{\partial z}\right|.$$

This proves Lemma 5 for f_{n+1}, and the sum in round brackets gives the desired estimate for G_{n+1}. The proof for $\hat{f}_{n+1}=f_{n+1}\circ G_n$ is similar. □

b) Consider $\mathscr{I}(\delta_{q(n+1)})=\{\lambda: F_n\circ h(\lambda,\frac{1}{2})\in\delta_{q(n+1)}(\lambda)\}=[\lambda_0,\lambda_1]$. (11.4) gives $|\mathscr{I}(\delta_{q(n+1)})|<\frac{(1+\varepsilon_{11.4})\cdot 16\cdot\lambda_0^{1+s}}{(2\cdot\lambda_0^{c_0+s-2\alpha})^{n+1}}$. Let $\Delta(\lambda)$ be any interval in $L_{n+1}\bigcap\delta_{q(n+1)}$. The comparison of velocities (11.1) and (11.2) shows that the time it takes for $F_n\circ h(\lambda,\frac{1}{2})$ to traverse $\Delta(\lambda)$ equals $\frac{|\Delta(\lambda)|}{v_n(\lambda)}\cdot(1+o(\lambda^{-c_0 n}))$, where λ is any moment of passing by. We want to reduce all these moments (for different $\Delta(\lambda)$) to the same one, namely to λ_0, and then use the relation (11.3) for λ_0. This can be done for given $\Delta(\lambda)$ if for any $\lambda\in\mathscr{I}(\delta_{q(n+1)})$,

$$\frac{|\Delta(\lambda)|}{|\Delta(\lambda_0)|}>(1-\alpha_{n+1}),\qquad \alpha_{n+1}=o(\lambda_0^{-t(n+1)}).$$

Let $N=N(\Delta f_k)=N(f_k)$. If $N<\lambda_0^{s(n+1)}$, Lemma 5 and the estimate of $\mathscr{I}(\delta_{q(n+1)})$ imply

$$|\Delta(\lambda_0)|-|\Delta(\lambda)|<\frac{\lambda_0^{2s(n+1)}}{8\lambda_0}\cdot\lambda_0^{s(n+1)}\cdot\frac{16\cdot\lambda_0^{1+s}\cdot(1+\varepsilon_{11.9})}{(2\lambda_0^{c_0+s-2\alpha})^{n+1}}\cdot|\Delta(\lambda_0)|. \tag{11.9}$$

Thus for such Δ, $\alpha_{n+1}=O(\lambda_0^{[c_0-2(s+\alpha)](n+1)})$.

Lemma 7 of Sect. 12 gives the following relation between $N(\Delta)$ and $|\Delta|$ for $\Delta\in[0,1]\backslash\delta_n$:

$$N<\frac{\sqrt{n}\cdot 2s}{c_0}\,|\log_{\lambda_0}|\Delta||.$$

Thus $N<\lambda_0^{s(n+1)}$, if $\frac{2s}{c_0}\sqrt{n+1}\,|\log_{\lambda_0}|\Delta||<\lambda_0^{s(n+1)}$. Lemma 7 also gives the following estimate for a domain $\Delta(F_{n-1}\circ h)$ of the central branch $F_{n-1}\circ h(\lambda,x)$. If $\Gamma_p\supset\Delta(F_{n-1}\circ h)\supset\Gamma_{p+1}$, and $N=N(F_{n-1}\circ h)=N(F_{n+1})+1$, then

$$N<\frac{4s}{c_0}\sqrt{n}\cdot p.$$

When constructing L_{n+1} in Sect. 10, we had $L_{n+1}\bigcap(\delta_n\backslash\delta_{n+1})\subset\bigcup\Delta f_{n+1\,1}$. Using this fact one can check inductively following the proofs of Lemmas 2 and 4 that the following construction gives a set $\mathscr{X}'_{n+1}\subset\mathscr{X}_{n+1}$ with

$$\operatorname{mes}(L_{n+1}^{-k}\bigcap\mathscr{X}'_{n+1})>(1-\varepsilon'_0)\lambda^{2\alpha(n+1)}\operatorname{mes}\delta_{n+1}^{-k}$$

for every δ_{n+1}^{-k}.

We begin by constructing at step 2 the maps $f_1 \circ g$ and the holes $g^{-1}\delta_1$. Then at step $n+1$, $n \geqq 2$, we reproduce on each interval inside δ_n the structure obtained after step n on $[0,1]\backslash\delta_n$, and on each hole δ_i^{-k} [$1 \leqq i \leqq n-1$ and $k \leqq k_0(n)$ here, contrary to $i=n$ and $1 \leqq k < \infty$ in the construction of Sect. 3] we reproduce the structure of $\delta_i \backslash \delta_n$ obtained after step n. Ignoring $N(F_{n-1} \circ h)$ this construction gives for $N(\mathscr{X}'_n) = \max\{N(\Delta f_k), \Delta f_k \in \mathscr{X}'_n\}$ the upper estimate 2^n. Taking into account $N(F_{n-1} \circ h)$ estimated above, we obtain

$$N(\mathscr{X}'_n) < n \cdot 2^n .$$

This implies the following

Lemma 6.

$$\frac{\operatorname{mes}\{\Delta(f_k) \in \hat{L}_{n+1}(\lambda) : N(f_k) < n \cdot 2^n\}}{\operatorname{mes} \hat{\delta}_{n+1}(\lambda)} > (1-\varepsilon)\lambda^{\alpha(n+1)} .$$

Lemma 6 implies the predominance of Δ satisfying (11.9) in $\hat{L}_{n+1}$. Thus (11.3) implies

$$\frac{\operatorname{mes}\{\lambda \in \mathscr{J}(\delta_{q(n+1)}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{L}_{n+1}(\lambda)\}}{\operatorname{mes}\{\lambda \in \mathscr{J}(\delta_{q(n+1)}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{\delta}_{n+1}(\lambda)\}} > (1-\varepsilon_{11.10})\lambda_0^{\alpha(n+1)} . \tag{11.10}$$

c) Let $\mathscr{J} = \mathscr{J}(\delta_{q(n+1)}^{-k}) = [\lambda_2, \lambda_3] = \{\lambda : F_n \circ h(\lambda, \tfrac{1}{2}) \in \delta_{q(n+1)}^{-k}(\lambda)\}$. (11.4) and Lemma 5 imply that for any $\lambda \in \mathscr{J}$

$$\left\| \frac{\partial G_{n+1\,\lambda}^{-1}}{\partial z}\right|_{z=1/2} - \left.\frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z}\right|_{z=1/2} \Bigg\| < \frac{\left(\left.\dfrac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z}\right|_{z=1/2}\right)^2 \cdot N(G_{n+1}) \cdot 2\lambda_2^s(1+\varepsilon_{11.11})}{(2\lambda_2^{c_0-(s+2\alpha)})^{n+1}} . \tag{11.11}$$

(11.11) and the estimate b_{4n+1} of $\mu(G_{n+1})$ give for any $\Delta(\lambda) \subset (\hat{L}_{n+1} \cap \mathscr{X}'_{n+1})^{-k}$

$$\begin{aligned}\frac{\operatorname{mes} G_{n+1\,\lambda}^{-1}\Delta(\lambda)}{\operatorname{mes} G_{n+1\,\lambda_2}^{-1}\Delta(\lambda_2)} &> (1-\alpha_{n+1})\left(1 - \frac{\left|\dfrac{\partial G_{n+1\,\lambda}^{-1}}{\partial z} - \dfrac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z}\right|_{z=1/2}}{\left|\dfrac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z}\right|_{z=1/2}}\right) \\ &\quad \cdot \exp(\mu(G_{n+1})) > \left(1 - \frac{N(G_{n+1}) \cdot \left|\dfrac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z}\right| 2\lambda_2^s(1+\varepsilon_{11.12})}{(2\lambda_2^{c_0-(s+2\alpha)})^{(n+1)}}\right) \\ &\quad \cdot (1-\alpha_{n+1})(1 - O(\lambda^{-\alpha(n+1)})) .\end{aligned} \tag{11.12}$$

As $N(G) \cdot \left|\dfrac{\partial G^1}{\partial z}\right| = o(1)$, we obtain from (11.12) and (11.3)

$$\frac{\operatorname{mes}\{\lambda \in \mathscr{J}(\delta_{q(n+1)}^{-k}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{L}_{n+1}^{-k}\}}{\operatorname{mes}\{\lambda \in \mathscr{J}(\delta_{q(n+1)}^{-k}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{\delta}_{n+1}^{-k}\}} > (1-\varepsilon_{11.13}) \cdot \lambda_2^{\alpha(n+1)} \tag{11.13}$$

Using $\hat{L}_{n+1}^{-k}(\lambda) \cap \hat{L}_{n+1}^{-\ell}(\lambda) = \emptyset$, if $\hat{\delta}_{n+1}^{-k} \neq \hat{\delta}_{n+1}^{-\ell}$, we obtain from (11.10) and (11.13)

Proposition 2. *Let* $\mathcal{I}_n=[\lambda_{0n},\lambda_{1n}]\subset\mathcal{M}_n$ *be any interval on the* λ*-axis constructed at Step n. Then*

$$\frac{\operatorname{mes}\left\{\lambda\in\mathcal{I}_n, F_n\circ h(\lambda,\tfrac{1}{2})\in\bigcup_{k=0}^{\infty}\bigcup\hat{\delta}_{n+1}^{-k}(\lambda)\right\}}{\operatorname{mes}\mathcal{I}_n}<\frac{1+\varepsilon_{11}}{\lambda_{0n}^{\alpha(n+1)}},$$

where $\varepsilon_{11}<\lambda^{-t}$.

We define

$$\mathcal{M}_{n+1}\bigcap\mathcal{I}_n=\left\{\lambda:F_n\circ h(\lambda,\tfrac{1}{2})\in[0,1]\Big\backslash\bigcup_{k=0}^{\infty}\bigcup\hat{\delta}_{n+1}^{-k}(\lambda)\right\},$$

and obtain

$$\operatorname{mes}(\mathcal{M}_{n+1}\bigcap\mathcal{I}_n)>1-\frac{1+\varepsilon_{11}}{\lambda_{0n}^{\alpha(n+1)}}$$

and consequently

$$\operatorname{mes}\mathcal{M}_{n+1}>\left(1-\frac{1+\varepsilon_{11}}{N_0^{\alpha(n+1)}}\right)\operatorname{mes}\mathcal{M}_n.$$

Remark XI/1. Any λ such that $F_n\circ h(\lambda,\tfrac{1}{2})\in[0,1]\Big\backslash\bigcup_{k=0}^{\infty}\bigcup\hat{\delta}_{n+1}^{-k}(\lambda)$ lies in one of the intervals $\mathcal{I}_{n+1}(\Delta_k)$ corresponding to the relation $F_n\circ h(\lambda,\tfrac{1}{2})\in\Delta_k(\lambda)$, or is a limit point of such intervals. One can apparently prove that

$$\operatorname{mes}\{\lambda:F_n\circ h(\lambda,\tfrac{1}{2})\in\bigcup\Delta_k(\lambda)\}>\left(1-\frac{1+\varepsilon'_{11}}{\lambda_{0n}^{t(n+1)}}\right)\operatorname{mes}\mathcal{I}_n,$$

but there is no reason to avoid λ lying in the limit set. They are even better in some sense (see Remark VIII/3).

12. Transition from T_λ-Invariant Measure to f_λ-Invariant Measure

The previous relations between $\operatorname{mes}\mathcal{M}_{n+1}$ and $\operatorname{mes}\mathcal{M}_n$, and the choice of the position of the top

$$F_{n-1}\circ h(\lambda,\tfrac{1}{2})\in[0,1]-\left(\tfrac{1}{2}-\frac{1}{\lambda^{s/2}},\ \tfrac{1}{2}+\frac{1}{\lambda^{s/2}}\right)$$

within the first steps $1,2,\ldots,n_0$, imply that there exists a set $\mathcal{M}=\bigcap_{n=1}^{\infty}\mathcal{M}_n$ on the λ-axis with measure

$$\operatorname{mes}\mathcal{M}>4\left[\prod_{n=1}^{n_0}\left(1-\frac{2(1+\varepsilon)}{N_0^{s/2}}\right)\right]\cdot\left[\prod_{n=n_0+1}^{\infty}\left(1-\frac{1+\varepsilon}{N_0^{\alpha n}}\right)\right]$$

such that for any $\lambda\in\mathcal{M}$ the partition ξ_λ of Sect. 1 exists.

Conditions i–iv of Sect. 1 imply that for T_λ defined by $T_\lambda|\Delta_i(\lambda)=f_\lambda^{n_i}$ there exists a unique T_λ – invariant probabilistic measure $\nu_\lambda < dx$ with a density $\varrho_\lambda(x)\in C_{[0,1]}$, $\varrho_\lambda > c > 0$. The endomorphism T_λ of the Lebesgue space $([0,1], \nu_\lambda)$ is exact and its natural extension is isomorphic to a Bernouli shift (see [12, 13]).

In order to finish the proof of Theorem A for the family $f_\lambda: x\to\lambda x(1-x)$ (mod 1) we have to construct an invariant measure $\mu_\lambda < dx$.

Let $f^{-k}(A)$ be the full preimage of $A\subset[0,1]$ under f^k, $f^{-k}A=\{x: f^k x\in A\}$. Suppose $\sum_{\Delta_i\in\xi_\lambda} n_i\nu_\lambda(\Delta_i)<\infty$. Then the measure defined for any dx-measurable set A by

$$\mu_\lambda(A)=\sum_{\Delta_i\in\xi_\lambda}\sum_{0\leqq j<n_i}\nu_\lambda(f^{-j}A\cap\Delta_i) \tag{12.1}$$

is absolutely continuous with respect to dx, by a theorem on integrability of a series of positive functions (see for example [15] Sect. 14).

We show μ_λ is f-invariant.

By definition

$$\mu_\lambda(f^{-1}A)=\sum_{\Delta_i\in\xi_\lambda}\sum_{0\leqq j<n_i}\nu_\lambda(f^{-j}\circ f^{-1}A\cap\Delta_i). \tag{12.2}$$

If $j<n_i-1$, every term $f^{-j}\circ f^{-1}A\cap\Delta_i$ in (12.2) coincides with $f^{-(j+1)}A\cap\Delta_i$ in (12.1). After excluding these terms, there remain in (12.1) terms with $j=0$, which give $\sum_{\Delta_i\in\xi_\lambda}\nu_\lambda(A\cap\Delta_i)=\nu_\lambda(A)$, and in (12.2) terms with $j=n_i-1$, which give

$$\sum_{\Delta_i\in\xi}\nu_\lambda(f^{-n_i}A\cap A)=\sum_{\Delta_i\in\xi}\nu_\lambda(T_\lambda^{-1}A\cap\Delta_i)=\nu_\lambda T_\lambda^{-1}A.$$

Thus (12.1) equals (12.2) because of the T_λ-invariance of ν_λ.

Let $\beta=\frac{2}{5}$. The following proposition implies $\sum n(\Delta_i)|\Delta_i|<\infty$.

Proposition 3. $\displaystyle\sum_{\Delta_i\in\delta_n\setminus\delta_{n-1}} n(\Delta_i)|\Delta_i|<\frac{n^{3/2}}{\lambda^{s(1-\beta)n}}$.

Proof. a) Consider step n of the induction construction of Sect. 3. If Φ_n is one of $f_n, \hat{f}_n, G_n, F_{n-1}$ obtained with N successive iterates of f_λ, we use an upper index so that $\Phi_n^N=f_\lambda\circ f_\lambda\circ\ldots\circ f_\lambda$, and $\Phi_n^{-N}=(\Phi_n^N)^{-1}|\mathrm{Im}\,\Phi_n^N$.

Lemma 7. $|Df_n^N|>\lambda^{\frac{c_0N}{2\sqrt{n}}+s}$

Let

$$\Gamma_\ell=\begin{cases}\delta_\ell & \text{if } \ell\leqq n\\ [\frac{1}{2}-\lambda^{-s\ell},\frac{1}{2}+\lambda^{-s\ell}] & \text{if } \ell\geqq n+1.\end{cases}$$

We prove Lemma 7 by induction and assume that for $k=1,\ldots,n$ Lemma 7 holds together with the following properties:

i) Let $\delta_n^{-N}=G^{-N}\delta_n\subset[0,1]\setminus\Gamma_\ell$, and let $r=\max(1,n)$. Then $|DG^N|\,\delta_n^{-N}|>\lambda^{\frac{c_0N}{2\sqrt{r}}+s}$.

ii) Let $\Delta\hat{f}_n^N\subset[0,1]\setminus\Gamma_\ell$. Then $|D\hat{f}_n^N|>\lambda^{\frac{c_0N}{2\sqrt{\ell}}}$.

Consider $k=n+1$. Notice that if $x\in[0,1]\setminus\Gamma_2$, then $|Df_\lambda|>\lambda^{1-2s}>\lambda^{\frac{c_0}{2}+s}$. If $x\in\Gamma_2$, then ℓ, $r\geqq 3$ in i), ii) and $\frac{c_0}{2\sqrt{3}}+s<\frac{c_0}{2}$. As $|Df_\lambda|>\lambda^{\frac{c_0}{2}}$ on any hole δ_1^{-1} we obtain i) for $n=1$. ii) for $n=1$ holds because of i) and $|Df_1|>\lambda^{c_0}>\lambda^{\frac{c_0}{2}+s}$.

Let $F_{n-1}\circ h(\lambda,x)$ be the central branch, $F_{n-1}=f_{i_{n-1}}\circ\ldots\circ f_{i_1}$, $1\leqq i_k\leqq k$, $N(f_{i_k})=N_k$ the number of iterations of f_λ corresponding to f_{i_k}, $k\in[1,n-1]$, $\delta=G^{-N_n}\delta_n$, $M=1+\sum_{k=1}^{n}N_k$. Then $(F_{n-1}\circ h)^{-1}\delta=\delta_n^{-M}$. (In the notation of Sect. 3, $G^M:\delta_n^{-M}\to\delta_n$ is one of the G_n' constructed at the beginning of step n.)

Let D_{n-1} be the domain of $F_{n-1}\circ h$, and let $p_0=\min\{p|\delta_n^{-M}\subset[0,1]\setminus\Gamma_p\}$. Then (see (6.12))

$$p_0>\frac{c_0}{2s}n. \tag{12.3}$$

According to the construction of Sect. 3,

$$\mathrm{dist}(\delta,F_{n-1}\circ h(\lambda,\tfrac{1}{2}))>\frac{\delta}{2}\lambda^{\alpha n}(1-\varepsilon),$$

which implies (see (6.4)) that

$$|DG^M|>\frac{\sqrt{\lambda}\left[|DG^{N_n}|\prod_{k=1}^{n-1}|Df_{i_k}|\right]^{1/2}}{\lambda^{(s-\alpha)n/2}}, \tag{12.4}$$

where DG^M is evaluated on δ_n^{-M} and DG^{N_n} on δ. By the induction hypotheses we have

$$|Df_{ik}|>\lambda^{[c_0N_k/2\sqrt{k}]+s}\qquad 1\leqq k\leqq n-1$$
$$|DG^{N_n}|>\lambda^{[c_0N_n/2\sqrt{n}]+s}.$$

Hence, on δ_n^{-M},

$$|DG^M|>\lambda^\theta,$$

where

$$\theta=\tfrac{1}{2}+\frac{s(n-1)}{2}+\left(\sum_{i=1}^{n}N_i\right)\frac{c_0}{4\sqrt{n}}-\frac{(s-\alpha)}{2}n. \tag{12.5}$$

We have to prove

$$|DG^M|>\lambda^{[c_0M/2\sqrt{p_0}]+s}. \tag{12.5a}$$

Now, $M=\left(\sum_{i=1}^{n}N_i\right)-1$,

$$\frac{1-s+n\alpha}{2}>s+\frac{c_0}{2\sqrt{p_0}}$$

and $c_0/2s > 4$ imply

$$2\sqrt{n} < \sqrt{c_0 n/2s} < \sqrt{p_0}$$

and (12.5a) follows from (12.5).

So i) is proved for the holes $\delta_n^{-M} = (F_{n-1} \circ h)^{-1} \delta_n^{-N}$. Any branch $f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g$ is some composition of the form $f_{i_{n-1}} \circ \ldots \circ f_{i_k} \circ (f_{i_{k-1}} \circ \ldots \circ f_{i_1} \circ h)$, where $f_{i_{k-1}} \circ \ldots \circ f_{i_1} \circ h$ is a central branch of some previous step. Thus the same arguments prove i) for $\delta_n^{-M} = (F_{n-1} \circ g)^{-1} \delta_n^{-N}$ (the estimates are better in this case). If $\delta_n^{-M} = \hat{f}_n^{-1} \delta_n^{-N}$, i) follows from i) and ii) of Step $n-1$.

Let $G^{-N}\delta_n = \delta_n^{-N} \subset [0,1] \backslash \delta_{n+1}$. Then we have $\max(\ell, n) = n+1$. Now i) follows for $G_{n+1}^M : \delta_{n+1}^{-M} \to \delta_{n+1}$ with $\delta_{n+1}^{-M} \subset [0,1] \backslash \delta_{n+1}$ because they are compositions of maps satisfying i) with $r \leqq n+1$. Similarly for $G_{n+1}^M : \delta_{n+1}^{-M} \to \delta_{n+1}$, $\delta_{n+1}^{-M} \subset [0,1] \backslash \Gamma_\ell$, $\ell > n+1$. This proves i_{n+1}.

Let $f_{n+1\,1} = f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g_\lambda | [0,1] \backslash \delta_{n+1}$. The induction conditions on $|Df_{i_k}|$ imply that $|Df_{n+1\,1}| = \prod_{k=1}^{n} |Df_{i_k}| \cdot 2\lambda |x - \frac{1}{2}|$ satisfies Lemma 7. The same is true for $f_{n+1\,1} = f_{i_k} \circ \hat{f}_n$, because of ii). Taking into account i_{n+1}, we obtain Lemma 7 for $f_{n+1\,k}$ with $k > 1$. Finally ii) at Step $n+1$ follows from i) and the assertion of Lemma 7 for f_{n+1}. □

b) We shall use the following estimates for compositions of maps.

Let $g: B \to J$ be given by $g(x) = ax^2$, where $B = [x_{\min}, x_{\max}]$ and $J = \bigcup \Delta$, where $\operatorname{int}\Delta_1 \bigcap \operatorname{int}\Delta_2 = \emptyset$ if $\Delta_1 \neq \Delta_2$. Let $\Delta = [h_\Delta, h_\Delta + |\Delta|]$ and denote by $n(\Delta)$ the number of iterations corresponding to Δ. Then $B = \bigcup g^{-1}\Delta$, where

$$|g^{-1}\Delta| = \frac{1}{\sqrt{a}}\left(\sqrt{h_\Delta + |\Delta|} - \sqrt{h_\Delta} = \frac{|\Delta|}{\sqrt{a}(\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}}\right)$$

and

$$n(g^{-1}\Delta) = 1 + n(\Delta).$$

Hence

$$\sum n(g^{-1}\Delta)|g^{-1}\Delta| = \frac{1}{\sqrt{a}} \sum \frac{(1 + n(\Delta))|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}}$$

$$= \frac{1}{\sqrt{a}} \sum \frac{|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}} + \frac{1}{\sqrt{a}} \sum \frac{n(\Delta)|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}}. \tag{12.6}$$

Let us now consider $\{\Delta', f', n'\}$, where $\operatorname{int}\Delta'_1 \bigcap \operatorname{int}\Delta'_2 = \emptyset$, $n' = n'(\Delta') = n'(f')$. Suppose every f' maps its domain onto the same interval, $f': \Delta' \to J$, and $\mu(f', \Delta') < c$. Let $\{\Delta, f, n\}$ be so that $\Delta \subset J$, $\operatorname{int}\Delta_1 \bigcap \operatorname{int}\Delta_2 = \emptyset$, $n = n(\Delta) = n(f)$. Then

$$\sum_{\Delta, \Delta'} n(f'^{-1}(\Delta))|f'^{-1}\Delta| < \left(\sum_{\Delta, \Delta'} (n + n')|\Delta||\Delta'|\right) \frac{\exp(c)}{|J|}$$

$$= [(\sum n|\Delta|)(\sum |\Delta'|) + (\sum n'|\Delta'|)(\sum |\Delta|)] \frac{\exp(c)}{|J|}.$$

c) When estimating $\sum n(\Delta)|\Delta|$ after step n of the induction construction we shall attribute to any preimage δ_n^{-N} mapped onto δ_n by G_n^N the number of iterations N, ignoring the structure inside δ_n. But when considering δ_n itself, we take into account this structure. This gives the estimate of $\sum n(\Delta)|\Delta|$ on any domain inside δ_n. Then according to the construction of Sect. 3 we introduce at step $n+1$ the structure from $\delta_n \backslash \delta_{n+1}$ inside every domain $(\delta_n \backslash \delta_{n+1})^{-N}$.

Before formulating the induction hypotheses of Proposition 3 we introduce a new notation. Let $\Delta_0 = \Delta f \subset [0,1] \backslash \delta_n$ be a domain of some f, constructed after Step n. We define a "block" $B(\Delta_0)$ as a maximal interval containing Δ_0, which doesn't contain any hole δ_n^{-k}. If $B(\Delta_0) \bigcap \delta_n = \emptyset$, then $B(\Delta_0) = \bigcup \Delta_i$, where any $\Delta_i = [a_i, a_{i+1}]$, $i \in \mathbb{Z}$, is a domain of some Δf_ℓ, $\ell \leqq n$. If $B(\Delta_0) \bigcap \delta_n \neq \emptyset$, then a part of the Δ_i are as above and the others are $\Delta(F_{n-1} \circ g)$ or $\Delta \hat{f}_n$.

After Step 2 we obtain two exceptional one-side blocks B_0^*, which contains 0, and B_1^*, containing 1, and for any $\tilde{B} \neq B_0^*$, B_1^*, $\tilde{B} = \tilde{B}_1 \bigcup \tilde{B}_2$, where $\tilde{B}_1 = g^{-n} B_0^*$, $\tilde{B}_2 = g^{-n} B_1^*$.

The structure of B_0^* is: $B_0^* = \bigcup B_{0i}$, $i = 1, 2, \ldots$, where $B_{0i} = \bigcup \Delta_{ik}$, $k \in [1, n_0]$, $n_0 = \operatorname{card}\{\Delta f_1 \subset [0, \frac{1}{2}]\}$, $\Delta_{1k} = \Delta f_1$, $\Delta_{ik} = \Delta f_{2i-1}$ for $i \geqq 2$, and the corresponding number of iterations $N(\Delta_{ik}) = i$. The structure of B_1^* is similar.

Let $\tilde{B}$ be some block of step $n+1$. Then either $\tilde{B} = \tilde{B}_1 \bigcup \tilde{B}_2$, where $\tilde{B}_1 \subset [a_1, a]$, $\tilde{B}_2 \subset [a, a_2]$, and $[a_1, a]$, $[a, a_2]$ are two adjacent intervals constructed at step n, $\tilde{B}_1 \bigcap \tilde{B}_2 = a$, and both $\tilde{B}_1$, $\tilde{B}_2$ are preimages of B_0^* or $\tilde{B}_1^*$, or $\tilde{B}$ is some preimage of such blocks constructed at previous steps.

When constructing δ_{n+1} we shall take the precaution to choose two adjacent intervals $\Delta' \subset [0,1] \backslash \delta_{n+1}$ and $\Delta'' \subset \delta_{n+1}$ which are the preimages of Δ_{ik} with the same i. This can be done by moving if necessary the point ξ_{1n} of Sect. 10 a distance less than $(2\lambda^{c_1})^{-(n+1)}$ and still having (10.10) true.

Let $B_+(\Delta_0) = \bigcup (\Delta_i \subset B(\Delta_0),\ i > 0)$, $B_-(\Delta_0) = \bigcup (\Delta_i \subset B(\Delta_0),\ i < 0)$. Then the preceding implies

$$\min(\operatorname{mes} B_+(\Delta_0), \operatorname{mes} B_-(\Delta_0)) > \frac{1-\lambda^{-t}}{2} |\Delta_0| . \tag{12.8}$$

(12.8) together with (4.6) imply the following

Property. Let $\Delta_0 = \Delta f_k \subset B(\Delta_0) \subset \operatorname{Im} F_{n-1} \circ h(\lambda, x)$ be so that $F_{n-1} \circ h(\lambda, \frac{1}{2}) \notin B(\Delta_0)$. Then

$$\mu(F_{n-1} \circ h(\lambda, x), \Delta_0) < 3 . \tag{12.9}$$

d) Let $\mathscr{D}_n = \Delta(F_{n-1} \circ h)$, $F_{n-1} = f_{i_{n-1}} \circ \ldots \circ f_{i_1}$, and let $\Delta_0^{(n)} = \Delta f_{i_n}$ be so that $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \Delta_0^{(n)}$. Then $\mathscr{D}_{n+1} = \Delta(f_{i_n} \circ F_{n-1} \circ h)$. Let $B_n = B(\Delta_0^{(n)})$ be the block of $\Delta_0^{(n)}$, $\mathscr{U}_n = (F_{n-1} \circ h)^{-1} B(\Delta_0^{(n)})$. Notice that $\mathscr{D}_{n+1}$ may be equal to $\mathscr{D}_n$ (it is, if $\operatorname{im} F_{n-1} \circ h(\lambda, x) \subset \Delta(0)$ – the first interval Δf_1 on $[0,1]$ (or $\subset \Delta(1)$ – the last one)), but always $\mathscr{U}_{n+1} \subsetneqq \mathscr{U}_n$.

We now formulate the induction hypotheses for the proof of Proposition 3. Let $R(n) = \max\{R : \mathscr{U}_{n-1} \subset \Gamma_R\}$ where $\Gamma_R = (\frac{1}{2} - \lambda^{-sR}, \frac{1}{2} + \lambda^{-sR})$. Let $\sum_n^k = \sum N(\Delta)|\Delta|$ after step n, where $\Delta \subset \delta_k \backslash \delta_{k+1}$ if $k < n$, or $\Delta \subset \Gamma_k \backslash \Gamma_{k+1}$ if $k \geqq n$, are either intervals Δf_i, $i \leqq n$, $\Delta F_{n-1} \circ g$, $\Delta \hat{f}_n$ or holes δ_n^{-M}. ($N(\delta_n^{-M}) = M$ for holes.) Then for $k \leqq R(n) - 1$,

i) $\sum_n^k < \dfrac{k^{3/2}}{\lambda^{s(1-\beta)k}} \sum\limits_{i=0}^{n-1} \lambda^{-it}$.

Consider any $x_0, x_1, x_2 \in \delta_{k-2} \backslash \delta_k$ (respectively $\Gamma_{k-2} \backslash \Gamma_k$), so that $x_1 \in [x_0, x_2]$. Then for $k \leqq R(n)$

$$\text{ii)} \sum_{\Delta \subset [x_0, x_2]} N(\Delta)|\Delta|/|x_0 - x_2| < \lambda^{s\beta k} \cdot \left(\sum_{\Delta \subset [x_0, x_1]} N(\Delta)|\Delta|/|x_0 - x_1| \right) \left(\sum_{i=0}^{n-1} \lambda^{-it} \right).$$

We have to prove (i) and (ii) for $n+1$ and $k \leqq R(n+1)-1$ (respectively $k \leqq R(n+1)$), where $R(n+1) = \max\{R | \mathcal{U}_n \subset \Gamma_R\}$.

We shall assume that the boundary points of δ_n, δ_{n+1}, $\mathcal{D}_n$ and $\mathcal{U}_n$ lie in $\{\lambda^{-sm}\}$, that is, $\delta_k = (\frac{1}{2} - \lambda^{-sk}, \frac{1}{2} + \lambda^{-sk})$ for $k = n$, $n+1$, and for some r, $p \in \mathbb{Z}$,

$$\mathcal{D}_n = (\tfrac{1}{2} - \lambda^{-sr}, \tfrac{1}{2} + \lambda^{-sr})$$

$$\mathcal{U}_n = (\tfrac{1}{2} - \lambda^{-s(r+p)}, \tfrac{1}{2} + \lambda^{-s(r+p)}).$$

In addition we suppose $\frac{c_0}{2s}$ and $\frac{\alpha}{s} n$ to be integers. The reader can check there is no loss of generality here.

Let $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \delta_{q-1} \backslash \delta_q$. According to the main construction $q \leqq \left(1 - \frac{\alpha}{s}\right) n$. Let $N(f_{i_k}) = N_k$, $k \in [1, n-1]$. Lemma 7 implies

$$|Df_{i_k}| > \lambda^{\frac{N_k c_0}{2\sqrt{k}} + s}.$$

As $\frac{1}{2} - \lambda^{-sr}$ is a root of the equation

$$F_{n-1} \circ [\lambda x(1-x)] = F_{n-1} \circ h(\lambda, \tfrac{1}{2}) \quad (\bmod 1)$$

we have

$$\frac{1}{\lambda^{sr}} < \frac{1}{\left[\lambda \prod_{k=1}^{n-1} |Df_{ik}|\right]^{1/2}} < \exp\left[-\tfrac{1}{2}\left(1 + \sum_{k=1}^{n-1} \left(\frac{N_k c_0}{2\sqrt{k}} + s\right)\right) \ell n \lambda\right].$$

Hence

$$sr > \tfrac{1}{2}\left[1 + (n-1)s + \frac{c_0}{2} \sum_{k=1}^{n-1} N_k / \sqrt{k}\right]$$

$$> \tfrac{1}{2}\left[1 + (n-1)s + \frac{c_0}{2\sqrt{n-1}} \sum_{k=1}^{n-1} N_k\right].$$

This implies that the number of iterations $N(F_{n-1} \circ h) = 1 + \sum_{k=1}^{n-1} N_k$ satisfies

$$N(F_{n-1} \circ h) < \frac{4s}{c_0} r \sqrt{n-1} - \frac{2s}{c_0}(n-1)^{3/2}.$$

Taking into account that $r > c_0 n / 2s$, we obtain

$$N(F_{n-1} \circ h) < 2\left(\frac{2sr}{c_0}\right)^{3/2} - \frac{2s}{c_0}(n-1)^{3/2}. \qquad (12.10)$$

We shall denote $\Delta_0^{(n)}$ by Δ_0 and B_n by B below. As $F_{n-1}\circ h(\lambda,\frac{1}{2})\in\Delta_0$, $\operatorname{im} F_{n-1}\circ h(\lambda,x)$ contains either $B_+(\Delta_0)$, or $B_-(\Delta_0)$. Suppose the former. The number of iterations $N(\Delta_i)$ are either increasing, or they decrease till some $N_{\min}$, and then increase up to infinity. Let $\mathscr{S}=\operatorname{im}F_{n-1}\circ h(\lambda,x)\cap B$. The properties of blocks are so that in the second case $|\mathscr{S}|=2n_0\cdot|\Delta_{\min}|(1+\varepsilon)$, where $\Delta_{\min}$ is any interval corresponding to $N_{\min}$, $n_0=\operatorname{card}\{\Delta f_1\subseteqq[0,\frac{1}{2}]\backslash\delta_1\}<\lambda$, $\varepsilon<\lambda^{-t}$. In the first case more than $1-\varepsilon$ of $|\mathscr{S}|$ consists of intervals with $N(\Delta_i)=N(\Delta_0)$, and $N(\Delta_i)=N(\Delta_0)+1$ (the distribution depends on the number of first Δ_i with $N(\Delta_i)=N(\Delta_0)$). In both cases we have

$$\sum_{\Delta\subset\mathscr{S}} N(\Delta)|\Delta|<(1+\varepsilon)\cdot N(\mathscr{S})\cdot|\mathscr{S}| \tag{12.11}$$

and

$$|\mathscr{S}|<2\cdot\lambda\cdot(1+\varepsilon)|Df_k^{-(N(\mathscr{S})-1)}|, \tag{12.12}$$

where $N(\mathscr{S})=N_{\min}$ in the second case, $N(\mathscr{S})=N(\Delta_0)+1$ in the first case. Taking into account that Δ_0, $\Delta_{\min}\subseteqq B\subseteqq[0,1]\backslash\delta\left(1-\frac{\alpha}{s}\right)n$ and thus $f_k^{N_{\min}}:\Delta_{\min}\to[0,1]$ (correspondingly $f_k^{N(\Delta_0)}$) satisfies Lemma 7 with $\left(1-\frac{\alpha}{s}\right)n$, and proceeding as above when deriving (12.10), we obtain for $\mathscr{U}_n=(F_{n-1}\circ h(\lambda,x))^{-1}\mathscr{S}=\Gamma_{r+p}$

$$N(F_{n-1}\circ h)+N(\mathscr{S})<2\left(\frac{2s}{c_0}(r+p)\right)^{3/2}-\frac{2s}{c_0}(n-1)^{3/2}. \tag{12.13}$$

e) As $\Gamma_r=\mathscr{D}_n=\Delta(F_{n-1}\circ h)$ consists of a unique Δ, after step n we have $\sum_n^k=N(F_{n-1}\circ h)\cdot 2(\lambda^{-sk}-\lambda^{-s(k+1)})$ for $k\geqq r$.

Let us estimate $\sum n(\Delta)|\Delta|$ after taking the first compositions $f_i\circ F_{n-1}\circ h$ on every $\Gamma_k\backslash\Gamma_{k+1}\overset{\text{def}}{=}\tilde{\Gamma}_k$, $r\leqq k<r+p$. We shall denote this sum by $\sum_{n+1\,1}^k$.

Let $\mathscr{S}_0=\mathscr{S}$, and let $\mathscr{S}_i$ be the λ^{2si}-enlargement of $\mathscr{S}_0$ with center $F_{n-1}\circ h(\lambda,\frac{1}{2})$. Then $\Gamma_{r+p}=(F_{n-1}\circ h)^{-1}\mathscr{S}_0$, $\tilde{\Gamma}_{r+p-i}=(F_{n-1}\circ h)^{-1}(\mathscr{S}_i\backslash\mathscr{S}_{i-1})$, $i=1,2,\ldots,p$. Applying (12.7) to $\{\Delta\subseteqq\mathscr{S}_i\backslash\mathscr{S}_{i-1}\}$ and $\Delta'=\Delta\mathrm{F}_{n-1}$ we obtain using b_{2n}

$$\begin{aligned}\sum_{\Delta\subset\mathscr{S}_i\backslash\mathscr{S}_{i-1}}&(N(F_{n-1})+n(\Delta))|F_{n-1}^{-1}\Delta|<(1+O(\lambda^{-\gamma}))|\Delta F_{n-1})|\\&\left(N(F_{n-1})|\mathscr{S}_i\backslash\mathscr{S}_{i-1}|+\sum_{\Delta\subset\mathscr{S}_i\backslash\mathscr{S}_{i-1}}n(\Delta)|\Delta|\right).\end{aligned} \tag{12.14}$$

Applying (12.6) to $F_{n-1}^{-1}(\mathscr{S}_i\backslash\mathscr{S}_{i-1})$ we have $h_\Delta>\lambda^{1-2s(r+p-(i-1))}$ and consequently

$$\begin{aligned}\sum_{n+1\,1}^{r+p-i}&=\sum_{(F_{n-1}\circ h)^{-1}\Delta}(N(F_{n-1})+n(\Delta)+1)|(F_{n-1}\circ h)^{-1}\Delta|\\&<2^{-1}\lambda^{-1+s(r+p-(i-1))}(1+O(\lambda^{-\gamma}))\Delta F_{n-1}\\&\quad\cdot\left(|\mathscr{S}_i\backslash\mathscr{S}_{i-1}|+N(F_{n-1})|\mathscr{S}_i\backslash\mathscr{S}_{i-1}|+\sum_{\Delta\subset\mathscr{S}_i\backslash\mathscr{S}_{i-1}}n(\Delta)|\Delta|\right).\end{aligned} \tag{12.15}$$

We shall assume $\operatorname{im}F_{n-1}\circ h(\lambda,x)\subseteqq[0,\frac{1}{2}]$, and leave to the reader the modifications corresponding to another position of $\operatorname{im}F_{n-1}\circ h(\lambda,x)$ in $[0,1]$. Let

$\ell = \max\{i : \mathcal{S}_i \subseteqq \delta_{q-2} \backslash \delta_q\}$. Then for $i \leqq \ell$ we can apply (ii). Together with (12.11) this gives

$$\sum_{\Delta \subset \mathcal{S}_i \backslash \mathcal{S}_{i-1}} n(\Delta)|\Delta| < \sum_{\Delta \subset \mathcal{S}_i} n(\Delta)|\Delta| < \lambda^{s\beta\left(1-\frac{\alpha}{s}\right)n} \cdot N(\mathcal{S}_0) \cdot |\mathcal{S}_0| \cdot \lambda^{2si} \cdot (1+\varepsilon)$$
$$= N(\mathcal{S}_0) \cdot |\mathcal{S}_i \backslash \mathcal{S}_{i-1}| \cdot (1-\lambda^{-2s})^{-1} \cdot (1+\varepsilon) \cdot \lambda^{\beta(s-\alpha)n}.$$

Substituting this estimate in (12.15), we obtain

$$\sum_{n+1\,1}^{r+p-1} < \frac{(1+O(\lambda^{-\gamma})) \cdot |\Delta F_{n-1}| \cdot |\mathcal{S}_i \backslash \mathcal{S}_{i-1}| (1+N(F_{n-1}) + N(\mathcal{S}_0)(1+\varepsilon_1) \cdot \lambda^{\beta(s-\alpha)n})}{2\lambda^{1-s(r+p-(i-1))}}.$$

We have by definition $\mathcal{S}_k \backslash \mathcal{S}_{k-1} = F_{n-1} \circ h(\tilde{\Gamma}_{r+p-k})$ and using b_{2n}, this implies

$$|\Delta F_{n-1}| \cdot |\mathcal{S}_k \backslash \mathcal{S}_{k-1}| = \lambda^{1-2s(r+p-k)} \cdot (1+O(\lambda^{-\gamma}))(1-\lambda^{-2s}). \tag{12.16}$$

Thus

$$\sum_{n+1\,1}^{r+p-i} < \frac{(1+\varepsilon_2) \cdot \lambda^s \cdot (1+N(F_{n-1}) + N(\mathcal{S}_0)\lambda^{\beta(s-\alpha)n})}{2\lambda^{s(r+p-i)}}.$$

Consequently, by (12.13) this implies

$$\sum_{n+1\,1}^{r+p-i} < \frac{(1+\varepsilon_{12.17}) \cdot \lambda^s \cdot \left(\frac{2s}{c_0}(r+p)\right)^{3/2} \lambda^{\beta(s-\alpha)n}}{\lambda^{s(r+p-i)}}$$
$$< (1+\varepsilon_{12.17}) \left(\frac{2s}{c_0}\right)^{3/2} \frac{(r+p-i)^{3/2}}{\lambda^{s(1-\beta)(r+p-i)}} \cdot \frac{\lambda^{s\beta\left(1-\frac{\alpha}{s}\right)n+s+2\beta}}{\lambda^{s\beta(r+p-i)}}. \tag{12.17}$$

Thus for $1 \leqq i \leqq \ell$ we have on $\tilde{\Gamma}_{r+p-i}$ the analogue of assumption (i) but with an additional factor less than $\left(\text{we use } n < \frac{2s}{c_0} r\right)$

$$\frac{\lambda^{s+2\beta}}{\lambda^{s\beta\left(r+p-i-\left(1-\frac{\alpha}{s}\right)n\right)}} < \frac{\lambda^{s+2\beta}}{\left[\lambda^{s\beta\; p-i+r\left(1-\frac{2(s-\alpha)}{c_0}\right)}\right]}.$$

In a general case we have $\ell < p$ (this is not so only if $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in [0,1] \backslash \delta_3$), and we have also to estimate $\sum_{n+1\,1}^{r+j}$, $0 \leqq j < p - \ell$. Let us consider

$$\mathcal{S}_{\ell+1} = \lambda^{2s(\ell+1)} \mathcal{S}_0, \ldots, \mathcal{S}_p = \lambda^{2sp} \cdot \mathcal{S}_0 = \operatorname{im} F_{n-1} \circ h(\lambda, x).$$

We have $\mathcal{S}_p \backslash \mathcal{S}_{p-1} = [0, a_{p-1}]$, where $\frac{1}{2} - \lambda^{-2s}/2 \approx a_{p-1} \in \delta_2 \backslash \delta_3$,

$$\mathcal{S}_{p-1} \backslash \mathcal{S}_{p-2} = [a_{p-1}, a_{p-2}], \quad \frac{1}{2} - \frac{1}{2\lambda^{4s}} \approx a_{p-2} \in \delta_4 \backslash \delta_5 \ldots,$$
$$\mathcal{S}_{\ell+1} \backslash \mathcal{S}_\ell = [a_{\ell+1}, a_\ell],$$
$$\tfrac{1}{2} - \tfrac{1}{2}\lambda^{-2s(p-\ell-1)}) \approx a_{\ell+1} \in \delta_{2(p-\ell-1)} \backslash \delta_{2(p-\ell-1)+1},$$
$$\tfrac{1}{2} - \tfrac{1}{2}\lambda^{-2s(p-\ell)} \approx a_\ell \in \delta_{2(p-\ell)} \backslash \delta_{2(p-\ell)+1}.$$

By construction $\mathscr{S}_\ell=[a_\ell, F_{n-1}\circ h(\lambda,\frac{1}{2})]$ is the last enlargement of $\mathscr{S}_0$ which lies in $\delta_{q-2}\backslash\delta_q$. Hence either $a_\ell\in\delta_{q-1}\backslash\delta_q$, or $a_\ell\in\delta_{q-2}\backslash\delta_{q-1}$. For definiteness let $a_\ell\in\delta_{q-1}\backslash\delta_q$. Then $q=2(p-\ell)+1$, $\ell=p-\dfrac{q-1}{2}$,

$$\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}\subseteqq\delta_{2(p-\ell-j)}\backslash\delta_{2(p-\ell-j)+3}, \qquad j\in[1,p-\ell],$$

where $\delta_0=[0,\frac{1}{2}]$.

(i) for δ_k with $k=2(p-\ell-j)$, $k+3\leqq q\leqq\left(1-\dfrac{\alpha}{s}\right)n$ implies

$$\sum_n^k+\sum_n^{k+1}+\sum_n^{k+2}<\frac{k^{3/2}}{\lambda^{s(1-\beta)k}}(1+\varepsilon_3).$$

By construction

$$\frac{1-\varepsilon_4}{2}\lambda^{-2s(p-\ell-j)}<|\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}|<\lambda^{-2s(p-\ell-j)}.$$

Hence using (12.7) we obtain similarly to (12.14) for $j\in[1,p-\ell]$

$$\begin{aligned}
&\sum_{\Delta\subset\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}}(N(F_{n-1})+n(\Delta))|F_{n-1}^{-1}(\Delta)| \\
&\quad<(1+O(\lambda^{-\gamma}))|\Delta F_{n-1}|\{N(F_{n-2})|\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}| \\
&\qquad+[2(p-\ell-j)]^{3/2}\lambda^{-s(1-\beta)2(p-\ell-j)}\} \\
&\quad<(1+\varepsilon_{12.19})|\Delta F_{n-1}||\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}| \\
&\qquad\cdot[N(F_{n-1})+2(2(p-\ell-j))^{3/2}\lambda^{2s\beta(p-\ell-j)}].
\end{aligned}\tag{12.19}$$

By construction

$$\lambda^{2s}\cdot\operatorname{dist}(a_\ell, F_{n-1}\circ h(\lambda,\tfrac{1}{2}))>\frac{1-\lambda^{-s}}{\lambda^{s(q-2)}}.$$

This implies

$$h_\Delta>(1-\lambda^{-s})|\Delta F_{n-1}|\lambda^{-s(2(p-\ell)+1)}$$

on $F_{n-1}^{-1}(\mathscr{S}_{\ell+1}\backslash\mathscr{S}_\ell)$ and

$$h_\Delta>\frac{(1-\lambda^{-s})|\Delta F_{n-1}|}{\lambda^{s(2(p-\ell-j)+3)}}$$

on $F_{n-1}^{-1}(\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1})$. Applying (12.6) we obtain from (12.19) that

$$\begin{aligned}
\sum_{n+1\,1}^{r+p-(\ell+j)}&<(1+\varepsilon_{12.20})|\Delta F_{n-1}|^{1/2}|\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}|[N(F_{n-1})+1 \\
&\quad+(2(p-\ell-j))^{3/2}2\lambda^{2s\beta(p-\ell-j)}]\lambda^{-1/2}\lambda^{-s(p-\ell-j+3/2)}.
\end{aligned}\tag{12.20}$$

Now, (12.16) with $k=\ell+j$ implies

$$|\Delta F_{n-1}||\mathscr{S}_{\ell+j}\backslash\mathscr{S}_{\ell+j-1}|<\frac{\sqrt{\lambda}\lambda^{-s(p-\ell-j)}(1+\varepsilon_5)}{\lambda^{s(r+p-\ell-j)}}.$$

Substituting this into (12.20) we obtain

$$\sum_{n+1\,1}^{r+p-(\ell+j)} < (1+\varepsilon_{12.21})\lambda^{3s/2}\,\frac{N(F_{n-1})+1+(2(p-\ell-j))^{3/2}\lambda^{2s\beta(p-\ell-j)}}{\lambda^{s(r+p-\ell-j)}}. \tag{12.21}$$

According to (12.10), $N(F_{n-1})+1<2(2sr/c_0)^{3/2}$. Because

$$2(p-\ell)<\left(1-\frac{\alpha}{s}\right)n<\left(1-\frac{\alpha}{s}\right)\frac{2sr}{c_0},$$

we can rewrite (12.21) as

$$\sum_{n+1\,1}^{r+p-\ell-j} < (1+\varepsilon_{12.22})\left(\frac{2s}{c_0}\right)^{3/2}\frac{r^{3/2}}{\lambda^{s(1-\beta)(r+p-\ell-j)}} \cdot \frac{2\lambda^{\frac{3s}{2}}}{\lambda^{s\beta(r+p-\ell-j)\left(1-\frac{2s}{c_0}\right)}}. \tag{12.22}$$

Thus for $0\leqq k<p-\ell$ we have on $\tilde{\Gamma}_{r+k}$ an additional exponentially small factor compared with the assumption (i), as well as for $p-\ell\leqq k<p$ (see (12.18)).

f) In order to estimate the contribution of terms in $(\delta_n-\delta_{n+1})^{-M}$ we first do it in $\delta_n\backslash\delta_{n+1}$.

Step $n+1$ on $\delta_n\backslash\delta_{n+1}$ divides into subspteps $\ell=1,2,\ldots$ corresponding to the construction of $f_{n+1\,\ell}$ (see Sect. 3).

We use the following notation: Δ is any interval $\Delta F_{n-1}\circ g$, $\Delta\hat{f}_n$, $\Delta f_{n+1\,\ell}$; δ is any hole δ_n^{-M}, $\delta_{n+1}^{-N}\subseteqq\delta_n\backslash\delta_{n+1}$; $n(\Delta)$, $n(\delta)$ are corresponding numbers of iterations.

Let $i_{n\ell}=\sum|\Delta|$ after substep ℓ of step $n+1$ and with the same meaning of indices n, ℓ

$$h_{n\ell}=\sum|\delta|;\qquad x_{n\ell}=\sum n(\Delta)|\Delta|;\qquad y_{n\ell}=\sum n(\delta)|\delta|.$$

We consider also the corresponding sums on $[0,1]\backslash\delta_n$ namely

$$I_n=\sum|\Delta|,\qquad \Delta\subseteqq[0,1]\backslash\delta_n,$$

after step n:

$$H_n=\sum|\delta|;\quad X_n=\sum n(\Delta)|\Delta|;\quad Y_n=\sum n(\delta)|\delta|.$$

Then $I_n+H_n+|\delta_n|=1$, $i_{n\ell}+h_{n\ell}+|\delta_{n+1}|=|\delta_n|$ for all ℓ. Besides, let $\ell=0$ correspond to i, h, x, y constructed after step n, and $\ell=\infty$ after step $n+1$, so that (n,∞) equals $(n+1,0)$.

We may assume all the compositions to be linear (see Remark XII/1 below) and thus using (12.7) we obtain

$$\left.\begin{aligned} i_{n1}&=i_{n0}(1-H_n-|\delta_n|)\\ h_{n1}&=h_{n0}+i_{n0}(H_n+|\delta_n|)\\ x_{n1}&=x_{n0}I_n+X_n\cdot i_{n0}\\ y_{n1}&=y_{n0}+x_{n0}(H_n+|\delta_n|)+Y_n\cdot i_{n0}. \end{aligned}\right\} \tag{12.23}$$

The holes and intervals of subsequent substeps $1=2,3,\dots$ are obtained using compositions of maps $\tilde{G}_n:\delta_n^{-M}\to\delta_n$, so that after substep ℓ the remaining preimages of δ_n are of the form $\tilde{G}_{n_\ell}^{-1}\circ\dots\circ\tilde{G}_{n_1}^{-1}\delta_n$ and preimages of δ_{n+1} are $\tilde{G}_{n_i}^{-1}\circ\dots\circ\tilde{G}_{n_1}^{-1}\delta_{n+1}$, $i=1,2,\dots,\ell-1$ (compare with (3.6), (3.7)). Let $\tilde{h}_{n\ell}$ and $\tilde{y}_{n\ell}$ correspond to preimages of δ_n and $\tilde{\tilde{h}}_{n\ell}$, $\tilde{\tilde{y}}_{n\ell}$ to preimages of δ_{n+1}. With this notation we have

$$\tilde{\tilde{y}}_{n1}=\tilde{\tilde{h}}_{n1}=0\,,\qquad h_{n\ell}=\tilde{h}_{n\ell}+\tilde{\tilde{h}}_{n\ell}\,,\qquad y_{n\ell}=\tilde{y}_{n\ell}+\tilde{\tilde{y}}_{n\ell}\,,$$

and for $\ell\geqq 2$

$$\left.\begin{aligned}
i_{n\ell}&=i_{n\ell-1}+i_{n\ell-1}\cdot\tilde{h}_{n\ell-1}\cdot|\delta_n|^{-1}\\
\tilde{h}_{n\ell}&=\tilde{h}_{n\ell-1}\cdot\tilde{h}_{n\ell-1}\cdot|\delta_n|^{-1}\,;\\
\tilde{\tilde{h}}_{n\ell}&=\tilde{\tilde{h}}_{n\ell-1}+\tilde{h}_{n\ell-1}\cdot\frac{\tilde{\tilde{h}}_{n\ell-1}+|\delta_{n+1}|}{|\delta_n|}\\
x_{n\ell}&=x_{n\ell-1}+(\tilde{y}_{n\ell-1}\cdot i_{n\ell-1}+x_{n\ell-1}\cdot\tilde{h}_{n\ell-1})\cdot|\delta_n|^{-1}\\
\tilde{y}_{n\ell}&=2\tilde{y}_{n\ell-1}\cdot\tilde{h}_{n\ell-1}\cdot|\delta_n|^{-1}\,;\\
\tilde{\tilde{y}}_{n\ell}&=\tilde{\tilde{y}}_{n\ell-1}+\frac{\tilde{y}_{n\ell-1}\cdot\tilde{\tilde{h}}_{n\ell-1}+\tilde{\tilde{y}}_{n\ell-1}\cdot\tilde{h}_{n\ell-1}+\tilde{y}_{n\ell-1}|\delta_{n+1}|}{|\delta_n|}\,.
\end{aligned}\right\}\tag{12.24}$$

According to Sect. 10, to any hole $\delta_n^{-M}=G_n^{-M}(\delta_n)$ there corresponds uniquely a set $L_n^{-M}=G_n^{-M}(L)$. As for any interval $\Delta\subseteqq L_n^{-M}$, $n(\Delta)>n(\delta_n^{-M})$; this implies

$$\tilde{h}_{n1}<(|\delta_n|-|\delta_{n+1}|)(1+\varepsilon)\cdot\lambda^{-2\alpha n}\,.$$

Using $\dfrac{|\delta_{n+1}|}{|\delta_n|}<(1+o(\lambda^{-s(n+1)})\lambda^{-s}$, the recurrent formulas (12.24) give

$$\begin{aligned}
x_{n\infty}&=x_{n+1\,0}<(x_{n1}+y_{n1})(1+O(\lambda^{-2\alpha n}))\,;\\
y_{n\infty}&=y_{n+1\,0}<y_{n0}\cdot\frac{1+o(\lambda^{-tn})}{\lambda^s}\,.
\end{aligned}$$

The induction hypotheses imply

$$\begin{aligned}
X_n&<1-\frac{2}{\lambda^s}+\left(\sum_{k=1}^{n}\frac{k^{3/2}}{\lambda^{s(1-\beta)k}}\right)\left(\sum_{i=0}^{n-1}\lambda^{-ti}\right)<1+\varepsilon_1\,;\\
x_{n0}&<\frac{n^{3/2}}{\lambda^{s(1-\beta)n}}\sum_{i=0}^{n-1}\lambda^{-ti}\,;\\
i_{n0}&<2(1+o(\lambda^{-sn}))\cdot(1-\lambda^{-s})\cdot\lambda^{-sn}\,;\\
I_n&<1-2\lambda^{-sn}\,;\\
H_n+|\delta_n|&<[\lambda^s\cdot(1+\varepsilon_2)]^{-n}\,.
\end{aligned}$$

By the above reasons

$$y_{n0}<x_{n0}\cdot\lambda^{-2\alpha n}\cdot(1+\varepsilon)\,;\qquad Y_n<X_n\cdot\lambda^{-2\alpha n}\cdot(1+\varepsilon)\,.$$

Using (12.23) we obtain

$$\sum_{n+1}^{n} = x_{n+10} + y_{n+10} < \left(x_{n0} + \frac{2}{\lambda^{sn}}\right)(1+O(\lambda^{-2\alpha n})) < \frac{n^{3/2}}{\lambda^{s(1-\beta)n}} \sum_{i=0}^{n} \lambda^{-ti} \tag{12.25}$$

which proves i_{n+1} for $\delta_n \backslash \delta_{n+1}$. The proof is similar for $\delta_k \backslash \delta_{k+1}$, $1 \leqq k < n$.

Now we can estimate the contribution of $\sum n(\Delta)|\Delta|$ in every hole $(\delta_n \backslash \delta_{n+1})^{-M}$ on $\tilde{\Gamma}_{r+i} = \Gamma_{r+i} \backslash \Gamma_{r+i+1}$, $i \in [0, p-1]$.

Though we cannot correspond the $\lambda^{2\alpha n}$-enlargement to any δ_n^{-M}, we can consider its $\lambda^{\alpha n/2}$-enlargement. The construction of Sect. 10 gives, as above,

$$\sum_{\delta \subset \tilde{\Gamma}_{r+i}} |\delta| < \frac{|\tilde{\Gamma}_{r+i}|(1+\varepsilon)}{\lambda^{\alpha n/2}}; \qquad \sum_{\delta \subset \Gamma_{r+i}} n(\delta)|\delta| < \left(\sum_{\Delta \subset \tilde{\Gamma}_{r+i}} n(\Delta)|\Delta|\right)\frac{1+\varepsilon}{\lambda^{\alpha n/2}}.$$

Using (12.7) with $\mu = O(\lambda^{-\alpha n/2})$ we obtain after step $(n+1)$

$$\sum_{n+1}^{r+i} < \sum_{n+1\,1}^{r+i} + \left(\sum_{n+1\,1}^{r+i} \cdot (1-\lambda^{-s})|\delta_n| \cdot \lambda^{-\alpha n/2} + \sum_{n+1}^{n} \cdot |\tilde{\Gamma}_{r+i}| \cdot \lambda^{-\alpha n/2}\right) \frac{1+O(\lambda^{-\alpha n/2})}{|\delta_n|}$$
$$< (1+\varepsilon_{12.26}) \left(\sum_{n+1\,1}^{r+i} + \frac{n^{3/2}}{\lambda^{s(r+i)}} \cdot \frac{\lambda^{s\beta n}}{\lambda^{\alpha n/2}}\right). \tag{12.26}$$

Thus we still have for $\sum_{n+1}^{r+i}$ an exponentially better estimate than that required by (i).

This proves (i_{n+1}) for $k \in [r, R(n+1)-1]$. Now $\tilde{\Gamma}_k$ $k \in [R(n), r-1]$ are contained in the union of preimages $(F_{n-2} \circ h)^{-1} \Delta_i^{(n-1)}$, where $\Delta_i^{(n-1)} \subset B_{n-1}(\Delta_0^{(n-1)})$. One obtains (i_{n+1}) for such $\tilde{\Gamma}_k$ in a similar way, using the construction of block B_{n-1} (the estimates are better in this case).

In order to obtain (i_{n+1}) for $n+1 \leqq k < R(n)-1$, we notice that at step $m(k)$ corresponding to the first consideration of $\Gamma_k \backslash \Gamma_{k+1}$, we have on $\Gamma_k \backslash \Gamma_{k+1}$ an exponential reserve by comparison with $(i_{m(k)})$. (12.7) and Property (12.9) imply that the nonlinearity at Step $(m(k)+1)$ gives an additional factor less than 3. Any of the subsequent steps implies the diminishing of the maximal interval $\Delta \subset \Gamma_k \backslash \Gamma_{k+1}$ at least $3\lambda^{c_0}$ times (because of taking compositions), and we obtain the following:

Remark XII/1. The total non-linear effect of steps $m(k)+1$, $m(k)+2, \ldots$ on $\Gamma_k \backslash \Gamma_{k+1}$ is less than

$$\exp\left(3 \cdot \sum_{n \subset 0}^{\infty} \lambda^{-c_0 n}\right).$$

In particular this shows that when proving (i) for $\Gamma_k \backslash \Gamma_{k+1}$ it suffices to consider only step $m(k)$.

g) In order to prove (ii_{n+1}) of Proposition 3 we consider three points x_0, x_1, $x_2 \in \Gamma_{r+i-2} \backslash \Gamma_{r+i}$ and their images under $F_{n-1} \circ h(\lambda, x)$. We may suppose x_2 to be closer to $\frac{1}{2}$ than x_0, (otherwise h_Δ for $\Delta \subset [x_1, x_2]$ is larger than for $\Delta \subset [x_0, x_1]$ and an estimate for x_0, x_1, x_2 is better than for their images).

Let $Q_1 = F_{n-1} \circ h[x_0, x_1]$, $Q_2 = F_{n-1} \circ h[x_0, x_2]$. Using (12.7) and (12.6) with $h_\Delta \geqq h(x_2) \geqq \lambda^{1-2s(r+i)}$ we obtain

$$\sum_{\Delta \subset [x_0, x_2]} n(\Delta)|\Delta| < \frac{1+O(\lambda^{-\gamma}))|\Delta F_{n-1}|\left((N(F_{n-1})+1)|Q_2| + \sum_{\Delta \subset Q_2} n(\Delta)|\Delta|\right)}{2\sqrt{\lambda}\lambda^{-s(r+i)}}. \tag{12.27}$$

For $\Delta\subset[x_0,x_1]$ $h_\Delta \leqq h(x_0)\leqq \lambda^{1-2s(r+i)+4s}$. Hence

$$\sum_{\Delta\subset[x_0,x_1]} n(\Delta)|\Delta| > \frac{(1-O(\lambda^{-\gamma}))|\Delta F_{n-1}|\left((N(F_{n-1})+1)|Q_1|+\sum_{\Delta\subset Q_1} n(\Delta)|\Delta|\right)}{2\sqrt{\lambda}\lambda^{-s(r+i)+2s}}. \tag{12.28}$$

As $\dfrac{|h[x_0,x_2]|}{|h[x_0,x_1]|}<\dfrac{|x_0-x_2|}{|x_0-x_1|}$, we have $\dfrac{|Q_2|}{|Q_1|}<\dfrac{|x_0-x_2|}{|x_0-x_1|}(1+O(\lambda^{-\gamma}))$. First let $Q_2\subset\delta_{q-2}\backslash\delta_q$. Then we can use (ii$_n$) for $F_{n-1}\circ h(x_0,x_1,x_2)\subset[0,1]\backslash\delta_{(1-\alpha/s)n}$. Applying (12.27), (12.28), we obtain

$$\frac{\sum_{\Delta\subset[x_0,x_2]} n(\Delta)|\Delta|}{\sum_{\Delta\subset[x_0,x_1]} n(\Delta)|\Delta|}<\frac{|x_0-x_2|}{|x_0-x_1|}\lambda^{(s-\alpha)\beta n+2s}.$$

If $F_{n-1}\circ h[x_0,x_2]$ is not contained in $\delta_{q-2}\backslash\delta_q$, we have $Q_2=Q_2'\cup Q_2''$ where $Q_2'\subset\delta_{q-2}\backslash\delta_q$, $Q_2''\subset[0,1]\backslash\delta_{q-2}$. We estimate $\sum_{\Delta\subset Q_2'} n(\Delta)|\Delta|$ as above, and $\sum_{\Delta\subset Q_2''} n(\Delta)|\Delta|$ using (i_n) similarly to (12.19)–(12.21), and obtain

$$\frac{\sum_{\Delta\subset[x_0,x_2]} n(\Delta)|\Delta|}{\sum_{\Delta\subset[x_0,x_1]} n(\Delta)|\Delta|}<\frac{|x_0-x_2|}{|x_0-x_1|}\sqrt{n}\lambda^{(s-\alpha)\beta n+2s}. \tag{12.29}$$

For large λ, $\sqrt{n}\ll\lambda^{\alpha\beta n}$. Comparing with the requirement (ii_{n+1}) for $k=r+i$, we obtain a sufficient condition on r

$$s\beta n+2s\leqq s\beta r.$$

As $r>\dfrac{c_0}{2s}n$, it suffices to have

$$n\geqq\tfrac{2}{5}\left(\frac{c_0}{2s}-1\right)^{-1} \tag{12.30}$$

which holds for $s\leqq\frac{1}{13}$, $\beta=\frac{2}{5}$, $n\geqq1$.

The account of $\Delta\subset\delta_n^{-M}$ gives an additional factor $(1+O(\lambda^{-\alpha n/2}))$ and one finishes the proof of (ii_{n+1}) as above (i_{n+1}). □

Remark XII/2. One can check that for $n\leqq n_0$, when

$$F_{n-1}\circ h(\lambda,\tfrac{1}{2})\in[0,1]\backslash[\tfrac{1}{2}-\lambda^{-s/2},\tfrac{1}{2}+\lambda^{-s/2}],$$

(ii) is satisfied with $\beta=0$ [$\lambda^{s\beta k}$ on the right side of (ii) can be replaced by a constant]. From Remark VI/5 and (12.30) it follows that one can take β arbitrarily small. It seems that more careful estimates should give Proposition 3 with $\beta=0$ and $k^{1+\varepsilon}$ ($\varepsilon>0$ small) instead of $k^{3/2}$.

Remark XII/3. Lemma 7 implies that for any $\lambda\in\mathcal{M}$ and for $\Delta f_k\in\xi_\lambda$ so that $f_k:\Delta f_k\to[0,1]$, $f_k=f_\lambda^N|\Delta f_k$,

$$|Df_k|>\lambda^{c_0\sqrt{N}/2}.$$

Collet and Eckmann [10] proved for a particular smooth family f_δ that the Liapunov exponent is positive on the trajectory of $\frac{1}{2}$ for a set of λ of positive measure.

13. Theorem A for a General Family. The Reduction of Theorem B to Theorem A

a) Let $f(x):[0,1]\to[0,1]$, $f(0)=f(1)=0$, be a C^3-map, c a single critical point of f. Consider a family $f_\lambda(x):x\to\lambda\cdot f(x)$ (mod 1). We take λ sufficiently large and imitate the construction used for $\lambda x(1-x)$.

We take $T_0=(f(c))^{-1}$ so as to make $\lambda\cdot f(c)$ traverse $[0,1]$ when λ crosses $[L, L+T_0]$.

Then we choose a small $\varepsilon>0$ and consider an ε-neighbourhood U of the critical point c. Using the Hadamard lemma we represent $f(x)$ and its derivatives in the form

$$\begin{aligned} f(x)&=f(c)-a(x-c)^2(1+(x-c)\theta_1(x))\\ f'(x)&=-2a(x-c)(1+(x-c)\theta_2(x))\\ f''(x)&=-2a(1+(x-c)\theta_3(x)), \end{aligned} \tag{13.1}$$

where $-2a=f''(c)<0$, $|\theta_i(x)|<c_1$. Using (13.1), one can check that (4.6) with $\frac{|\Delta|}{H}$ instead of $\frac{|\Delta|}{2H}$, and (4.7) with $\frac{|\Delta|}{ax^2}$ instead of $\frac{|\Delta|}{2ax^2}$ are still true in U.

Remark XIII/1. Notice that the condition $f''(c)\neq 0$ is not necessary, $f^{(n)}(c)\neq 0$ for some $n\geqq 2$ will do as well.

Then we consider

$$Df_\lambda=\lambda f'(x), \qquad \frac{D^2f_\lambda}{(Df_\lambda)^2}=\frac{1}{\lambda}\frac{f''(x)}{(f'(x))^2},$$

$$\frac{\partial f/\partial\lambda}{\partial f/\partial x}=\frac{1}{\lambda}\frac{f(x)}{f'(x)}, \qquad \frac{\frac{\partial}{\partial\lambda}\frac{\partial f_\lambda^{-1}}{\partial z}}{\frac{\partial f_\lambda^{-1}}{\partial z}}=-\frac{1}{\lambda}\left(1-\frac{f''\cdot f}{(f')^2}\right).$$

Let

$$A=\max_{x\in[0,1]\setminus U}\left\{\frac{1}{|f''(x)|},\ 1+\frac{|f''(x)|}{(f'(x))^2}\right\}.$$

We take s from Sect. 2, and we take a λ as a parameter. We choose λ so large that

$$\lambda^s>\max\left\{\frac{2}{\varepsilon a^s}, 2Aa^{1-s}\right\}.$$

Then we choose $\delta_1\approx(c-(\lambda a)^{-s}, c+(\lambda a)^{-s})$ as in Sect. 2, and define $f_1(\lambda, x)$ so that $\Delta f_1\subseteqq[0,1]\setminus\delta_1$. One can check that the branches f_1 and their derivatives satisfy the conditions of Step 1 with $\max\limits_{x\in[0,1]} f(x)$ instead of $\frac{1}{4}=\max\limits_{x\in[0,1]} x(1-x)$. Then for $a\lambda > N_0$ of Sect. 2, the inductive construction may be used, and we obtain Theorem A for the family $\lambda \cdot f(x)$.

Remark XIII/2. Theorem A holds also in the case of a family $\lambda \cdot f(x)$, $f(0)=0=f(1)$, $f'(0) \neq 0$, when $f(x)$ has several extremal points $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$. Then the construction can be generalized in the following manner. During step n we construct intervals $\delta_n^{(i)} \approx (c^{(i)} - (\lambda a_i)^{-sn}, c^{(i)} + (\lambda a_i)^{-sn})$, $1 \leqq i \leqq k$, their preimages $(\delta_n^{(i)})^{-m}$, and enlarged preimages $(\hat{\delta}_n^{(i)})^{-m}$; the constants a_i are defined according to the map f. The condition

$$F_{n-1}^{(i)} \circ h^{(i)}(\lambda, c^{(i)}) \in [0,1] \Big\backslash \bigcup_{j=1}^{k} \bigcup_{m=0}^{\infty} (\hat{\delta}_n^{(j)})^{-m}$$

defines on step n the set of admissible values of the parameter $\mathcal{M}_n^{(i)}$, the set $\mathcal{M}$ is defined as $\mathcal{M} = \bigcap_{i=1}^{k} \bigcap_{n=1}^{\infty} \mathcal{M}_n^{(i)}$.

b) We reduce the proof of Theorem B to the proof of Theorem A using the induced map studied in [5]. Let $f_\lambda(x) = \lambda x(1-x)$, $0 < \lambda \leqq 4$, and $t_\lambda = 1 - 1/\lambda$ its fixed point. We consider for $\lambda \in [4-\varepsilon, 4]$ the induced map T_λ on the interval $I_\lambda = [1/\lambda, 1-1/\lambda]$. T_λ has $2p$ monotone branches $T_{i\lambda}$, $i = \pm 1, \ldots, \pm p$ $(p = p(\lambda))$ and one middle branch S_λ. Furthermore, $T_{i\lambda} = f_\lambda^{i+1}$ on $\Delta T_{i\lambda}$ and $S_\lambda = f_\lambda^{p+2}$ on ΔS_λ. The interval $[4-\varepsilon, 4]$ is divided into a countable number of intervals $[\lambda_p, \lambda_{p+1}]$ such that for $\lambda \in [\lambda_p, \lambda_{p+1}]$ the number $p(\lambda)$ defined above is constant and as λ passes λ_p, the old parabolic branch S_λ breaks up into two branches T_λ, a new branch S_λ is born, and $p(\lambda)$ grows from p to $p+1$.

For some constants $c_1, c_2 > 0$ we have

$$\begin{gathered} 2^i c_2 < |\partial T_i/\partial x| < 2^i c_1 \quad 1 \leqq i \leqq p-1 \\ 4^{p+1} c_2 |x - \tfrac{1}{2}| < |\partial T_p/\partial x|, |\partial S/\partial x| < 4^{p+1} c_1 |x - \tfrac{1}{2}|. \end{gathered} \tag{13.2}$$

Applying (9.1) to $T_{i\lambda}$ we obtain

$$\begin{gathered} \left| \frac{\partial T_i/\partial \lambda}{\partial T_i/\partial x} \right| < 2^i c_3 \quad 1 \leqq i \leqq p-1, \\ \left| \frac{\partial T_p/\partial \lambda}{\partial T_p/\partial x} \right|, \left| \frac{\partial S/\partial x}{\partial S/\partial x} \right| < \frac{c_3}{|x-\frac{1}{2}|}. \end{gathered} \tag{13.3}$$

The estimate for the velocity of the top is

$$v_p(\lambda) = -4^p(1 + O(\lambda^{-p})). \tag{13.4}$$

We have

$$\begin{gathered} \frac{|D^2 T_{i\lambda}|}{|DT_{i\lambda}|^2} < c_4 \quad 1 \leqq i \leqq p-1 \\ \frac{|D^2 T_{p\lambda}|}{|DT_{p\lambda}|^2}, \frac{|D^2 S_\lambda|}{|DS_\lambda|^2} < c_4 \left(1 + \frac{1}{4^p (x - \frac{1}{2})^2}\right). \end{gathered} \tag{13.5}$$

Using (11.6) we obtain for all i and $z=T_i(\lambda,x)$, $x\in\Delta T_i$

$$\left|\frac{\frac{\partial}{\partial\lambda}\partial T_{i\lambda}^{-1}/\partial z}{\partial T_{i\lambda}^{-1}/\partial z}\right|<c_5\left(i+\frac{1}{(x-\frac{1}{2})^2}\right). \tag{13.6}$$

Now we use the following property of T_4 (see [5]). There exists $d>1$ and a positive integer q so that

$$|DT_4^q|>d. \tag{13.7}$$

Remark XIII/3. Apparently $q=1$ but it is not essential for our purpose.

For any fixed i, $T_i(\lambda,x)$ and its derivatives uniformly converge to $T_i(4,x)$ when $\lambda\to 4$. Thus for $i_s\in[0,i_0]$ and for λ sufficiently close to 4 we still have

$$|D(T_{\lambda i_1}\circ T_{\lambda i_2}\circ\ldots\circ T_{\lambda i_q})|>d. \tag{13.8}$$

Choose a very large k and some $p\gg k$, and consider $\lambda\in[\lambda_p,\lambda_{p+1}]$. Let

$$n=\left[\frac{k}{\log_2 d}+1\right].$$

Let us consider consecutive compositions of the form

$$T_{\lambda\tau_r}=T_{\lambda i_1}\circ\ldots\circ T_{\lambda i_r},\qquad i_s\in[1,k]$$

until we have on the domain of $T_{\lambda\tau}$

$$|DT_{\lambda\tau}|>2^k.$$

Because of (13.8), for any $T_{\lambda\tau_p}$, $r\leqq qn$ (really (13.2) implies $r\ll qn$ for many $T_{\lambda\tau_r}$). Let

$$\delta_1(\lambda)=\Delta S_\lambda\cup\left(\bigcup_{i>k}\Delta T_{\lambda i}\right),\qquad J_\lambda=I_\lambda\backslash\delta_1(\lambda).$$

Then we obtain the following partition of J_λ.

$$J_\lambda=(\bigcup\Delta T_{\lambda\tau})\cup\left(\bigcup_{m\leqq qn}\delta_1^{-m}(\lambda)\right). \tag{13.9}$$

(13.5), (13.8) and a modification of Lemma 1 imply

$$\frac{|D^2T_{\lambda\tau_r}|}{|DT_{\lambda\tau_r}|^2}<c_6$$

independent of k. Hence we obtain

$$\operatorname{mes}\bigcup\delta_1^{-m}(\lambda)<1-\left(1-\frac{c_7}{2^k}\right)^{qn}<c_8\frac{k}{2^k}. \tag{13.10}$$

Using (13.8), (9.1), and (11.6) we obtain

$$\left|\frac{\partial T_{\lambda\tau}/\partial\lambda}{\partial T_{\lambda\tau}/\partial x}\right|<2^k c_9,$$

$$\left|\frac{\frac{\partial}{\partial\lambda}\partial T_{\lambda\tau}^{-1}/\partial z}{\partial T_{\lambda\tau}^{-1}/\partial z}\right|<c_{10}(k+4^k). \tag{13.11}$$

Although the estimates (13.11) grow with k, we can choose p so large that the time that the top $S_\lambda(\frac{1}{2})$ spends inside the union of the enlarged domains $\bigcup \hat{\delta}_1^{-m}(\lambda)$ will still be proportional to its measure.

Now we are able to begin the inductive construction, with branches $T_{\lambda\tau}$ instead of f_1 and $\bigcup_{m \leq ck} \hat{\delta}_1^{-m}(\lambda)$ instead of $\hat{\delta}_1(\lambda)$. In particular, the intervals δ_n have the form $\delta_n \approx 2^{-sk(n-1)}\delta_1$. The estimates (13.2)–(13.10) allow the induction to continue, and if we denote by $\mathcal{M}_p$ the set of $\lambda \in [\lambda_p, \lambda_{p+1}]$ obtained by using an inductive construction similar to that in Sect. 3, we obtain that the induced map $T_\lambda : I_\lambda \to I_\lambda$ has a measure $\tilde{\mu}_\lambda$ absolutely continuous with respect to dx. Besides, for some constants c, $\alpha > 0$ independent of k and p we have

$$\frac{\operatorname{mes} \mathcal{M}_p}{|\lambda_p - \lambda_{p+1}|} > 1 - c\frac{k}{(2^k)^\alpha}. \tag{13.12}$$

The measure $\tilde{\mu}_\lambda$ induces an f_λ-invariant measure on $[0, 1]$ supported on $[f_\lambda^2(\frac{1}{2}), f_\lambda(\frac{1}{2})]$. Since the time of return to I_λ is finite for all $x \in I_\lambda$, μ_λ is certainly finite.

Let $\Lambda_1 = \bigcup_{p=p_0}^{\infty} \mathcal{M}_p$. We take $k \to \infty$ together with p, and obtain from (13.12) that $\lambda = 4$ is a Lebesgue point (from one side) of Λ_1. This proves Theorem B and the Remark of the introduction for $f_\lambda(x) = \lambda x(1-x)$ $0 < \lambda \leq 4$.

Remark XIII/4. The measures μ_λ certainly are ergodic, because the ν_λ are. It follows from the recent results by Ledrappier [16] that the natural extensions of (f_λ, μ_λ) are Bernoulli.

Remark XIII/5. One may conjecture that the densities μ_λ converge in L_1 to $\varrho_4(x) = (\pi\sqrt{x(1-x)})^{-1}$, when $\lambda \to 4$. Notice that the construction always gives measures supported on the maximal possible interval $[f_\lambda^2(\frac{1}{2}), f_\lambda(\frac{1}{2})]$ and thus avoids λ corresponding to measures supported by pairwise disjoint intervals permuted by f_λ.

c) Consider any $f(x) : [0, 1] \to [0, 1]$, $f(0) = f(1) = 0$, $f'(c) = 0$, lying in a sufficiently small C^3-neighbourhood of $x(1-x)$. Then for a family $\lambda \cdot f(x)$ there exists some λ_0 close to 4 so that $\lambda_0 f(c) = 1$. Considering for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$ the corresponding induced map $T_{f\lambda} : I_{f\lambda} \circlearrowleft$, we obtain that $T_{f\lambda}$ has on $I_{f\lambda}$ a structure similar to the one described above for $T_\lambda = T_{x(1-x)\lambda}$ and (13.7) still holds for $T_{f\lambda}$. This implies Theorem B for $f_\lambda = \lambda \cdot f(x)$.

Now, if for some $\lambda_0 \neq 4$, $f_{\lambda_0} = \lambda_0 x(1-x)$ or its iteration on some interval admits the induced map described above, the construction still goes and we obtain absolutely continuous measures invariant under f or under some iteration of f for a set of $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$ of positive measure.

One can check this is so for a countable set $\{\lambda_{0n} : f\lambda_{0n}(\frac{1}{2})$ falls into a periodic unstable orbit$\}$ and for a set $\Phi = \{\lambda : f_\lambda(\frac{1}{2}) \in K_\lambda =$ an invariant unstable Cantor set$\}$, card $\Phi =$ continuum (see [5–7]), thus all these λ are Lebesgue density points of $\mathcal{M}_1$.

Remark XIII/5. As Misiurewicz pointed out, for a family $f_\lambda = \lambda f(x)$ with unimodal $f(x) : [0, 1] \to [0, 1]$, $f(0) = f(1) = 0$, having negative Schwarzian derivative, and for λ_0 such that $f_{\lambda_0}(c)$ falls into an unstable periodic orbit or an invariant unstable Cantor set, the corresponding incuded map also satisfies (13.7). Thus the same construction implies that for a set of λ of positive measure f_λ admits an absolutely continuous invariant measure and λ_0 is a Lebesgue density point of this set.

Acknowledgements. I would like to thank Ya. G. Sinai and M. Misiurewicz for many helpful discussions, M. Misiurewicz and Z. Nitecki for attentive reading of the text and for making many useful remarks. I am very grateful to Z. Nitecki for his help in preparation of the manuscript for publication.

References

1. Ulam, S., von Neumann, J.: Bull. Am. Math. Soc. **53**, 1120 (1947)
2. Bunimovič, L.A.: Mat. Zametki **8**, 205–216 (1970)
3. Ruelle, D.: Commun. Math. Phys. **55**, 47–51 (1977)
4. Bowen, R.: Commun. Math. Phys. **69**, 1–17 (1979)
5. Jakobson, M.V.: Dokl. Akad. Nauk SSSR **243**, 866–869 (1978); Sov. Math. Dokl. **19**, 1452–1456 (1978)
6. Misiurewicz, M.: Absolutely continuous measures for certain maps of the interval. Publ. Math. IHES (to appear)
7. Szlenk, W.: Some dynamical properties of certain dfferentiable mappings of an interval, Part I. Bol. Soc. Mat. Mexico (to appear) Part II, Ergodic theory and dynamical systems, Vol. II, Katok, A. (ed.). Boston: Birkhäuser 1981
8. Ognev, A.: Metric properties of a class of maps of the interval. Mat. Zametki (1981) (to appear)
9. Ito, S., Tanaka, S., Nakada, H.: Tokyo J. Math. **2**, 221–239, 241–259 (1979)
10. Collet, P., Eckmann, J.-P.: On the abundance of aperiodic behaviour for maps of the interval. Commun. Math. Phys. **73**, 115–160 (1980)
11. Guckenheimer, J.: Commun. Math. Phys. **70**, 133–160 (1979)
12. Adler, R.L.: F-expansions revisited. J: Lecture Notes in Mathematics, Vol. **318**, pp. 1–5. Berlin, Heidelberg, New York: Springer 1973
13. Walters, P.: Trans. Am. Math. Soc. **236**, 121–153 (1978)
14. Brolin, H.: Ark. Mat. **6**, 103–144 (1966)
15. Kolmogorov, A.N., Fomin, S.V.: Elements of the theory of functions and of functional analysis. Moscow, 1960
16. Ledrappier, F.: Some properties of absolutely continuous measures on an interval. Paris, Preprint, 1980

Communicated by Ya. G. Sinai

Received May 5, 1980; in revised form February 25, 1981