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On the Bundle of Connections and the Gauge Orbit Manifold in Yang-Mills Theory

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Abstract. In an appropriate mathematical framework we supply a simple proof that the quotienting of the space of connections by the group of gauge transformations (in Yang-Mills theory) is a C^∞ principal fibration. The underlying quotient space, the gauge orbit space, is seen explicitly to be a C^∞ manifold modelled on a Hilbert space.

0. Introduction

In [1], Singer announced interesting results on the quotienting of the space of C^∞ connections of a principal G-bundle [on compact orientable Riemannian space without boundary by the group of gauge transformations under appropriate restrictions (essentially free group action)]. In particular [1], the quotienting is a principal fibration, and the underlying quotient space (gauge orbit space) is C^∞ manifold. In [2] Narasimhan and Ramadas prove independently that the quotienting in question is a principal fibration (for Sobolev spaces of connections). In [1, 2] it is proved that when $G=SU(N)$, and the initial base space S^d $(d=3,4)$, the corresponding fibration is nontrivial. The gauge orbit space is not contractible. Thus continuous global gauge fixing (section) is not possible.

These global results are of relevance to quantum gauge field theory where the dynamical variables are supplied by the gauge orbit space.

The present paper is motivated by the need, on the part of gauge field theorists, to understand better the geometry of the gauge orbit space, for reasons adduced below. We return to the quotienting of the space of irreducible connections by the group of gauge transformations (restricted to free group action) within the mathematical framework of [2], i.e. we work with Sobolev spaces of sections of various bundles. We prove that the gauge orbit space is a C^∞ manifold modelled on a Hilbert space. In order to prove this directly, and to exhibit the C^∞ structure, we give an alternative proof (to that of [2]), that we have a principal fibration, in fact a C^∞ fibration. Our strategy is to use the existence of local sections to give

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manifold structure to the orbit space and exploit the inverse function theorem to prove C^∞ local triviality. The proof automatically supplies C^∞ manifold structure to the gauge orbit space.

It is well known that the physical degrees of freedom in Yang-Mills theory are the space of connections modulo the group of gauge transformations, i.e. by the gauge orbit manifold. This is true in Euclidean field theory and particularly transparent in the canonical formalism where the gauge orbit manifold appears as the true configuration space for a non-singular dynamical system [11]. The gauge orbit manifold has a natural (weak) Riemannian structure [1, 3]. It has recently been shown [3] that the associated (formal) volume element evaluated in local coordinates gives rise to the Faddeev-Popov determinant (associated with Feynman-De Witt-Faddeev-Popov quantization). An important step in global quantization would be to give a meaning to this volume element. For these and related questions a deeper understanding of the gauge orbit manifold appears indispensable.

Contents

Remark. For Sobolev spaces of sections of fibre bundles, see Palais [4]. For manifolds of maps, see Palais [4] and Eells [5]. Section II is expository. In Sect. II–IV we set up the background for the main theorem of Sect. V.
See [10] for an introduction to geometry of gauge fields.

Note added. Since this article was submitted for publication, a Feynman-Kac integral with regularisation for continuum Yang-Mills theory has been rigorously constructed in [12] working directly in the gauge orbit space of this paper.

I. Preliminaries

A. Yang-Mills potentials will be identified with connections in a principal C^∞ G-bundle $P(M, G)$. The structure group G is taken to be a compact, connected semi-simple matrix Lie group. The base space M is taken to be a compact finite-dimensional oriented C^∞ Riemannian manifold without boundary.

We have two situations in mind:

(i) M is model of 4-dimensional Euclidean *space-time*

(ii) M is model of 3-dimensional Euclidean *space.*

The latter case corresponds to gauge field theory viewed as a (canonical) dynamical system [11]. Compactness is tantamount to a strong form of boundary conditions on fields, necessary for topological field configurations and boundedness of the action integral. It is a "volume cutoff".

B. Gauge Transformations. A C^∞ gauge transformation is a C^∞ equivariant automorphism of $P(M,G)$ which induces an identity transformation on M. It is necessarily fibre preserving.

Let $f:P\to P$, $u\to f(u)$

$$f(ua)=f(u)a\,,\qquad a\in G \tag{1.1}$$

be a gauge transformation.

Since each gauge transformation f is a fibre preserving automorphism, it may be realised as:

$$u\to f(u)=ug(u)\,, \tag{1.2}$$

where $g:P\to G$

$$g(ua)=a^{-1}g(u)a\,,\qquad a\in G\,. \tag{1.3}$$

If A is C^∞ connection 1-form in P, then f^*A is the C^∞ gauge transformed connection.

C^∞ gauge transformations form a group, also a transformation group on the space of connections of P.

We make contact with the usual definition of gauge transformations. Let $\{\mathcal{U}_\alpha,\varphi_\alpha\}:C^\infty$ bundle atlas. $\{\mathcal{U}_\alpha\}$ system of neighbourhoods covering M, and

$$\varphi_\alpha:\pi^{-1}(\mathcal{U}_\alpha)\to\mathcal{U}_\alpha\times G \tag{1.4}$$

diffeomorphisms ($\pi:P\to P/G=M$; canonical projection). Let

$$\sigma_\alpha:\mathcal{U}_\alpha\to P\,,\qquad \pi\cdot\sigma_\alpha=\mathrm{id}|_{\mathcal{U}_\alpha} \tag{1.5}$$

be a system of local sections.

If $x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$, we have:

$$\begin{gathered}\sigma_\beta(x)=\sigma_\alpha(x)\cdot\psi_{\alpha\beta}(x)\,,\\ \psi_{\alpha\beta}:\mathcal{U}_\alpha\cap\mathcal{U}_\beta\to G\end{gathered} \tag{1.6}$$

smooth transition functions satisfying the co-cycle condition.

From (1.2),

$$f(\sigma_\alpha(x))=\sigma_\alpha(x)\cdot g_\alpha(x)\,, \tag{1.7}$$

where $g_\alpha(x)\equiv g(\sigma_\alpha(x))$.

Thus under a gauge transformation:

$$\sigma_\alpha(x)\to\sigma_\alpha(x)\cdot g_\alpha(x)\,. \tag{1.8}$$

Let $A^{(\alpha)}=\sigma_\alpha^*A$, connection form on $\mathcal{U}_\alpha$. Then we have gauge transformation:

$$A^{(\alpha)}\to A^{(\alpha)}\cdot g_\alpha=\mathrm{Ad}_{g_\alpha^{-1}}\cdot \mathrm{A}^{(\alpha)}+\mathrm{g}_\alpha^{-1}\mathrm{dg}_\alpha \tag{1.9}$$

$$(1.6,\ 1.7)\Rightarrow g_\beta=\mathrm{ad}_{\psi_{\alpha\beta}^{-1}}\cdot\mathrm{g}_\alpha \tag{1.10}$$

pointwise for $x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$.

Thus the gauge transformation group $\mathscr{G}$ may be identified with the set of families $\{g_\alpha\}$, $g_\alpha:\mathscr{U}_\alpha\to G$ satisfying (1.9), with pointwise group operations, or equivalent with C^∞ sections of some bundle (see later). (1.9) may also be written:

$$A^{(\alpha)}\to A^{(\alpha)}\cdot g_\alpha = A^{(\alpha)} + g_\alpha^{-1} d_{A^{(\alpha)}} g_\alpha \,. \tag{1.11}$$

$$d_{A^{(\alpha)}} = a + [A^{(\alpha)},\ ]\,. \tag{1.12}$$

where $d_A(\alpha)$ is exterior covariant derivative in some bundle (see (1.23)).

C. Global Transcription. Following [6], we introduce the associated gauge bundle (bundle of groups)

$$E_G = P\times_G G \tag{1.13}$$

with G having adjoint action on G (second factor).

We shall identify the group of gauge transformations $\mathscr{G}$ with the space of all C^∞ sections (with pointwise group operation)

$$\mathscr{G} = \Gamma(E_G)\,. \tag{1.14}$$

We define the normal subgroup $\mathscr{G}^0$:

$$\mathscr{G}^0 = \{g\in\mathscr{G}, g(x_0) = e\}\,, \tag{1.15}$$

where x_0 is some definite point of M, chosen once for all. Note that

$$\mathscr{G}/\mathscr{G}^0 = G\,. \tag{1.16}$$

We also define the subgroup:

$$\bar{\mathscr{G}} = \mathscr{G}/\mathscr{Z}\,, \tag{1.17}$$

where $\mathscr{Z}$ is the center of $\mathscr{G}$.

We introduce the adjoint bundle (as in [6])

$$E_{\mathrm{ad}} = P\times_G L(G)\,, \tag{1.18}$$

where G has adjoint action on its Lie algebra $L(G)$.

We define

$$\mathscr{L} = \Gamma(E_{\mathrm{ad}})\,. \tag{1.19}$$

and

$$\mathscr{L}_0 = \{\xi\in\mathscr{L} \mid \xi(x_0) = 0\}\,. \tag{1.20}$$

Later on, once Lie group structure has been introduced in $\mathscr{G}$ (respectively $\mathscr{G}^0$), $\mathscr{L}$ (respectively $\mathscr{L}_0$) will be identified with its Lie algebra.

Let

$$\mathscr{A} = \text{space of all } C^\infty \text{ connections on } P(M,G) \tag{1.21}$$

and

$$\bar{\mathscr{A}}\subset\mathscr{A} = \text{subspace of irreducible connections}\,. \tag{1.22}$$

For $A\in\mathscr{A}$,

$$d_A:\Gamma(E_{\mathrm{ad}})\to\Gamma(E_{\mathrm{ad}}\otimes\Lambda^1) \tag{1.23}$$

is exterior covariant derivative (local expression is (1.12)). We have a right $\mathcal{G}$-action on $\mathcal{A}$:

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A},$$
$$(A, g) \to A \cdot g = A + g^{-1} d_A g. \tag{1.24}$$

It is useful to introduce one more structure.

Recall that the structure group G is a matrix Lie group, hence a subset of $M(n, \mathbb{C})$ (algebra of $n \times n$ complex matrices).

We introduce the associated bundle of matrices:

$$E_{M(n,\mathbb{C})} = P \times_G M(n, \mathbb{C}), \tag{1.25}$$

where G has adjoint action on $M(n, \mathbb{C})$. We also introduce the space of C^∞ sections

$$\mathcal{R} = \Gamma(E_{M(n,\mathbb{C})}), \tag{1.26}$$

where $\mathcal{R}$ has the structure of an infinite dimensional algebra (pointwise the structure of a matrix algebra). We have both:

$$\mathcal{G} \subset \mathcal{R}, \qquad \mathcal{L} \subset \mathcal{R} \tag{1.27}$$

II. The Group of Gauge Transformations as an Infinite Dimensional Lie Group

A. Topological Group

We obtain from $\mathcal{G}$ a group with the structure of a Hausdorf topological space, in fact a complete metric space, and verify it is a topological group.

Since [see Eq. (1.27)]

$$\mathcal{G} \subset \mathcal{R}$$

we shall give a topology on $\mathcal{R}$ and to $\mathcal{G}$ the induced topology. To give the topology on $\mathcal{R}$ we exploit the fact that $\mathcal{R}$ has pointwise the structure of a matrix algebra.

Let $\{\mathcal{U}_\alpha, \varphi_\alpha\} : C^\infty$ bundle atlas for $P(M, G)$ as in (1.4), and $\{\mathcal{U}_\alpha, f_\alpha\} : C^\infty$ atlas for M. Then for $g_1, g_2 \in \mathcal{R}$, we introduce the norms:

$$\|g_1 - g_1\|_k = \Big(\sum_\alpha \|\varphi_\alpha(g_1 - g_2) f_\alpha^{-1}\|_k^2\Big)^{1/2}, \tag{2.1}$$

where

$$\|\varphi_\alpha(g_1 - g_2) f_\alpha^{-1}\|_k^2 = \|g_{1,\alpha} - g_{2,\alpha}\|_k^2 = \int_{f_\alpha(\mathcal{U}_\alpha)} d\,\mathrm{vol} \sum_{\ell=0}^{k} |D^\ell(g_{1,\alpha} - g_{2,\alpha})|^2 .$$

Here $g_{i,\alpha}$ is the fibre coordinate of the section g_i over $\mathcal{U}_\alpha$, and

$$|f|^2 = (f, f) = \mathrm{tr}\, f^* f.$$

$d(\mathrm{vol})$ is the volume element with respect to Riemannian metric on M. This gives an admissible norm on $\mathcal{R}$. (See Palais [4], Sect. 4.)

$\mathscr{R}_k$ is the completion of $\mathscr{R}$ using (2.1). $\mathscr{G}_k$ is the completion of $\mathscr{G}$ in the induced metric (Sobolev space of sections). $\mathscr{G}_k$ is closed in $\mathscr{R}_k$ for $k > \dim M/2$ (Eells [5], Sect. 6), using the Sobolev embedding theorem and the fact that the structure group G is closed in $M(n, \mathbb{C})$. See also [2]. Using the Sobolev inequality:

$$\|f \cdot g\|_k \leqq \text{const} \, \|f\|_k \cdot \|g\|_k, \tag{2.2}$$

valid for $k > (\dim M)/2$, it follows that the group operations in $\mathscr{G}_k$ are continuous in the above topology (exactly as for finite dimensional matrix groups). Thus $\mathscr{G}_k$ (and also $\mathscr{G}_k^0$, $\bar{\mathscr{G}}_k$) are topological groups.

B. Lie Group Structure

We have $\mathscr{L} = \Gamma(E_{\text{ad}}) \subset \mathscr{R}$. Let $\xi_1, \xi_2 \in \mathscr{L}$, $\xi_{i,\alpha}$ $(i = 1, 2)$ the fibre coordinates of the sections of E_{ad}, over $\mathscr{U}_\alpha \subset M$. Then [similar notation as before, (2.1)] we can introduce the distance $\| \; \|_k$ in $\mathscr{L}$:

$$\|\xi_1 - \xi_2\|_k^2 = \sum_\alpha \int_{f\alpha(\mathscr{U}_\alpha)} d(\text{vol}) \sum_{\ell=0}^{k} |D^\ell(\xi_{1,\alpha} - \xi_{2,\alpha})|_{L(G)}^2, \tag{2.3}$$

where $|X|_{L(G)}^2 = (X, X)_{L(G)}$, the bi-invariant metric on the Lie algebra $L(G)$ of the structure group G. We have the completed Sobolev space of sections of the adjoint bundle:

$$\mathscr{L}_k = \Gamma_k(E_{\text{ad}}) \subset \mathscr{R}_k. \tag{2.4}$$

It is a Hilbert space.

Let $V_k(0) \subset \mathscr{L}_k$ be a sufficiently small neighbourhood of the origin. Since $L(G)$ is the Lie algebra of the (compact) Lie group G we can introduce, *pointwise*, the exponential map:

$$\begin{aligned} &\exp : V_k(0) \to \mathscr{G}_k \subset \mathscr{R}_k \\ &\xi \to \exp \xi = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}. \end{aligned} \tag{2.5}$$

Using the Campbell-Haussdorf formula and the inequality (2.2), we have:

$$\exp(\xi + h) \cdot \exp(-\xi) = \exp(\mu(\xi) \cdot h + r(h, \xi)), \tag{2.6}$$

where

(i) $$\mu(\xi) : \mathscr{L}_k \to \mathscr{L}_k$$

is *linear* and *continuous*. Explicitly:

$$\begin{aligned} \mu(\xi)h &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \underbrace{[\ldots[[h, \xi], \xi], \ldots, \xi]}_{n \text{ factors}} \\ &= h + \mathcal{O}(\xi) \end{aligned}$$

$$\mu(\xi)_*|_{\xi=0} = \text{id}$$

$$\text{Ker}\, \mu(\xi) = \Phi, \text{ sufficiently small } \xi \text{ in } \| \; \|_k$$

(ii) $$\|r(h, \xi)\|_k \leqq \text{const} \, \|h\|_k^2 \cdot c(\xi).$$

Thus the exponential map is differentiable (easily generalised to C^∞). Moreover, using the inverse function theorem, the exponential map provides a local diffeomorphism. Thus we have:

Proposition 2.7. *There exists a sufficiently small neighbourhood of the identity e, $N_k(e)\subset\mathcal{G}_k$ for which the exponential map* (2.5) *provides a chart in $\mathcal{L}_k$.*

Let $M_k(e)\subset N_k(e)$ (neighbourhoods of identity) such that

$$M_k(e)\cdot M_k(e)\subset N_k(e).$$

Let $g_1=\exp\xi_1$, $g_2=\exp\xi_2\in M_k(e)$. By the Campbell-Haussdorf formula:

$$g_1\cdot g_2=\exp f(\xi_1,\xi_2), \tag{2.8}$$

where the Campbell-Haussdorf power series:

$$f(\xi_1,\xi_2)=\xi_1+\xi_2+\tfrac{1}{2}[\xi_1,\xi_2]+\dots \tag{2.9}$$

converges absolutely in $\mathcal{L}_k$ for sufficiently small ξ_1,ξ_2 [use (2.2)*]. Thus we have coordinates by the exponential map:*

$$g_1\cdot g_2\to f(\xi_1,\xi_2), \tag{2.10}$$

where f is C^∞. Thus we have:

Proposition 2.11. *The topological group $\mathcal{G}_k$ is a local Lie group, moreover $\mathcal{L}_k$ is its Lie algebra.*

We shall now transport the C^∞ structure [provided by the exponential map in $N_k(e)\subset\mathcal{G}_k$] everywhere in $\mathcal{G}_k$ by right translation by the following standard method: consider a neighbourhood of identity $M_k'(e)$ such that

$$M_k'(e)^{-1}\cdot M_k'(e)\subset M_k(e). \tag{2.11}$$

Then

$$M_k'(e)\subset M_k(e)\subset N_k(e). \tag{2.12}$$

Let $a\in\mathcal{G}_k$. Then $M_k'(e)a$ provides a neighbourhood of a.

We now define a chart on $M_k'(e)\cdot a$:

$$g\in M_k'(e)\cdot a. \tag{2.13}$$

Then,

$$g=g_\xi a,\qquad g_\xi=\exp\xi\in M_k'(e). \tag{2.14}$$

Then

$$g\to\xi \tag{2.15}$$

gives a homeomorphism of $M_k'(a)=M_k'(e)\cdot a\to$ neighbourhood of origin in $\mathcal{L}_k$ [since $M_k'(e)\to M_k'(e)\cdot a$ is a homeomorphism]. Thus (2.15) gives a chart for $M_k'(e)\cdot a$ in $\mathcal{L}_k$. Finally, if $g\in M_k'(a_1)\cap M_k'(a_2)$

$$g=g_{\xi_1}\cdot a_1=g_{\xi_2}\cdot a_2 \tag{2.16}$$

whence [using (2.11)], $a_1 a_2^{-1}$, $a_2 a_1^{-1} \in M'_k(e)$. We have:

$$g_{\xi_1} = g_{\xi_2} a_2 a_1^{-1},$$

whence

$$\xi_1 = f(\xi_2, a_2 a_1^{-1}). \tag{2.17}$$

Similarly

$$\xi_2 = f(\xi_1, a_1 a_2^{-1}).$$

Hence, the charts are C^∞ related, and $\mathscr{G}_k$ is a C^∞ manifold modelled on the Hilbert space $\mathscr{L}_k$.

Thus we have proved:

Theorem 2.18. *$\mathscr{G}_k$ (and also $\mathscr{G}_k^0$, $\bar{\mathscr{G}}_k$) has the structure of a Lie group. $\mathscr{L}_k$ may be canonically identified as the Lie algebra of $\mathscr{G}_k$.*

Remark 2.19. For $k > \dfrac{\dim M}{2}$, by the Sobolev embedding theorem, $\mathscr{G}_k$ is contained in the space of continuous sections of the gauge bundle. Convergence in $\| \;\|_k$ implies uniform convergence, (Eells, [5], Sect. 6]. It follows that $\mathscr{G}_k^0$ is closed in $\mathscr{G}_k$. $\mathscr{G}_k^0$ is a closed Lie subgroup of $\mathscr{G}_k$.

Remark 2.20. As a consequence of Remark 2.19, the exact sequence [1]

$$0 \to \mathscr{G}_k^0 \to \mathscr{G}_k \to G \to 0$$

is a principal fibration.

III. The Action of the Lie Group $\mathscr{G}_{k+1}$ on the Space of Connections $\mathscr{A}_k$

A. Preliminaries

Let $\Gamma_{k+1-p}(E_{\text{ad}} \otimes \Lambda^p)$: Sobolev space of sections of $E_{\text{ad}} \otimes \Lambda^p$ (i.e. p-forms on M with values in E_{ad}) in class $(k+1-p)$. First introduce a pointwise inner product using Riemannian metric on space of forms and invariant metric on Lie algebra $L(G)$. Sobolev norms are then introduced as in Sect. II. These are Hilbert spaces.

$\mathscr{A}_k$ is the Sobolev space of connections of $P(M,G)$ in class k. I.e., if $A_1, A_2 \in \mathscr{A}_k$, then $A_1 - A_2 = \tau \in \Gamma_k(E_{\text{ad}} \otimes \Lambda^1)$ and $\mathscr{A}_k$ is an affine space. $\bar{\mathscr{A}}_k \subset \mathscr{A}_k$ is the subspace of irreducible connections. For $A \in \mathscr{A}_k$ or $\bar{\mathscr{A}}_k$, we have the exterior covariant derivative:

$$d_A : \Gamma_{k+1}(E_{\text{ad}}) \to \Gamma_k(E_{\text{ad}}) \otimes \Lambda^1), \tag{3.1}$$

a continuous linear operator.

$$\|d_A \xi\|_k \leqq \text{const} \, \|\xi\|_{k+1}. \tag{3.2}$$

For $\omega_i \in \Gamma_{k+1-p}(E_{\text{ad}} \otimes \Lambda^p)$, $p = 0, 1$ we have Sobolev inequalities

$$\|[\omega_1, \omega_2]\|_{k+1-p} \leqq \text{const} \, \|\omega_1\|_{k+1-p} \|\omega_2\|_{k+1-p} \tag{3.3}$$

valid for $k>(\dim M)/2$. We shall hold $k>\frac{\dim M}{2}+1$, hereafter held fixed. By the Sobolev embedding theorem, $\mathscr{A}_k$ is embedded in the space of C^1 connections.

B. Group Action on $\mathscr{A}_k$

We have the gauge transformations

$$\begin{gathered}\mathscr{A}_k\times\mathscr{G}_{k+1}\to\mathscr{A}_k\\ (A',g)\to A'\cdot g=A'+g^{-1}d_{Ag}.\end{gathered}\tag{3.4}$$

The group action is differentiable (in fact C^∞). Let

$$A'=A+\tau,\qquad \tau\in\Gamma_k(A_{\mathrm{ad}}\otimes\Lambda^1)\tag{3.5}$$

$A'\to\tau$ coordinatizes $\mathscr{A}_k$.

From Sect. II [(2.11) et seq.], $g\in M'_{k+1}(e)\cdot a$ for some $a\in\mathscr{G}_{k+1}$.

$$g=g_\xi a=(\exp\xi)a\tag{3.6}$$

$g\to\xi$ coordinatizes $M'_{k+1}(e)\cdot a$.

Then the map:

$$(A',g)\to A'\cdot g$$

reads in coordinates

$$\begin{gathered}\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)\times\mathscr{L}_{k+1}\to\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)\\ (\tau,\xi)\to\Phi(\tau,\xi)=a^{-1}d_Aa+a^{-1}(\exp(-\xi)\cdot d_A\exp\xi)a+a^{-1}(\exp(-\xi)\tau\cdot\exp\xi)a\end{gathered}\tag{3.7}$$

Using (2.6), and the inequalities (3.2) and (3.3), we obtain:

$$\begin{aligned}&\Phi(\tau+\eta,\xi+h)-\Phi(\tau,\xi)\\ &=(\Phi_*)_\xi(\tau,\xi)h+(\Phi_*)_\tau(\tau,\xi)\eta+r_1(\eta,h;\xi,\tau)+r_2(h;\xi,\tau),\end{aligned}\tag{3.8}$$

where:

(i) $(\Phi_*)_\xi:\mathscr{L}_{k+1}\to\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)$, linear, continuous

$$\begin{gathered}(\Phi_*)_\xi h=a^{-1}(\exp(-\xi)\cdot d_{A'}(\mu(\xi)h)\exp(\xi))a\\ A'=A+\tau\end{gathered}\tag{3.9}$$

(ii) $(\Phi_*)_\tau:\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda')\to\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)$, linear, continuous

$$(\Phi_*)_\tau\eta=a^{-1}(\exp(-\xi)\cdot\eta\cdot\exp(\xi))a\tag{3.10}$$

(iii)

$$\left.\begin{aligned}&\lim_{\|h\|_{k+1}\to 0}\frac{\|r_2(h;\xi,\eta)\|_k}{\|h\|_{k+1}}=0\\ &\lim_{\|h\|_{k+1},\|\eta\|_k\to 0}\frac{\|r_1\|_k}{\|h\|_{k+1}\cdot\|\eta\|_k}=c(\xi,\tau),\end{aligned}\right\}\tag{3.11}$$

where C is a constant (depending on ξ,τ).

We have:

Proposition 3.12. *The group action* $\mathscr{A}_k \times \mathscr{G}_{k+1} \to \mathscr{A}_k$ *given by* (3.4) *is* C^1 *(it is easy to generalize to* C^∞*).*

C. Free Action

We now take special cases of the above.

(a) Take $\mathscr{A}_k \times \mathscr{G}^0_{k+1} \to \mathscr{A}_k$.

(b) Take $\bar{\mathscr{A}}_k \times \bar{\mathscr{G}}_{k+1} \to \bar{\mathscr{A}}_k$.

In both cases the group action is free:

If $A' \cdot g = A'$. Then $d_{A'} g = 0$, i.e. g is covariant constant. For case (a) $g(x_0) = e$. Hence by parallel transport $g = e$ everywhere. For case (b), $g = e$ follows by applying $d_{A'}$ again and using irreducibility and "absence" of centre.

D. From now on we restrict ourselves to the two cases of *C*, where we have C^∞ free action. We shall concentrate on $\bar{\mathscr{A}}_k$ (subspace of irreducible connections) and $\bar{\mathscr{G}}_{k+1} = \mathscr{G}_{k+1}/\mathscr{Z}, k > \frac{\dim M}{2} + 1$. Let

$$\Delta_A = d_A^* d_A = \text{covariant laplacian}$$
$$\Delta_A : \Gamma_{k+1}(E_{\mathrm{ad}}) \to \Gamma_{k-1}(E_{\mathrm{ad}}),$$

a continuous, linear operator.

For $A \in \bar{\mathscr{A}}_k$, $\operatorname{Ker} \Delta_A = 0$, using positivity (in L^2 of the scalar product on $\mathscr{G}_{k+1}$) and irreducibility. From Proposition 3.3 [2], Δ_A is also surjective. Thus Δ_A is an isomorphism.

Let $G_A = \Delta_A^{-1}$, the Green's operator. We have the fundamental inequality which will play an important role in the following:

$$\begin{gathered} \|G_A d_A^* \tau\|_{k+1} \leqq \text{const} \, \|\tau\|_k \\ A \in \bar{\mathscr{A}}_k, \qquad \tau \in \Gamma_k(E_{\mathrm{ad}} \otimes \Lambda^1). \end{gathered} \tag{3.13}$$

We shall use another fact: for $A \in \bar{\mathscr{A}}_k \subset \mathscr{A}_k$

$$T_A(\mathscr{A}_k) = T_A^v(\mathscr{A}_k) \oplus T_A^h(\mathscr{A}_k) \tag{3.14}$$

is a splitting. Here $T_A^v(\mathscr{A}_k)$ may be identified with $d_A \mathscr{L}_{k+1}$ (tangent space to orbit through A) and $T_A^h(\mathscr{A})$ with $\operatorname{Ker} d_A^*$ (see Proposition 3.3 [2]).

Now $\bar{\mathscr{A}}_k \subset \mathscr{A}_k$ is open in $\mathscr{A}_k$. Hence we also have:

$$T_A(\bar{\mathscr{A}}_k) = T_A^v(\bar{\mathscr{A}}_k) \oplus T_A^h(\bar{\mathscr{A}}_k) \tag{3.15}$$

is also splitting.

IV. Manifold Structure for the Gauge Orbit Space

A. We consider the C^∞ free action of $\bar{\mathscr{G}}_{k+1}$ on $\bar{\mathscr{A}}_k$ and consider the quotient space

$$\bar{\mathscr{A}}_k \xrightarrow[\pi]{} \bar{\mathscr{A}}_k / \bar{\mathscr{G}}_{k+1} = \mathfrak{M}_k, \tag{4.1}$$

where π stands for the canonical projection. We give the quotient space the quotient topology; so $\mathfrak{M}_k$ is a topological space and π is a continuous map.

By the structure of a manifold on $\mathfrak{M}_k$ we mean a system of neighbourhoods covering $\mathfrak{M}_k$, homeomorphic to open sets in a Hilbert space (model space). In this section we show how a manifold structure is given. In Sect. V, we will see that the structure is C^∞, i.e. coordinate changes on overlaps are C^∞.

B. Local Gauge Sections. Let

$$N_k(A)=\{A'\in\bar{\mathscr{A}}_k \,|\, \|A'-A\|_k=\|\tau\|_k<c\} \tag{4.2}$$

be a neighbourhood in $\bar{\mathscr{A}}_k$, centred at $A\in\bar{\mathscr{A}}_k$.

Let

$$H_k(A)=\{A'\in\mathscr{A}_k \,|\, d_A^*(A'-A)=d_A^*\tau=0\}.$$

We define:

$$\mathscr{S}_k(A)=H_k(A)\cap N_k(A). \tag{4.3}$$

Proposition 4.4. *For sufficiently small c, the set $\mathscr{S}_k(A)$ is* (i) *locally complete and* (ii) *globally effective.*

(i) *means: given $A'\in N_k(A)$, $\exists$ unique (small) $g\in\mathscr{G}_{k+1}$ such that $A'\cdot g\in\mathscr{S}_k(A)$*

(ii) *means: given A', $A''\in\mathscr{S}_k(A)$, $A'\neq A''$, there does not exist any $g\in\bar{\mathscr{G}}_{k+1}$ s.t. $A'=A''\cdot g$.*

Proposition 4.4 is proved along the lines in [Sect. 6, [6] and [9]). The proof goes through for Sobolev spaces connections because the inequality (3.13) is valid. $\mathscr{S}_k(A)$ will be called a local gauge section.

C. Coordinate Neighbourhoods in Orbit Space

$$\bar{\mathscr{A}}_k \xrightarrow[\pi]{} \bar{\mathscr{A}}_k/\bar{\mathscr{G}}_{k+1}=\mathfrak{M}_k \tag{4.5}$$

and π is continuous in quotient topology. Then, by virtue of Proposition (4.4),

$$\pi_A=\pi|_{\mathscr{S}_k(A)}:\mathscr{S}_k(A)\to\eta_k(A)\subset\mathfrak{M}_k \tag{4.6}$$

is a homeomorphism [$\eta_k(A)$ is the image of $\mathscr{S}_k(A)$ under π_A]. We define:

$$\sigma_A:\eta_k(A)\to\mathscr{S}_k(A)\subset\bar{\mathscr{A}}_k$$
$$\pi_A\cdot\sigma_A=\mathrm{id}_{\eta_k(A)}.$$

σ_A is a continuous local section of (4.5). $\{\eta_k(A)\}$ provides a system of (coordinate) neighbourhoods covering $\mathfrak{M}_k$.

Claim. $\mathfrak{M}_k$ has a manifold structure modelled on a Hilbert space $\mathscr{H}_k(E_{\mathrm{ad}}\otimes\Lambda^1)$.

In fact σ_A gives a chart in $\eta_k(A)$ as follows. We define:

$$\tau(m)=A-\sigma_A(m), \qquad m\in\eta_k(A)$$

and $\mathscr{H}_k(E_{\mathrm{ad}}\otimes\Lambda^1)$ as the Kernel of d_A^* in $\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)$ (it is independent of A up to an isomorphism). $\mathscr{H}_k(E_{\mathrm{ad}}\otimes\Lambda^1)$ is a closed subspace of $\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)$. It is a Hilbert space. Clearly $\mathscr{S}_k(A)$ is isomorphic to an open set in $\mathscr{H}_k(E_{\mathrm{ad}}\otimes\Lambda^1)$ and $\tau\in\mathscr{H}_k(E_{\mathrm{ad}}\otimes\Lambda^1)$.

V. $\bar{\mathscr{A}}_k \xrightarrow{\pi} \bar{\mathscr{A}}_k/\bar{\mathscr{G}}_{k+1} = \mathfrak{M}_k$ as a C^∞ Principal Fibre Bundle

In Sects. II–IV, we have shown that the Lie group $\bar{\mathscr{G}}_{k+1}$ has C^∞ free action on $\bar{\mathscr{A}}_k$ and the topological space $\mathfrak{M}_k$ has been given manifold structure [system of neighbourhoods homeomorphic to open sets in $\mathscr{H}_k(E_{\mathrm{ad}} \otimes \Lambda^1)$]. We shall now prove C^∞ local triviality.

Let $\eta_k(A)$ be a coordinate neighbourhood in $\mathfrak{M}_k$, coordinates being supplied by $\mathscr{S}_k(A)$ (Sect. IV, B, C). We consider the map

$$\begin{aligned}
&\Phi_A : \eta_k(A) \times \mathscr{G}_{k+1} \to \pi^{-1}(\eta_k(A))\\
&(m', g) \to \Phi_A(m', g) = \sigma_A(m') \circ g\\
&\begin{cases} \sigma_A(m') = A + \tau \in \mathscr{S}_k(A) \\ \pi(A+\tau) = m' . \end{cases}
\end{aligned} \tag{5.1}$$

It is easy to check that Φ_A is an isomorphism. To this end, define first the map:

$$\begin{aligned}
g_A : \pi^{-1}(\eta_k(A)) &\to \mathscr{G}_{k+1}\\
A' &\to g_A(A')
\end{aligned} \tag{5.2}$$

by:

$$A' \cdot g_A(A')^{-1} = \sigma_A(\pi(A')) = \sigma_A(m') . \tag{5.3}$$

Such a g_A exists because $\sigma_A(m')$ is a point on the orbit through A'. g_A is uniquely defined because $\bar{\mathscr{G}}_{k+1}$'s action is free. We have:

$$g_A(A' \cdot g) = g_A(A') \cdot g . \tag{5.4}$$

Next we define the map:

$$\begin{aligned}
&\chi_A : \pi^{-1}(\eta_k(A)) \mapsto \eta_k(A) \times \bar{\mathscr{G}}_{k+1}\\
&A' \mapsto \chi_A(A') = (\pi A', g_A(A')) = (m', g_A(A')) .
\end{aligned} \tag{5.5}$$

Then

$$\begin{aligned}
&\Phi_A(\chi_A(A')) = \Phi_A(m', g_A(A')) = \sigma_A(m') \cdot g_A(A')\\
&\Phi_A(\chi_A(A')) = A' .\\
&\chi_A(\Phi_A(m', g)) = \chi_A(\sigma_A(m') \cdot g) = (m', g_A(\sigma_A(m') \cdot g))\\
&\chi_A(\Phi_A(m', g)) = (m', g) .
\end{aligned}$$

Hence Φ_A is an isomorphism and $\Phi_A^{-1} = \chi_A$.

We shall now prove:

Proposition 5.6. *The application Φ_A of (5.1) is a C^∞ diffeomorphism (local triviality).*

Proof. We have already checked that Φ_A is an isomorphism. Hence it is sufficient to check that Φ_A is a local diffeomorphism.

Let

$$(m', g) \in \eta_k(A) \times \bar{\mathscr{G}}_{k+1}. \tag{5.7}$$

Then

$$\sigma_A : m' \to A + \tau \in \mathscr{S}_k(A) \tag{5.8}$$

provides coordinates for m'.

Let $g \in M'_{k+1}(e) \cdot a$. ($M'_{k+1}(e) \cdot a$ is a neighbourhood of a in $\bar{\mathscr{G}}_{k+1}$) [see Sect. IIB, especially (2.11) et seq.].

Then

$$g = g_\xi a = \exp(\xi) \cdot a$$

and (5.9)

$$g \to \xi \in V_{k+1}(0) \subset \mathscr{L}_{k+1}$$

provides coordinates for g. $V_{k+1}(0)$ is a neighbourhood of the origin in $\mathscr{L}_{k+1}$, the Lie algebra of $\bar{\mathscr{G}}_{k+1}$.

Let:

$$\Phi_A(m', g) \in \mathscr{U} \subset \pi^{-1}(\eta_k(A)). \tag{5.10}$$

Here $\mathscr{U}$ is isomorphic to an open set in $\Gamma_k(E_{\mathrm{ad}} \otimes \Lambda^1)$, $\mathscr{S}_k(A)$ is isomorphic to an open set in $\mathscr{H}_k(E_{\mathrm{ad}} \otimes \Lambda^1)$, and $V_{k+1}(0)$ to an open set in $\Gamma_{k+1}(E_{\mathrm{ad}})$. $\mathscr{H}_k(E_{\mathrm{ad}} \otimes \Lambda^1)$ and $\Gamma_{k+1}(E_{\mathrm{ad}})$ are Hilbert spaces.
Hence we have to show, for sufficiently small $\mathscr{U}$, $\mathscr{S}_k(A)$, $V_{k+1}(0)$,

$$\Phi_A : \mathscr{S}_k(A) \times V_{k+1}(0) \to \mathscr{U} \tag{5.11}$$

is C^∞ and also Φ_A^{-1} is C^∞.

That Φ_A is C^∞ follows from Sect. IIIB leading to Proposition 3.12. That Φ_A^{-1} is C^∞ will follow from the inverse function theorem if we can show that the differential $(\Phi_A)_*$ is an isomorphism of tangent spaces on both sides of (5.11).

This we now show:

$$(\Phi_A)_* : T_{A'}(\mathscr{S}_k(A) \oplus T_\xi(V_{k+1}(0)) \to T_{A' \cdot g}(\mathscr{U}), \tag{5.12}$$

using (3.8)–(3.10)

$$(\Phi_A)_*(\eta, h) = a^{-1} \cdot \exp(-\xi) \cdot (d_{A'}(\mu(\xi)h) + \eta) \cdot \exp(\xi) \cdot a, \tag{5.13}$$

where we also have:

$$d_A^* \eta = 0. \tag{5.14}$$

See (2.6) for definition of $\mu(\xi)$.

Claim. $\mathrm{Ker}(\Phi_A)_* = 0$.

Proof. Suppose $(\Phi_A)_*(\eta, h) = 0$.

Then from (5.13) we obtain:

$$d_{A'}(\mu(\xi)h) + \eta = 0 \tag{5.15}$$

and using (5.13)

$$d_A^* d_{A'}(\mu(\xi)\cdot h)=0$$
$$A'=A+\tau\in\mathcal{S}_k(A).$$

Thus

$$\Delta_A(\mu(\xi)\cdot h)+d_A^*[\tau,\mu(\xi)\cdot h]=0$$

or

$$\mu(\xi)\cdot h+G_A d_A^*[\tau,\mu(\xi)\cdot h]=0. \tag{5.16}$$

Here $G_A=\Delta_A^{-1}$.

Then, using (3.13)

$$\begin{aligned}\|\mu(\xi)\cdot h\|_{k+1} &\leqq \text{const}\|[\tau,\mu(\xi)\cdot h]\|_k\\ &\leqq \text{const}\,\|\tau\|_k\cdot\|\mu(\xi)\cdot h\|_k\\ \Rightarrow \|\mu(\xi)\cdot h\|_{k+1} &\leqq \text{const}\,\|\tau\|_k\cdot\|\mu(\xi)\cdot h\|_{k+1}.\end{aligned} \tag{5.17}$$

(5.17) implies that (5.16) has the unique solution

$$\mu(\xi)h=0 \tag{5.18}$$

for sufficiently small $\mathcal{S}_k(A)$. From (5.18) and (2.6)(i) we have $h=0$ for sufficiently small $V_{k+1}(0)$ (exponential map is a local diffeomorphism).

From (5.18) and (5.15) using (3.2) we have $\eta=0$. The claim has been proved.

Claim. $(\Phi_A)_*$ is surjective.

Writing $g=(\exp\xi)\cdot a$ (5.9) we can express (5.13) as:

$$(\Phi_A)_*(\eta,h)=d_{A'\cdot g}(\mathrm{Ad}_{g^{-1}}(\mu(\xi)\cdot h)+G_{A'\cdot g}\cdot d_{A'\cdot g}^*\cdot\mathrm{Ad}_{g^{-1}}\cdot\eta)+\Pi_{A'\cdot g}(\mathrm{Ad}_{g^{-1}}\cdot\eta), \tag{5.19}$$

where

$$\Pi_{A'}=1-d_{A'}\cdot G_{A'}\cdot d_{A'}^*. \tag{5.20}$$

Then $(\Phi_A)_*$ induces a surjective map:

$$T_{A'}(\mathcal{S}_k(A))\oplus T_\xi(V_{k+1}(0))\to T_{A'\cdot g}^v(\mathcal{U})\oplus T_{A'\cdot g}^h(\mathcal{U}), \tag{5.21}$$

where $T_{A'\cdot g}^v(\bar{\mathcal{A}}_k)$ is the tangent space to the orbit through $A'\cdot g$ and $T_{A'\cdot g}^h(\bar{\mathcal{A}}_k)$ is the orthogonal complement [in the natural Riemannian metric on $\Gamma_k(E_{\mathrm{ad}}\otimes\Lambda^1)$].

Let us prove surjectivity of (5.21).

Let

$$d_{A'\cdot g}h_0\in T_{A'\cdot g}^v(\mathcal{U}),\qquad \eta_0\in T_{A'\cdot g}^h(\mathcal{U})$$

and suppose:

$$(\Phi_A)_*(\eta,h)=d_{A'\cdot g}\cdot h_0+\eta_0 \tag{5.22}$$

shall show there exists (unique) η,h satisfying (5.22).

From (5.19) and (5.22), we have:

$$\left.\begin{aligned}&\text{(i)}\quad \eta_0=\Pi_{A'\cdot g}(\mathrm{Ad}_{g^{-1}}\cdot\eta)=\mathrm{Ad}_{g^{-1}}(\Pi_{A'}(\eta))\\&\text{(ii)}\quad d_{A'\cdot g}h_0=d_{A'\cdot g}(\mathrm{Ad}_{g^{-1}}(\mu(\xi)h+G_{A'}d^*_{A'}\eta))\,.\end{aligned}\right\}\tag{5.23}$$

From (5.23)(i), (5.20) and using $d^*_A\eta=0$, we have:

$$F(\tau,\eta)\equiv\eta+d_{A'}G_{A'}*[\tau,*\eta]=\mathrm{Ad}_g\eta_0\,.\tag{5.24}$$

From (3.2), (3.3), and exploiting (3.13), we have F is continuous. Moreover $F_{*,\eta}|_{\tau=0}=\mathrm{id}$. Hence by the implicit function theorem, for sufficiently small $\mathscr{S}_k(A)$, (5.24) admits a unique continuous solution.

$$\eta=\eta(\tau,\eta_0,g)\,.\tag{5.25}$$

Returning to (5.23)(ii) we have, using irreducibility, "absence" of centre in $\bar{\mathscr{G}}_{k+1}$, and that exponential map is a local diffeomorphism (Proposition 2.7), the unique solution:

$$h=-\mu(\xi)^{-1}(G_{A'}d^*_{A'}\eta(\tau,\eta_0,g)-\mathrm{Ad}_gh_0)\,.\tag{5.26}$$

Again using (3.13) and (Theorem 2.7) h is continuous. Thus surjectivity of (5.21) has been proved (in the process we have constructed a continuous inverse).

On the other hand, by virtue of (3.15)

$$T_{A'\cdot g}(\mathscr{U})\to T^v_{A'\cdot g}(\mathscr{U})\oplus T^h_{A'\cdot g}(\mathscr{U})\tag{5.27}$$

is an isomorphism. Combining (5.21) and (5.27) our claim is proved. Hence we have proved that $(\Phi_A)_*$ is an isomorphism. Thus by inverse function theorem Φ_A^{-1} is C^∞. The proof of Proposition 5.6 is complete.

By virtue of opening remarks of Sect. V and Proposition (5.6), we have proved the main theorem:

Theorem 5.28. *$\bar{\mathscr{A}}_k\to\bar{\mathscr{A}}_k/\bar{\mathscr{G}}_{k+1}=\mathfrak{M}_k$ is a C^∞ principal fibre bundle with the Lie group $\bar{\mathscr{G}}_{k+1}$ as structure group.*

Theorem 5.29. *$\mathfrak{M}_k$ is a C^∞ Hilbert manifold.*

Proof. Let $m'\in\eta_k(A_1)\cap\eta_k(A_2)$.

We have:

$$\sigma_{A_i}(m')=A_i'=A_i+\tau_i\in\mathscr{S}_k(A_i)\,,\qquad i=1,2\,.$$

Then we have coordinates:

$$m'\to\tau_i\,,\qquad i=1,2$$

in local charts. From (5.3) we have the coordinates change formula:

$$\sigma_{A_1}(m')=\sigma_{A_2}(m')\cdot g_{A_1}(\sigma_{A_2}(m'))^{-1}$$

or

$$\begin{aligned}&\tau_1=(A_2-A_1)+g_{A_1}(\tau_2)\cdot d_{A_2}(g_{A_1}(\tau_2)^{-1})+\mathrm{Ad}_{g_{A_1}(\tau_2)}\cdot\tau_2\\&\tau_1=F_{A_1A_2}(\tau_2)\,.\end{aligned}\tag{5.30}$$

We have proved $\Phi_A^{-1} = \chi_A$ (5.5) is C^∞ and hence the map g_A of (5.2) is C^∞. The map F [between open sets of $\mathscr{H}_k(E_{\mathrm{ad}} \otimes \Lambda^1)$] of (5.30) is the composition of the C^∞ map g_A with a gauge transformation which is also C^∞ by virtue of Theorem 3.12. Hence the coordinate change map F is C^∞.

Remark 5.31. $\mathfrak{M}_k$ is separable since $\bar{\mathscr{A}}_k$ is seperable. It can be shown that $\mathfrak{M}_k$ is metrizable. Hence it is Haussdorf and para compact and has a countable topological base.

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