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On the Construction of Quantized Gauge Fields

III. The Two-Dimensional Abelian Higgs Model Without Cutoffs

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Abstract. In this paper the construction of the two-dimensional abelian Higgs model begun in two earlier articles is completed. First we show how to remove the remaining ultraviolet cutoff on the gauge field, then we construct the infinite volume limit and verify the axioms of Osterwalder and Schrader for the expectation values of gauge invariant local fields. Finally it is shown that an auxiliary gauge field mass that was introduced to avoid infrared problems can be safely removed.

1. Introduction and Notation

In this paper we continue our investigation of quantized gauge fields begun in [1, 2] by constructing a cutoff free version of the abelian Higgs model in two dimensions obeying all the Osterwalder-Schrader axioms (except possibly clustering) and therefore corresponding to a Wightman theory. From our study of the theory on the lattice we have reason to believe that this theory in fact does have exponential clustering for gauge invariant observables; this is the well known Higgs mechanism. In order to verify this for the continuum theory, one would have to work harder than we do in the present paper and construct a convergent expansion around some mean field theory in the spirit of [3]: the mean field configurations would presumably be configurations of vertices.

The plan of this paper is as follows: After fixing notation we show stability of the theory in a finite volume by an expansion that is, of course, inspired by earlier work in constructive quantum field theory, in particular [4]. This is done in Sect. 2; some technical matters concerning Feynman graphs are deferred to an Appendix. The difficulty of the problem lies somewhere between the two dimensional Yukawa model and the three-dimensional ϕ^4 theory; the fields (in particular the gauge field) have to be localized only in momentum space, not in phase space. It is important to preserve gauge invariance in the form of the Ward

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identities at each step of the expansion in order to keep the cancellations between divergent graphs and counterterms simple. A crucial rôle is also played by the diamagnetic inequality proven in [1, 2] (we use it to prove what is usually called a "Wick bound"). Since the expansion requires estimating a large number of Feynman graphs, we prove a general power counting lemma in the appendix; it covers the two kinds of boundary conditions that are used in the sequel to construct the infinite volume limit: Periodic and mixed (half-Dirichlet for the matter and free for the gauge field); free b.c. could be treated too but are not needed.

In Sect. 3 we prove, using the stability expansion of Sect. 2, exponential (in the cutoff volume) lower and upper bounds on the partition function. Some of the methods used there might be slightly novel and of some limited independent interest, but the main results of Sect. 3 have a technical flavor and the reader might prefer to skip this section in a first reading. In Sect. 4 we prove existence and Euclidean invariance of the thermodynamic limit of gauge invariant Euclidean Green's (Schwinger) functions. A verification of all Osterwalder-Schrader axioms except clustering concludes that section.

In Sect. 2 through 4, the bare mass of the gauge field A_μ is chosen strictly positive (since the gauge group is abelian and A_μ couples to a conserved current, the introduction of a bare mass in the gauge field propagator does not destroy superrenormalizability. This is due to Ward identities which permit one to choose the longitudinal part of the propagator arbitrarily). The mass in the A_μ-propagator clearly prevents any (spurious) infrared divergences. In Sect. 5 we apply the correlation inequalities and the infrared bounds of Paper I in conjunction with an adaptation of the stability expansion to prove that the limit in which the bare mass of the gauge field tends to zero exists and that the physical (gauge invariant) Green's functions are *free of infrared* divergences. This might suggest that the Higgs mechanism is at work. It is another confirmation of the experience that constructive field theory methods seem to be particularly apt at avoiding artificial infrared divergences.

It may be interesting to note that we can construct not only correlation functions of gauge invariant local fields such as $:|\phi|^2:$ and $F_{\mu\nu}$ but also of so-called "string" and "loop" observables such as $:\bar{\phi}(x)\left(\exp\int_x^y A_\mu dx'_\mu\right)\phi(y):$ and $:\exp\oint A_\mu dx_\mu:$ which might be more natural objects in gauge theories than local fields even though an axiomatic framework for them is only beginning to emerge [5].

Let us now introduce some notation: Λ is a bounded open set in $\mathbb{R}^2$, typically a rectangle,

$$\{(x, x_2)\in\mathbb{R}^2 \,|\, |x_\mu|<\tfrac{1}{2}a_\mu,\ \mu=1,2\}\,.$$

A_μ is an abelian gauge field, ϕ a complex scalar ("Higgs") field.

Covariances. C_A is the kernel of $(m^2-\Delta_A)^{-1}$

$$\left(-\Delta_A\equiv\sum_{\mu=1}^{2} D^*_{A,\mu}D_{A,\mu}\,;\ D_{A,\mu}\equiv\partial_\mu-ieA_\mu,\ \mu=1,2\right)$$

considered as an operator on $L^2(\mathbb{R}^2)$. $C_{D,\Lambda}$, $C_{P,\Lambda}$ are the corresponding objects with 0 Dirichlet and Periodic b.c. respectively

$$C^t_{\lambda\nu,\Lambda}(x,y)=\chi_\Lambda(x)\chi_\Lambda(y)(2\pi)^{-2}$$
$$\cdot\int e^{ip(x-y)}\left(\delta_{\mu\nu}-\frac{p_\lambda p_\nu}{p^2+\mu^2}\right)\frac{1}{P^2+\mu^2}e^{-tp_1^2}d^2p$$

is the covariance for the gauge field with "free" b.c. t parametrizes an ultraviolet cutoff

$$C^t_{\lambda\nu,P,\Lambda}(x,y)=\frac{1}{(2\pi)^2}\sum_{(n_1,n_2)\in\mathbb{Z}^2}e^{ip^{(n)}\cdot(x-y)}\left(\delta_{\lambda\nu}-\frac{p^{(n)}_\lambda p^{(n)}_\nu}{p^{(n)2}+\mu^2}\right)$$
$$\cdot(p^{(n)2}+\mu^2)^{-1}e^{-tp_1^{(n)2}}\frac{a_1}{2\pi}\frac{a_2}{2\pi}$$

is the gauge field covariance with periodic b.c.; $p^{(n)}=\left(\frac{2\pi n_1}{a_1},\frac{2\pi n_2}{a_2}\right)$.

Gaussian Measures. $d\nu_\Lambda(\phi)$ is the (normalized, centered) Gaussian measure on $S'(\mathbb{R}^2)$ with covariance C_Λ.

$dm^{(k)}(A^{(k)})$ is the Gaussian measure with covariance

$$C^{(t_k)}_{\lambda\nu,\Lambda}-C^{(t_{k-1})}_{\lambda\nu,\Lambda}\quad(k=1,2,3,\ldots).$$

($C^{(t_0)}\equiv 0$, $t_1,t_2\ldots\to 0$ is a monotonically decreasing sequence of positive numbers – "cutoffs".)

$dm(A)$ is the product Gaussian measure

$$dm(A)=\prod_k dm^{(k)}(A^{(k)}).$$

We will use the same notation for the Gaussian measure with covariance $C^{(0)}_{\lambda\nu,\Lambda}$ because

$$\int dm_{C^{(0)}_{\lambda\nu,\Lambda}}(A)f(A)=\int\prod_k dm^{(k)}(A^{(k)})f(A),$$

where in the right hand side

$$A=\sum_k A^{(k)}$$

Let

$$A_{\mu,s(l)}\equiv\sum_{i=1}^{l}(s_1\ldots s_i)^{1/2}A^{(i)}_\mu\qquad(s_i\in[0,1],\ i=1,2,3,\ldots).$$

$dm_t(A)$ is the Gaussian measure with covariance $C^t_{\lambda\nu,\Lambda}$.

Interactions, Counterterms, Partition Functions, etc. $V_\Lambda\equiv\int_\Lambda :V(|\phi|^2):d^2x$; $V(|\phi|^2)\geqq 0$. $V(|\phi|^2)\geqq 0$ is a polynomial of degree at least 2

$$\delta m_t^2=e^2\int A^2_\mu(0)dm_t(A)=C^{(t)}_{\mu\mu,\Lambda}(0).$$

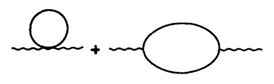
$\delta m^2_{s(l)} = e^2 \int A^2_{\mu, s(l)} dm(A)$. (These counterterms differ by an irrelevant finite term from the ones defined in [2].)

$$E_\Lambda^{(t)} = e^2 \int_{x, y \in \Lambda} d^2x d^2y \int A_\mu(x) A_\nu(y) \Pi_{\mu\nu}(x-y) dm_t(A).$$

$$E_{\Lambda, s(l)} = e^2 \int_{x, y \in \Lambda} d^2x d^2y \int A_{\mu, s(l)}(x) A_{\nu, s(l)}(y) \Pi_{\mu\nu}(x-y) dm(A),$$

where $\Pi_{\mu\nu}$ is the vacuum polarization tensor in second order, discussed in detail in Paper II.

Graphically,

$\Pi_{\mu\nu} =$ 

$E_{P,\Lambda}^{(t)}$, $E_{D,\Lambda}^{(t)}$ are defined analogously by replacing $\Pi_{\mu\nu}$ by the corresponding object with periodic or 0-Dirichlet b.c., respectively.

$$z(A) = \det^{-1}((m^2 - \Delta)^{1/2}(m^2 - \Delta_A)^{-1}(m^2 - \Delta)^{1/2})$$

(this object was discussed in detail in [2]).

$$\begin{aligned} d\omega_A(\phi) &\equiv z(A) d\nu_A(\phi), \\ d\mu_{A,\Lambda,t}(\phi) &\equiv d\omega_A(\phi) e^{-V_\Lambda} e^{1/2\,\delta m^2 \int_\Lambda :|\phi|^2: d^2x} \times e^{E_\Lambda^t}, \\ Z_{\Lambda,t} &\equiv \int dm_t(A) \int d\mu_{A,\Lambda,t}(\phi). \end{aligned}$$

Trace Norms. $I_p (p \geqq 1)$ is the space of compact operators A on a Hilbert space such that

$$\|A\|_p \equiv (\mathrm{Tr}\,(A^*A)^{p/2})^{1/p} < \infty.$$

For more details see [6].

2. Stability in a Finite Volume

We develop a rather simple expansion that reduces the proof of stability in a finite volume to certain plausible estimates on a finite (not particularly small) number of Feynman graphs; their proof requires some machinery, however, and is therefore relegated to the Appendix. The expansion as such is independent of boundary conditions but in the appendix we prove the necessary bounds for periodic and mixed (free-half-Dirichlet) b.c. since no other b.c. are needed. The volume, Λ, is held fixed in this section, so we drop all subscripts, etc.

1. The Stability Expansion

The purpose of this expansion is to prove uniform upper bounds on unnormalized expectations of observables, i.e., expressions like

$$\langle P \rangle_N Z_N \equiv \int P(A, \phi) d\mu_A(\phi) dm_{t_N}(A) \equiv F_N,$$

where P is a polynomial in the fields $\phi, \bar{\phi}, A$. The idea is very simple: We introduce a suitable sequence of cutoffs $t_1, t_2, \ldots (t_N \to 0)$ and write F_N as a telescopic sum:

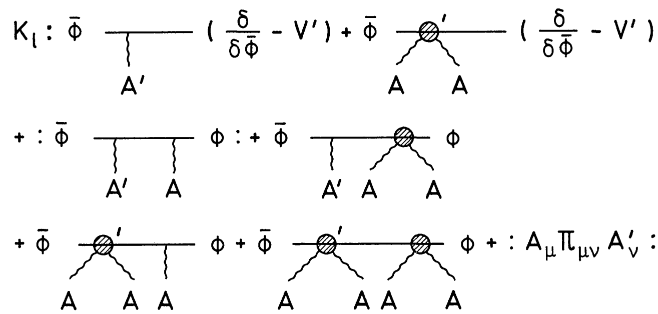
$$F_N = \sum_{k=1}^{N} (F_k - F_{k-1}) + F_0 \tag{2.1}$$

$F_0 \equiv \langle P \rangle_0 Z_0 \equiv \int P(A, \phi) d\mu_0(\phi) dm(A)$. Then we bound each term in this sum in such a way that we obtain absolute convergence as $N \to \infty$. The differences $F_k - F_{k-1}$ are estimated by interpolation, using the interpolating fields $A_{s(l)}$. The first result is

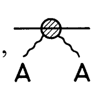
Lemma 2.1. *Let* $k \geq 1$. *Then*

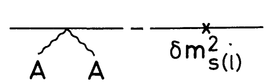
$$F_k - F_{k-1} = \int_0^1 ds_1 \ldots \int_0^1 ds_k \int dm(A) \cdot \int d\mu_{A_{s(k)}} K_k \ldots K_1 P, \tag{2.2}$$

where $K_1, \ldots, K_k$ *are functional differential operators acting on* P *(their action is to be understood in the obvious "algebraic" sense). They can be represented graphically as follows:*



(2.3)

(*A' stands for* $\frac{\partial}{\partial s_l} A_{s(l)}$, 

means 

. *The graphical notation is discussed in detail in Paper II [2]; we do not repeat this discussion since most readers are probably familiar with it.*)

Proof. This is essentially an exercise in the application of the fundamental theorem of calculus. The following formula can be proven by induction. For $l < k$:

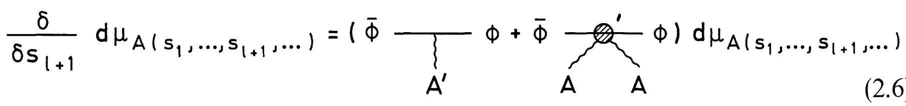
$$F_k - F_{k-1} = \int_0^1 ds_1 \ldots \int_0^1 ds \int dm(A) \cdot \Big(\int d\mu_{A(s_1, \ldots, s_l, \underbrace{1, \ldots, 1}_{k-l})} - \int d\mu_{A(s_1, \ldots, s_l, \underbrace{1, \ldots, 1}_{k-l-1}, 0)}\Big) \cdot K_l \ldots K_1 P.$$

The formula is trivial for $l=0$; to go from l to $l+1<k$ we write the difference of the measures in (2.4) as

$$\int_0^1 ds_{l+1} \frac{\partial}{\partial s_{l+1}} \Big(d\mu_{A(s_1, \ldots, s_{l+1}, \underbrace{1, \ldots, 1}_{k-l-1})} - d\mu_{A(s_1, \ldots, s_{l+1}, \underbrace{1, \ldots, 1}_{k-l-2}, 0)}\Big) \tag{2.5}$$

(note that the expression in brackets vanishes at $s_{l+1}=0$ due to the choice of interpolating fields A_{s_l}).

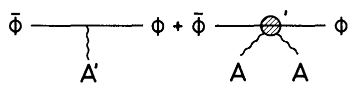
Now



(2.6)

where the prime stands for $\dfrac{\partial}{\partial s_{l+1}}$.

Inserting (2.5) and (2.6) into (2.4) and integrating by parts with respect to the free Gaussian measure $d\nu_0(\phi)$ replaces

 by K_{l+1}

acting on the integrand; this gives (2.4) with l replaced by $l+1$.

Remark. We sketched the change of covariance and integration by parts here in a slightly formal way; the procedure is justified in more detail in Sect. VI, [2].

To complete the proof of the lemma, note that for $l+1=k$ the second term in (2.5) has to be omitted, so the last inductive step produces (2.2). □

From Lemma 2.1 and (2.1) we obtain the following expansion:

Theorem 2.2.

$$\langle P\rangle_N Z_N = \langle P\rangle_0 Z_0 + \sum_{l=1}^{N} \int_0^1 ds_1 \ldots \int_0^1 ds_l \int dm(A) \int d\mu_{A_{s(l)}} K_l \ldots K_1 P\,, \tag{2.7}$$

where K_r $(r\in\mathbb{Z})$ *is defined in* (2.3)*; the prime appearing there stands for* $\dfrac{\partial}{\partial s_r}$.

2. *Convergence of the Expansion*

The goal of this subsection (together with the Appendix) is to prove the following bound on the terms of the expansion (2.1) or (2.7):

$$|F_k - F_{k-1}| \leqq C_1 |\log t_k|^{kr} \prod_{j=1}^{k} t_j^{\varepsilon} (k!)^p e^{c_2(\log t_k)^2} \tag{2.8}$$

for some constants $C_1, c_2, \varepsilon, r, p>0$. This will imply convergence:

Proposition 2.3. *Let* $t_j = \text{const}\, e^{-j^\gamma}$; $(j=1,2,\ldots; 0<\gamma<1)$. *Then* (2.8) *implies convergence of the expansion* (2.7) *as* $N\to\infty$.

Proof (sketch). Under the assumption about $\{t_j\}$:

$$\begin{aligned} |F_k - F_{k-1}| &\leqq C_1 k^{\gamma k r} e^{-\varepsilon \sum_{j=1}^{k} j^\gamma} k^{pk} e^{c_2 k^{2\gamma}} \\ &\leqq C_1 \exp\left\{(r\gamma + p) k \log k + c_2 k^{2\gamma} - \frac{\varepsilon}{\gamma+1} k^{\gamma+1}\right\} \\ &= \exp\{-O(k^{\gamma+1})\} \Rightarrow \sum_{k=1}^{\infty} |F_k - F_{k-1}| < \infty\,. \quad \square \end{aligned}$$

The remainder of Sect. 2 is devoted to reducing the proof of (2.8) to certain bounds on Feynman graphs which in turn are proven in the appendix.

First we use Schwarz's inequality to obtain

$$|\int d\mu_{A_{s(l)}} K_l \dots K_1 P| \leqq |z(A)|^{1/2}$$
$$\cdot (\int d\nu_{A_{s(l)}} |K_l \dots K_1 P|^2)^{1/2} (\int d\omega_{A_{s(l)}} e^{-2V + \delta m^2_{s(l)} \int :|\phi|^2:})^{1/2} e^{E_{s(l)}}. \tag{2.9}$$

By the diamagnetic inequality (I, Theorems 2.3, 4.1 and Sect. 3, see also [7])

$$|z(A)| \leqq 1. \tag{2.10}$$

So (2.8) will be a consequence of the following three lemmas:

Lemma 2.4. $E(s_1, \dots, s_l, 0, \dots) \leqq a_1 (\log t_l)^2$.

Lemma 2.5. $\int d\omega_A e^{-2V + \delta m^2_{s(l)} \int :|\phi|^2:} \leqq \exp a_2 (\delta m^2_{s(l)})^2$ *provided* V *contains a term* $\lambda \int :(\bar{\phi}\phi)^2:$, $\lambda > 0$. *Also*

$$\delta m^2_{s(l)} \equiv \int A^2(s_1, \dots, s_l, 0, \dots, 0) dm(A) \leqq a_3 |\log t_l|.$$

Lemma 2.6. $\int dm(A) \int d\nu_A |K_l \dots K_1 P|^2 \leqq a_4^l \left(\prod_{j=1}^{l} t_j^\delta \right) (l!)^p |\log t_l|^{lr}$ *for some* $\delta > 0$, $p > 0$, $r > 0$.

Remark. The assumption in Lemma 2.5 that V contain a quadratic term could be replaced by the requirement that V contain an even power of $|\phi|$ greater than two. Lemma 2.5 would then hold with some other power of δm^2 appearing on the right hand side. (2.8) would still converge for an appropriate choice of t_j.

Lemma 2.4 is a simple estimate on Feynman graphs, Lemma 2.5 is a consequence of the diamagnetic bound [see (2.10)] and an easy $P(\phi)_2$ estimate, whereas Lemma 2.6 contains the technical core of this paper. It should be rather plausible, though, since $\int dm(A) \int d\nu_A \left| \prod_{i=1}^{l} K_i P \right|^2$ contains only strongly (powerlike) converging Feynman graphs – notice that all cancellations of divergent graphs with counterterms have already been accomplished by the integration by parts in subsection (1) ([2], Sect. II).

We shall not give a proof of Lemma 2.4 since it can be easily deduced from

$$\Pi_{\mu\nu}(k) = O(\log k^2) \qquad (k^2 \to \infty).$$

(Appendix A, [2]).

Proof of Lemma 2.5. By the diamagnetic inequality [1, 2]:

$$\int d\omega_A e^{-2V + \delta m^2_{s(l)} \int :|\phi|^2:} \leqq \int d\nu_0 e^{-2V + \delta m^2_{s(l)} \int :|\phi|^2:}. \tag{2.11}$$

So what remains to be shown is

Proposition 2.7. *Let*

$$V = \lambda \int :|\phi|^4: d^2x - \alpha \int :|\phi|^2: d^2x.$$

Then

$$\int d\nu_0 e^{-V} \leqq \exp O(\alpha^2)$$

for α *large positive. (Similar results may be found in* [8, 9].*)*

Proof. Without loss of generality we assume that $|\Lambda|=1$. We split $V: V=V_1+V_2$ with

$$V_1=\tfrac{1}{2}\lambda(\int :|\phi|^2: d^2x)^2-\alpha\int :|\phi|^2: d^2x$$
$$V_2=\lambda\int :|\phi|^4: d^2x-\tfrac{1}{2}\lambda(\int :|\phi|^2: d^2x)^2 .$$

Claim 1. $V_1 \geqq -\frac{1}{2}\dfrac{\alpha^2}{\lambda}$.

The proof of this is trivial.

Claim 2. $\int dv_0 e^{-V_2}<\infty$.

Proof. This follows by Nelson's argument [10]: In reference to this note that

$$\begin{aligned}
\text{(a)}\quad \frac{1}{\lambda}V_{2,\kappa} &\equiv \int :|\phi_\kappa|^4: d^2x-\tfrac{1}{2}(\int :|\phi_\kappa|^2: d^2x)^2\\
&=\int|\phi_\kappa|^4 d^2x-8C_\kappa\int|\phi_\kappa|^2 d^2x+6C_\kappa^2\\
&\quad -\tfrac{1}{2}(\int|\phi_\kappa|^2 d^2x)^2+2C_\kappa\int|\phi_\kappa|^2 d^2x-\tfrac{1}{2}C_\kappa^2\\
&=\tfrac{1}{2}\int(|\phi_\kappa|^2-\int|\phi_\kappa|^2)^2 d^2x+\tfrac{1}{2}\int|\phi_\kappa|^4 d^2x-6C_\kappa\int|\phi_\kappa|^2 d^2x+\tfrac{11}{2}C_\kappa^2\\
&\geqq -18C_\kappa^2+\tfrac{11}{2}C_\kappa^2\\
&=-O(\log^2\kappa)\qquad (C_\kappa=\tfrac{1}{2}\int|\phi_\kappa(0)|^2 dv_0);
\end{aligned}$$

in the last inequality we used

$$ax^2-bx\geqq-\frac{b^2}{4a}.$$

$$\text{(b)}\quad \|V_{2,\kappa}-V_2\|^2=O(\kappa^{-\varepsilon})$$

for some $\varepsilon>0$. This is a standard fact [16].

(a) and (b) together imply Claim 2 as in Nelson's proof [10, 16] of stability in $P(\phi)_2$. □

Remark. A similar argument works for more general interaction polynomials.

Proof of Lemma 2.6. We begin by developing some notation to organize all the terms that arise when the functional derivatives in K_i are performed. First we split K_i as follows

$$K_i=q_i+a_i \qquad (i=1,\dots,l), \tag{2.12}$$

where

$$q_i=\int\overline{\phi(x)}f_i(x,y)\frac{\delta}{\delta\overline{\phi}(y)}dx\,dy \tag{2.13}$$

whereas a_i is a multiplication operator (i.e. it does not contain functional derivatives). Explicit expressions for q_i, a_i follow by comparing (2.12) and (2.3). We also define

$$K_0\equiv q_0+a_0;\qquad q_0=0\quad (f_0=0);\qquad a_0\equiv P$$

because then we can write

$$\prod_{i=1}^{l} K_i P = \prod_{i=0}^{l} K_i \equiv \left(\prod_{i=0}^{l} K_i\right) 1$$

which will simplify many of the ensuing formulas.

Next we use Leibniz's rule to write

$$\prod_{i=0}^{l} K_i = \sum_{\alpha,\beta} \left(\prod_{i=0}^{l} (q^{\alpha_i} a_i^{\beta_i}) \right), \tag{2.14}$$

where $\beta_i \in \{0,1\}$ for $i=0,\ldots,l$ and

$$q^{\alpha_i} \equiv q_l^{\alpha_l(i)} \ldots q_i^{\alpha_i(i)}; \qquad \alpha_r(i) \in \{0,1\}.$$

i runs from 0 to l, r from i to l. Also

$$\sum_{s=1}^{r} \alpha_r(s) = 1 - \beta_r.$$

All products are ordered according to the index of the factors, e.g.

$$\prod_{i=0}^{l} K_i \equiv K_l \ldots K_0.$$

In (2.14) $q_i^{\alpha_i}$ is acting only on $a_i^{\beta_i}$.

Notice that the degree of $q^{\alpha_i} a_i^{\beta_i}$ as a polynomial in $\bar{\phi}$ and ϕ is bounded uniformly in $i=1,\ldots,l$.

We defer the integration over $dm(A)$ in Lemma 2.6 and estimate first

$$\left(\int dv_A(\phi) \left| \prod_{k=0}^{l} K_i \right|^2 \right)^{1/2} \equiv \|\prod K_i\|_2. \tag{2.15}$$

By Hölder's inequality and (2.14)

$$\|\prod K_i\|_2 \leqq \sum_{\alpha,\beta} \prod_{i=0}^{l} \|q^{\alpha_i} a^{\beta_i}\|_{2(l+1)}. \tag{2.16}$$

Now we use the well known "hypercontractive" estimate [10] for Gaussian measures: If Q is a polynomial of degree q, then

$$\|Q\|_p \leqq (p-1)^{q/2} \|Q\|_2. \tag{2.17}$$

This allows us to bound (2.15) by

$$\sum_{\beta} (l!)^r C^l \sup_{\alpha} \prod_{i=0}^{l} \|q^{\alpha_i} a_i^{\beta_i}\|_2. \tag{2.18}$$

The supremum is over α consistent with β, where r, C are some constants. Here we used that the degree of $q^{\alpha_i} a_i^{\beta_i}$ is bounded uniformly in $i=1,\ldots,l$ with an upper bound depending only on deg V.

Expressions like $\|q^{\alpha_i} a_i^{\beta_i}\|_2^2$ correspond to possibly large Feynman graphs (their size depends mainly on α_i) with external A-lines and internal lines corresponding to C_A and C_0. The next step, familiar in constructive quantum field theory, is to bound large graphs in terms of a finite number of small ones. Because of our more complicated interaction this requires some thought.

If we let an operator

$$q(f) \equiv \int \bar{\phi}(x) f(x, y) \frac{\delta}{\delta \bar{\phi}(y)} dx dy \tag{2.19}$$

act on a Wick ordered monomial

$$P_{n,m} \equiv \int : \prod_{i=1}^{l} \phi(x_i) \prod_{k=1}^{m} \bar{\phi}(y_k) : p_{n,m}(\underset{\sim}{x}; \underset{\sim}{y}) d\underset{\sim}{x} d\underset{\sim}{y} \tag{2.20}$$

(Wick ordering with respect to C_A) it produces two terms:

$$qP_{n,m} = \tilde{P}_{n,m} + \tilde{P}_{n-1,m-1}, \tag{2.21}$$

where $\tilde{P}_{n,m}$, $\tilde{P}_{n-1,m-1}$ are again Wick monomials of the form (2.20), but with new kernel functions

$$\tilde{p}_{n,m}(\underset{\sim}{x}; \underset{\sim}{y}) = \sum_{l=1}^{m} \int f(y_l, y_l') p_{n,m}(\underset{\sim}{x}; y_1, \ldots, y_l', \ldots, y_m) dy_l' \tag{2.22}$$

$$\begin{aligned} \tilde{p}_{n-1,m-1}(\underset{\sim}{x}; \underset{\sim}{y}) &= 2nm \int (C_A f)(x', y') \\ &\cdot S_x S_y p_{n,m}(x', x_2, \ldots, x_n; y', y_2, \ldots, y_m) dx' dy'. \end{aligned} \tag{2.23}$$

S_x (S_y) symmetrizes over the x (y) variables and

$$(C_A f)(x, y) \equiv \int C_A(x, z) f(z, y) dz.$$

According to (2.21) q splits into two parts:

$$q = r + s, \tag{2.24}$$

$$rP_{n,m} = \tilde{P}_{n-1,m-1}, \tag{2.25}$$

$$sP_{n,m} = \tilde{P}_{n,m} \tag{2.26}$$

(i.e. r reduces the degree of Wick monomials, s leaves it the same; for $m=0$: $qP_{n,m} \equiv 0$, for $n=0$: $rP_{n,m}=0$).

The following estimates are straightforward:

Proposition 2.8.

$$\|sP_{n,m}\|_2 \leqq m \|C_A^{1/2} f C_A^{-1/2}\|_{L^2 \to L^2} \|P_{n,m}\|_2, \tag{2.27}$$

$$\|rP_{n,m}\|_2 \leqq \sqrt{nm} \|C_A^{1/2} f C_A^{-1/2}\|_{\mathrm{H.S.}} \|P_{n,m}\|_2. \tag{2.28}$$

Proof. Equation (2.27) is obvious for $n=0$, $m=1$. The left hand side is

$$(p_{0,1}, f C_A f p_{0,1})_{L^2}^{1/2} = \|C_A^{1/2} f C_A^{-1/2} C_A^{1/2} p_{0,1}\|_{L^2} \leqq \|C_A^{1/2} f C_A^{-1/2}\|_{L^2 \to L^2} \|P_{0,1}\|_2.$$

The restriction $n=0$ is clearly irrelevant; the generalization to $m>1$ follows from the "functorial properties of second quantization" [11, 12]; it is also not difficult to verify it directly.

Equation (2.28) follows for $m=n=1$ simply from Schwarz's inequality: The left hand side is

$$|\int (C_A f)(x,y)p_{1,1}(x,y)dxdy| \leqq \|C_A^{1/2} f C_A^{-1/2}\|_{L^2} \|C_A^{1/2} p_{1,1}\|_{L^2}$$
$$= \|C_A^{1/2} f C_A^{-1/2}\|_{\mathrm{H.S.}} \|P_{1,1}\|_2 . \tag{2.29}$$

For general m,n the proof requires in addition very simple combinatorics which we leave to the reader. □

Unfortunately (2.28) is not very suitable for our purpose because the Hilbert-Schmidt norm will in general not exist for the operators q_i we have to consider. To get a finite estimate, we have to "borrow" something from P. In order to systematize this we need some new definitions:

If $P_{n,m}$ is a Wick monomial as before we define $s_k(f)P_{n,m}$ to be the Wick monomial of the same degree with kernel function

$$(s_k(f)p_{n,m})(\underset{\sim}{x},\underset{\sim}{y}) = \binom{m}{k} S_y \int \prod_{i=1}^{k} f(y_i,y_i')p_{n,m}(\underset{\sim}{x};y_1',\dots,y_k',y_{k+1},\dots,y_m)dy_1'\dots dy_k', \tag{2.30}$$

where S_y denotes symmetrization over the y-variables. Note that $s_1(f) \equiv s(f) \equiv s$ (2.26). If $k>m$ we make the convention that

$$s_k(f)P_{n,m} = 0 .$$

We also define operators $\bar{s}_k(f)$ by interchanging the role of x and y variables in (2.30): then obviously

$$[s_k(f), \bar{s}_l(g)] = 0 \tag{2.31}$$

for any f, g, k, l.

We now have the following estimate:

Proposition 2.9. *For,* $n,m \geqq 1$

$$\|\bar{s}_k(g^*)r(f)P_{n,m}\|_2 \leqq \sqrt{\frac{m}{n}}(k+1)\|C_A^{-1/2}g^{-1}C_A f C_A^{-1/2}\|_{\mathrm{H.S.}} \|\bar{s}_{k+1}(g^*)P_{n,m}\|_2 ,$$

where g^* denotes the adjoint of g considered as a kernel.

Proof. Again the proof reduces for $n=m=1$, $k=0$ to a simple Schwarz inequality; for general n,m,k the left hand side contains $\binom{n-1}{k} nm\sqrt{(n-1)(m-1)}$ terms (all equal) which are bounded by Schwarz's inequality by the equal number of terms on the right hand side. □

Remark. g will have to be chosen appropriately to make $C_A^{-1/2}g^{-1}C_A f C_A^{-1/2}$ Hilbert-Schmidt; a suitable choice is for instance $g^{-1} = C_A^{1/2} C^{\varepsilon+1/2} C_A^{-1}$ $(\varepsilon>0)$.

Next we combine Propositions 2.7 and 2.8 to obtain

Proposition 2.10.

$$\left\|\prod_{i=0}^{N-1} q(f_i)P_{n,m}\right\|_2 \leqq N!\, m^N \sup_k \|\bar{s}_k(g^*)P\|_2$$
$$\cdot \prod_{i=0}^{N-1} \{\|C_A^{1/2} f_i C_A^{-1/2}\|_{L^2\to L^2} + \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}}\} .$$

Proof.

$$\prod_{i=0}^{N-1} q(f_i) = \prod_{i=0}^{N-1} (s(f_i) + r(f_i)) = \sum_{\{\gamma\} \in \{0,1\}^N} \prod_{i=0}^{N-1} \{s(f_i)^{1-\gamma_i} r(f_i)^{\gamma_i}\}.$$

We claim

$$\left\| \prod_{i=0}^{N-1} (s(f_i)^{1-\gamma_i} r(f_i)^{\gamma_i}) P_{n,m} \right\|_2$$
$$\leq N!\, m^N \prod_{i=0}^{N-1} \{\|C_A^{1/2} f_i C_A^{-1/2}\|_{L^2 \to L^2}^{1-\gamma_i} \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}}^{\gamma_i}\}$$
$$\cdot \|\bar{s}_{\gamma(N)}(g^*) P_{n,m}\|_2 \qquad \left(\gamma_i \in \{0,1\},\ i=0,\dots,N-1;\ \gamma(N) = \sum_{i=0}^{N} \gamma_i\right). \tag{2.32}$$

The proof is by induction with respect to N, using Propositions 2.8 and 2.9. For $N=1$ it follows from (2.27) and Proposition 2.9. We assume (2.32) true for $N=N_0$. We obtain

$$\left\| \prod_{i=0}^{N_0} (s(f_i)^{1-\gamma_i} r(f_i)^{\gamma_i}) P_{n,m} \right\|_2$$
$$\leq N_0!\, m^{N_0} \prod_{i=1}^{N_0} \{\|C_A^{1/2} f_i C_A^{-1/2}\|_{L^2 \to L^2}^{1-\gamma_i} \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}}^{\gamma_i}\}$$
$$\cdot \|\bar{s}_{\gamma(N_0)}(g^*) s(f_0)^{1-\gamma_0} r(f_0)^{\gamma_0} P_{n,m}\|_2 \,. \tag{2.33}$$

We commute $\bar{s}$ through s and use Propositions 2.8 and 2.9 to estimate the last factor; we obtain

$$\|C_A^{1/2} f_0 C_A^{-1/2}\|_{L^2 \to L^2}^{1-\gamma_0} (m-\gamma_0)^{1-\gamma_0} \left(\sqrt{\frac{m}{n}} \sum_{i=0}^{N_0} \gamma_i \right)^{\gamma_0}$$
$$\cdot \|C_A^{-1/2} g^{-1} C_A f_0 C_A^{-1/2}\|_{\mathrm{H.S.}}^{\gamma_0} \|\bar{s}_{\gamma(N_0)}(g^*) P_{n,m}\|_2 \,. \tag{2.34}$$

Using the inequalities

$$(m-\gamma_0)^{1-\gamma_0} \left(\frac{m}{n}\right)^{\gamma_0} \leq m$$
$$\left(\sum_{i=0}^{N_0} \gamma_i\right)^{\gamma_0} \leq N_0 + 1$$

and inserting (2.34) in (2.33) proves (2.32) for $N = N_0 + 1$.

Using

$$\|\bar{s}_{\gamma(N_0)}(g^*) P_{n,m}\|_2 \leq \sup_k \|\bar{s}_k(g^*) P_{n,m}\|_2$$

and summing (2.32) over $\{\gamma\}$ completes the proof of Proposition 2.10. $\square$

Now we are ready to estimate (2.18) and thereby (2.15):

Proposition 2.11.

$$\prod_{i=0}^{l} \|q^{\alpha_i} a_i^{\beta_i}\|_2 \leqq c^{l+1}(l+1)!$$
$$\cdot \prod_{i=0}^{l} \{\|C_A^{1/2} f_i C_A^{-1/2}\|_{L^2 \to L^2} + \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}}\}^{1-\beta_i}$$
$$\cdot \prod_{i=0}^{l} \sup_k \|\bar{s}_k(g^*) a_i\|_2^{\beta_i}$$

a_i is to be understood as having been expanded in normal ordered (with respect to C_A) polynomials.

Proof. This follows directly from Proposition 2.10 if we recall that

$$q^{\alpha_i} \equiv q_l^{\alpha_l(i)} \dots q_i^{\alpha_i(i)}; \qquad \sum_{s=1}^{r} \alpha_r(s) = 1 - \beta_r. \quad \square$$

Corollary 2.12. *For some constants $c, r > 0$,*

$$\left\| \prod_{i=1}^{l} K_i P \right\|_2 \leqq C^l (l!)^r \prod_{i=0}^{l+1}$$
$$\cdot \left\{ \|C_A^{1/2} f_i C_A^{-1/2}\|_{L^2 \to L^2} + \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}} + \sup_k \|\bar{s}_k(g^*) a_i\|_2 \right\}.$$

a_i is understood as having been expanded in normal ordered (with respect to C_A) polynomials.

Proof. This follows from Proposition 2.11, (2.18), and the identity

$$\sum_{\{\beta\} \in \{0,1\}^N} \prod_{i=1}^{N} a_i^{1-\beta_i} b_i^{\beta_i} = \prod_{i=1}^{N} (a_i + b_i). \quad \square$$

We have now achieved the objective of bounding large graphs by small graphs because Corollary 2.12 only involves L_2 norms of polynomials of low degree in ϕ [occurring in $\bar{s}_k(g^*) a_i$]. Unfortunately we have to normal order the a_i with respect to C_A and calculate L_2 norms with respect to $d\nu_A(\phi)$. This means our Feynman graphs have C_A propagators as well as C_0. We now develop some operator bounds to control C_A by C_0. The end result is found in Proposition 2.16.

We specialize now to the choice

$$g = C_A C_0^{-\varepsilon - 1/2} C_A^{-1/2}.$$

Proposition 2.13.

$$\text{(a)} \quad \|C_A^{1/2} f_i C_A^{-1/2}\| \leqq \|C_A^{1/2} C_0^{-1/2}\| \, \|C_A^{-1/2} C_0^{1/2}\| \, \|C_0^{1/2} f_i C_0^{-1/2}\|. \tag{2.34}$$

$$\text{(b)} \quad \|C_A^{-1/2} g^{-1} C_A f_i C_A^{-1/2}\|_{\mathrm{H.S.}} = \|C_0^{1/2+\varepsilon} f_i C_A^{-1/2}\|_{\mathrm{H.S.}}$$
$$\leqq \|C_0^{1/2} C_A^{-1/2}\| \, \|C_0^{1/2+\varepsilon} f_i C_0^{-1/2}\|_{\mathrm{H.S.}}. \tag{2.35}$$

$$\text{(c)} \quad \|\bar{s}_k(g^*) P_{n,m}\|_{2,C_A} \leqq \|C_A^{1/2} C_0^{-1/2}\|^{n-k}$$
$$\cdot \|C_0^{-1/2-\varepsilon} C_A C_0^{-1/2+\varepsilon}\|^k \|\bar{s}_k(C_0^{-\varepsilon}) P_{n,m}\|_{2,C_0}. \tag{2.36}$$

Remark. $\|\cdot\|$ denotes the operator norm denoted before by $\|\cdot\|_{L^2\to L^2}$, $\|\cdot\|_{2,C_A}$ stands for the L^2-norm with respect to the Gaussian measure with covariance C_A, i.e.,

$$\|F\|_{2,C_A}=\left(\int dv_A(\phi)|F(\phi)|^2\right)^{1/2}$$

(denoted before simply by $\|\cdot\|_2$).

Proof. (a) and (b) are trivial; (c) can be reduced to the case $k=n=1$, $m=0$; in this case it simply says

$$(p_{1,0},C_AC_0^{-1-2\varepsilon}C_Ap_{1,0})\leqq(p_{1,0},C_0^{1-2\varepsilon}p_{1,0})\|C_0^{-1/2-\varepsilon}C_AC_0^{-1/2+\varepsilon}\|^2. \quad \square$$

Proposition 2.14.

(a) $\|C_A^{-1/2}C_0^{1/2}\|\leqq 1+e\|AC_0^{1/2}\|_4$.

(b) $\|C_0^{-1/2}C_A^{1/2}\|\leqq 1+e\|AC_0^{1/2}\|_4$.

(c) $$\|C_0^{-1/2-\varepsilon}C_AC_0^{-1/2+\varepsilon}\|\leqq(1+e\|AC_0^{1/2}\|_4)\cdot\{(1+em^{-2\varepsilon}\|AC_0^{1/2-\varepsilon}\|_4)^2+em^{-4\varepsilon}\|C_0^{1/2-\varepsilon}|(\partial\cdot A)|C_0^{1/2-\varepsilon}\|_2\},$$

where $\|\cdot\|_p$ $(p\geqq 1)$ is the I_p norm (for operators on $L^2(\mathbb{R}^2)$).

Proof.

(a) $$C_0^{1/2}C_A^{-1}C_0^{1/2}=1-C_0^{1/2}(\Delta_A-\Delta)C_0^{1/2}=1-C_0^{1/2}(-ieA\partial-ie\partial A+e^2A^2)C_0^{1/2}.$$

Taking norms and using $\|\partial C_0^{1/2}\|\leqq 1$ we obtain

$$\begin{aligned}\|C_A^{-1/2}C_0^{1/2}\|^2&\leqq 1+2e\|AC_0^{1/2}\|+e^2\|C_0^{1/2}A^2C_0^{1/2}\|\\&\leqq 1+2e\|AC_0^{1/2}\|_4+e^2\|C_0^{1/2}A^2C_0^{1/2}\|_2=(1+e\|AC_0^{1/2}\|_4)^2.\end{aligned}$$

(b) $$\begin{aligned}C_A^{1/2}C_0^{-1}C_A^{1/2}&=1-C_A^{1/2}(\Delta-\Delta_A)C_A^{1/2}=1+C_A^{1/2}(-ieA\partial-ie\partial A+e^2A^2)C_A^{1/2}\\&=1-C_A^{1/2}(+ieAD_A+ieD_AA-e^2A^2)C_A^{1/2}.\end{aligned}$$

Taking norms as above and using in addition the diamagnetic inequality $C_A(x,y)\leqq C_0(x,y)$ we obtain the same bound as above.

(c) Using $C_A=C_0(\Delta_A-\Delta)C_A+C_0$ we obtain

$$\begin{aligned}C_0^{-1/2-\varepsilon}C_AC_0^{-1/2+\varepsilon}&=1+C_0^{1/2-\varepsilon}(\Delta_A-\Delta)C_AC_0^{-1/2+\varepsilon}\\&=1+C_0^{1/2-\varepsilon}(-2ieAD_A-ie(\partial A)+e^2A^2)C_AC_0^{-1/2+\varepsilon};\end{aligned}$$

taking norms

$$\begin{aligned}\|C_0^{-1/2-\varepsilon}C_AC_0^{-1/2+\varepsilon}\|&\leqq 1+2e\|C_0^{1/2-\varepsilon}A\|\,\|C_A^{1/2}C_0^{-1/2+\varepsilon}\|\\&\quad+[e^2\|C_0^{1/2-\varepsilon}A^2C_A^{1/2-\varepsilon}\|+e\|C_0^{1/2-\varepsilon}(\partial A)C_A^{1/2-\varepsilon}\|]\,\|C_A^{1/2+\varepsilon}C_0^{-1/2+\varepsilon}\|\\&\leqq 1+2e\|C_0^{1/2-\varepsilon}A\|_4(1+e\|AC_0^{1/2}\|_4)m^{-2\varepsilon}\\&\quad+[e^2\|C_0^{1/2-\varepsilon}A\|_4^2+e\|C_0^{1/2-\varepsilon}(\partial A)C_A^{1/2-\varepsilon}\|_2](1+e\|AC_0^{1/2}\|_4)m^{-4\varepsilon}\\&\leqq(1+m^{-2\varepsilon}e\|AC_0^{1/2-\varepsilon}\|_4)^2(1+e\|AC_0^{1/2}\|_4)\\&\quad+e\|C_0^{1/2-\varepsilon}|(\partial A)|C^{1/2-\varepsilon}\|_2(1+e\|AC_0^{1/2}\|_4)m^{-4\varepsilon}.\end{aligned}$$

(here again we used the diamagnetic bound). $\square$

We now improve Proposition 2.13, Part (c) by removing the requirement that $P_{n,m}$ be C_A normal ordered – an obstacle to applying the lemma to a_i. We temporarily conflict with previous notation by taking $P_{n,m}$ to be *un-normal ordered* and use $::_0$, $::_A$ to distinguish C_0, C_A normal ordering.

A version of Wicks theorem says

$$:P_{n,m}:_0 = \sum_{j=0}^{\infty} \frac{1}{j!} r_A^j :P_{n,m}:_A ,$$

where r_A is defined in the same way as r (2.25), (2.23) but $C_A f$ is replaced by $C_A - C_0$ in (2.23). By convention r_A^j annihilates $P_{n,m}$ if $j>n$ or m. Repeated application of Proposition 2.9 shows that we can bound each term on the right hand side according to

$$\begin{aligned}\|\bar{s}_k(g^*) r_A^j :P_{n,m}:_A\|_{2,C_A} \leqq c \|C_A^{-1/2} g^{-1}(C_A - C_0) C_A^{-1/2}\|_2^j \\ \cdot \|\bar{s}_{k+j}(g^*) :P_{n,m}:_A\|_{2,C_A} .\end{aligned}$$

c is a constant depending only on the degree of $P_{n,m}$. By substituting our choice for g and applying operator bounds very similar to those used in the proof of Proposition 2.13, we bound the I_2 norm by a polynomial in

$$\|C_0^\varepsilon A C_0^{1/2}\|_2, \ \|C_0^{\varepsilon+1/2} A\|_2, \ \|C_0^{1/2} A\|_4 . \tag{2.37}$$

What we have obtained so far is that

$$\|\bar{s}_k(g^*) :P_{n,m}:_0\|_{2,C_A} \leqq \sup_k \|\bar{s}_k(g^*) :P_{n,m}:_A\|_{2,C_A} Q(A) , \tag{2.38}$$

where Q is a polynomial in the above norms (2.37). We wish to apply this bound to a_i [the sum of terms not involving functional derivatives in (2.3)]. Therefore we write

$$a_i = :a_i:_0 + :t_0 a_i:_0$$

(which defines $:t_0 a_i:_0$). We take $P_{n,m} = a_i$ and $t_0 a_i$ in (2.38) and estimate the right hand side using Propositions 2.13 [Part (c)] and 2.14. The conclusion is:

Proposition 2.15. *For* $i=0, \ldots, l$, $k=0,1, \ldots$

$$\|\bar{s}_k(g^*) a_i\|_{2,C_A} \leqq \left\{\sup_k \|\bar{s}_k(C_0^\varepsilon) :a_i:\|_{2,C_0} + \sup_k \|\bar{s}_k(C_0^\varepsilon) :t_0 a_i:\|_{2,C_0}\right\} Q(A).$$

$Q(A)$ is a polynomial in

$$\|A C_0^{1/2}\|_4, \ \|C_0^{1/2-\varepsilon} A\|_4, \ \|C_0^{1/2-\varepsilon} |\partial A| C_0^{1/2-\varepsilon}\|_{\mathrm{H.S.}}, \ \|C_0^{1/2+\varepsilon} A\|_2, \ \|C_0^\varepsilon A C_0^{1/2}\|_2 .$$

In this proposition and from this point on until the end of the section, ": :" *denotes C_0 normal ordering.*

Now we combine Propositions 2.15, 2.14, and Corollary 2.12 to obtain

Proposition 2.16. *For some constants* $c, r>0$

$$\begin{aligned}\left\|\prod_{i=1}^{l} K_i P\right\|_{2,C_A} \leqq c^l (l!)^r \prod_{i=0}^{l} \\ \cdot \Big\{\|C_0^{1/2} f_i C_0^{-1/2}\| + \|C_0^{1/2+\varepsilon} f_i C_0^{-1/2}\|_2 + \sup_k \|\bar{s}_k(C_0^{-\varepsilon}) :a_i:\|_{2,C_0} \\ + \sup_k \|\bar{s}_k(C_0^{-\varepsilon}) :t_0 a_i:\|_{2,C_0}\Big\} Q(A)^{l+1}\end{aligned}$$

where $Q(A)$ is a polynomial in

$$\|AC_0^{1/2}\|_4,\ \|AC_0^{1/2-\varepsilon}\|_4,\ \|C_0^{1/2-\varepsilon}|\partial A|C_0^{1/2-\varepsilon}\|_2,\ \|C_0^{1/2+\varepsilon}A\|_2,\ \|C_0^{\varepsilon}AC_0^{1/2}\|_2$$

which depends on P. ($f_0=0$, $a_0=P$; f_i, a_i defined in (2.12), (2.13).*)*

Proof. Bound the right hand side of Corollary 2.12 using Propositions 2.15 and 2.14. The common factor $Q(A)^l$ can be extracted by taking $Q(A)\geqq 1$. □

Now we can substitute this bound into the left hand side of Lemma 2.6 and start to consider the $dm(A)$ integration. We use Schwarz's inequality to separate off the $Q(A)$ factors.

Also recall that A has to be read as $A(\mathbf{s}_l)\equiv A(s_1,\dots,s_l,0,0,\dots)$, therefore has a cutoff t_l.

Proposition 2.17

$$\int dm(A)(Q(A(s_1,\dots,s_l,0,\dots))^{2(l+1)}\leqq C^l|\log t_l|^{rl}(l!)^p$$

with some integers r,p.

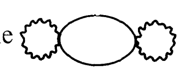
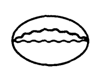
Note. We use the letter C for various constants appearing in our estimates, i.e., it can change its meaning from estimate to estimate.

Proof. By Nelson's hypercontractive estimate (2.17), we only have to estimate the integrals with respect to $dm(A)$ of

$$\begin{gathered}\|A(\underset{\sim}{s})C_0^{1/2}\|_4^4,\ \|A(\underset{\sim}{s})C_0^{1/2-\varepsilon}\|_4^4,\ \|C_0^{1/2-\varepsilon}|\partial A(\underset{\sim}{s})|C_0^{1/2-\varepsilon}\|_2^2,\\ \|C_0^{1/2+\varepsilon}A(\underset{\sim}{s})\|_2^2,\ \|C_0^{\varepsilon}A(\underset{\sim}{s})C_0^{1/2}\|_2^2.\end{gathered}\tag{2.39}$$

In the third norm, use the bound,

$$|\partial A|\leqq\tfrac{1}{2}(1+(\partial A)^2).$$

(2.39) corresponds to small graphs like  ,  . Some of these graphs, such as the second, may be estimated by the power counting lemma of the Appendix. Others, like the first, are to be estimated directly. All are (for small enough ε) less than $O(\log^2 t_l)|A|$. □

For the other factors in Proposition 2.16, we use again Hölder's inequality and the hypercontractive bound (2.17). This reduces our task to the estimation of expressions like

$$\int dm(A)\|C_0^{1/2}f_iC_0^{-1/2}\|_4^4\quad\text{etc.}$$

We claim:

Proposition 2.18. *For some $\delta>0$*

$$\begin{gathered}\int dm(A)\|C_0^{1/2}f_iC_0^{-1/2}\|_4^4,\\ \int dm(A)\|C_0^{1/2+\varepsilon}f_iC_0^{-1/2}\|_2^2,\\ \sup_k\int dm(A)\|\bar{s}_k(C_0^{-\varepsilon}):a_i:\|_{2,C_0}^2,\\ \sup_k\int dm(A)\|\bar{s}_k(C_0^{-\varepsilon}):t_0a_i:\|_{2,C_0}^2,\end{gathered}$$

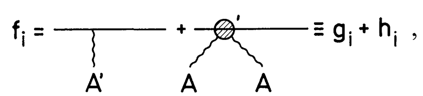
are all bounded by expressions of the form Ct_i^δ, uniformly in i and in the interpolation parameters s_i that appear in them through the dependence on $A(\underset{\sim}{s}_l)$.

Proof. All these expressions reduce to a finite (not particularly small) number of convergent Feynman graphs; each of them contains at least one $\frac{\partial}{\partial s_i}A$ which enforces an upper t-cutoff t_i; by a power counting argument given in the appendix the proof is completed.

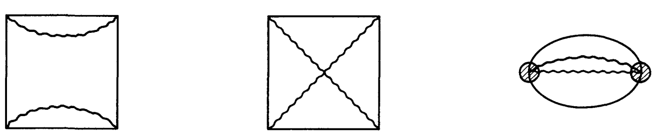
Combining Propositions 2.16, 2.17, and 2.18 finally completes the proof of Lemma 2.6 and in (2.8) and therefore proves convergence of the stability expansion. □

We close this section by showing in more detail the graphs that arise from the expressions appearing in Proposition 2.18.

Recall that graphically, for $i \neq 0$, $(f_0 = 0)$,



where the prime stands for $\frac{\partial}{\partial s_i}$. Inserting this in $\int dm(A)\|C_0^{1/2}f_iC_0^{-1/2}\|_4^4$ produces many topologically distinct graphs which, however, may be estimated in terms of the following three:



(2.40)

if we use

$$\int dm(A)\|C_0^{1/2}f_iC_0^{-1/2}\|_4^4 \leqq 8\int dm(A)(\|C_0^{1/2}g_iC_0^{-1/2}\|_4^4 + \|C_0^{1/2}h_iC_0^{-1/2}\|_4^4) \tag{2.41}$$

and estimate the second term by a constant times

$$(\int dm(A)\|C_0^{1/2}h_iC_0^{-1/2}\|_2^2)^2 \tag{2.42}$$

using $\|\cdot\|_4 \leqq \|\cdot\|_2$ and the hypercontractive estimate (2.17).

Note that in each graph of (2.40) at least one of the lines is differentiated with respect to s_i. The quantity

$$\int dm(A)\|C_0^{1/2+\varepsilon}f_iC_0^{-1/2}\|_2^2$$

produces the graphs

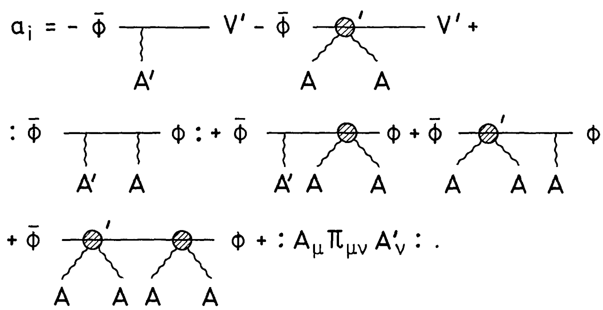
(2.43)

where —‖— stands for $C_0^{1+2\varepsilon}$.

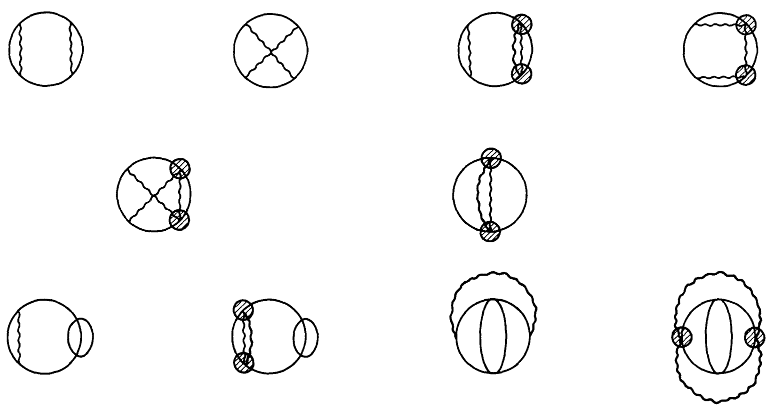
The most complicated expression in Proposition 2.18 is

$$\sup_k \int dm(A) \|\bar{s}_k(C_0^{-\varepsilon}) : a_i : \|_{2,C_0}^2 . \tag{2.44}$$

Recall that graphically, for $i \neq 0$, $(a_0 = P)$,

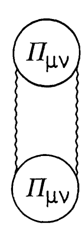


For the special (but typical) case $V(|\phi|) = \lambda |\phi|^4$ we have listed below the graphs that arise from (2.44). We have used the bound $\| \ \|_4 \leqq \| \ \|_2$ together with hypercontractivity (2.17) which accounts for the appearance of the 6th graph instead of four graphs with four vertices.



(2.45)

where in each graph the lines ——— have to be interpreted as either $C_0^{1-2\varepsilon}$ or C_0. In addition there is the term $\| : A_\mu \Pi_{\mu\nu} A'_\nu : \|_2^2$ which may be represented as

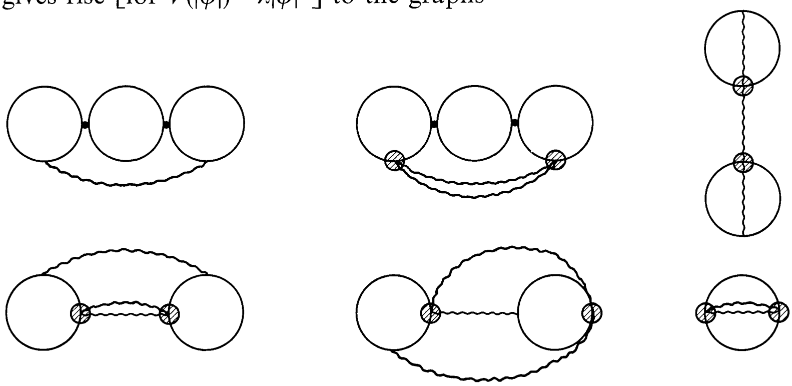


(2.46)

In Proposition 2.18 we also have

$$\sup_k \int dm(A) \|\bar{s}_k(C_0^{-\varepsilon}) : t_0 a_i : \|_{2,C_A}^2 .$$

This gives rise [for $V(|\phi|)=\lambda|\phi|^4$] to the graphs



(2.47)

It is now easy to apply the power counting lemma of the Appendix to the list of graphs contained in (2.45), (2.46), and (2.47).

3. Volume Dependent Bounds

In this section we employ periodic and mixed (free-half-Dirichlet) boundary conditions and their convexity properties in the volume to prove upper and lower bounds on partition functions of the form $\exp O(|\Lambda|)$. It is convenient to slightly change the energy counterterm used in the stability expansion to the so-called matched counterterm introduced in [13] which has the advantage of being

a) independent of boundary conditions

b) exactly proportional to the volume $|\Lambda|$.

In the introduction we defined the energy counterterm

$$E^{(t)}_{X,\Lambda}=\int_{x,y\in\Lambda} d^2x d^2y \Pi^X_{\mu\nu}(x,y)C^{(t)}_{\mu\nu}(x-y) \tag{3.1}$$

(X stands for the boundary conditions F, P or D). The "matched counterterm" is

$$\tilde{E}^{(t)}_{\Lambda}=|\Lambda|\varepsilon^{(t)}\equiv|\Lambda|\int_{\mathbb{R}^2} d^2x \Pi^F_{\mu\nu}(x)C^{(t)}_{\mu\nu}(x) \tag{3.2}$$

[where we wrote $\Pi^F_{\mu\nu}(x)$ for $\Pi^F_{\mu\nu}(x,0)$]. We can replace E by $\tilde{E}$ because we have the following

Lemma 3.1. *For $X=F$, D or P*

(1) $|E^{(t)}_{X,\Lambda}-\tilde{E}^{(t)}_{\Lambda}|\leqq C_\Lambda$,

where C_Λ is a constant dependent on Λ but independent of ε.

(2) $\lim_{t\to 0}(E^{(t)}_{X,\Lambda}-\tilde{E}^{(t)}_{\Lambda})$ *exists.*

Proof (Essentially in [13]).

a) $X=F$

$$E^{(t)}_{F,\Lambda}=\int_{\mathbb{R}^2}\Pi^F_{\mu\nu}(x)C^{(t)}_{\mu\nu}(x)g_\Lambda(x)dx,$$

where

$$g_\Lambda(x)=\int \chi_\Lambda(x+y)\chi_\Lambda(y)dy$$
$$=a_1a_2\left(1-\frac{|x_1|}{a_1}\right)\left(1-\frac{|x_2|}{a_2}\right)\chi_{2\Lambda}(x),$$

$$E^{(t)}_{F,\Lambda}-a_1a_2\varepsilon^{(t)}=a_1a_2\int_{\mathbb{R}^2\setminus 2\Lambda}\Pi^F_{\mu\nu}(x)C^{(t)}_{\mu\nu}(x)dx$$
$$+a_1a_2\int_{2\Lambda}\Pi^F_{\mu\nu}(x)C^{(t)}_{\mu\nu}(x)\left(\frac{|x_1x_2|}{a_1a_2}-\frac{|x_1|}{a_1}-\frac{|x_2|}{a_2}\right)$$

which is easily seen to be bounded independently of t because

$$|\Pi^F_{\mu\nu}(x)|\leqq \mathrm{const}\,|x|^{-2}$$

(see [2], Appendix A). Convergence follows easily by the dominated convergence theorem.

b) $X=D,P$.

In Appendix B of [2] it is proven that

$$(\Pi^P_{\mu\nu}-\Pi^F_{\mu\nu})(x,y)\quad\text{and}$$
$$(\Pi^D_{\mu\nu}-\Pi^F_{\mu\nu})(\mathrm{x},\mathrm{y})\quad\text{are in}\quad L^1(\Lambda\times\Lambda),$$

The discussion there actually shows that the above expressions are in $L^{1+\delta}$ for some $\delta>0$ (by a direct computation with image charges the reader may convince himself that this is true by virtue of

$$|(\Pi^X_{\mu\nu}-\Pi^F_{\mu\nu})(x,y)|\leqq \mathrm{const}\frac{1}{|x-y|}\,\frac{1}{|x-\tilde{y}|},$$

where $\tilde{y}$ is the location of the image charge closest to y). By the dominated convergence theorem it is then easy to see that $E^{(t)}_{X,\Lambda}-E^{(t)}_{F,\Lambda}$ converges as $t\to 0$ which is sufficient to complete the proof of b). □

From now on we understand the partition functions $Z_{D,\Lambda}$ and $Z_{P,\Lambda}$ to be defined with the energy counterterm $\tilde{E}_\Lambda$ instead of $E_{X,\Lambda}$. Then the following theorem holds:

Theorem 3.2. *Let Λ be a rectangle of sides $L>3\delta$ and $T>3\delta$ ($\delta>0$). Then there are constants $c,c_+,c_-\in\mathbb{R}$; $K,K_+,K_->0$ such that*

a) $Z_{D,\Lambda}\geqq K_-e^{c_-LT}$,

b) $Z_{P,\Lambda}\leqq K_+e^{c_+LT}$,

c) $Z_{D,\Lambda}\leqq Ke^{c(L+T)}Z_{P,\tilde{\Lambda}}$,

where $\tilde{\Lambda}$ is a rectangle of sides $L-\delta$, $T-\delta$.

Proof. a) For

$$L,T\geqq 3\delta,\qquad Z_{P,\Lambda}\leqq \mathrm{const}\,e^{c_+LT}. \tag{3.3}$$

The lattice analogue of (3.3) is proven in [1], Corollary 2.9 and follows from the fact that the periodic partition function can be written as a trace of a power of the transfer matrix. In the continuum we have to be careful to use the right normalization for the partition function. Formally

$$Z_{P,A} = \frac{\mathrm{Tr} e^{-TH_{P,L}}}{\mathrm{Tr} e^{-TH^0_{P,L}}} = \frac{\mathrm{Tr} e^{-LH_{P,T}}}{\mathrm{Tee}^{-LH^0_{P,T}}}, \tag{3.4}$$

where $H_{P,L}$ is the Hamiltonian with periodic boundary conditions on the interval $\left[-\frac{L}{2}, \frac{L}{2}\right]$; $H^0_{P,L}$ the corresponding free Hamiltonian. We give some arguments to show how a formula like (3.4) can be justified [8, 14].

As in the proof of O.S. positivity in [2] we have to introduce a somewhat complicated lattice approximation: We use two rectangular lattices; for ϕ the lattice constants are ε_s in space and ε_t in time direction, for A they are ε'_s and ε'_t and we assume that the A-lattice is a refinement of the ϕ-lattice. Then it is straightforward to see that the lattice partition function with the appropriate normalization for the continuum limit can be written as

$$Z^{\varepsilon,\varepsilon'}_{P,A} = \frac{\mathrm{Tr}(T_{L,\varepsilon,\varepsilon'})^{T/\varepsilon_t}}{\mathrm{Tr}(T^0_{L,\varepsilon,\varepsilon'})^{T/\varepsilon_t}} \tag{3.5}$$

$$(\varepsilon = (\varepsilon_s, \varepsilon_t; \varepsilon' = (\varepsilon'_s, \varepsilon'_t)),$$

where $T_{L,\varepsilon,\varepsilon'}$ is the lattice transfer matrix for translation by ε_t with periodic boundary conditions; $T^0_{L,\varepsilon,\varepsilon'}$ the corresponding operator for the free theory. T and T^0 are defined up to a normalization factor which will be chosen in a way that makes the continuum limit easy.

By Gaussian integration

$$\mathrm{Tr}(T^0_{L,\varepsilon,\varepsilon'})^{T/\varepsilon_T} = (\det(-\Delta_\varepsilon + m^2))^{-1} f_m(\varepsilon)^{T/\varepsilon_t}$$

$$\cdot \left[\det\left(\frac{\delta_{\nu\lambda} - \dfrac{\partial^{\varepsilon'*}_\nu \partial^{\varepsilon'}_\lambda}{-\Delta_{\varepsilon'} + \mu^2}}{-\Delta_{\varepsilon'} + \mu^2}\right)\right]^{1/2} g_\mu(\varepsilon')^{T/\varepsilon'_t}, \tag{3.6}$$

where f_m and g_μ are the above mentioned normalization factors to be chosen below; ∂^ε_μ, Δ_ε are the (periodic) finite difference gradient and Laplacean, respectively. Computing the 2×2 determinant indexed by ν, λ we obtain

$$\mathrm{Tr}(T^0_{L,\varepsilon,\varepsilon'})^{T/\varepsilon_t} = (\det(-\Delta_\varepsilon + m^2))^{-1} (\det(-\Delta_{\varepsilon'} + \mu^2))^{-3/2}$$

$$\cdot (\mu)^{LT/\varepsilon'_s\varepsilon'_t} f_m(\varepsilon)^{T/\varepsilon_t} g_\mu(\varepsilon')^{T/\varepsilon'_t}. \tag{3.7}$$

In view of this, we will choose

$$g_\mu(\varepsilon') = f_m^{3/2}(\varepsilon')(\mu)^{-L/\varepsilon'_s}. \tag{3.8}$$

Limits should be taken in the following sequence: We should modify the determinant coming from the A-integration by introducing an ultraviolet cutoff of the kind used in the stability expansion; then we send $\varepsilon'_t \to 0$, $\varepsilon'_s \to 0$, $\varepsilon_t \to 0$, $\varepsilon_s \to 0$ (in that order); finally the ultraviolet cutoff is removed. This ultraviolet cutoff is irrelevant here and we ignore it in the sequel to avoid overly clumsy formulas.

In [2] we had $\varepsilon_s' = \varepsilon_t'$; $\varepsilon_s = \varepsilon_t$, but by the methods used there it is straightforward to establish that the partition functions with the correct continuum normalization have the same limit if we remove the lattice in the order just described.

So we are reduced to studying

$$\det(-\Delta_\varepsilon + m^2)^{-1} f(\varepsilon)^{T/\varepsilon_t} \qquad (f \equiv f_m). \tag{3.9}$$

By explicit diagonalization

$$\det(-\Delta_\varepsilon + m^2) = \prod_{r=-L/2\varepsilon_s+1}^{L/2\varepsilon_s} \prod_{n=-T/2\varepsilon_t+1}^{T/2\varepsilon_t} \cdot \left(m^2 + \frac{2}{\varepsilon_s^2}\left(1-\cos\frac{2\pi r\varepsilon_s}{L}\right) + \frac{2}{\varepsilon_t^2}\left(1-\cos\frac{2\pi n\varepsilon_t}{T}\right)\right) \tag{3.10}$$

$\left(\text{we assumed } \frac{L}{2\varepsilon_s} \text{ and } \frac{T}{2\varepsilon_t} \text{ to be integers}\right)$. We now claim:

$$\lim_{\varepsilon\to 0} e^{-\omega T}\varepsilon^{T/\varepsilon} \prod_{n=-T/2\varepsilon+1}^{T/2\varepsilon} \left(\omega^2 + \frac{2}{\varepsilon^2}\left(1-\cos\frac{2\pi n\varepsilon}{T}\right)\right)^{1/2}$$
$$= \omega T e^{-\omega T} \prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega T}{2\pi n}\right)^2\right) = 1 - e^{-\omega T}. \tag{3.11}$$

Proof. The last identity is just the canonical product decomposition of $1-e^{-\omega T}$ (see [26]). To prove the first identity, note that

$$\prod_{n=-T/2\varepsilon+1}^{T/2\varepsilon} \left(\varepsilon^2\omega^2 + 2 - 2\cos\frac{2\pi n\varepsilon}{T}\right)^{1/2}$$
$$= \varepsilon\omega\sqrt{4+\varepsilon^2\omega^2} \prod_{n=1}^{T/2\varepsilon-1} \left(1 + \frac{\varepsilon^2\omega^2}{2-2\cos\frac{2\pi n\varepsilon}{T}}\right) \prod_{n=1}^{T/2\varepsilon-1} \left(2-2\cos\frac{2\pi n\varepsilon}{T}\right). \tag{3.12}$$

The first factor is easily seen to converge to $\prod_{n=1}^{\infty}\left(1+\left(\frac{\omega T}{2\pi n}\right)^2\right)$ by taking the logarithm and using the dominated convergence theorem. The second factor is equal to $T/2\varepsilon$ because of the identity

$$\prod_{n=1}^{N-1} \left(2 - 2\cos\frac{\pi n}{N}\right) = N. \tag{3.13}$$

(3.13) can be proven in various ways [15]. One way is to note that

$$\prod_{n=1}^{N-1}\left(2-2\cos\frac{\pi n}{N}\right)^{-1/2} = (2\pi)^{-\frac{N-1}{2}} \int d^{N-1}x\, e^{-1/2\sum_{n=2}^{N-1}(x_n - x_{n-1})^2 - 1/2(x_1^2 + x_{N-1}^2)}$$

which equals $\frac{1}{\sqrt{N}}$ by the semigroup property of the heat kernel. (3.13) inserted in (3.12) completes the proof of (3.11). □

If we now set

$$\omega_{\varepsilon_s}(k_r^{(L)})^2 \equiv m^2 + \frac{2}{\varepsilon_s^2}\left(1 - \cos\frac{2\pi r \varepsilon_s}{L}\right).$$

[If we were making the U.V. cutoff explicit, it would modify this and ensuing ω's] and

$$f(\varepsilon) \equiv \prod_{r=-L/2\varepsilon_s+1}^{L/2\varepsilon_s} \varepsilon_t^{-2} e^{2\varepsilon_t \omega_{\varepsilon_s}(k_r^{(L)})} \tag{3.14}$$

it follows from (3.10) and (3.11) that

$$\begin{aligned}\lim_{\varepsilon_s \to 0} \lim_{\varepsilon_t \to 0} & f(\varepsilon)^{T/\varepsilon_t} \det(-\Delta_\varepsilon + m^2)^{-1} \\ &= \prod_{r=-\infty}^{\infty} (1 - e^{-T\omega(k_r^{(L)})})^{-2} \\ &= \operatorname{Tr} e^{-TH_{P,L}^0(m)},\end{aligned}$$

where $\omega(k_r^{(L)})^2 = m^2 + \left(\frac{2\pi r}{L}\right)^2$ and $H_{P,L}^0(m)$ is the Hamiltonian of the complex free field of mass m with periodic b.c. on $\left[-\frac{L}{2}, \frac{L}{2}\right]$.

Remark. It is not hard to see that

$$\sum_{r=-L/2\varepsilon+1}^{L/2\varepsilon} \omega_\varepsilon(k_r^{(L)}) = c_1 L \varepsilon^{-2} + c_2 L \log \varepsilon + O(1)$$

for $\varepsilon \to 0$; this implies that

$$\varepsilon_t^{-2LT/\varepsilon_t \varepsilon_s} e^{2c_1 LT \varepsilon_s^{-2} + 2c_2 LT \log \varepsilon_s} \det(-\Delta_\varepsilon + m^2)^{-1}$$

has a finite limit as $\varepsilon_t \to 0$ and then $\varepsilon_s \to 0$. The asymmetry of this expression in ε_s and ε_t shows that the order of limits is essential. This asymmetry was also noted in the context of dual string theory [15] as "noncovariance of divergent parts".

We have not quite established (3.4) since we did not construct $H_{P,L}$; this could be done, but at this point we only need the following two facts which follow from what we have proven, namely that after all limits have been taken

$$\frac{1}{T}(\log Z_{P,\Lambda} + \log \operatorname{Tr} e^{-TH_{P,L}^0})$$

is decreasing in T and

$$\frac{1}{L}(\log Z_{P,\Lambda} + \log \operatorname{Tr} e^{-LH_{P,T}^0})$$

is decreasing in L (the second statement follows from the first by Nelson's symmetry proven in the next section).

These two facts imply (3.3) ($\log Z_{P,\Lambda} \leqq c_+ LT + \text{const}$) for $L, T \geqq 1$; for $3\delta \leqq L$, $T \leqq 1$). (3.3) follows directly from the stability expansion which produces bounds uniform in $3\delta \leqq L$, $T \leqq 1$. This completes the proof of (3.3).

b)
$$Z_{D,\Lambda} \geqq \text{const}\, e^{C_- LT}. \tag{3.15}$$

This follows by a similar method. Formally we have, using "Nelson's symmetry" [8, 16]

$$\begin{aligned} Z_{D,\Lambda} &= \frac{(\eta_L, e^{-TH_{D,L}}\eta_L)}{(\eta_L^0, e^{-TH_{D,L}^0}\eta_L^0)} \\ &= \frac{(\eta_T, e^{-LH_{D,T}}\eta_T)}{(\eta_T^0, e^{-LH_{D,T}^0}\eta_T^0)} \end{aligned} \tag{3.16}$$

with some "idealized vectors" η_L, η_L^0 etc. (products of δ functions enforcing boundary conditions, see [12]). This means that e.g. $\eta_L(\varepsilon) \equiv e^{-\varepsilon H_{D,L}}\eta_L$ (formally) is a bona fide vector in the physical Hilbert space for $\varepsilon > 0$, but $\lim_{\varepsilon \to 0} \|\eta_L(\varepsilon)\| = \infty$. The justification of (3.16) goes along the same lines as before; the crucial facts are:

$$\log Z_{D,\Lambda} + \log(\eta_L^0, \mathrm{e}^{-TH_{D,L}^0}\eta_L^0)$$

is convex in T and

$$\log Z_{D,\Lambda} + \log(\eta_T^0 e^{-LH_{D,T}^0}\eta_T^0)$$

is convex in L, where

$$(\eta_L^0, e^{-TH_{D,L}^0}\eta_L^0) = \prod_{r=1}^{\infty} (1 - e^{-T\omega(k_r^{(L)})})^{-2} \geqq 1 \tag{3.17}$$

[16, 17]

$$\omega(k_r^{(L)})^2 = m^2 + \left(\frac{r\pi}{L}\right)^2 \quad (r = 1, 2, 3, \ldots).$$

Next we claim that for sufficiently small rectangles Λ $Z_{D,\Lambda} > 0$ (the stability expansion for $Z_{D,\Lambda}$ converges uniformly in $L, T \leqq 1$, cf. Appendix): We introduce a t-cutoff in the covariance of A; then obviously $\lim_{|\Lambda| \searrow 0} Z_{D,\Lambda}^{(t)} = 1$. On the other hand the stability expansion shows that by choosing t small enough,

$$|Z_{D,\Lambda}^{(t)} - Z_{D,\Lambda}| < \varepsilon \quad \text{for any} \quad \varepsilon > 0$$

(uniformly in Λ for $|\Lambda| \leqq 1$); therefore $Z_{D,\Lambda} > 0$ for small $|\Lambda|$ and $Z_{D,\Lambda} \to 1$ as $|\Lambda| \to 0$.

A simple argument using convexity then shows that $Z_{D,\Lambda} > 0$ for all rectangles and

$$\begin{aligned} \log Z_{D,\Lambda} \geqq{} & \frac{LT}{\delta^2} \log Z_{D;\delta,\delta} - \log(\eta_L^0, e^{-TH_{D,L}^0}\eta_L^0) \\ & + \frac{L}{\delta}\log(\eta_\delta^0, e^{-TH_{D,\delta}^0}\eta_\delta^0) - \frac{L}{\delta}\log(\eta_T^0, e^{-\delta H_{D,T}^0}\eta_T^0) \\ & + \frac{LT}{\delta^2}\log(\eta_\delta^0, e^{-\delta H_{D,\delta}^0}\eta_\delta^0) \end{aligned} \tag{3.18}$$

from which (3.15) follows if we use (3.17) and the fact

$$(\eta_L^0, e^{-TH_{D,L}^0}\eta_L^0) \leqq (\eta_L^0, e^{-\delta H_{D,L}^0}\eta_L^0) = e^{0(L)}$$

[13] which can be easily deduced from (3.17).

c)
$$Z_{D,\Lambda} \leqq K Z_{P,\tilde{\Lambda}} e^{c(L+T)}. \tag{3.19}$$

This is essentially the fact that the trace is bigger than any expectation value. Note that all the manipulations in the following use only O.S. positivity which follows from the lattice approximation [2]; we use the formal objects like η_L etc. only to make the argument more transparent.

$$\begin{aligned} Z_{D,\Lambda} &= (\eta_L(\delta), e^{-(T-2\delta)H_{D,L}}\eta_L(\delta)) \\ &\quad \cdot (\eta_L^0, e^{-TH_{D,L}^0}\eta_L^0)^{-1} \\ &\leqq \operatorname{Tr} e^{-(T-2\delta)H_{D,L}} \|\eta_L(\delta)\|^2, \end{aligned} \tag{3.20}$$

where we used (3.17). The trace in (3.20) can be expressed in terms of a partition function $Z_{D,P;L,T-2\delta}$ which has periodic b.c. in time and (half)-Dirichlet b.c. in space or its "Nelson transform" (time and space interchanged) $Z_{P,D;T-2\delta,L}$:

$$\begin{aligned} \operatorname{Tr} e^{-(T-2\delta)H_{D,L}} &= \operatorname{Tr} e^{-(T-2\delta)H_{D,L}^0} \times Z_{D,P;L,T-2\delta} \\ &= \operatorname{Tr} e^{-(T-2\delta)H_{D,L}^0} Z_{P,D;T-2\delta,L} \\ &= \operatorname{Tr} e^{-(T-2\delta)H_{D,L}^0} (\eta_{T-2\delta}, e^{-LH_{P,T-2\delta}}\eta_{T-2\delta}) \\ &\quad \cdot (\eta_{T-2\delta}^0, e^{-LH_{P,T-2\delta}^0}\eta_{T-2\delta}^0)^{-1}. \end{aligned} \tag{3.21}$$

If we estimate this in a way analogous to (3.20) we obtain

$$\begin{aligned} \operatorname{Tr} e^{-(T-2\delta)H_{D,L}} &\leqq \|\eta_{T-2\delta}(\delta)\|^2 \\ &\quad \cdot \operatorname{Tr} e^{-(L-2\delta)H_{P,T-2\delta}^0} Z_{P;L-2\delta,T-2\delta}. \end{aligned} \tag{3.22}$$

Finally, using (3.20) once more for the rectangle with sides L, δ we obtain

$$\begin{aligned} \|\eta_L(\delta)\|^2 &= Z_{D,L,\delta}(\eta_L^0, e^{-2\delta H_{D,L}^0}\eta_L^0) \\ &\leqq \|\eta_\delta(\delta)\|^2 \operatorname{Tr} e^{-(L-2\delta)H_{D,\delta}}(\eta_L^0, e^{-2\delta H_{D,L}^0}\eta_L^0), \end{aligned} \tag{3.23}$$

where we also again used Nelson's symmetry. Now by standard properties of the trace [used already under a)]

$$\operatorname{Tr} e^{-(L-2\delta)H_{D,\delta}} \leqq (\operatorname{Tr} e^{-\delta H_{D,\delta}})^{\frac{L-2\delta}{\delta}} \tag{3.24}$$

and by the explicit formula (3.17) it is seen that

$$(\eta_L^0, e^{-2\delta H_{D,L}^0}\eta_L^0) = e^{0(L)} \tag{3.25}$$

[13] so that

$$\|\eta_L(\delta)\|^2 \leqq \|\eta_\delta(\delta)\|^2 c_1 e^{c_2 L} \tag{3.26}$$

with some constants c_1, c_2. Inserting (3.26) together with (3.22) into (3.20) gives

$$Z_{D,\Lambda} \leqq Z_{P,\tilde{\Lambda}} c_1^2 e^{c_2(L+T)}$$

which is (3.19).

This completes the proof of Theorem 3.2 □

4. Thermodynamic Limit and Osterwalder-Schrader Axioms

In this section we first prove some properties of the finite volume theory, namely Euclidean covariance (that is independence of what was called the 1 direction in the cutoff for A_μ) and bounds on expectations of observables that are independent of the volume.

The correlation inequalities of [1] are then used to construct a unique infinite volume limit of the Half-Dirichlet expectations of a certain class of observables. The verification of the Osterwalder-Schrader axioms [18] for the Schwinger functions of $:|\phi|^2:$ and $F\equiv\varepsilon_{\mu\nu}F_{\mu\nu}$ follows as a corollary.

1. Euclidean Covariance

In the stability expansion we used a sequence of cutoff covariances with Fourier transforms

$$\hat{D}^{t,0}_{\mu\nu}(p)=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2+\mu^2}\right)\frac{1}{p^2+\mu^2}e^{-t(p_1^2+\mu^2)}. \tag{4.1}$$

We want to show that we obtain the same expectations in the limit $t\to 0$ if we use instead

$$\hat{D}^{t,\theta}_{\mu\nu}(p)=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2+\mu^2}\right)\frac{1}{p^2+\mu^2}e^{-t((p_1\cos\theta+p_2\sin\theta)^2+\mu^2)}. \tag{4.2}$$

This implies in particular Nelson's symmetry as used in the previous section. As in Sect. 2 we consider unnormalized expectations of an observable $P(A,\phi)$. Let us denote by $Z_{t,\theta}\langle P\rangle_{t,\theta}$ the unnormalized expectation with respect to the measure using (4.2) for the A-covariance. Then

Theorem 4.1. $\lim\limits_{t\to 0}(Z_{t,\theta}\langle P\rangle_{t,\theta}-Z_{t,0}\langle P\rangle_{t,0})=0.$

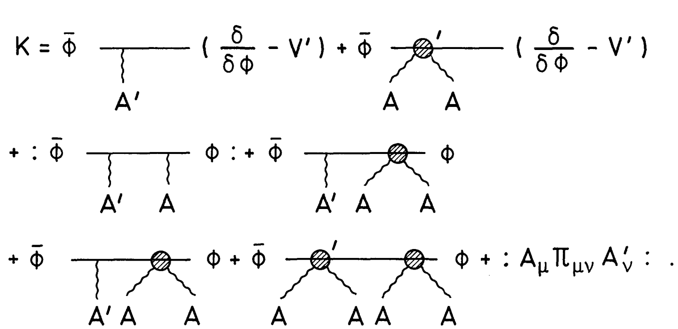
Proof. We introduce an interpolating covariance

$$sD^{0,t}_{\mu\nu}+(1-s)D^{\theta,t}_{\mu\nu}\equiv D_{\mu\nu}(s). \tag{4.3}$$

and use the fundamental theorem of calculus, the change of covariance formula (see also [2]) and an integration by parts to obtain, exactly as in Lemma 2.1

$$Z_{t,\theta}\langle P\rangle_{t,\theta}-Z_{t,0}\langle P\rangle_{t,0}=\int_0^1 ds\int dm(A)\int d\mu_{A(s)}KP, \tag{4.4}$$

where K is given by an expression analogous to (2.3), i.e.,



The prime now means $\frac{d}{ds}$ or $\frac{\delta}{\delta\phi}$; $A(s)$ is the Gaussian random field with covariance (4.3). (4.4) is the unnormalized expectation of a new "observable" KP. So if we do a stability expansion for this expectation we obtain a bound, as in Sect. 2 (Proposition 2.16), of the form

$$\text{const}\left\{\sup_k \|\bar{s}_k(C_0^{-\varepsilon}):KP:\|_{2,C_0} + \sup_k \|\bar{s}_k(C_0^{-\varepsilon}):t_0 KP:\|_{2,C_0}\right\}. \tag{4.5}$$

We claim that this goes to 0 as $t\downarrow 0$. The reason is that (4.5) gives rise to a number of Feynman graphs with good power counting, but at least one of of the photon lines is $\frac{d}{ds}D_{\mu\nu}(s)$ which is the Fourier transform of

$$\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2+\mu^2}\right)\frac{e^{-t\mu^2}}{p^2+\mu^2}\left(e^{-tp_1^2} - e^{-t(p_1\cos\theta + p_2\sin\theta)^2}\right)$$

and it is easy to see that

$$|e^{-tp_1^2} - e^{-t(p_1\cos\theta+p_2\sin\theta)^2}| \leqq (tp^2)^\alpha \tag{4.6}$$

for $1 \geqq \alpha \geqq 0$, (consider separately the cases $tp^2 \geqq 1$, $tp^2 < 1$). Choosing $\alpha > 0$ small enough, so that the power counting of the Feynman graphs is still good, we see that (4.5) goes to 0 as t^α. □

2. Volume Independent Bounds

There is a well known machine for establishing such bounds [19] based on the chessboard estimate (see [19] and [1]). There are, however, a few extra subtleties in our case.

We define the norm

$$|||f||| = \sum_i \left(\int \chi_{\Delta_i} f^2\right)^{1/2}$$

on measurable functions on $\mathbb{R}^2$. $\{\Delta_i\}$ is a set of disjoint open unit cubes filling $\mathbb{R}^2$.

Theorem 4.2. *Let $f \in L^2(\mathbb{R}^2)$, g, with $|||g||| < \infty$, be real or complex valued and supported in Λ. Then for $X = P$ (periodic* b.c.*) or $X = D$ (half-Dirichlet* b.c.*)*

$$\langle e^{F(f) + :|\phi|^2:(g)}\rangle_{X,\Lambda} \leqq e^{1/2\|f\|_2^2 + a|||g|||^2},$$

where a is some (Λ independent) constant.

Proof. Without loss we may assume f, g real valued. By the infrared bound of [1] (Theorem 4.3 and remarks following it) which carries over to the continuum,

$$\langle e^{F(f) + :|\phi|^2:(g)}\rangle_{X,\Lambda} \leqq e^{1/2\|f\|_2^2}\langle e^{:|\phi|^2:(g)}\rangle_{X,\Lambda}$$

so that we may assume $f = 0$. By the correlation inequalities of [1] [Theorem 6.2, (1) and (3)]

$$\langle e^{:|\phi|^2:(g)}\rangle_{D,\Lambda} \leqq \langle e^{:|\phi|^2:(g)}\rangle_{P,\Lambda} \tag{4.7}$$

if $g \geqq 0$

$$\langle e^{:|\phi|^2:(g)}\rangle_{D,\Lambda} \geqq \langle e^{:|\phi|^2:(g)}\rangle_{P,\Lambda} \tag{4.8}$$

if $g \leqq 0$.

So if $g=g_+-g$; $g_+, g_-\geqq 0$

$$\langle e^{:|\phi|^2:(g)}\rangle_{X,\Lambda}\leqq\langle e^{2:|\phi|^2:(g_+)}\rangle_{P,\Lambda}^{1/2}\langle e^{-2:|\phi|^2:(g_-)}\rangle_{X,\Lambda}^{1/2} \tag{4.9}$$

for $X=P,D$.

We claim that

$$\langle e^{2:|\phi|^2:(g_+)}\rangle_{P,\Lambda}\leqq e^{\int(\alpha_{P,\Lambda}(2g_+(x))-\alpha_{P,\Lambda}(0))dx}, \tag{4.10}$$

where $\alpha_{P,\Lambda}(\sigma)\equiv\frac{1}{|\Lambda|}\log Z_{P,\Lambda}(\sigma)$, $Z_{P,\Lambda}(\sigma)$ is the periodic partition function with the action modified by replacing m^2 by $m^2-2\sigma$. (4.10) is a standard application of the chessboard bound (see [14]).

We now claim:

$$\alpha_{P,\Lambda}(\sigma)-\alpha_{P,\Lambda}(0)\leqq a\sigma^2+b|\sigma| \tag{4.11}$$

with constants a,b that are independent of Λ (for $L,T\geqq 1$). This may be seen as follows:

Firstly for fixed Λ

$$\alpha_{P,\Lambda}(\sigma)\leqq a_\Lambda\sigma^2+c_\Lambda. \tag{4.12}$$

This follows essentially from Proposition 2.7. It has to be remembered, however, that σ will also enter the graphs of the stability expansion for $Z_{P,\Lambda}(\sigma)$ $\equiv\exp\{\Lambda\alpha_{P,\Lambda}(\sigma)\}$. This dependence on σ can be tracked from Proposition 2.16 (where it occurs in a_i) and bounded by including an extra factor σ^k on the right hand side of 2.8 and then 4.12 follows from Propositions 2.3 and 2.7.

Secondly, as already used in Sect. 3

$$\alpha_{P,\Lambda}(\sigma)+\frac{1}{LT}\log \mathrm{Tr} e^{-TH^0_{P,L}}$$

is decreasing in T and

$$\alpha_{P,\Lambda}(\sigma)+\frac{1}{LT}\log \mathrm{Tr} e^{-LH^0_{P,T}}$$

is decreasing in L, hence for L, $T\geqq 1$

$$\alpha_{P,\Lambda}(\sigma)\leqq\alpha_{P,\Delta}+\mathrm{const}. \tag{4.13}$$

(Δ is a unit square) because $0\leqq\frac{1}{LT}\log \mathrm{Tr}\, e^{-TH^0_{P,L}}\leqq\mathrm{const}$ for L, $T\geqq 1$ (Sect. 3).

Equation (4.13) shows that in (4.12) a_Λ, c_Λ may be chosen independent of Λ.

Finally we use the fact that $\alpha_{P,\Lambda}(\sigma)$ is convex in σ (which is well known and easy to prove). This implies that for $|\sigma|\leqq 1$

$$\alpha_{P,\Lambda}(\sigma)-\alpha_{P,\Lambda}(0)\leqq|\sigma|(\alpha_{P,\Lambda}(1)-\alpha_{P,\Lambda}(0))$$

or, using Theorem 3.2

$$\alpha_{P,\Lambda}(\sigma)-\alpha_{P,\Lambda}(0)\leqq\mathrm{const}|\sigma| \tag{4.14}$$

for $|\sigma|\leqq 1$ (const independent of Λ). Combining this with (4.12) (a_Λ, c_Λ independent of Λ) gives (4.11).

Now we can insert (4.11) into (4.10) to obtain

$$\langle e^{2:|\phi|^2:(g_+)}\rangle_{X,\Lambda} \leqq e^{a|||g_+|||^2}. \tag{4.15}$$

Note that the $|||\ |||$ norm dominates $\|\ \|_2$ and $\|\ \|_1$.

It remains to estimate

$$\langle e^{-2:|\phi|^2:(g_-)}\rangle_{X,\Lambda}.$$

We claim

$$\langle e^{-2:|\phi|^2:(g_-)}\rangle_{X,\Lambda} \leqq e^{a|||g_-|||^2} \tag{4.16}$$

for some (new) constant a.

Because of (4.8) we only have to consider $X=D$. We decompose $\mathbb{R}^2$ into the unit squares Δ_i and write

$$g_- = \sum_{i=1}^{\infty} \chi_{\Delta_i} g_-. \tag{4.17}$$

Now let $\{p_i\}_{i=1}^{\infty}$ be the sequence of (possibly infinite) numbers $|||g_-|||/|||g_-\chi_{\Delta_i}|||$. Obviously

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = 1, \qquad p_i \geqq 1 \qquad (i=1,2,\ldots). \tag{4.18}$$

So we may use Hölder's inequality to deduce

$$\langle e^{-2:|\phi|^2:(g_-)}\rangle_{D,\Lambda} \leqq \prod_{i=1}^{\infty} \langle e^{-2p_i:|\phi|^2:(\chi_{\Delta_i} g_-)}\rangle_{D,\Lambda}^{1/p_i}. \tag{4.19}$$

By a correlation inequality of ([1], Corollary 6.3, (2)) we may replace Λ by Δ_i on the right hand side of (4.19):

$$\langle e^{-2:|\phi|^2:(g_-)}\rangle_{D,\Lambda} \leqq \prod_{i=1}^{\infty} \langle e^{-2p_i:|\phi|^2:(\chi_{\Delta_i} g_-)}\rangle_{D,\Delta_i}^{1/p_i}. \tag{4.20}$$

From the stability expansion we can deduce (see below)

$$\langle e^{-2p:|\phi|^2:(\chi_{\Delta_i} g_-)}\rangle_{D,\Delta_i}^{1/p_i} \leqq \exp O(p_i \|\chi_{\Delta_i} g_-\|_2^2). \tag{4.21}$$

Now

$$\sum_{i=1}^{\infty} p_i \|\chi_{\Delta_i} g_-\|_2^2 = |||g_-|||^2 \tag{4.22}$$

so that (4.20) and (4.21) imply our claim (4.16).

We add a few remarks about the proof of (4.21):

We consider the term $p_i:|\phi|^2:(\chi_{\Delta_i} g_-)$ as part of the interaction V. This produces changes in Lemma 2.5 (and Proposition 2.7) as well as the graphs of the stability expansion.

The appropriate generalization of Proposition 2.7 is

Proposition 2.7. *Let*

$$V=\lambda\int :|\phi|^4: d^2x-\alpha\int_{\Lambda} :|\phi|^2: d^2x+\int :|\phi|^2: g dx \qquad (g\geqq 0).$$

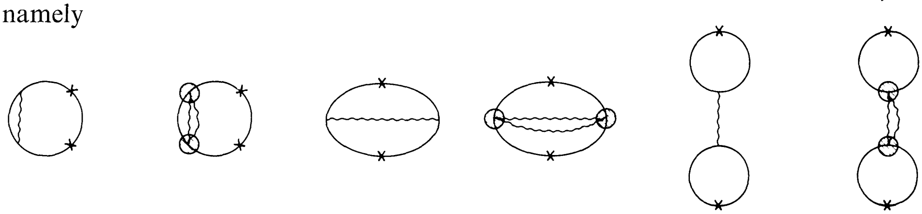
Then $\int d\nu_0 e^{-V}\leqq \exp[O(\alpha^2)+O(\|g\|_2^2)]$.

Proof. Using Schwarz's inequality this follows from Proposition 2.7 and

$$\int d\nu_0 e^{-2\int :|\phi|^2: g dx}\leqq \exp O(\|g\|_2^2). \tag{4.23}$$

(4.23) is true because Gaussian integration gives for the left hand side $[\det_2(1+4C^{1/2}gC^{1/2})]^{-1}$ which is bounded by $\exp O(\|C^{1/2}gC^{1/2}\|_{\mathrm{H.S.}}^2)$ by a well known determinant inequality (see for instance [20]) and $\|C^{1/2}gC^{1/2}\|_{\mathrm{H.S.}} \leqq \mathrm{const}\,\|g\|_2$.

In the stability expansion there will be some extra graphs involving $\chi_{\Lambda_i} g_-$, namely



where —×— stands for $p_i\chi_{\Lambda_i} g_i$ [the last line of (2.45)]. It is easy to see that these graphs may be bounded by $p_i^2\|\chi_{\Lambda_i} g_-\|_2^2$ times some other graphs with good power counting, and this dependence is also majorized by the factor $\exp O(\|p_i\chi_{\Lambda_i} g_-\|_2^2)$ in the bound for $\langle e^{-2p_i:|\phi|^2:(\chi_{\Lambda_i} g_-)}\rangle_{D,\Lambda_i}$.

This completes the proof of Theorem 4.2. □

Corollary 4.3. *For* $f_i\in L^2(\mathbb{R}^2)$, g_k *with* $|||g_k|||<\infty$, $\mathrm{supp} f_i\subset\Lambda$, $\mathrm{supp} g_k\subset\Lambda$ $(i=1,\dots,n;$ $k=1,\dots,m)$

$$\left\langle \prod_{i=1}^n F(f_i)\prod_{k=1}^m :|\phi|^2:(g_k)\right\rangle_{X,\Lambda}\leqq C^{n+m}(n!)^{1/2}(m!)^{1/2}\times\prod_{i=1}^n\|f_i\|\prod_{k=1}^m |||g_k|||.$$

Proof. This is a standard consequence of Theorem 4.2 which follows by a Cauchy estimate. □

3. Infinite Volume Limit and Osterwalder-Schrader Axioms

In [1] it was shown that for $g\geqq 0$

$$\langle e^{-:|\phi|^2:(g)}e^{F(f)}\rangle_{D,\Lambda}$$

is decreasing in Λ (this follows directly from Corollary 6.3 of [1] by taking the continuum limit). A simple and standard consequence is

Theorem 4.4. *For an arbitrary sequence of rectangles* $\Lambda_n\uparrow\mathbb{R}^2$; $f,g\in\mathscr{S}(\mathbb{R}^2)$

$$\lim_{n\to\infty}\langle e^{-:|\phi|^2:(g)+F(f)}\rangle_{D,\Lambda}\equiv\langle e^{-:|\phi|^2:(g)+F(f)}\rangle$$

exists and is independent of the sequence (Λ_n).

Corollary 4.5. $\langle e^{-:|\phi|^2:(g)+F(f)}\rangle$ *is Euclidean invariant.*

Corollary 4.6. *The infinite volume Schwinger functions*

$$\left\langle \sum_{k=1}^{m} :|\phi|^2:(g_k) \prod_{i=1}^{n} F(f_i)\right\rangle$$

obey all the Osterwalder-Schrader axioms except possibly clustering.

The proof of the theorem follows from the uniform bounds of Theorem 4.2 and the monotonicity result quoted above together with Vitali's theorem by well known arguments (see [21]).

Writing $g=g_+-g_-$, $g_+, g_-\geqq 0$ we see that

$$\{F_n(\xi_1,\xi_2,\xi_3)\}\equiv\{\langle e^{\xi_1 F(f)+\xi_2:|\phi|^2:(g_+)-\xi_3:|\phi|^2:(g_-)}\rangle_{D,\Lambda_n}\}$$

is a normal family of entire functions that converges as $n\to\infty$ for $\operatorname{Im}\xi_1=0$, $\xi_2\leqq 0$, $\xi_3\geqq 0$, therefore by Vitali's theorem for $\xi_1, \xi_2, \xi_3\in\mathbb{C}$ uniformly on compacts to a limit independent of the sequence $\{\Lambda_n\}$. Corollary 4.5 follows then from this independence of the sequence $\{\Lambda_n\}$ and Theorem 4.1. Corollary 4.6 is also a standard consequence:

Temperedness is Corollary 4.3 (the restriction on the supports of f_i, g_k can be eliminated by a density argument);

Symmetry is trivial;

Euclidean invariance has just been proven;

Osterwalder-Schrader positivity has been proven in [1] (Theorem 5.5) for the cutoff theory and carries over by taking limits. □

We close by remarking that we can also construct infinite volume expectations of so-called loop observables

$$e^{i\int_{\partial G} A_\mu dx^\mu}$$

and string observables

$$\bar{\phi}(x)e^{i\int_x^y A_\mu(x')dx'^\mu}\phi(y)$$

The loop observables have in fact already been constructed because $\int_{\partial G} A_\mu dx^\mu=\int_G Fd^2x$ (G a reasonable region). The string observables can be treated by methods analogous to the ones above, i.e., chessboard bounds and pressure estimates coming from the stability expansion. We leave the details to the reader.

5. The Limit $\mu^2\to 0$

Here again the correlation inequalities of [1] come in handy. It was proven there [Corollary 6.3 (1)] that for f, g real,

$$\langle e^{-:|\phi|^2:(g)+F(f)}\rangle \quad (g\geqq 0) \tag{5.1}$$

is increasing and

$$\langle e^{:|\phi|^2:(g)+iF(f)}\rangle \quad (g\geqq 0) \tag{5.2}$$

is decreasing in the covariance of the measure $dm(A)$ (this requires of course to choose the counterterm δm^2 independent of the covariance). Therefore these expressions will have limits as $\mu^2 \to 0$ (because as μ^2 decreases the transverse part of the A-covariance increases; the longitudinal part is irrelevant because of Ward identities), provided there is a uniform upper bound on (5.1). It suffices to prove such an upper bound for

$$\langle e^{-:|\phi|^2:(g)} \rangle_{D,\Lambda_0} \quad (g \geqq 0), \tag{5.3}$$

where Λ_0 is some suitably chosen rectangle, because by the infrared bound of [1] (Theorem 4.3)

$$\langle e^{-:|\phi|^2:(g)+F(f)} \rangle_{D,\Lambda} \leqq e^{1/2\|f\|_2^2} \langle e^{-:|\phi|^2:(g)} \rangle_{D,\Lambda}$$

and $\langle e^{-:|\phi|^2:(g)} \rangle_\Lambda$ decreases in Λ (for $g \geqq 0$) by correlation inequalities, as noted in the previous section.

Not surprisingly, an upper bound on (5.3) independent of μ^2 can be proven by the stability expansion. There is a subtlety, however, because

$$C_{\lambda\nu}(x;\mu^2) \equiv \left(\frac{1}{2\pi}\right)^2 \int e^{-ipx} \frac{\delta_{\lambda\nu} - \dfrac{p_\lambda p_\nu}{p^2+\mu^2}}{p^2+\mu^2} d^2p$$

diverges as $\mu^2 \to 0$.

In order to get an upper bound on expectations we need an upper bound on unnormalized expectations and a lower bound on the partition function. The upper bound is easy, since

$$\begin{aligned} &\langle e^{-:|\phi|^2:(g)} \rangle_{D,\Lambda} Z_{D,\Lambda} \\ &= \int e^{-:|\phi|^2:(g)} d\mu_{A,\Lambda,D}(\phi) dm C_{\mu\nu}(A) \end{aligned}$$

is decreasing when the covariance $C_{\mu\nu}$ of $dmC_{\mu\nu}(A)$ is increasing ([1], Corollary 4.2; we should regard $:|\phi|^2:(g)$ as part of the interaction because Corollary 4.2 is stated for partition functions).

It only remains to show that for a suitable Λ_0

$$Z_{D,\Lambda_0} \geqq \varepsilon > 0, \tag{5.4}$$

where ε is independent of μ^2 for, say, $\mu^2 \leqq 1$.

There are three principles that facilitate the proof of (5.4). Denote by $Z_{D,\Lambda_0}(C_{\mu\nu})$ the partition function with half-Dirichlet b.c. and A-covariance $C_{\mu\nu}$.

(a) $Z_{D,\Lambda_0}(C_{\mu\nu})$ is decreasing if the covariance $C_{\mu\nu}$ of the Gaussian measure $dmC_{\mu\nu}(A)$ is increasing ([1], Corollary 4.2).

(b) $Z_{D,\Lambda_0}(C_{\mu\nu})$ depends only on the values of $C_{\mu\nu}(x)$ for $x \in \Lambda_0$.

(c) $Z_{D,\Lambda_0}(C_{\mu\nu})$ does not change if $C_{\mu\nu}(x,y)$ is replaced by $C_{\mu\nu}(x,y) + \alpha\delta_{\mu\nu}$. (a) and (b) are clear; (c) follows from gauge invariance if we note that the change

$$A_\mu \to A_\mu + \sqrt{\alpha}\,\partial_\mu(c \cdot x), \tag{5.5}$$

where (c_1, c_2) is a pair of independent centered normalized Gaussian random variables just produces the desired change of covariance, by (b) the function $(c \cdot x)$

may be cut off outside Λ_0. Now let

$$\Lambda_0 = \left\{(x,y)\in \mathbb{R}^2 \,\Big|\, |x| < \frac{L}{2}, |y| < \frac{T}{2}\right\},$$

and let Δ_N be the Laplacean with Neumann b.c. on $\partial(2\Lambda_0)$; P the projection in $L^2(\mathbb{R}^2)$ on the orthogonal complement of the null space of Δ_N. From gauge invariance and (a), (b), (c) we obtain the following string of (in)equalities:

$$\begin{aligned} Z_{D,\Lambda_0}(C_{\mu\nu}) &= Z_{D,\Lambda_0}(\delta_{\mu\nu}(-\Delta+\mu^2)^{-1}) \geqq Z_{D,\Lambda_0}(\delta_{\mu\nu}(-\Delta_N+\mu^2)^{-1}) \\ &= Z_{D,\Lambda_0}(\delta_{\mu\nu}(-\Delta_N+\mu^2)^{-1}P) \quad \text{[by (c)]} \\ &\geqq Z_{D,\Lambda_0}(\delta_{\mu\nu}(-\Delta_N)^{-1}P) \\ &= Z_{D,\Lambda_0}((\delta_{\mu\nu}+(-\Delta_N+1)^{-1}\partial_\mu\partial_\nu)(-\Delta_N)^{-1}P). \end{aligned} \tag{5.6}$$

The last expression contains a covariance that is already suitable for the stability expansion. It might be somewhat easier instead to use the bound

$$\chi_{\Lambda_0}(-\Delta_N)^{-1}P\chi_{\Lambda_0} \leqq \chi_{\Lambda_0}(-\Delta+1)^{-1}\chi_{\Lambda_0} + c\chi_{\Lambda_0}(-\Delta+1)^{-2}\chi_{\Lambda_0}. \tag{5.7}$$

With this bound we obtain from (5.6) [using (b), (c) and gauge invariance]:

$$\begin{aligned} Z_{D,\Lambda_0}(C_{\mu\nu}) &\geqq Z_{D,\Lambda_0}(\delta_{\mu\nu}(-\Delta+1)^{-1} + c(-\Delta+1)^{-2}) \\ &= Z_{D,\Lambda_0}((\delta_{\mu\nu}+(-\Delta+1)^{-1}\partial_\mu\partial_\nu)(-\Delta+1)^{-1} + c\delta_{\mu\nu}(-\Delta+1)^{-2}). \end{aligned} \tag{5.8}$$

The last expression contains a covariance that is, up to more regular terms, identical to the one used in Sect. 2. The bound $Z_{D,\Lambda_0}(C_{\mu\nu}) \geqq \varepsilon > 0$ follows now as in Sect. 3. The bound (5.7) is not very hard to prove: By explicit diagonalization

$$(-\Delta_N)^{-1}P \leqq (-\Delta_N+1)^{-1} + c(-\Delta_N+1)^{-2} \tag{5.9}$$

with some constant c that is uniform for $\Lambda_0 \subseteqq \Delta$, Δ a unit square

$$(-\Delta_N+1)^{-1} = (-\Delta+1)^{-1} + R, \tag{5.10}$$

where R has a kernel that is C^∞ in $\frac{3}{2}\Lambda_0$ [possible singularities lie on $\partial(2\Lambda_0)$]. Integration by parts shows that for $\phi\in L^2$, $\operatorname{supp}\phi \subset \Lambda_0$

$$(\phi, R\phi) \leqq \text{const}\, \|(-\Delta+1)^{-1}\phi\|_2^2$$

or

$$\chi_{\Lambda_0} R \chi_{\Lambda_0} \leqq \text{const}\,(-\Delta+1)^{-2}. \tag{5.11}$$

Similarly

$$(-\Delta_N+1)^{-2} = (-\Delta+1)^{-2} + \tilde{R} \tag{5.12}$$

with a $\tilde{R}$ that has a kernel that is smooth in $\frac{3}{2}\Lambda_0$; therefore again

$$\chi_{\Lambda_0} R \chi_{\Lambda_0} \leqq \text{const}\,(-\Delta+1)^{-2}. \tag{5.13}$$

(5.9)–(5.13) obviously imply (5.7). The fact that there are no infrared divergences, as shown in this subsection, may be taken to be a hint of mass generation by the

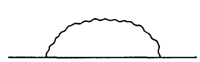
Higgs mechanism; note, however, that we did not use any special "double hump" form of the potential. To really show the existence of a mass gap will require the use of expansion methods. But we want to stress that we have here another instance in which Counstructive Quantum Field Theory shows its aptness at dealing with mass zero situations that are tricky in perturbation theory; earlier examples are the Sine-Gordon theory [22] (bare mass zero) and the critical $P(\phi)_2$ theory [23] (physical mass zero).

Appendix

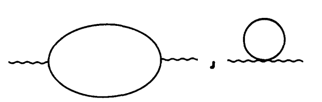
Estimation of Feynman Graphs

In this Appendix we prove the estimates on Feynman graphs that are used in Sect. 2 to prove convergence of the stability expansion; see the end of that section for a list of the graphs in question. The Appendix is organized as follows: First we sketch how to estimate the graphs corresponding to periodic and mixed (free-half-Dirichlet) boundary conditions in terms of graphs which can be written in momentum space with continuous momentum (momentum is discrete for periodic b.c.'s) and momentum conservation at vertices. The main part of the appendix proves a power counting lemma for graphs of this type, making use of the machinery developed by Nakanishi [24] and Speer [25].

At the outset we need to make it clear that our estimates as stated will only be finite when applied to graphs with the property that every subgraph is convergent according to "power counting", i.e. the quantity $\tilde{K}(G)$ defined in Lemma A.4 and (A.17) below must be strictly negative. In our stability expansion we have three graphs or subgraphs which violate this condition, namely



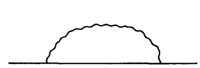
and



This pair is always to be added together. The result, denoted $\Pi_{\mu\nu}$, has been discussed in Appendices A and B of [2], to which the reader is referred for the estimate

$$\Pi_{\mu\nu}(k^2) = O(\log k^2)$$

by which the graphs in (2.46) can be estimated directly. The first graph is finite only because we work in a gauge wherein the A propagator is approximately transverse. To see it is finite one can use the principle of "shifting" derivatives which the reader will find described under (1) (b) below. We leave it to the reader to verify that all graphs occurring in (2.40)–(2.47) which have



as a subgraph are convergent according to naive power counting after the derivatives have been shifted. We require this operation to be performed on all such subgraphs before applying the estimates described below.

1. Reduction to Standard Feynman Graphs

a) Periodic boundary conditions: Here the momentum space Feynman integrals ([2], Sect. VI) become sums; we will, however, still interpret them as integrals where the momentum space covariances (= propagators) and factors of p coming from derivative couplings are replaced by piecewise constant functions.

To be more specific, let Λ be the rectangle $\{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1|<1/2a_1, |x_2|<1/2a_2\}$. Furthermore let χ be the characteristic function of the interval $[-1/2, 1/2]$. We then replace the Fourier transform of the periodic covariance of the matter ("Higgs") field by

$$\tilde{C}_p(p)=\frac{1}{(2\pi)^2}\sum_{(n_1,n_2)\in\mathbb{Z}^2}\chi\left(\frac{a_1p_1}{2\pi}-n_1\right)\chi\left(\frac{a_2p_2}{2\pi}-n_2\right) \cdot\left[\left(\frac{2\pi n_1}{a_1}\right)^2+\left(\frac{2\pi n_2}{a_2}\right)^2+m^2\right]^{-1}. \tag{A.1}$$

We also have to consider photon lines with t-cutoffs; for periodic b.c. we may replace their Fourier transforms by

$$\tilde{C}^t_{\lambda\nu,p}(p)=\frac{1}{(2\pi)^2}\sum_{(n_1,n_2)\in\mathbb{Z}^2}\chi\left(\frac{a_1p_1}{2\pi}-n_1\right)\chi\left(\frac{a_2p_2}{2\pi}-n_2\right) \cdot P_{\lambda\nu}\left[\left(\frac{2\pi n_1}{a_1}\right)^2+\left(\frac{2\pi n_2}{a_2}\right)^2+\mu^2\right]^{-1} e^{-t\left(\left(\frac{2\pi n}{a_2}\right)^2+\mu^2\right)}, \tag{A.2}$$

where

$$P_{\lambda\nu}=\delta_{\lambda\nu}-4\pi^2\frac{n_\lambda}{a_\lambda}\frac{n_\nu}{a_\nu}\left[\left(\frac{2\pi n_1}{a_1}\right)^2+\left(\frac{2\pi n_2}{a_2}\right)^2+\mu^2\right]^{-1} \tag{A.3}$$

in a similar way we also make factors of p piecewise constant. Obviously the periodic expressions (A.1), (A.2) differ from their free analogs only by a shift in the arguments; the shifts are at most $\frac{2\pi}{a_1}$ in p_1 and $\frac{2\pi}{a_2}$ in p_2.

The graphs we have to estimate also contain photon lines differentiated with respect to an interpolation parameter s_l, therefore we also have to compare $\tilde{C}^t_{\lambda\nu,p}-\tilde{C}^{t'}_{\lambda\nu,p}$ with the corresponding "free" expressions.

The relevant bounds are contained in

Lemma A.1. *For* $0\leqq t\leqq 1$

(1) $\tilde{C}_p(p)\leqq \text{const}\,\hat{C}(p)=\text{const}\dfrac{1}{(2\pi)^2}\dfrac{1}{p^2+m^2}$.

(2) $\tilde{C}^t_p(p)-\tilde{C}^{t'}_p(p)\leqq \text{const}\,(\hat{C}^t-\hat{C}^{t'})(p)\,(t\leqq t')$ *where we (fudging the difference between* μ *and* m*) put* $\tilde{C}^t_{\mu\nu,p}\equiv P_{\lambda\nu}\tilde{C}^t_p$ *etc.*

Remark. The constants in this lemma depend on a_1, a_2.

This lemma is a direct consequence of

Proposition A.2. *For* $0 \leqq t \leqq 1$ *and*

$$|\delta_\mu| \leqq \frac{2\pi}{a_\mu} (\mu = 1, 2).$$

(1) $\hat{C}(p+\delta) \leqq \text{const}\, \hat{C}(p)$.

(2) $e^{-t((p_1+\delta_1)^2+\mu^2)} - e^{-t'((p_1+\delta_1)^2+\mu^2)}$
$\leqq \text{const}\,(e^{-1/2t(p_1^2+\mu^2)} - e^{-1/2t'(p_1^2+\mu^2)})\,(t \leqq t')$.

(3) $e^{-t((p_1+\delta_1)^2+\mu^2)} \leqq \text{const}\, e^{-1/2t(p_1^2+\mu^2)}$.

Proof. (1) follows from the fundamental theorem of calculus and the fact that the logarithmic derivative of the right hand side is uniformly bounded.

(3) follows from the obvious fact that for $0 \leqq t \leqq 1$

$$t((p_1+\delta_1)^2+\mu^2) - \frac{t}{2}(p_1^2+\mu^2)$$

is bounded below by a constant independent of t and p_1.

(2) can be seen as follows: The left hand side is

$$((p_1+\delta_1)^2+\mu^2) \int_t^{t'} d\tau e^{-\tau((p_1+\delta)^2+\mu^2)}$$

$$\leqq 2\frac{(p_1+\delta_1)^2+\mu^2}{p_1^2+\mu^2}(e^{-1/2t(p_1^2+\mu^2)} - e^{-1/2t'(p_1^2+\mu^2)}),$$

where we used (3) and the fundamental theorem of calculus. $\frac{(p_1+\delta_1)^2+\mu^2}{p_1^2+\mu^2}$ is bounded uniformly, as can be seen from the fact that $\left|\frac{\partial}{\partial p_1}\log(p_1^2+\mu^2)\right| \leqq \text{const}$. □

Lemma A.1 shows that any absolutely convergent periodic Feynman graph may be estimated in terms of a free one with half the value of the t-cutoff. Vacuum graphs are automatically proportional to the volume $|\Lambda|$ and this property is preserved by the estimate that replaces periodic by free propagators.

b) Mixed (free-half-Dirichlet) boundary conditions: If there were no derivative couplings in the model, we could eliminate the Dirichlet b.c. simply by the remark that

$$C_D \leqq C_{F,\Lambda} \equiv \chi_\Lambda C_F \chi_\Lambda \tag{A.4}$$

both in the pointwise sense for the kernels and in the sense of quadratic forms (C_F is the covariance with free b.c.).

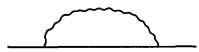
To deal with the derivative couplings we need in addition

$$\|\partial_\mu C_D^{1/2}\| \leqq 1 \tag{A.5}$$

and

$$\|C_D^\alpha C_0^{-\alpha}\| \leqq 1 \quad \text{for} \quad 1 \geqq \alpha \geqq 0. \tag{A.6}$$

(A.5) follows from (A.4); (A.6) follows from (A.4) combined with the fact that operator inequalities such as (A.4) are preserved by the operation of taking fractional powers of both sides. To see that these three estimates suffice we have to make use of the "principle of shifting derivatives" through vertices which comes from the fact that we use a covariance $C_{\mu\nu}$ for A_μ that is "essentially transverse", i.e., $\partial \cdot A$ has a covariance that is so well behaved that it doesn't give rise to any divergent graphs. To see how this works, let us look at the following example, which is actually essential for the functioning of our stability proof: The graph



(A.7)

contains the somewhat dangerous looking expression

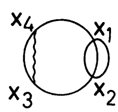
$$\int f(x)g(y)C_{\mu\nu}(x-y)\partial_{\mu,x}\partial_{\nu,y}C_D(x,y)dxdy\,.$$

However, integration by parts removes the derivatives from C_D and moves them onto the functions f, g, in addition producing some terms involving the harmless covariance of $(\partial \cdot A)$.

In short, the "shifting of derivatives" is nothing but the application of the trivial identity (to be read as an operator identity);

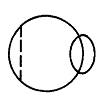
$$\partial_\mu A_\mu + A_\mu \partial_\mu = 2A_\mu \partial_\mu + (\partial \cdot A) = -(\partial \cdot A) + 2\partial_\mu A_\mu\,.$$

Let us apply this principle to the more complicated graph



.

Shifting the derivatives onto the horizontal lines produces the harmless graph:



– where ---- stands for the covariance of $\partial \cdot A$ – and the expression

$$\begin{aligned}&\int dx_1 \ldots dx_4 C_D(x_1,x_2)^3(\partial_\mu C_D(x_2,x_3))\\&(\partial_\nu C_D(x_1,x_4))C_D(x_3,x_4)C_{\mu\nu}(x_3-x_4)\end{aligned} \tag{A.8}$$

which can be interpreted as the trace of a product of 4 operators with kernels C_D^3, $\partial_\mu C_D$, $C_D C_{\mu\nu}$, $\partial_\nu C_D$. Now we can use (A.5), (A.6) and then (A.4) to bound (A.8) by

$$\left(\int_{A\times A} dx_1 dx_2 C(x_1-x_2)^6\right)^{1/2}\left(\int_{A\times A} dx_3 dx_4 C^2(x_3-x_4)C_{\mu\mu}^2(x_3-x_4)\right)^{1/2}$$

which has good power counting.

As a last example, which requires a slightly different argument we consider

$$\|C_D^{1/2}\{\partial, A\}C_D^{1/2}\|_4^4$$

[which gives rise to two of the graphs of (2.40)]. We claim that this again can be bounded by the analogous expression with free instead of Dirichlet boundary conditions:

$$\begin{aligned}\|C_D^{1/2}\{\partial,A\}C_D^{1/2}\|_4^4 &= \operatorname{Tr} C_D^{1/2}\{\partial,A\}C_D\{\partial,A\}C_D\{\partial,A\}C_D\{\partial,A\}C_D\{\partial,A\}C_D^{1/2}\\ &\leqq \operatorname{Tr} C_D^{1/2}\{\partial,A\}C_D\{\partial,A\}C_{F,A}\{\partial,A\}C_D\{\partial,A\}C_D^{1/2}\\ &= \operatorname{Tr} C_{F,A}^{1/2}\{\partial,A\}C_D\{\partial,A\}C_D\{\partial,A\}C_D\{\partial,A\}C_{F,A}^{1/2}\\ &\leqq \operatorname{Tr} C_{F,A}^{1/2}\{\partial,A\}C_D\{\partial,A\}C_{F,A}\{\partial,A\}C_D\{\partial,A\}C_{F,A}^{1/2}\\ &= \operatorname{Tr} C_D^{1/2}\{\partial,A\}C_{F,A}\{\partial,A\}C_D\{\partial,A\}C_{F,A}\{\partial,A\}C_D^{1/2}\\ &\leqq \operatorname{Tr} C_D^{1/2}\{\partial,A\}C_{F,A}\{\partial,A\}C_{F,A}\{\partial,A\}C_{F,A}\{\partial,A\}C_D^{1/2}\\ &\leqq \|C_{F,A}^{1/2}\{\partial,A\}C_{F,A}^{1/2}\|_4^4,\end{aligned}$$

where we used cyclicity of the trace and (A.4) four times in the quadratic form sense.

These fairly typical examples should suffice to indicate how all our graphs with Dirichlet lines may be estimated by similar ones containing only free propagators. We leave it to the reader to check that this can be done for all the graphs occuring in the list of Sect. 2.

There is another point, however, that has to be discussed: We estimated the mixed b.c. graphs in terms of graphs with free propagators and a volume cutoff χ_A at each vertex. Put differently, these graphs do not have momentum conservation at the vertices because χ_A acts like an external field. They correspond to expressions of the form

$$\begin{gathered}\int \hat{G}(P_1,\ldots,P_V)\hat{\chi}_A(P_1)\ldots\hat{\chi}_A(P_V)\\ \delta\left(\sum_{i=1}^{V} P_i\right)d^{2V}P,\end{gathered} \tag{A.9}$$

where $\hat{G}$ is the standard Feynman amplitude with external momenta $P_1,\ldots,P_V$ flowing in at the vertices.

In the case of periodic b.c. the estimates involved simply $\hat{G}(0,\ldots,0)|A|$. Here we use instead

Proposition A.3. $\left|\int \hat{G}(P_1,\ldots,P_V)\prod_{i=1}^{V}\hat{\chi}_A(P_i)\delta\left(\sum_{i=1}^{V} P_i d^{2V}P\right|=\text{const}\,\|\hat{G}\|_\infty|A|.$

Proof. It suffices to show that

$$\int \hat{\chi}(P_1)\ldots\hat{\chi}_A(P_V)\delta\left(\sum_{i=1}^{V} P_i\right)d^{2V}P=\text{const}\,|A|\,.$$

This follows from a simple scaling argument. □

The rest of this Appendix is concerned with estimating $\|\hat{G}\|_\infty$; the main result is contained in Lemma A.4 below.

2. Estimation of the Feynman Amplitude $\hat{G}(P_1, \ldots, P_V)$

For the sake of estimates we may eliminate all internal indices by Schwarz's inequality, replacing e.g.

$$\frac{\delta_{\lambda\nu} - (k^2+\mu^2)^{-1} k_\lambda k_\nu}{k^2+\mu^2} \quad \text{by} \quad \frac{2}{k^2+\mu^2}$$

or

$$\frac{P_v}{p^2+m^2} \quad \text{by} \quad \frac{1}{\sqrt{p^2+m^2}}.$$

In the course of the stability expansion we had to introduce lines corresponding to $C_0^{1+\varepsilon}$, $C_0^{1-\varepsilon}$; so we consider now more generally Feynman graphs composed of "Higgs lines" corresponding to C_0^α [or $(p^2+m^2)^{-\alpha}$ in momentum space] with $0<\alpha\leqq 1$ and "photon lines" with t-cutoffs, corresponding to either sums of terms of the form

$$\hat{C}_T^{\beta,t}(p) \equiv (p^2+\mu^2)^{-\beta}\left(e^{-T(p_1^2+\mu^2)} - e^{-t(p_1^2+\mu^2)}\right) (0\leqq T<t; 0<\beta\leqq 1) \qquad \text{(A.10)}$$

or

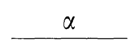
$$\hat{C}^{\beta,t}(p) \equiv -\hat{C}_\infty^{\beta,t}(p) = (p^2+\mu^2)^{-\beta} e^{-t(p_1^2+\mu^2)} \qquad \text{(A.11)}$$

depending on whether the photon line had a derivative with respect to an interpolation parameter s_i or not.

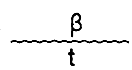
For the sake of estimates we may set $T=0$ in (A.10) and $t=0$ in (A.11). If we also assume for simplicity $\mu^2=m^2$ (obviously no real loss of generality) we are left with two kinds of lines: Higgs lines and undifferentiated photon lines corresponding to $\hat{C}^\alpha$ $(0<\alpha\leqq 1)$ and differentiated photon lines corresponding to

$$\hat{C}_t^{\beta,0} = (p^2+\mu^2)^{-\beta}\left(1-e^{-t(p_1^2+\mu^2)}\right). \qquad \text{(A.12)}$$

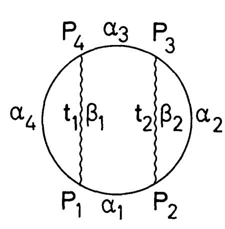
We represent now

$\hat{C}^\alpha$ by 

and

$\hat{C}_t^{\beta,0}$ by 

A typical graph would be for instance



where $P_1, \ldots, P_4$ are the momenta flowing into the graph at the four vertices.

We need a little bit of graph theory which can be found in the book by Nakanishi [24] (see also [25]).

A graph G is a collection of vertices $\{v_1, \ldots, v_V\}$ and lines $\{l_1, \ldots, l_L\}$ such that for each line l_k there is an initial vertex $v_i(k)$ and a final vertex $v_f(k)$ (we actually have two subsets of lines: $\{l_1, \ldots, l_p\}$ are the lines corresponding to $\hat{C}_t^{\beta, 0}$; $\{l_{p+1}, \ldots, l_L\}$ are the lines corresponding to $\hat{C}^\alpha$).

We have occasion to use Euler's formula [24]: If $C(G)$ is the number of connected components of G, $V(G)$, and $L(G)$ the number of vertices and lines, respectively, and $h(G)$ the number of independent loops (i.e., the first Betti number), then

$$h(G) = L(G) - V(G) + C(G). \tag{A.13}$$

If G is connected, a tree T in G is a subgraph that is connected, has $V(G)$ vertices, and has no loops ($h(T) = 0$). A tree contains $V(G) - 1$ lines.

We also need the concept of the *circuit matrix* $C = (c_{ik})$ of a graph. This is a $L(G) \times h(G)$ matrix defined as follows: Pick h independent (oriented) loops[1]. Then

$$c_{li} = \begin{cases} 1 & \text{if the } i\text{th loop contains line } l \text{ with positive orientation} \\ -1 & \text{negative orientation} \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define the momentum space amplitude corresponding to a graph G as follows: To each independent loop c_i we assign a loop momentum k_i, to each $\hat{C}_t^{\beta, 0}$ line l_i a momentum p_i $(i = 1, \ldots, p)$, to each $\hat{C}^\alpha$ line l_i a momentum p_i $(i = p+1, \ldots, L)$ and to each vertex v_i an external momentum (ingoing) $P_i (i = 1, \ldots, V)$ such that

$$p_r = \sum_{i=1}^{h} c_{ri} k_i + \sum_{m=1}^{V} \alpha_{rm} P_m \tag{A.14}$$

or in matrix notation

$$p = C_k + AP. \tag{A.15}$$

(We have changed our notation somewhat: We use now lower indices to label the different momenta and upper indices for the components.)

The matrix $A = (\alpha_{rm})$ is partially determined by the requirement that the sum of all line momenta and external momenta going into a vertex is zero (for details see [24]). The amplitude corresponding to the graph is then given by

$$\begin{aligned} &\hat{G}(P_1, \ldots, P_V; t_1, \ldots, t_p) \\ &\equiv \int \prod_{i=1}^{h(G)} d^2 k_i \prod_{i=p+1}^{L} (p_i^2 + m^2)^{-\alpha_i} \prod_{i=1}^{p} (p_i^2 + \mu^2)^{-\beta_i} \prod_{i=1}^{p} (1 - e^{-t(p_i^{(1)2} + \mu^2)}) \end{aligned} \tag{A.16}$$

Remark. When we do the stability expansion for the proof of Theorem 4.1 we will encounter graphs where $p_i^{(1)}$ is replaced by $\cos\theta p_i^{(1)} + \sin\theta p_i^{(2)}$ $(i = 1, \ldots, P)$; and $p_i^{(2)}$ is also rotated. A glimpse at (A.16) shows that there is no θ dependence, so we may safely replace θ by 0.

---

1 I.e., a basis of the first homology group of the graph

The naive power counting (usually called the "superficial divergence") $K(G)$ of the graph G is given by

$$K(G)=2h(G)-2\sum_{i=p+1}^{L}\alpha_i-2\sum_{i=1}^{p}\beta_i. \tag{A.17}$$

We can now state the main result of this Appendix:

Lemma A.4. *Let G be a connected graph. Assume that for each subgraph H of $G, K(H)<0$, and let $\tilde{K}(G)\equiv\sup_{H\subset G}K(H)$. Furthermore assume that* $\max(t_1,\dots,t_p$ $\leqq\min(t_1,\dots,t_p)\times\text{const}$, $0\leqq t_1,\dots,t_p<\frac{1}{2}$ *and* $\beta_i>0$ $(i=1,\dots,p)$, $\alpha_i>0$ $(i=p+1,\dots,L)$. *Then*

$$|\hat{G}(P_1,\dots,P_V;t_1,\dots,t_p)|\leqq\text{const}\,t_1^{1/2\varepsilon},$$

where $\varepsilon<-\frac{1}{2}\tilde{K}(G)$.

Proof of Lemma A.4. First we rewrite (A.12):

$$\hat{C}_t^{\beta,0}(p)=\frac{1}{\Gamma(\beta)}\int_0^t dt'\frac{\partial}{\partial t'}e^{-t'(p^{(1)2}+\mu^2)}\int_0^\infty du\,u^{\beta-1}e^{-u(p^2+\mu^2)} \tag{A.18}$$

and we also write

$$\hat{C}_0^{\alpha}(p)=\frac{1}{\Gamma(\alpha)}\int_0^\infty ds\,s^{\alpha-1}e^{-s(p^2+m^2)}. \tag{A.19}$$

Postponing the t', u, s integrations, we have to consider amplitudes where the differentiated photon lines are interpreted as

$$\frac{\partial}{\partial t_i'}e^{-t_i'(p_i^{(1)2}+\mu^2)-u_i(p_i^2+m^2)}\qquad(i=1,\dots,p) \tag{A.20}$$

and the remaining lines as

$$e^{-s(p_i^2+m^2)}\qquad(i=p+1,\dots,L). \tag{A.21}$$

The corresponding amplitude is

$$\begin{aligned}&H(P_1,\dots,P_V;t_1',\dots,t_p';u_1,\dots,u_p;s_{p+1},\dots,s_L)\\&\quad\equiv\prod_{i=1}^{p}\frac{\partial}{\partial t_i'}\int d^{2h(G)}k\,e^{-\sum_{r=p+1}^{L}s_r(p_r^2+\mu^2)}e^{-\sum_{r=1}^{p}u_r(p_r^2+\mu^2)}e^{-\sum_{r=1}^{p}t_r'(p_r^{(1)2}+\mu^2)}.\end{aligned} \tag{A.22}$$

Inserting (A.14) or (A.15) and using the Gaussian integration formula

$$\int d^n x\exp\{-\tfrac{1}{2}(\mathbf{x},A\mathbf{x})+\mathbf{y}\cdot\mathbf{x}\}=\pi^{-n/2}(\det A)^{-1/2}\exp\{\tfrac{1}{2}(\mathbf{y},A^{-1}\mathbf{y})\}$$

twice, once with $\mathbf{x}=(k_1^{(1)},k_2^{(1)},\dots)$ and once with $\mathbf{x}=(k_1^{(2)},k_2^{(2)},\dots)$ we obtain

$$\begin{aligned}H=\pi^{-h(G)}&\left(\prod_{i=1}^{P}\frac{\partial}{\partial t_i'}\right)(\det C^TSC)^{-1/2}(\det C^T(S+T')C)^{-1/2}e^{-(P,MP)}\\&\cdot e^{-\mu^2\sum_{r=p+1}^{L}s_r-\mu^2\sum_{r=1}^{p}(u_r+t_r')},\end{aligned} \tag{A.23}$$

where

$$S \equiv 2\begin{bmatrix} u_1 & & & & & & \\ & \ddots & & & & & \\ & & u_p & & & & \\ & & & s_{p+1} & & & \\ & & & & \ddots & & \\ & & & & & & s_L \end{bmatrix}; \quad T' \equiv 2\begin{bmatrix} t'_1 & & & & & \\ & \ddots & & & & \\ & & t'_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}. \tag{A.24}$$

$$\begin{aligned} M \equiv & \tfrac{1}{2}A^T(S - SC(C^TSC)^{-1}C^TS)A \\ & \oplus \tfrac{1}{2}A^T[S+T' - (S+T')C(C^T(S+T')C)^{-1}C^T(S+T')]A. \end{aligned} \tag{A.25}$$

In [24] the following remarkable formula is proven:

Proposition A.5. $\det(C^TSC) = \sum_T \prod_{l,l' \notin T} (s_l u_{l'})$ *where the sum is over all trees* T *of* G; l' *labels the lines corresponding to* $\hat{C}_t^{\beta,0}$, l *the* $\hat{C}^\alpha$ *lines.*

We need two more propositions:

Proposition A.6. *For* $0 \leqq t'_1, \ldots, t'_p \leqq \frac{1}{2}$, $q \in \mathbb{R}$, $u_1, \ldots, u_p$, $s_{p+1}, \ldots, s_L \geqq 0$

$$\begin{aligned} & \left| \frac{\partial^k}{\partial t'_{i_1} \ldots \partial t'_{i_k}} (\det C^T(S+T')C)^{-q} \right| \\ & \quad \leqq \text{const} \frac{1}{(u_{i_1} + t'_{i_1}) \ldots (u_{i_k} + t'_{i_k})} (\det C^T(S+T')C)^{-q}. \end{aligned}$$

Proposition A.7. *For* $0 \leqq t'_1, \ldots, t'_p \leqq \frac{1}{2}$, $u_1, \ldots, u_p$, $s_{p+1}, \ldots, s_L > 0$

$$\left| \frac{\partial^k}{\partial t'_{i_1} \ldots \partial t'_{i_k}} e^{-(P,MP)} \right| \leqq \text{const} \frac{1}{(u_{i_1} + t'_{i_1}) \ldots (u_{i_k} + t'_{i_k})}.$$

Two obvious corollaries are

Corollary A.8.

$$\begin{aligned} & \left| \prod_{i=1}^{p} \frac{\partial}{\partial t'_i} e^{-(P,MP)} (\det C^T(S+T')C)^{-1/2} \right| \\ & \quad \leqq \text{const} \prod_{i=1}^{p} \frac{1}{u_i + t'_i} (\det C^T(S+T')C)^{-1/2}. \end{aligned}$$

Corollary A.9.

$$\begin{aligned} |H| & \leqq \text{const} (\det C^TSC)^{-1/2} (\det C^T(S+T')C)^{-1/2} \prod_{i=1}^{p} \frac{1}{u_i + t'_i} \\ & = \text{const} \prod_{i=1}^{p} \frac{1}{u_i + t'_i} \Big(\sum_T \prod_{l,l' \notin T} s_l u_{l'} \Big)^{-1/2} \\ & \quad \cdot \Big(\sum_T \prod_{l,l' \notin T} s_l (u_{l'} + t'_{l'}) \Big)^{-1/2} e^{-\mu^2 \sum_{i=p+1}^{L} s_i - \mu^2 \sum_{i=1}^{p} (u_i + t'_i)}. \end{aligned}$$

Proof of Proposition A.6. By Leibniz' rule

$$\frac{\partial^k}{\partial t'_{i_1} \ldots \partial t'_{i_k}} (\det C^T(S+T')C)^{-q}$$
$$\cdot \sum_{\{P\}} c_{\{P\}} \prod_{k=1}^{n_{\{P\}}} \left[\prod_{r \in P_k} \frac{\partial}{\partial t_r} \det(C^T(S+T')C) \right] (\det C^T(S+T')C)^{-q-n_{\{P\}}}, \tag{A.26}$$

where the sum is over partitions $\{P\} \equiv \{P_1, \ldots, P_{n\{P\}}\}$ of the set $\{i_1, \ldots, i_k\}$. Now since by Proposition A.5 $\det C^T(S+T')C$ is a polynomial in $u_1 + t'_1, \ldots, u_p + t'_p$ with positive coefficients

$$\left| \prod_{r \in P_k} \frac{\partial}{\partial t'_r} \det C^T(S+T')C \right| \leqq \text{const} \prod_{r \in P_k} (u_r + t'_r)^{-1} \det(C^T(S+T')C)$$

which, inserted into (A.26), yields Proposition A.6. □

Proof of Proposition A.7. From the definition (A.25) of M it is clear that $M \geqq 0$ (using the polar decomposition of $\sqrt{SC}$ it is seen that $\sqrt{SC}(C^TSC)^{-1}C^T\sqrt{S}$ is the projection onto the image of $\sqrt{SC}$). Furthermore

$$Q(t') \equiv M \det C^T(S+T')C$$

is a polynomial of at most first degree in each variable $t'_{i_1}, \ldots, t'_p$; it follows (by taking expectations) that all its coefficients must be positive semidefinite matrices.

Because of Leibniz's rule it suffices to prove

$$\left| \frac{\partial^k}{\partial t'_{i_1} \ldots \partial t'_{i_k}} (P, MP) \right| \leqq \text{const} \prod_{r=1}^{k} \frac{1}{u_{i_r} + t'_{i_r}} (P, MP).$$

Again by Leibniz's rule and Proposition A.6 this will follow from

$$\left| \frac{\partial^k}{\partial t'_{i_1} \ldots \partial t'_{i_k}} (P, QP) \right| \leqq \text{const} \prod_{r=1}^{k} \frac{1}{u_{i_r} + t'_{i_r}} (P, QP).$$

But this is true because (P, QP) is a polynomial with positive coefficients in $u_1 + t'_1, \ldots, u_p + t'_p$. □

Now note that the relation between the amplitudes $\hat{G}$ and H is

$$\hat{G} = \left(\prod_{i=p+1}^{L} \frac{1}{\Gamma(\alpha_i)} \right) \left(\prod_{i=1}^{p} \frac{1}{\Gamma(\beta_i)} \right) \left(\prod_{i=p+1}^{L} \int_0^\infty ds_i s_i^{\alpha_i - 1} \right)$$
$$\cdot \left(\prod_{i=1}^{p} \int_0^\infty du_i u_i^{\beta_i - 1} \right) \left(\prod_{i=1}^{p} \int_0^{t_i} dt'_i \right) H. \tag{A.27}$$

Insertion of Corollary A.9 into this formula produces a bound for $\hat{G}$. It is convenient, however, to break up the region of integration over $u_1, \ldots, u_p \equiv \underset{\sim}{u}$, $s_{p+1}, \ldots, s_L \equiv \underset{\sim}{s}$ as follows: Let π be a permutation of $\{1, \ldots, L\}$.

Define

$$\pi(s_r) \equiv \begin{cases} s_{\pi(r)} : \pi(r) > p \\ u_{\pi(r)} : \pi(r) \leqq p \end{cases} \qquad (r = p+1, \ldots, L)$$

$$\pi(u_r) \equiv \begin{cases} s_{\pi(r)} : \pi(r) > p \\ u_{\pi(r)} : \pi(r) \leqq p \end{cases} \qquad (r = 1, \ldots, p)$$

$$E_\pi \equiv \{(\underset{\sim}{u}, \underset{\sim}{s}) \in \mathbb{R}^L \mid 0 \leqq \pi(u_1) \leqq \ldots \leqq \pi(u_p) \leqq \pi(s_{p+1}) \leqq \ldots \leqq \pi(s_L)\}. \tag{A.28}$$

It is clear that $\bigcup_\pi E_\pi = \mathbb{R}^L_+$; $E_\pi \cap E_{\pi'}$ is a null set for $\pi \neq \pi'$.

By Corollary A.9 and (A.27) we now have

$$|\hat{G}| \leqq \text{const} \sum_\pi F_\pi \tag{A.29}$$

with

$$F_\pi \equiv \prod_{i=p+1}^{L} \frac{1}{\Gamma(\alpha_i)} \prod_{i=1}^{p} \frac{1}{\Gamma(\beta_i)} \int_{E_\pi} d\underset{\sim}{u} d\underset{\sim}{s}$$

$$\cdot \prod_{i=p+1}^{L} s_i^{\alpha_i - 1} \prod_{i=1}^{p} u_i^{\beta_i - 1} \prod_{i=1}^{p} \int_0^{t_i} dt'_i \prod_{i=1}^{p} \frac{1}{u_i + t'_i}$$

$$\equiv \cdot \Big(\sum_T \prod_{l,l' \notin T} s_l u_{l'}\Big)^{-1/2} \Big(\sum_T \prod_{l,l' \notin T} s_l (u_{l'} + t'_l)\Big)^{-1/2} e^{-\mu^2 \sum_{i=p+1}^{L} s_i - \mu^2 \sum_{i=1}^{p} (u_i + t'_i)}. \tag{A.30}$$

The idea is now to estimate the sum over trees by the contribution of a single "leading" tree T which leads to integrals that are easy to estimate. The possibility of finding such a "leading" tree is the content of

Proposition A.10. *To each permutation π of $\{1, \ldots, L\}$ there is a tree T_π such that for $\underset{\sim}{u}, \underset{\sim}{s} \in E_\pi$*

$$\prod_{l,l' \in T} s_l u_{l'} \leqq \prod_{l,l' \in T} s_l u_{l'}. \tag{A.31}$$

Proof. There is an obvious choice of a tree with the smallest possible values of s_l, $u_{l'}$; it is easy to see that it obeys (A.31). We leave the details to the reader. □

For $(\underset{\sim}{u}, \underset{\sim}{s}) \in E_\pi$ we now use the estimate

$$\sum_T \prod_{l,l' \notin T} s_l (u_{l'} + t'_{l'}) \geqq \prod_{l,l' \notin T_\pi} s_l (u_{l'} + t'_{l'}) \tag{A.32}$$

and similarly

$$\sum_T \prod_{l,l' \notin T} s_l u_{l'} \geqq \prod_{l,l' \notin T_\pi} s_l u_{l'}. \tag{A.33}$$

Inserting this into (A.22) gives

$$F_\pi \leqq \prod_{i=p+1}^{L} \frac{1}{\Gamma(\alpha_i)} \prod_{i=1}^{p} \frac{1}{\Gamma(\beta_i)} \int_{E_\pi} d\underset{\sim}{u} d\underset{\sim}{s}$$

$$\cdot \prod_{i=p+1}^{L} s_i^{\alpha_i - 1} \prod_{i=1}^{p} u_i^{\beta_i - 1} \prod_{i=1}^{p} \int_0^{t_i} dt'_i \prod_{i=1}^{p} \frac{1}{u_i + t'_i} \tag{A.34}$$

$$\cdot \prod_{l,l' \notin T_\pi} (s_l u_{l'})^{-1/2} \prod_{l,l' \notin T_\pi} s_l^{-1/2} (u_{l'} + t'_{l'})^{-1/2}$$

$$\cdot e^{-\mu^2 \sum_{i=p+1}^{L} s_i - \mu^2 \sum_{i=1}^{p} (u_i + t'_i)}.$$

Next we estimate the result of the t'-integration using

Proposition A.11.

$$\int_0^t \left(\frac{1}{u+t'}\right)^{1+\delta} dt' \leqq \text{const} \left(\frac{1}{u}\right)^{\delta+\varepsilon} t^\varepsilon$$

if $\varepsilon > 0$, $\delta \geqq 0$, $u \geqq 0$, $t \geqq 0$.

We omit the easy proof. Insertion of this proposition in (A.34) yields

$$F_\pi \leqq \text{const} \int_{E_\pi} d\underset{\sim}{u} d\underset{\sim}{s} e^{-\mu^2(\sum u_i + \sum s_j)}$$
$$\cdot \prod_j s_j^{\alpha_j - 1} \prod_i u_i^{\beta_i - 1 - \varepsilon_i} t_i^{\varepsilon_i} \prod_{l,l' \notin T_\pi} (s_l u_{l'})^{-1} \tag{A.35}$$

for some $\varepsilon_i > 0$ $(i = 1, \dots, p)$ to be chosen presently.

The right hand side has the form

$$\text{const} \left(\int_{0 \leqq v_1 \leqq v_2 \leqq \dots \leqq v_L} d\underset{\sim}{v} \prod_{i=1}^{L} \left(\frac{1}{v_i}\right)^{1-q_i} e^{-\mu^2 v_i} \right) t^{\sum_{i=1}^{p} \varepsilon_i} \tag{A.36}$$

after the variables $u_{\pi(1)}, \dots, s_{\pi(L)}$ are relabelled as $v_1, \dots, v_L$. By discarding all exponentials except $e^{-\mu^2 v_L}$ and performing the integrals in the order $v_1, v_2, \dots$, we see that (A.36) is convergent provided

$$\inf_{r \leqq L} \sum_{i=1}^{r} q_i > 0. \tag{A.37}$$

We compare (A.36) with (A.35) in order to find the q_i's and thereby see that (A.37) reads

$$\inf_{r \leqq L} \left(\sum \alpha_j + \sum (\beta_i - \varepsilon_i) - L(H_r \backslash T_\pi)\right) > 0, \tag{A.38}$$

where H_r is the subgraph of G that contains the lines associated to the first r variables in the list $u_{\pi(1)}, u_{\pi(2)}, \ldots, s_{\pi(L)}$ and the sums extend over the α's and β's associated to the lines in H_r. We relate this criterion to power counting by noting that

$$L(H_r \setminus T_\pi) = L(H_r) - V(H_r) + C(H_r)$$

because by the definition of T_π (see Proposition A.10) T_π intersects each connected component of H_r in a tree of that component [below (A.13)]. By the Euler relation (A.13), $L(H_r \setminus T_\pi) = h(H_r)$ and so (A.38) can be rephrased as

$$\inf_{r \leqq L} \left[-\tfrac{1}{2} K(H_r) - \sum \varepsilon_i \right] > 0 . \tag{A.38'}$$

So we collect (A.35) to (A.38'), use our hypothesis on $t_1, \ldots, t_p$ and find that

$$F_\pi \leqq \text{const}\, t_1^\varepsilon \quad \text{if} \quad \varepsilon < -\tfrac{1}{2}\tilde{K}(G) .$$

Summing this over π (A.29) completes the proof of Lemma A.4. □

From the reduction carried out earlier in this Appendix it follows that Lemma A.4 implies the following theorem (that contains what was used in the proof of Proposition 2.18):

Theorem A.13. *Let G_Λ be a connected vacuum Feynman graph with free, periodic or mixed* b.c. *in a rectangle Λ. Assume it is convergent according to power counting* ($\tilde{K}(G) < 0$) *and contains at least one covariance of $\frac{\partial}{\partial s_i} A$. Then*

$$|G_\Lambda| \leqq \text{const}\, |\Lambda| t_i^\delta$$

for some $\delta > 0$.

Remark. The reader should refer to the discussion of "shifting derivatives" at the beginning of this Appendix before applying this theorem to the graphs produced by our stability expansion.

Proof. This is just a compressed formulation of the content of this Appendix; note that condition

$$\max(t_1, \ldots, t_p) \leqq \min(t_1, \ldots, t_p) \times \text{const}$$

occurring in Lemma A.4 is trivially fulfilled if we choose $p = 1$ (if there is more than one differentiated line we may estimate it by $\hat{C}^\alpha$). The condition $0 \leqq t_1, \ldots, t_p \leqq \frac{1}{2}$ also occurring there is irrelevant here because G_Λ is certainly bounded uniformly in t_i. □

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