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Tetrahedron Equations and the Relativistic S-Matrix of Straight-Strings in $2+1$-Dimensions

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Abstract. The quantum S-matrix theory of straight-strings (infinite one-dimensional objects like straight domain walls) in $2+1$-dimensions is considered. The S-matrix is supposed to be "purely elastic" and factorized. The tetrahedron equations (which are the factorization conditions) are investigated for the special "two-colour" model. The relativistic three-string S-matrix, which apparently satisfies this tetrahedron equation, is proposed.

1. Introduction

The progress of the last decade in studying two-dimensional exactly solvable models of quantum field theory and lattice statistical physics was motivated to some extent by using *the triangle equations*. These equations were first discovered by Yang [1]; they appeared in the problem of non-relativistic $1+1$-dimensional particles with δ-function interaction, as the self-consistency condition for Bethe's ansatz. Analogous (at least formally) relations were derived by Baxter [2], who had investigated the eight-vertex lattice model. These relations restrict the vertex weights and are of great importance for exact solvability. In particular, for the rectangular-lattice model they guarantee the commutativity of transfer-matrices with different values of the anisotropy parameter v. In the case of Baxter's general nonregular lattice $\mathscr{L}$ [3], the triangle relations for the vertex weights ensure the remarkable symmetry of the statistical system (the so-called Z-invariance): the partition function is unchanged under the deformations of the lattice, generated by the arbitrary shifts of the lattice axes. Z-invariant model on the lattice $\mathscr{L}$ is exactly solvable [3] (see also [4]).

Recently Faddeev, Sklyanin, and Takhtadjyan [5, 6] have developed a new general method of studying the exactly solvable models in $1+1$-dimensions – the quantum inverse scattering method. The triangle equations are the significant constituent of this method; they are to be satisfied by the elements of the R-matrix which determine the commutation relations between the elements of the monodromy matrix.

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The triangle equations are also the central part of the theory of the relativistic purely elastic ("factorized") S-matrix in $1+1$-dimensions (for a review, see [7] and references therein). These equations (the "factorization equations") connect the elements of the two-particle S-matrix; they represent the conditions which are necessary for the factorization of the multiparticle S-matrix into two-particle ones. For the scattering theory including n different kinds of particles A_i; $i=1,2,\ldots,n$ the factorization equations have the form [7, 4]

$$\begin{aligned} &S_{i_1 i_2}^{k_1 k_2}(\theta) S_{i_3 k_1}^{k_3 j_1}(\theta+\theta') S_{k_2 k_3}^{j_2 j_3}(\theta') \\ &\quad = S_{i_2 i_3}^{k_2 k_3}(\theta') S_{i_1 k_3}^{k_1 j_3}(\theta+\theta') S_{k_1 k_2}^{j_1 j_2}(\theta), \end{aligned} \tag{1.1}$$

where, for instance, $S_{i_1 i_2}^{k_1 k_2}(\theta)$ is the two-particle S-matrix, (i_1, i_2), (k_1, k_2) are the kinds of the initial (final) particles having the rapidities[1] θ_1 and θ_2, respectively; $\theta=(\theta_1-\theta_2)$. This equation has the following meaning. In the purely elastic scattering theory the three-particle S-matrix is factorized into three two-particle ones, as if the three-particle scattering were the sequence of successive pair collisions. If the rapidities $\theta_1, \theta_2, \theta_3$ of the initial particles are given, the two alternatives for the successions of these pair collisions are possible. The two different (in general) formal expressions for the three-particle S-matrix in terms of two-particle ones [the right- and left-hand sides of (1.1)] correspond to these alternatives. The conservation of the individual particle momenta requires the two "semifronts" of outgoing wave, which correspond to these two alternatives, to be coherent. The Eq. (1.1) expresses this requirement. The diagrammatic representation of the triangle Eq. (1.1) is given in Fig. 1, where the straight lines represent the "world lines" of three particles moving with the rapidities $\theta_1, \theta_2, \theta_3$. The two-particle S-matrices correspond to the intersection points of the lines; $i_a(j_a)$; $a=1,2,3$ are the kinds of the initial (final) particles; the summing over the kinds k_a of the "intermediate" particles is implied.

In [8] the version of factorized scattering theory in $2+1$-dimensions was proposed. In this theory the scattered objects are not the particles but one-dimensional formations like infinite straight-lined domain walls, which are characteristic of some models of $2+1$-dimensional field theory. We shall consider the quantum objects of this type and call them *the straight strings*. The stationary state of a moving straight string is characterized by the uniform momentum distribution along its length; its kinematics can be described completely by the direction of the string and by the transversal velocity. We assume also that the stationary states of any number of arbitrarily directed (intersecting, in general) moving straight strings are realizable[2]. The intersection points divide each string into segments, each being assumed to carry some internal quantum number i which will be called "colour". The relativistic case of the straight-string kinematics will be implied.

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1 The rapidity of the relativistic $1+1$-dimensional particle is defined by the formulae

$$p_a^0 = m\cosh\theta_a\,; \qquad p_a^1 = m\sinh\theta_a\,,$$

where p_a^μ is the two-momentum; $p^2=m^2$

2 Solutions of this type are likely in some completely integrable classical models in $2+1$-dimensions (S. Manakov, private communication)

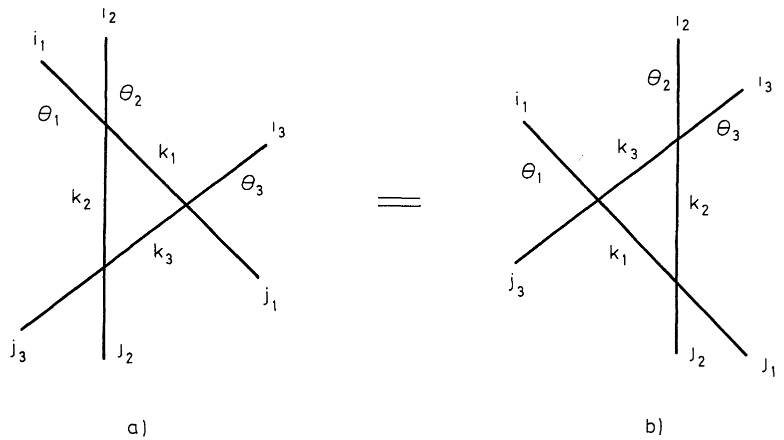

Fig. 1a and b. Diagrammatic representation of the triangle Eq. (1.1)

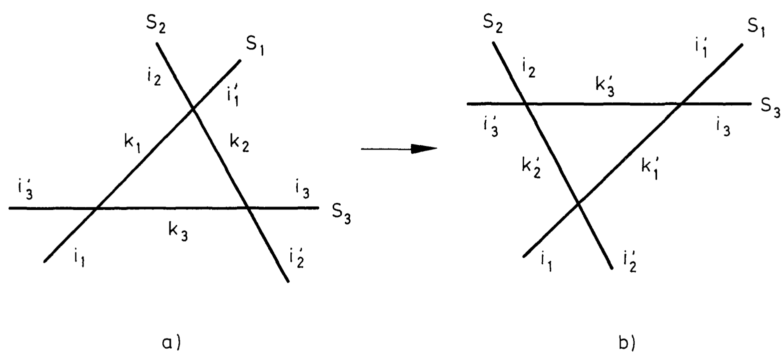

Fig. 2a and b. The initial **a** and the final **b** states of a three-string scattering

In fact, if the number L of straight-strings is less than three, the nontrivial "scattering" is impossible. The three-string scattering is "elementary". The nature of this process is illustrated in Fig. 2. The initial configuration of three-strings s_1, s_2, s_3 is shown in Fig. 2a. The indices $\{i\} = \{i_1, i_2, i_3, i'_1, i'_2, i'_3\}$ denote the colours of six "external" segments while $\{k\} = \{k_1, k_2, k_3\}$ are the colours of the "internal" ones. The motion of the strings S_a; $a = 1, 2, 3$ is such that the triangle in Fig. 2a shrinks with time. Shrinking and then "turning inside out" this triangle is the three-string scattering. After scattering, only states of the type shown in Fig. 2b appear. (This is, essentially, the meaning of the assumption of the "purely elastic" character of scattering.) The directions and velocities of the outgoing strings s_1, s_2, s_3 (Fig. 2b) coincide with those of the initial ones. The "internal" segments of strings, however, can be recoloured (in general $\{k'\} \neq \{k\}$).

In quantum theory the process shown in Fig. 2 is described by *the three-string scattering amplitude*

$$S^{i_1 k_1 i'_1 k'_1}_{\substack{i_2 k_2 i'_2 k'_2 \\ i_3 k_3 i'_3 k'_3}}(\theta_1, \theta_2, \theta_3), \tag{1.2}$$

where the variables $\theta_1, \theta_2, \theta_3$ ("interfacial angles", see below) describe the scattering kinematics.

One can imagine the three-string scattering as the intersection of three planes in $2+1$-dimensional space-time. These planes represent the "world sheets" swept out by the moving straight-strings. Let n_1, n_2, n_3 $[n_a^2 = (n_a^1)^2 + (n_a^2)^2 - (n_a^0)^2 = 1]$ be the normal unit vectors of the planes corresponding to the strings s_1, s_2, s_3, respectively. The mutual orientation of three planes, and, hence, the kinematics of the three-string scattering, is described completely by three invariants

$$n_1 n_2 = -\cos\theta_3 ; \quad n_1 n_3 = -\cos\theta_2 ; \quad n_2 n_3 = -\cos\theta_1 . \tag{1.3}$$

The two-plane intersection lines divide every plane s_a $(a=1,2,3)$ into four parts which will be called plaquettes. The colours of string segments, denoted by the indices i_a, k_a, i'_a, k'_a in (1.2) can be obviously attached to twelve plaquettes joined to the three-plane intersection point. In what follows this point will be called the vertex while the angles $\theta_1, \theta_2, \theta_3$, defined by (1.3) – the vertex variables.

The L-string scattering for $L>3$ has similar properties: the directions and velocities of all the strings s_a; $a=1, 2, \ldots, L$ remain unchanged after the scattering, the "internal" segments being, in general, recoloured. We assume the factorization of the multistring S-matrix: the L-string S-matrix is the product of $L(L-1)(L-2)/6$ three-string ones (1,2), according to the idea that the L-string scattering can be thought of as the sequence of three-string collisions. The succession of this three-string collision is not determined uniquely by the directions and velocities of all the strings s_a but depends also on their "initial positions". Like the $1+1$-dimensional case, the self-consistency of the factorization condition for the straight-string S-matrix requires the equality of different formal expressions for the L-string S-matrix in terms of three-string amplitudes, corresponding to the different successions of three-string collisions. It is easy to note that this requirement is equivalent to *the tetrahedron equation* shown in Fig. 3. In this figure the "world planes" of four strings s_a, $a=1,2,3,4$ (undergoing the four-string scattering) are shown. These planes form the tetrahedron in $2+1$ space-time. The vertices of the tetrahedron represent the "elementary" three-string collisions; the corresponding S-matrices (1.2) are the multipliers in the expression for the 4-string S-matrix. The tetrahedra shown in Fig. 3a and 3b (which differ from each other by some parallel shift of the planes s_a) represent two possible successions of three-string collisions constituting the same four-string scattering process. The colours of the "external" plaquettes are fixed and respectively equal in the right- and the left-hand sides of the equality in Fig. 3; the summing over all possible colourings of the "internal" plaquettes (which are the faces of the tetrahedra) is implied. This tetrahedron equation should be satisfied at any mutual orientations of the planes s_1, s_2, s_3, s_4.

The $1+1$-dimensional factorized S-matrix can be interpreted, after euclidean continuation, as the Z-invariant statistical model on the planar Baxter's lattice $\mathscr{L}$ (see [4]). The 2+1-dimensional factorized S-matrix of straight-strings admits similar interpretation [8]. The natural three-dimensional analog of Baxter's lattice $\mathscr{L}$ is the lattice formed by a large number L of arbitrarily directed intersecting planes in three-dimensional euclidean space. The fluctuating variables ("colours")

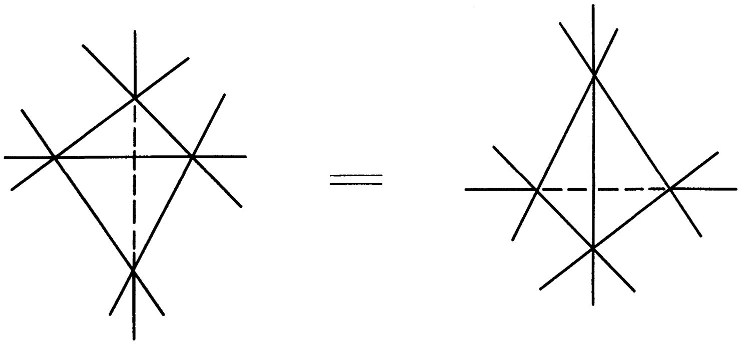

Fig. 3a and b. Diagrammatic representation of the tetrahedron equations

are attached to the lattice plaquettes. The partition function is defined as the sum over all possible colourings of all the plaquettes, each colour configuration being taken with the weight equal to the product of the vertex weights over all the vertices (the vertices are the points of triple intersections of the planes). The vertex weights are assumed to be the functions (common for all the vertices) of the mutual orientation of three planes intersecting in a given vertex. Identifying the vertex weights with the elements of the three-string S-matrix (1.2) (continued to the euclidean domain), one can note that, due to the tetrahedron equation (Fig. 3), the statistical system thus defined possesses Z-invariance.

The tetrahedron equation (Fig. 3) turns out to be highly overdefined system of functional equations; even in the simplest models the independent equations outnumber (by several hundredfold) the independent elements of the three-string S-matrix (1.2). Therefore, the compatibility of these equations is extremely crucial for the scattering theory of straight-strings. In [8] the *two-colour model* of straight-string scattering theory was proposed, and the explicit solution of the corresponding tetrahedron equations was found in the special "static limit" which corresponds to the case $v_a \to 0$, where v_a are the velocities of all the strings. In this paper we construct the relativistic three-string S-matrix for the two-colour model, which is apparently the solution of the "complete" tetrahedron equations. Although the complete evidence of the last statement is unknown we present some nontrivial checks.

The qualitative aspects of the factorized straight-string scattering theory have been described briefly in this Introduction; more detailed discussion can be found in [8]. In Sect. 2 the formulation of the two-colour model is given for the relativistic case. The corresponding tetrahedron equations are discussed in Sect. 3. In Sect. 4 the explicit formulae for the elements of the three-string S-matrix are proposed and the arguments that this S-matrix satisfies the tetrahedron equations are presented. In Sect. 5 it is shown that the obtained S-matrix is in agreement with the unitarity condition for the straight-string S-matrix.

2. Two-colour Model of Straight-strings Scattering Theory

Consider the relativistic scattering theory of straight-strings (see the Introduction) in which the strings' segments can carry only two colours – "white" or "black". Further, let us allow only the states satisfying the following requirement: the even number (i.e., 0, 2 or 4) of black segments can join in each point of two-string intersection. In other words, in any allowed state the black segments form continuous polygonal lines (which may intersect) without ends. Certainly, all the elements of the three-string S-matrix converting the allowed states into unallowed ones (and vice versa) are implied to be zero.

As it is explained in the Introduction, the three-string scattering kinematics can be represented by means of three intersecting "world planes" s_1, s_2, s_3 in $2+1$ space time, the vertex being the "place of collision". In the two-colour model each of the twelve plaquettes joining the vertex can be coloured into black or white so that the black plaquettes form the continuous broken surfaces without boundaries.

Each allowed coluring of these twelve plaquettes corresponds to some nonvanishing element of the three-string S-matrix.

It is convenient to perform the considerations in terms of the euclidean space-time: the "world planes" s_a can be treated as imbedded in the 3-dimensional euclidean space; each of the variables θ, defined by (1.3) being some interfacial angle. The "physical" amplitudes of scattering in the Minkowski space-time can be obtained from the euclidean formulae by means of analytical continuation.

Let us picture the "colour configuration" of the twelve plaquettes joining the vertex as follows. Consider the sphere with the vertex as its centre. The planes s_1, s_2, s_3 draw three great circles on this sphere; the variables θ_1, θ_2, θ_3 [see (1.3)] are exactly the intersection angles of these circles. The intersection points divide each of the circles into four segments; the colours of the plaquettes can be obviously attached to these segments. Performing the stereographic projection one can map these three circles on the plane as shown in Fig. 4. This picture can be interpreted as follows. The spherical triangle I_1 in Fig. 4 corresponds to the triangle in Fig. 2a and represents the initial state of some three-string scattering process. The final state of this process (shown in Fig. 2b) is represented by the spherical triangle F_1. The variables θ_1, θ_2, θ_3 are the interior angles of the triangles I_1 and F_1 (obviously, these triangles are equal on the sphere). Alternatively, one could consider, for instance, the triangle I_2 as representing the initial (and F_2 as the final) state of some other three-string process. This is just the cross-channel. Evidently, the transfer to this cross-channel is associated with the variable transformation

$$\theta_1 \to \pi - \theta_1; \quad \theta_2 \to \pi - \theta_2; \quad \theta_3 \to \theta_3. \tag{2.1}$$

As it is clear from Fig. 4, each three-string scattering process has four cross-channels $I_1 \to F_1$, $I_2 \to F_2$, $I_3 \to F_3$, $I_4 \to F_4$.

We shall assume the P and T invariances of the straight-string scattering theory [8], and also its symmetry under the simultaneous recolouring of all black segments into white and vice versa ("colour symmetry"). Then the three-string S-matrix contains 8 independent amplitudes which are shown (together with the adopted notations) in Fig. 5.

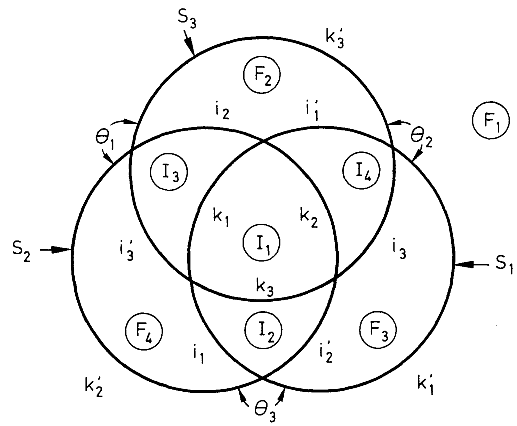

Fig. 4. Stereographic projection of the sphere surrounding the vertex

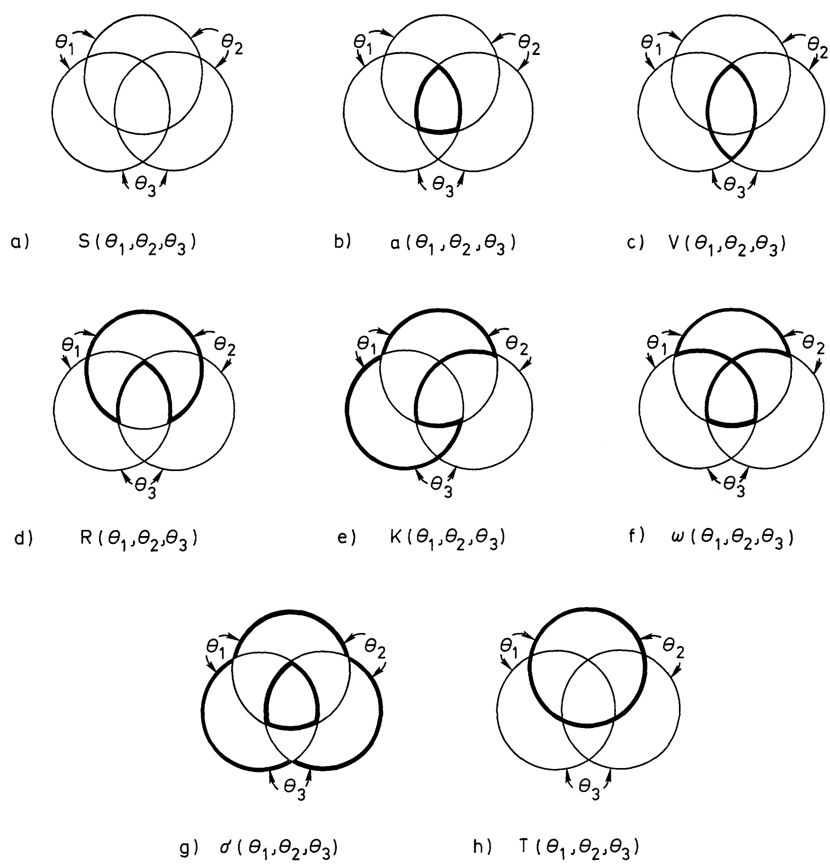

Fig. 5a–h. Eight "colour configurations" of twelve plaquettes joining the vertex, and the notations for corresponding three-string scattering amplitudes. The white (black) circular segments are represented by the ordinary (solid) lines

It is convenient to introduce, apart from the amplitudes defined in Fig. 5 the following 5 functions

$$U(\theta_1,\theta_2,\theta_3)=\mathcal{a}(\pi-\theta_1,\pi-\theta_2,\theta_3), \tag{2.2a}$$
$$L(\theta_1,\theta_2,\theta_3)=V(\pi-\theta_1,\theta_2,\pi-\theta_3), \tag{2.2b}$$
$$\Omega(\theta_1,\theta_2,\theta_3)=R(\pi-\theta_1,\pi-\theta_2,\theta_3), \tag{2.2c}$$
$$H(\theta_1,\theta_2,\theta_3)=R(\pi-\theta_1,\theta_3,\pi-\theta_2), \tag{2.2d}$$
$$W(\theta_1,\theta_2,\theta_3)=\sigma(\pi-\theta_1,\pi-\theta_2,\theta_3), \tag{2.2e}$$

which describe the cross-channels of the processes shown in Fig. 5. The three-string amplitudes should possess the following symmetries, which are the consequences of P, T, "colour" and crossing symmetries

$$\begin{aligned}S(\theta_1,\theta_2,\theta_3)&=S(\theta_2,\theta_1,\theta_3)=S(\theta_1,\theta_3,\theta_2)\\&=S(\pi-\theta_1,\pi-\theta_2,\theta_3).\end{aligned} \tag{2.3a}$$

$$\begin{aligned}&\mathcal{a}(\theta_1,\theta_2,\theta_3)=\mathcal{a}(\theta_2,\theta_1,\theta_3)=\mathcal{a}(\theta_1,\theta_3,\theta_2),\\&U(\theta_1,\theta_2,\theta_3)=U(\theta_2,\theta_1,\theta_3).\end{aligned} \tag{2.3b}$$

$$\begin{aligned}&V(\theta_1,\theta_2,\theta_3)=V(\theta_2,\theta_1,\theta_3)=V(\pi-\theta_1,\pi-\theta_2,\theta_3),\\&L(\theta_1,\theta_2,\theta_3)=L(\theta_2,\theta_1,\theta_3).\end{aligned} \tag{2.3c}$$

$$\begin{aligned}&R(\theta_1,\theta_2,\theta_3)=R(\theta_2,\theta_1,\theta_3),\qquad H(\theta_1,\theta_2,\theta_3)=H(\theta_2,\theta_1,\theta_3),\\&\Omega(\theta_1,\theta_2,\theta_3)=\Omega(\theta_2,\theta_1,\theta_3).\end{aligned} \tag{2.3d}$$

$$\begin{aligned}\omega(\theta_1,\theta_2,\theta_3)&=\omega(\theta_2,\theta_1,\theta_3)=\omega(\pi-\theta_1,\pi-\theta_2,\theta_3)\\&=\omega(\pi-\theta_1,\theta_2,\pi-\theta_3).\end{aligned} \tag{2.3e}$$

$$\begin{aligned}K(\theta_1,\theta_2,\theta_3)&=K(\theta_2,\theta_1,\theta_3)=K(\pi-\theta_2,\pi-\theta_1,\theta_3)\\&=K(\pi-\theta_1,\theta_2,\pi-\theta_3).\end{aligned} \tag{2.3f}$$

$$\begin{aligned}&\sigma(\theta_1,\theta_2,\theta_3)=\sigma(\theta_2,\theta_1,\theta_3)=\sigma(\theta_1,\theta_3,\theta_2),\\&W(\theta_1,\theta_2,\theta_3)=W(\theta_2,\theta_1,\theta_3).\end{aligned} \tag{2.3g}$$

$$\begin{aligned}T(\theta_1,\theta_2,\theta_3)&=T(\theta_2,\theta_1,\theta_3)=T(\pi-\theta_1,\pi-\theta_2,\theta_3)\\&=T(\pi-\theta_1,\theta_2,\pi-\theta_3).\end{aligned} \tag{2.3h}$$

The analytic properties of the three-string amplitudes will be considered in Sects. 3 and 4.

3. The Tetrahedron Equations

The hardest restrictions for the three-string S-matrix come from the tetrahedron equations, which are shown schematically in Fig. 3. Here we shall choose the four "world planes" s_1, s_2, s_3, s_4 shown in this figure to be placed into the euclidean space (see Sect. 2). The three-string S-matrices associated with the vertices of the tetrahedra in Fig. 3 are the functions of corresponding vertex variables. In the two-

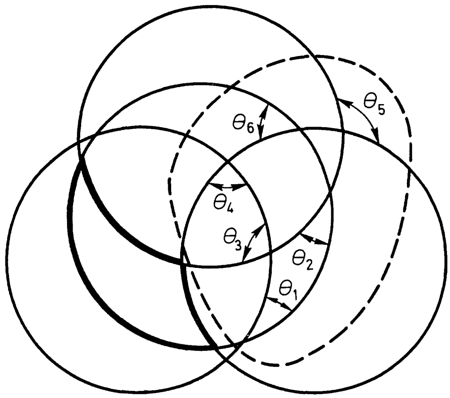

Fig. 6. Stereographic projection of large sphere surrounding the tetrahedron

colour model each plaquette in Fig. 3 can be black or white. Recall that the colours of 24 "external" plaquettes are fixed and equal in the right- and left-hand sides of the tetrahedron equation while the independent summing is performed over all colourings of "internal" plaquettes. Obviously, each allowed colouring of the "external" plaquettes gives rise to some functional equation connecting the three-string amplitudes.

To describe the colourings of the "external" plaquettes it is convenient to introduce again the large sphere (its radius is much larger than the size of the tetrahedra), taking some point near the vertices as the centre, and consider 4 great circles on the sphere corresponding to the planes s_1, s_2, s_3, s_4 (certainly, the tetrahedra in Figs. 3a and 3b are indistinguishable from this point of view). Stereographic projection of this sphere is shown in Fig. 6. The angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$, shown in this figure are just the interior interfacial angles (i.e., the angles between the planes s_a) of the tetrahedra.

Any allowed colouring of the "external" plaquettes in Fig. 3 corresponds in obvious manner to some allowed colouring of the 24 circular segments in Fig. 6 into black and white. In Fig. 6 some colouring of this type is shown as an example. This colouring gives rise, as it is evident from simple consideration, to the following functional equation

$$\begin{aligned} &S(\theta_1,\theta_2,\theta_3)S(\theta_1,\theta_4,\theta_6)S(\theta_5,\theta_4,\theta_3)\mathfrak{a}(\theta_2,\theta_5,\theta_6)\\ &+\mathfrak{a}(\theta_1,\theta_2,\theta_3)\mathfrak{a}(\theta_1,\theta_4,\theta_6)\mathfrak{a}(\theta_4,\theta_3,\theta_5)\sigma(\theta_2,\theta_5,\theta_6)\\ &=U(\theta_1,\theta_3,\theta_2)U(\theta_1,\theta_4,\theta_6)U(\theta_4,\theta_3,\theta_5)S(\theta_2,\theta_5,\theta_6)\\ &+V(\theta_1,\theta_3,\theta_2)V(\theta_1,\theta_4,\theta_6)V(\theta_3,\theta_4,\theta_5)\mathfrak{a}(\theta_2,\theta_5,\theta_6)\,. \end{aligned} \tag{3.1}$$

The equation (3.1) is only one representative of the system of functional tetrahedron equations which arises if one considers all possible allowed colourings of the circular segments in Fig. 6. This system includes hundreds of independent equations and we are not able to present it here; the equation (3.1) is written down mainly for illustration.

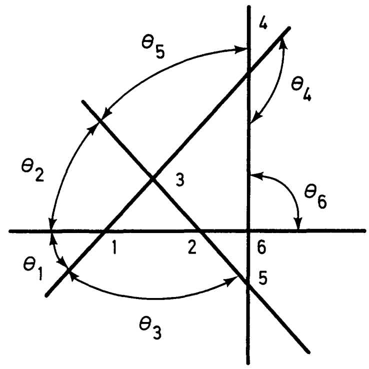

Fig. 7. Fragment of the diagram in Fig. 6 which is enclosed with the dotted curve

It is essential that the variables $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ in the tetrahedron equations are not completely independent. Since the mutual orientation of four planes in three-dimensional space is determined completely by only five parameters, there is one relation between these six angles[3]. This relation can be derived, for instance, from the spherical trigonometry. To do so, concentrate on the fragment of Fig. 6, surrounded by dotted curve. This fragment is shown separately in Fig. 7, where the circular arcs are drawn schematically as the straight lines. Using the formulae of spherical trigonometry one can express the lengths of segments ℓ_{32} and ℓ_{25} in terms of the interior angles of the spherical triangles (123) and (256), respectively. On the other hand, the length of the segment $\ell_{35}=\ell_{32}+\ell_{25}$ may be expressed independently in terms of the interior angles of the triangle (354). This allows one to write

$$\begin{aligned}&\left[\cos\frac{\theta_1+\theta_3-\theta_2}{2}\cos\frac{\theta_1+\theta_2-\theta_3}{2}\cos\frac{\theta_6+\theta_5-\theta_2}{2}\cos\frac{\theta_6+\theta_2-\theta_5}{2}\right]^{1/2}\\ &-\left[\cos\frac{\theta_1+\theta_2+\theta_3}{2}\cos\frac{\theta_2+\theta_3-\theta_1}{2}\cos\frac{\theta_2+\theta_5+\theta_6}{2}\cos\frac{\theta_2+\theta_5-\theta_6}{2}\right]^{1/2}\\ &=\sin\theta_2\left[\cos\frac{\theta_3+\theta_5-\theta_4}{2}\cos\frac{2\pi-\theta_3-\theta_4-\theta_5}{2}\right]^{1/2}. \end{aligned} \tag{3.2}$$

Equation (3.2) is a variant of the desired relation.

4. The Solution of the Tetrahedron Equations

The relation (3.2) connecting the interior angles of the tetrahedron essentially complicates the direct investigation of the tetrahedron equations. However, one can concentrate at first on the special limiting case. Namely, consider the variables

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3 Certainly, this relation is the imbedding condition of four vectors n_1, n_2, n_3, n_4 into the three-dimensional space. Its general form is $\det|n_a^\mu|=0$, where four vectors n_a^μ are treated formally as four-dimensional; $a, \mu=1,2,3,4$

$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ satisfying the relation

$$\theta_1+\theta_2+\theta_3=\theta_2+\theta_5+\theta_6=\theta_4+\theta_3-\theta_5=\pi, \tag{4.1}$$

which corresponds to the limit of coplanar vectors n_1, n_2, n_3, n_4. In this case all the spherical triangles in Fig. 7 can be treated as planar ones and the relation (3.2) is certainly satisfied. From the viewpoint of straight-strings kinematics, the relation (4.1) corresponds to the limit of "infinitely slow" strings; therefore we call this case the "static limit". In the static limit the variables θ_1, θ_2, θ_3 (which are the arguments of the three-string amplitudes) are just the planar angles between the directions of three strings s_1, s_2, s_3. They satisfy the relation $\theta_1+\theta_2+\theta_3=\pi$. Hence the "static" three-string amplitudes are the functions, not of three, but of two variables, θ_1, θ_2.

Most of the tetrahedron equations do not become identities in the static limit [as it happens for the Eq. (3.1)]. Actually, considering the static limit, the number of the independent tetrahedron equations even increase, since the different cross-channels of the same "complete" equations give rise to the different "static" tetrahedron equations.

In [8] the solution of the static-limit tetrahedron equation was constructed; it has the form

$$\begin{aligned}
&S^{st}(\theta_1,\theta_2)=\sigma^{st}(\theta_1,\theta_2)=T^{st}(\theta_1,\theta_2)=W^{st}(\theta_1,\theta_2)=1;\\
&a^{st}(\theta_1,\theta_2)=R^{st}(\theta_1,\theta_2)=0;\\
&L^{st}(\theta_1,\theta_2)=\omega^{st}(\theta_1,\theta_2)=-K^{st}(\theta_1,\theta_2)=\varepsilon_1 V^{st}(\theta_1,\theta_2)=\left[\mathrm{tg}\frac{\theta_1}{2}\,\mathrm{tg}\frac{\theta_2}{2}\right]^{1/2};\\
&H^{st}(\theta_1,\theta_2)=U^{st}(\theta_1,\theta_2)=-\varepsilon_1\Omega^{st}(\theta_1,\theta_2)=\varepsilon_2\left[\frac{\cos\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}\right)}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}\right]^{1/2},
\end{aligned} \tag{4.2}$$

where the notations for the three-string amplitudes are the same as in Fig. 5 and in (2.2), the θ_3 being set equal to $\pi-\theta_1-\theta_2$; for instance,

$$U^{st}(\theta_1,\theta_2)\equiv U(\theta_1,\theta_2,\pi-\theta_1-\theta_2).$$

In (4.2) ε_1 and ε_2 are arbitrary signs; $\varepsilon_1^2=\varepsilon_2^2=1$. Expressions (4.2) satisfy all the "static" tetrahedron equations. We do not insist that (4.2) is the general solution; rather we think that it is not so.

Let us search for the solution of the "complete" tetrahedron equations which corresponds to the static limit (4.2). First consider the power expansion around the static limit. Namely, let the velocities of the scattered strings be not exactly zero but small. In this case the three-string scattering amplitudes can be conveniently considered as the functions of two angles θ_1, θ_2 (which determine the space directions of the strings s_1, s_2, s_3, see Fig. 2) and "symmetrical velocity" $w=\frac{1}{2}dr/dt$ where r is the radius of the circle inscribed in the triangle in Fig. 2. At small velocities of the strings s_1, s_2, s_3, s_4 the nonrelativistic kinematics is valid, and the "velocities" w, corresponding to four triangles (123), (256), (146), (453) in Fig. 7, are connected as follows:

$$w_{146}\frac{\sin\frac{\theta_{125}}{2}\sin\frac{\theta_2}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_{25}}{2}}=w_{123}\frac{\theta_{12}}{\sin\frac{\theta_1}{2}}+w_{256}\frac{\cos\frac{\theta_5}{2}}{\cos\frac{\theta_{25}}{2}};$$

$$w_{453}\frac{\sin\frac{\theta_{125}}{2}\sin\frac{\theta_2}{2}}{\sin\frac{\theta_5}{2}\sin\frac{\theta_{12}}{2}}=w_{123}\frac{\cos\frac{\theta_1}{2}}{\cos\frac{\theta_{12}}{2}}+w_{256}\frac{\sin\frac{\theta_{25}}{2}}{\sin\frac{\theta_5}{2}},\tag{4.3}$$

where the notations $\theta_{12}=\theta_1+\theta_2$; $\theta_{25}=\theta_2+\theta_5$; $\theta_{125}=\theta_1+\theta_2+\theta_5$ are used. The investigation of the tetrahedron equations in the linear approximation (in w) leads to the result

$$\mathcal{A}(\theta_1,\theta_2,w)=-\varepsilon_1 R(\theta_1,\theta_2,w)=\varepsilon_2\lambda w\left[\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}\right]^{-1/2}+O(w^2),\tag{4.4a}$$

$$S(\theta_1,\theta_2,w)=T(\theta_1,\theta_2,w)=1-\lambda w+O(w^2),\tag{4.4b}$$

$$\mathcal{O}(\theta_1,\theta_2,w)=W(\theta_1,\theta_2,w)=1+\lambda w+O(w^2),\tag{4.4c}$$

$$\begin{aligned}U(\theta_1,\theta_2,w)&=H(\theta_1,\theta_2,w)=-\varepsilon_1\Omega(\theta_1,\theta_2,w)\\&=U^{st}(\theta_1,\theta_2)(1+O(w^2)),\end{aligned}\tag{4.4d}$$

$$\begin{aligned}\omega(\theta_1,\theta_2,w)&=-K(\theta_1,\theta_2,w)\\&=L^{st}(\theta_1,\theta_2)\left(1+\lambda w\operatorname{ctg}\frac{\theta_1}{2}\operatorname{ctg}\frac{\theta_2}{2}+O(w^2)\right)\end{aligned}\tag{4.4e}$$

$$\begin{aligned}L(\theta_1,\theta_2,w)&=\varepsilon_1 V(\theta_1,\theta_2,w)\\&=L^{st}(\theta_1,\theta_2)\left(1-\lambda w\operatorname{ctg}\frac{\theta_1}{2}\operatorname{ctg}\frac{\theta_2}{2}+O(w^2)\right),\end{aligned}\tag{4.4f}$$

where $\theta_3=\pi-\theta_1-\theta_2$, L^{st} and U^{st} are given by Eqs. (4.2), and λ is an arbitrary constant.

In studying the complete relativistic tetrahedron equations it is convenient to introduce the variables (spherical excesses)

$$\begin{aligned}2\alpha&=\theta_1+\theta_2+\theta_3-\pi,\\2\beta&=\pi+\theta_3-\theta_1-\theta_2,\\2\gamma&=\pi+\theta_1-\theta_2-\theta_3,\\2\delta&=\pi+\theta_2-\theta_1-\theta_3,\end{aligned}\tag{4.5}$$

obeying the relation

$$\alpha+\beta+\gamma+\delta=\pi.\tag{4.6}$$

Any transmutations of θ_1, θ_2, θ_3, and also any crossing transformations of the type of (2.1) lead, as one can easily verify, to some transmutations among the variables α, β, γ, δ. In fact, the quantity 2α is the area of the spherical triangle I_1 (and F_1) in Fig. 4, while 2β, 2γ, 2δ are the areas of the triangles I_2, I_3, I_4, respectively. Therefore the four cross-channels $I_1 \to F_1$, $I_2 \to F_2$, $I_3 \to F_3$, $I_4 \to F_4$ of the three-string scattering will be called α, β, γ, δ-channels, respectively.

In the static limit $\theta_1 + \theta_2 + \theta_3 \to \pi$, and we have

$$\alpha \to 0; \quad \beta \to \theta_3; \quad \gamma \to \theta_1; \quad \delta \to \theta_2. \tag{4.7}$$

The following relation is valid up to the main order in w

$$w = \sqrt{\frac{\alpha}{2}} \left[\mathrm{tg}\frac{\theta_1}{2} \mathrm{tg}\frac{\theta_2}{2} \mathrm{tg}\frac{\theta_3}{2} \right]^{1/2}. \tag{4.8}$$

Therefore, as it is seen from (4.4), the three-string amplitudes have the square-root branching plane $\alpha = 0$, which will be called the α-channel threshold. The crossing symmetry requires the amplitudes to possess the branching planes (also square-root) $\beta = 0$; $\gamma = 0$; $\delta = 0$, which are the thresholds of the β, γ, δ-channels.

These reasons allow one to write down the following formulae

$$\mathcal{a}(\theta_1, \theta_2, \theta_3) = R(\theta_1, \theta_2, \theta_3) = \varepsilon_2 \left[\frac{\sin\frac{\alpha}{2}}{\cos\frac{\beta}{2} \cos\frac{\gamma}{2} \cos\frac{\delta}{2}} \right]^{1/2}; \tag{4.9a}$$

$$-\mathrm{V}(\theta_1, \theta_2, \theta_3) = \left[\mathrm{tg}\frac{\gamma}{2} \mathrm{tg}\frac{\delta}{2} \right]^{1/2} - \left[\mathrm{tg}\frac{\alpha}{2} \mathrm{tg}\frac{\beta}{2} \right]^{1/2}; \tag{4.9b}$$

$$\omega(\theta_1, \theta_2, \theta_3) = -K(\theta_1, \theta_2, \theta_3) = \left[\mathrm{tg}\frac{\gamma}{2} \mathrm{tg}\frac{\delta}{2} \right]^{1/2} + \left[\mathrm{tg}\frac{\alpha}{2} \mathrm{tg}\frac{\beta}{2} \right]^{1/2}; \tag{4.9c}$$

$$S(\theta_1, \theta_2, \theta_3) = T(\theta_1, \theta_2, \theta_3) = 1 - \left[\mathrm{tg}\frac{\alpha}{2} \mathrm{tg}\frac{\beta}{2} \mathrm{tg}\frac{\gamma}{2} \mathrm{tg}\frac{\delta}{2} \right]^{1/2}; \tag{4.9d}$$

$$\sigma(\theta_1, \theta_2, \theta_3) = 1 + \left[\mathrm{tg}\frac{\alpha}{2} \mathrm{tg}\frac{\beta}{2} \mathrm{tg}\frac{\gamma}{2} \mathrm{tg}\frac{\delta}{2} \right]^{1/2}, \tag{4.9e}$$

which are in accordance with the expansion (4.4) provided $\varepsilon_1 = -1$ and $\lambda = 1$, and entirely satisfy the crossing relations (2.3). Therefore we suppose that the expressions (4.9) give the exact solution of the "complete" tetrahedron equations for the two-colour model.

Unfortunately, rigorous verification of this supposition is rather difficult. Direct substitution of (4.9) into the tetrahedron equation is complicated because of the relation (3.2), not to speak of the large number of equations to be verified. However, we have performed some simplified verifications; an example is given in the Appendix. Moreover, our supposition has been confirmed in various numerical checks.

Note that, like the triangle equations (1.1), the tetrahedron equations are homogeneous; the three-string S-matrix is determined by the equations only up to the overall factor which can be a function of the variables θ. The formulae (4.9) should be considered as expressions giving the ratios of different elements of the three-string S-matrix; the right-hand sides of all the equalities (4.9) are implied to be multiplied by some function

$$[Z(\alpha,\beta,\gamma,\delta)]^{-1}, \tag{4.10}$$

which is symmetric under arbitrary transmutations of the variables α, β, γ, δ [this is forced by the crossing symmetry requirements (2.3)]. This function will be determined by the unitarity condition for the straight-strings S-matrix, studied in the next section.

5. Unitarity Condition

In the euclidean domain the variables α, β, γ, δ [connected by (4.6)] are real and non-negative, and all the amplitudes (4.9) are real. The "physical" scattering of the strings s_a in Minkowski space-time corresponds to real negative values of α (provided the velocities of the strings s_a are not too large[4]. Here the amplitudes acquire the imaginary parts. Let us introduce the cutting hyperplane $\operatorname{Im}\alpha=\operatorname{Im}\beta=\operatorname{Im}\gamma=\operatorname{Im}\delta=0$; $\operatorname{Re}\alpha<0$ (corresponding to the branching plane $\alpha=0$) in the three-dimensional complex space of the variables α, β, γ, δ. Then the "upper" edge ($\operatorname{Im}\alpha=+0$; $\operatorname{Re}\beta>0$; $\operatorname{Re}\gamma>0$; $\operatorname{Re}\delta>0$) of this hyperplane represents the "physical" domain of α-channel. Continuing some amplitude to the "lower" edge $\operatorname{Im}\alpha=-0$, one obtains the complex-conjugated amplitude of reversed process (here we imply the T-invariance so that the amplitudes of direct and reversed processes are equal).

In the physical domain of α-channel the three-string unitarity condition should be satisfied, i.e.,

$$\sum_{k_1'k_2'k_3'} S^{i_1k_1i_1'k_1'}_{\substack{i_2k_2i_2'k_2'\\ i_3k_3i_3'k_3'}}(\theta_1,\theta_2,\theta_3)\, S^{i_1\ell_1i_1'k_1'}_{\substack{i_2\ell_2i_2'k_2'\\ i_3\ell_3i_3'k_3'}}(\theta_1,\theta_2,\theta_3)^*=\delta^{\ell_1}_{k_1}\delta^{\ell_2}_{k_2}\delta^{\ell_3}_{k_3}, \tag{5.1}$$

where S is the amplitude of the process shown in Fig. 2 and the star denotes the complex conjugation. If the second multiplier in the left-hand side of (5.1) is treated not as the complex-conjugated amplitude but the result of analytical continuation around the branching plane $\alpha=0$, the relation (5.1) becomes valid at any complex θ.

The requirement (5.1) for the two-colour string model leads, using (4.9), to the single equation for the "unitarizing factor" (4.10)

$$Z(\alpha,\beta,\gamma,\delta)Z^{(\alpha)}(\alpha,\beta,\gamma,\delta)=\frac{\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha+\gamma}{2}\cos\dfrac{\alpha+\delta}{2}}{\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}\cos\dfrac{\delta}{2}}, \tag{5.2}$$

where the suffix (α) denotes the continuation around the branching plane $\alpha=0$. This equation together with the requirement of symmetry under arbitrary

4 Actually, this is true unless the velocities of two-string intersection points exceed the speed of light

transmutation of the variables, determines the factor (4.10). The investigation of this equation is the subject of our further work.

It can be shown that, due to the factorization of the multistring S-matrix into three-string ones, the three-string unitarity condition (5.1) guarantees the unitarity of the total S-matrix of straight-strings.

6. Discussion

In a recent paper by Belavin [10] the remarkable symmetry of the triangle Eqs. (1.1) was discovered. This symmetry reveals the reasons for the compatibility of the overdefined system of functional Eqs. (1.1) and throws some light upon the nature of the general solution of these equations. It would be extremely interesting to find something like this symmetry in the tetrahedron equations.

As explained in the Introduction, the factorized S-matrix of straight-strings in the euclidean domain can be interpreted as the three-dimensional lattice statistical model which possesses Z-invariance and apparently is exactly solvable. Unfortunately, for the solution found in this paper some of the vertex weights turn out to be negative; therefore the existence of the thermodynamic limit of the corresponding lattice system becomes problematic. We suppose that there are solutions of tetrahedron equations which are free from this trouble. On the other hand, if the thermodynamic limit exists, there is the hypothesis that the partition function of a Z-invariant statistical system on the infinite lattice is simply connected to the "unitarizing factor" (4.10). (The two-dimensional analog of this hypothesis is discussed in [4].)

Appendix

Consider the Eq. (3.1) under the following condition

$$\theta_2+\theta_5+\theta_6=\pi. \tag{A.1}$$

Since $\mathcal{a}^{st}=0$, $S^{st}=\sigma^{st}=1$, the Eq. (3.1) acquires the form

$$\begin{aligned}&\mathcal{a}(\theta_1,\theta_2,\theta_3)\mathcal{a}(\theta_4,\pi-\theta_2-\theta_5,\theta_1)\mathcal{a}(\theta_4,\theta_3,\theta_5)\\&\quad=U(\theta_4,\theta_3,\theta_5)U(\theta_1,\theta_4,\pi-\theta_2-\theta_5)U(\theta_1,\theta_3,\theta_2).\end{aligned} \tag{A.2}$$

After the substitution of the explicit expressions (4.9) into (A.2) it can be rewritten

$$\begin{aligned}&\cos\frac{\theta_1+\theta_2+\theta_3}{2}\sin\frac{\theta_1+\theta_4-\theta_2-\theta_5}{2}\cos\frac{\theta_3+\theta_4+\theta_5}{2}\\&\quad=\cos\frac{\theta_4+\theta_3-\theta_5}{2}\sin\frac{\theta_1+\theta_2+\theta_4+\theta_5}{2}\cos\frac{\theta_1+\theta_3-\theta_2}{2}.\end{aligned} \tag{A.3}$$

The validity of this equality, assuming (3.2), remains to be proved.

The degeneration of the diagram in Fig. 7, corresponding to the case (A.1), is shown in Fig. 8. The length of the circular segment can be expressed independently in terms of the interior angles of two triangles in Fig. 8: either (123) or (124).

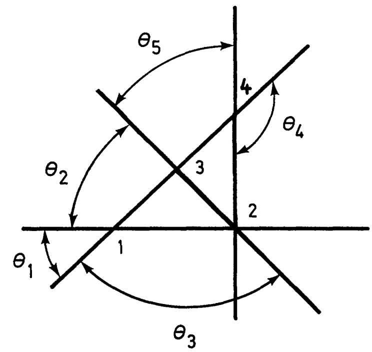

Fig. 8. Degeneration of diagram shown in Fig. 7, corresponding to the relation (A.1)

Comparing the results one obtains the relation

$$\left[\cos\frac{\theta_1+\theta_2+\theta_3}{2}\cos\frac{\theta_1+\theta_2-\theta_3}{2}\sin\frac{\theta_2+\theta_5-\theta_1-\theta_4}{2}\sin\frac{\theta_2+\theta_5+\theta_4-\theta_1}{2}\right]^{1/2}$$
$$=\left[\cos\frac{\theta_1+\theta_3-\theta_2}{2}\cos\frac{\theta_2+\theta_3-\theta_1}{2}\sin\frac{\theta_1+\theta_2+\theta_5-\theta_4}{2}\sin\frac{\theta_1+\theta_2+\theta_5+\theta_4}{2}\right]^{1/2}. \tag{A.4}$$

Doing the same with the segments ℓ_{34} and ℓ_{42} one gets two more relations

$$\left[\cos\frac{\theta_1+\theta_2+\theta_3}{2}\cos\frac{\theta_2+\theta_3-\theta_1}{2}\cos\frac{\theta_5+\theta_3-\theta_1}{2}\cos\frac{\theta_3+\theta_4+\theta_5}{2}\right]^{1/2}$$
$$=\left[\cos\frac{\theta_1+\theta_2-\theta_3}{2}\cos\frac{\theta_1+\theta_3-\theta_2}{2}\cos\frac{\theta_4+\theta_3-\theta_5}{2}\cos\frac{\theta_5+\theta_4-\theta_3}{2}\right]^{1/2}; \tag{A.5}$$

$$\left[\cos\frac{\theta_5+\theta_4-\theta_3}{2}\cos\frac{\theta_5+\theta_4+\theta_3}{2}\sin\frac{\theta_1+\theta_2+\theta_5-\theta_4}{2}\sin\frac{\theta_1+\theta_4-\theta_2-\theta_5}{2}\right]^{1/2}$$
$$=\left[\cos\frac{\theta_4+\theta_3-\theta_5}{2}\cos\frac{\theta_3+\theta_5-\theta_4}{2}\sin\frac{\theta_1+\theta_2+\theta_4+\theta_5}{2}\sin\frac{\theta_1-\theta_2-\theta_4-\theta_5}{2}\right]^{1/2}, \tag{A.6}$$

which are certainly equivalent to (A.4). Taking the products of the right- and left-hand sides of (A.4), (A.5), (A.6), one obtains exactly the equality (A.3).

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