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A Method of Integration over Matrix Variables

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Abstract. The integral over two $n \times n$ hermitan matrices $Z(g, c) = \int dA dB \exp\left\{ - \mathrm{tr}\left[A^2 + B^2 - 2cAB + \frac{g}{n}(A^4 + B^4) \right] \right\}$ is evaluated in the limit of large n. For this purpose use is made of the theory of diffusion equation and that of orthogonal polynomials with a non-local weight. The above integral arises in the study of the planar approximation to quantum field theory.

1. Introduction

In their study of planar diagrams some authors [1, 3] have discussed integrals of the form

$$Z = \int \prod_i dM^{(i)} \exp\left\{ - \sum_i V(M^{(i)}) + \sum_{i<j} C_{ij} \,\mathrm{tr}\, M^{(i)} M^{(j)} \right\} \tag{1.1}$$

$$V(M) = \mathrm{tr}\, M^2 + \frac{g}{n} \mathrm{tr}\, M^4 \tag{1.2}$$

where $M^{(1)}, M^{(2)}, \ldots$ are hermitian matrices of order $n \times n$. The integral is taken over all independent real parameters entering the matrix elements,

$$\int dM = \int_{-\infty}^{\infty} \ldots \int \prod_{i=1}^{n} dM_{ii} \prod_{1 \leqq i < j \leqq n} d(\mathrm{Re}\, M_{ij}) d(\mathrm{Im}\, M_{ij}). \tag{1.3}$$

The case of one matrix is the simplest. There are no cross terms containing C_{ij}. The integral reduces to that over the eigenvalues [4],

$$\begin{aligned} Z(g) &= dM \exp\left\{ - \mathrm{tr}\, M^2 - \frac{g}{n} \mathrm{tr}\, M^4 \right\} \\ &= \text{const.} \int \exp\left\{ - \sum_{i=1}^{n} \left(x_i^2 + \frac{g}{n} x_i^4 \right) \right\} |\Delta(X)|^{\beta} \prod^{n} dx_i, \end{aligned} \tag{1.4}$$

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where

$$\Delta(X) = \prod_{1 \leqq i < j \leqq n} (x_i - x_j) \tag{1.5}$$

and $\beta = 2$. It is now known [1, 2] that

$$-\frac{1}{n^2} \ln \frac{Z(g)}{Z(0)} = E_0(g) + \frac{1}{n^2} E_1(g) + 0(n^{-4}), \tag{1.6}$$

with

$$E_0(g) = -\tfrac{1}{2} \ln a^2 + \tfrac{1}{24}(a^2 - 1)(9 - a^2), \tag{1.7}$$

$$E_1(g) = \tfrac{1}{12} \ln(2 - a^2), \tag{1.8}$$

$$a^2 \equiv a^2(g) = \frac{1}{3\beta g}\{-1 + \sqrt{1 + 6\beta g}\}, \beta = 2. \tag{1.9}$$

In stead of hermitian matrices one could have taken matrices which are real symmetric or which are quaternion self-dual. The corresponding integrals reduce again to Eq. (1.4) where the parameter β is 1 for real symmetric matrices and it is 4 for quaternion self dual matrices. These integrals can again be evaluated in the large n limit, and give the same $E_0(g)$ except that β is now 1 or 4. The correction term $E_1(g)$ may be different. The details of this calculation being of no interest are omitted.

The next difficult case of two matrices was discussed by Itzykson and Zuber [3]. They reduced the integral to that over the eigenvalues. However, the expressions given by them are too complicated. Below we will reinvestigate this case

$$Z(g, c) = \int dA dB \exp\left\{-\operatorname{tr}(A^2 + B^2) - \frac{g}{n} \operatorname{tr}(A^4 + B^4) + 2c \operatorname{tr} AB\right\} \tag{1.10}$$

where A and B are $n \times n$ hermitian matrices. We will show that

$$\frac{1}{n^2} \ln \frac{Z(g, c)}{Z(0, c)} = \int_0^1 (1 - x)\left\{\ln f(x) - \ln \frac{cx}{2(1 - c^2)}\right\} dx + 0(n^{-2}) \tag{1.11}$$

where $f(x)$ is given by an algebraic equation of the fifth degree

$$f(x)\left\{\left(1 - 6\frac{g}{c} f(x)\right)^{-2} - c^2\right\} + 12g^2 f^3(x) - \tfrac{1}{2}cx = 0, \tag{1.12}$$

and the root to be taken equals $\frac{1}{2}cx(1 - c^2)^{-1}$ when $g = 0$.

2. The Method of Diffusion Equation

Consider the partial differential equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_i D_i \frac{\partial^2 \xi}{\partial x_i^2} \tag{2.1}$$

where the constants D_i may be unequal for different directions. The unique solution satisfying the initial condition $\xi(X;0) = \eta(X)$ is known to be [5]

$$\xi(X;t) = \int K(X, Y;t)\eta(Y)dY, \tag{2.2}$$

$$K(X, Y;t) = \prod_i (2\pi D_i t)^{-1/2} \exp\left\{ -\sum_i (x_i - y_i)^2/(2D_i t) \right\}. \tag{2.3}$$

Now let A be an $n \times n$ hermitian matrix with elements A_{ij}. The A_{ii} are real, while the real and imaginary parts of A_{ij} for $i < j$ are denoted by $\mathrm{Re}\, A_{ij}$ and $\mathrm{Im}\, A_{ij}$ respectively. Similarly for the matrix B. Then

$$\xi(A;t) = \int K(A, B;t)\eta(B)\, dB, \tag{2.4}$$

$$\begin{aligned} K(A, B;t) &= (2\pi t)^{-n^2/2} \exp\left\{ -\frac{1}{2t} \mathrm{tr}(A - B)^2 \right\} \\ &\equiv (2\pi t)^{-n^2/2} \exp\left\{ -\frac{1}{2t} \left[\sum_i (A_{ii} - B_{ii})^2 + 2 \sum_{i<j} (\mathrm{Re}\, A_{ij} - \mathrm{Re}\, B_{ij})^2 \right.\right. \\ &\quad \left.\left. + 2 \sum_{i<j} (\mathrm{Im}\, A_{ij} - \mathrm{Im}\, B_{ij})^2 \right] \right\}, \end{aligned} \tag{2.5}$$

$$\int dB \equiv \int_{-\infty}^{\infty} \cdots \int \prod_i dB_{ii} \prod_{i<j} d(\mathrm{Re}\, B_{ij}) d(\mathrm{Im}\, B_{ij}), \tag{2.6}$$

satisfies the equation

$$\frac{\partial \xi}{\partial t} = \tfrac{1}{2} \nabla_A^2 \xi, \tag{2.7}$$

$$\nabla_A^2 = \sum_i \frac{\partial^2}{\partial A_{ii}^2} + \frac{1}{2} \sum_{i<j} \left\{ \frac{\partial^2}{\partial (\mathrm{Re}\, A_{ij})^2} + \frac{\partial^2}{\partial (\mathrm{Im}\, A_{ij})^2} \right\}, \tag{2.8}$$

and the initial condition

$$\xi(A;0) = \eta(A). \tag{2.9}$$

As A and B are hermitian, we can choose unitary matrices U_A and U_B such that

$$A = U_A^+ X U_A, \quad B = U_B^+ Y U_B, \tag{2.10}$$

where $X = [x_i \delta_{ij}]$ and $Y = [y_i \delta_{ij}]$ are diagonal matrices. The x_i are the eigenvalues of A and the y_i are those of B. Changing the variables from matrix elements to the n eigenvalues and $n(n-1)$ angle parameters on which U_A and U_B depend, we have [4]

$$dB = \Delta^2(Y) dY d\Omega_B, \quad dY = \prod_{i=1}^{n} dy_i. \tag{2.11}$$

so that

$$\xi(A;t) = \xi(X, \Omega_A;t)$$

$$= (2\pi t)^{-n^2/2} \int \exp\left\{-\frac{1}{2t}\operatorname{tr}(X - U^+ YU)^2\right\}\eta(Y, \Omega_B)\Delta^2(Y)dY d\Omega_B, \tag{2.12}$$

where

$$U = U_B U_A^+ \tag{2.13}$$

Observe that if $\eta(B)$ is independent of Ω_B, then $\xi(A;t)$ is also independent of Ω_A as can be seen by a change of variables from Ω_B to Ω (depending on U),

$$\xi(X;t) = \text{const.}\, t^{-n^2/2} \int \exp\left\{-\frac{1}{2t}\operatorname{tr}(X - U^+ YU)^2\right\}\eta(Y)\Delta^2(Y)dY d\Omega. \tag{2.14}$$

Seperating[1] the Laplacian into parts depending on X and on U_A,

$$\nabla_A^2 = \frac{1}{\Delta^2(X)}\sum_i \frac{\partial}{\partial x_i}\Delta^2(X)\frac{\partial}{\partial x_i} + \nabla_{\Omega_A}^2, \tag{2.15}$$

one sees that $\xi(X;t)$ satisfies the (diffusion) equation

$$\frac{\partial \xi}{\partial t} = \frac{1}{2}\frac{1}{\Delta^2(X)}\sum_i \partial_i \Delta^2(X)\partial_i \xi, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \tag{2.16}$$

and has the initial value

$$\xi(X;0) = \eta(X) \tag{2.17}$$

Set

$$F(X;t) = \Delta(X)\xi(X;t) \tag{2.18}$$

Then

$$\sum_i \partial_i^2 F = \sum_i \{\Delta\partial_i^2 \xi + 2(\partial_i \Delta)(\partial_i \xi) + \xi\partial_i^2 \Delta\}$$

$$= \Delta(X)\left\{\frac{1}{\Delta^2(X)}\sum_i \partial_i(\Delta^2(X)\partial_i \xi)\right\} \tag{2.19}$$

Thus $F(X;t)$ satisfies the (diffusion) equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}\sum_i \frac{\partial^2 F}{\partial x_i^2}, \tag{2.20}$$

and has the initial value

$$F(X;0) = \Delta(X)\xi(X;0) = \Delta(X)\eta(X). \tag{2.21}$$

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1 The Jacobian of the transformation from matrix elements to the eigenvalues and angle variables for a hermitian matrix is $\Delta^2(X)f(\Omega_A)$, where f is independent of the x_i [4]. Therefore the Laplacian is given by Eq. (2.15) ([5], end of Chap. 1)

Therefore $F(X;t)$ is given by [5]

$$F(X;t) = \text{const.}\, t^{-n/2} \int \exp\left\{ -\frac{1}{2t} \sum_{i=1}^{n} (x_i - y_i)^2 \right\} \Delta(Y)\eta(Y)dY. \tag{2.22}$$

choose

$$\eta(B) = \exp\{ -V(B) + c \operatorname{tr} B^2 \} = \exp\left\{ -V(Y) + c \sum_i y_i^2 \right\}, \tag{2.23}$$

where $V(M)$ depends only on the eigenvalues of M. Setting $c = \frac{1}{2t}$ in Eqs. (2.4)–(2.6), (2.18) and (2.22) one gets

$$\begin{aligned} &\Delta(X) \int dB \exp\{ -V(B) + c \operatorname{tr} B^2 - c \operatorname{tr}(A - B)^2 \} \\ &= \left(\frac{\pi}{2c} \right)^{n(n-1)/2} \int dY \exp\left\{ -V(Y) + c \sum_i y_i^2 - c \sum_i (x_i - y_i)^2 \right\} \Delta(Y). \end{aligned} \tag{2.24}$$

The constant is obtained by choosing $V(B) = c \operatorname{tr} B^2$ and performing the gaussian integrals on both sides. Therefore

$$\begin{aligned} &\int \exp\{ -V(A) - V(B) + 2c \operatorname{tr} AB \} dA dB \\ &= \text{const.} \int dX d\Omega_A \Delta^2(X) \exp\left\{ -V(X) + c \sum_i x_i^2 \right\} \\ &\quad \cdot \int dB \exp\{ -V(B) + c \operatorname{tr} B^2 - c \operatorname{tr}(A - B)^2 \} \\ &= \text{const.} \int dX dY \Delta(X) \Delta(Y) \exp\left\{ -V(X) - V(Y) + 2c \sum_i x_i y_i \right\}. \end{aligned} \tag{2.25}$$

This is essentially the result of Itzykson–Zubar [3] expressed in a simpler form. The constant can be fixed by considering $V(A) = \operatorname{tr} A^2 = \sum x_i^2$. The gaussian integral on the left hand side is then straight forward, while that on the right hand side is given in the appendix. As a result the unknown constant is

$$\pi^{n(n-1)} (2c)^{-(1/2)n(n-1)} \left(\prod_1^n i! \right)^{-1}. \tag{2.26}$$

3. Orthogonal Polynomials Revisited

To get the asymptotic behavior of $Z(g, c)/Z(0, c)$.
where

$$\begin{aligned} Z(g, c) &= \int \exp\left\{ -\operatorname{tr}(A^2 + B^2) - \frac{g}{n} \operatorname{tr}(A^4 + B^4) + 2c \operatorname{tr} AB \right\} dA dB \\ &= \text{const.} \int \exp\left\{ -\sum_i (x_i^2 + y_i^2) - \frac{g}{n} \sum_i (x_i^4 + y_i^4) + 2c \sum_i x_i y_i \right\} \\ &\quad \cdot \Delta(X) \Delta(Y) \prod_i dx_i dy_i, \end{aligned} \tag{3.1}$$

we will use orthogonal polynomials with a non-local weight.

Writting $\Delta(X) = \prod_{i<j}(x_i - x_j)$ as the Vandermonde determinant, one sees that

$$\Delta(X) = \det[x_i^{j-1}] = \det[P_{j-1}(x_i)]_{i,j=1,2,\ldots,n}, \tag{3.2}$$

where

$$P_j(x) = x^j + \sum_0^{j-1} a_k x^k, \tag{3.3}$$

is an arbitrary polynomial of degree j with the coefficient of x^j equal to 1. Similarly,

$$\Delta(Y) = \det[Q_{j-1}(y_i)]_{i,j=1,2,\ldots,n}, \tag{3.4}$$

where $Q_j(x)$ is another set of similar polynomials.

Since

$$w(x, y) = \exp\left\{-(x^2 + y^2) - \frac{g}{n}(x^4 + y^4) + 2cxy\right\} \tag{3.5}$$

is symmetric in x and y, we will choose $P_i(x) = Q_i(x)$ and such that

$$\int_{-\infty}^{\infty}\int w(x, y)P_i(x)P_j(y)dxdy = h_i\delta_{ij} \tag{3.6}$$

where the Kronecker symbol δ_{ij} is 1 or 0 according as $i = j$ or $i \neq j$. Such a choice is possible. In fact

$$P_i(x) = \text{const. det}\begin{bmatrix} m_{00} & m_{01} & \ldots & m_{0i} \\ m_{10} & m_{11} & \ldots & m_{1i} \\ m_{(i-1)0} & m_{(i-1)1} & \ldots & m_{(i-1)i} \\ 1 & x & \ldots & x^i \end{bmatrix} \tag{3.7}$$

where

$$m_{ij} = \int_{-\infty}^{\infty}\int w(x, y)x^i y^j dxdy, \tag{3.8}$$

are the moments of $w(x, y)$. In particular, since y^j can be expressed as a linear combination of $P_k(y)$ with $k \leqq j$, one has

$$\int_{-\infty}^{\infty}\int w(x, y)P_i(x)y^j dxdy = 0, \text{ for } i > j. \tag{3.9}$$

With such a choice of $P_i(x)$ we expand the two Vandermonde determinants, multiply and use the orthogonal property (3.6) to integrate various products. The only terms which contribute have equal indices of the polynomials in x and in y, they contribute the same quantity, and they are $n!$ in number. Thus

$$Z(g, c) = \text{const. } n! \prod_0^{n-1} h_j(g, c), \tag{3.10}$$

and we need to know the asymptotic behaviour of the product of h_j. For this purpose, we proceed as with the usual orthogonal polynomials.

As $w(-x,-y)=w(x,y)$, $m_{ij}=0$ for $i+j$ odd, and $P_i(x)$ has a definite parity,

$$P_i(-x)=(-1)^iP_i(x) \tag{3.11}$$

Let

$$xP_i(x)=P_{i+1}(x)+R_iP_{i-1}(x)+S_iP_{i-3}(x), \tag{3.12}$$

where R_i and S_i are certain coefficients. Iterating thrice we get

$$\begin{aligned}x^3P_i(x)&=P_{i+3}(x)+(R_i+R_{i+1}+R_{i+2})P_{i+1}(x)\\&\quad+\{R_i(R_{i-1}+R_i+R_{i+1})+(S_i+S_{i+1}+S_{i+2})\}P_{i-1}(x)+\dots.\end{aligned} \tag{3.13}$$

Thus expressing $x^kP_i(x)$ as linear combinations of $P_j(x), j\leqq i+k$, and using equation (3, 6) we get

$$\begin{aligned}&\int_{-\infty}^{\infty}\int P_{i-1}(x)P_i(y)\left(x-cy+2\frac{g}{n}x^3\right)w(x,y)dxdy\\&\quad=h_i\left\{1+\frac{2g}{n}(R_{i-1}+R_i+R_{i+1})\right\}-cR_ih_{i-1}.\end{aligned} \tag{3.14}$$

Also integrating on x by parts, the left hand side of the above equation is

$$\frac{1}{2}\int_{-\infty}^{\infty}\int\frac{dP_{i-1}(x)}{dx}P_i(y)w(x,y)dxdy=0, \tag{3.15}$$

because of equation (3.9). From the last two equations we get

$$h_i\left\{1+\frac{2g}{n}(R_{i-1}+R_i+R_{i+1})\right\}=cR_ih_{i-1}. \tag{3.16}$$

Similarly by integrating

$$P_i(x)P_{i-1}(y)\left(x-cy+\frac{2g}{n}x^3\right)w(x,y) \tag{3.17}$$

and

$$P_{i-3}(x)P_i(y)\left(x-cy+\frac{2g}{n}x^3\right)w(x,y) \tag{3.18}$$

in two different ways, we get the relations

$$ch_i=h_{i-1}\left\{-\frac{i}{2}+R_i\left[1+\frac{2g}{n}(R_{i-1}+R_i+R_{i+1})\right]+\frac{2g}{n}(S_i+S_{i+1}+S_{i+2})\right\}, \tag{3.19}$$

and

$$2\frac{g}{n}h_i=cS_ih_{i-3}. \tag{3.20}$$

4. Asymptotic Evaluation of $Z(g, c)$

Let us write $f_i=h_i/h_{i-1}$, so that Eqs. (3.16), (3.19) and (3.20) can be rewritten as

$$f_i = cR_i\left\{1 + \frac{2g}{n}(R_{i-1} + R_i + R_{i+1})\right\}^{-1}, \tag{4.1}$$

$$cf_i = -\frac{i}{2} + R_i\left\{1 + \frac{2g}{n}(R_{i-1} + R_i + R_{i+1})\right\} + \frac{2g}{n}(S_i + S_{i+1} + S_{i-2}), \tag{4.2}$$

and

$$cS_i = \frac{2g}{n} f_i f_{i-1} f_{i-2}, \tag{4.3}$$

For large i and n, the f_i, R_i and S_i can be replaced by continuous functions. Thus

$$\begin{aligned} f_i &\sim nf(x), \quad f_{i\pm1} \sim nf(x \pm \varepsilon), \\ R_i &\sim nR(x), \quad R_{i\pm1} \sim nR(x \pm \varepsilon), \\ S_i &\sim n^2 S(x), \quad S_{i\pm1} \sim n^2 S(x \pm \varepsilon), \end{aligned} \tag{4.4}$$

$$x = \frac{i}{n}, \quad \varepsilon = \frac{1}{n}. \tag{4.5}$$

Making these substitutions, we get to the leading order,

$$f(x) = cR(x)\{1 + 6gR(x)\}^{-1}, \tag{4.6}$$

$$cf(x) = -\frac{x}{2} + R(x)(1 + 6gR(x) + 6gS(x), \tag{4.7}$$

$$cS(x) = 2gf^3(x). \tag{4.8}$$

Eliminating $R(x)$ and $S(x)$ from the last three equations, one gets

$$f(x)\left\{(1 - 6\frac{g}{c}f(x))^{-2} - c^2\right\} + 12g^2 f^3(x) = \tfrac{1}{2}cx. \tag{4.9}$$

When $g = 0$, the value of $f(x)$ will be denoted by $f_0(x)$. From (4.6) and (4.7)

$$f_0(x) = \tfrac{1}{2}cx(1 - c^2)^{-1}. \tag{4.10}$$

Now from Eq. (3.10) we have

$$\ln \frac{Z(g, c)}{Z(0, c)} = \sum_{i=0}^{n-1} \ln \frac{h_i(g, c)}{h_i(0, c)}, \tag{4.11}$$

$$\sum_{i=0}^{n-1} \ln h_i = n \ln h_0 + \sum_{i=1}^{n} (n - i) \ln f_i, \tag{4.12}$$

$$\frac{1}{n^2} \sum_{i=1}^{n} (n - i) \ln f_i(g, c) = \int_0^1 (1 - x) \ln (nf(x)) dx + 0(n^{-2}). \tag{4.13}$$

Therefore

$$\frac{1}{n^2}\ln\frac{Z(g,c)}{Z(0,c)}=\frac{1}{n}\ln\frac{h_0(g,c)}{h_0(0,c)}+\int_0^1(1-x)\ln\frac{f(x)}{f_0(x)}\,dx+0(n^{-2}). \tag{4.14}$$

But

$$h_0(g,c)=\int_{-\infty}^{\infty}dxdy\exp\left[-\left\{x^2+y^2-2cxy+\frac{g}{n}(x^4+y^4)\right\}\right]$$
$$=\frac{\pi}{\sqrt{1-c^2}}\left\{1-\frac{3}{2}\frac{g}{n}(1-c^2)^{-2}+0(n^{-2})\right\} \tag{4.15}$$

Hence

$$\ln\frac{h_0(g,c)}{h_0(0,c)}=0(n^{-1}). \tag{4.16}$$

Eqs. (4.14), (4.16), (4.9) and (4.10) together give the result announced in the introduction, Eqs. (1.11) and (1.12).

5. Some Remarks

5.1. Formula (2.25) looks trivial, but it is not. To be honest, we have no shorter way to derive it.

Itzykson and Zuber [3] derive a formula equivalent to (2.25) in a different way as well. They introduce the decomposition of unity into characters of irreducible representations of the unitary group. Using the orthogonality of these characters they can perform the angular integrations. The final result is a series containing eigenvalues of A and B, characters of irreducible representations of the unitary group, their dimensions and the number of times an irreducible representation occurs in various Kronecker powers of the initial matrix. This method can be adapted to deal with real symmetric or quaternion self-dual matrices; one has only to replace the unitary group by the orthogonal or the symplectic group. The formulas however, do not seem to be simple.

5.2. The same method adapted to evaluate the integral over a chain of matrices

$$\int\exp\left\{-\sum_{i=1}^{j}V(M^{(i)})+2\sum_{i=1}^{j-1}c_iM^{(i)}M^{(i+1)}\right\}\prod_1^j dM^{(i)}$$

in the limit of large n will be considered elsewhere [7]

5.3. An expansion in powers of g gives

$$\frac{f(x)}{f_0(x)}=1-6gx(1-c^2)^{-2}+3g^2x^2(1-c^2)^{-4}(c^4+8c^2+15)+0(g^3) \tag{5.1}$$

so that

$$-E_0(g)=\int_0^1(1-x)\ln\frac{f(x)}{f_0(x)}\,dx$$
$$=-g(1-c^2)^{-2}+\tfrac{1}{4}g^2(1-c^2)^{-4}(c^4+8c^2+9)+0(g^3) \tag{5.2}$$

5.4. Denoting by $\langle\ \rangle$ the average with respect to any positive measure, one has the inequality [16].

$$\langle e^F \rangle \geqq e^{\langle F \rangle}$$

Taking

$$\langle \ldots \rangle = \int dA dB \exp\{-\operatorname{tr}(A^2 + B^2 - 2cAB)\} \ldots, \tag{5.3}$$

or

$$\langle \ldots \rangle \equiv \int dX dY \Delta(X)\Delta(Y) \exp\left\{-\sum_1^b (x_i^2 + y_i^2 - 2cx_i y_i)\right\} \ldots \tag{5.4}$$

and

$$F = -\frac{g}{n}\operatorname{tr}(A^4 + B^4) = -\frac{g}{n}\sum_1^n (x_i^4 + y_i^4), \tag{5.5}$$

we get the inequality (see the appendix)

$$\begin{aligned} \frac{Z(g,c)}{Z(0,c)} &= \exp\{-n^2 E_0(g) - E_1(g) + 0(n^{-2})\} \\ &\geqq \exp\{-g(1-c^2)^{-2}(n^2 + \tfrac{1}{2})\} \end{aligned} \tag{5.6}$$

Thus one sees that in agreement with Eq. (5.2),

$$\begin{aligned} E_0(g) &= +g(1-c^2)^{-2} + 0(g^2), \\ E_1(g) &= +\tfrac{1}{2}g(1-c^2)^{-2} + 0(g^2). \end{aligned} \tag{5.7}$$

In general, let D be the $p \times p$ matrix $[\delta_{ij} - C_{ij}]$ and D_k the same matrix with its k^{th} row and k^{th} column removed. Observe that

$$\begin{aligned} &\int \exp\left\{-\sum_{i=1}^p \operatorname{tr} A_i^2 + 2 \sum_{1 \leqq i < j \leqq p} C_{ij} \operatorname{tr} A_i A_j\right\} dA_1 \ldots dA_{k-1} dA_{k+1} \ldots dA_p \\ &= \left(2^n \left(\frac{\pi}{2}\right)^{n^2}\right)^{(p-1)/2} a_k \exp(-b_k \operatorname{tr} A_k^2), \end{aligned}$$

with

$$a_k = (\det D_k)^{-n^2/2}, \quad a_k b_k^{-n^2/2} = (\det D)^{-n^2/2}.$$

A power series expansion in g gives

$$\begin{aligned} Z(g) &\equiv \int \exp\left\{-\sum_{i=1}^p \operatorname{tr}\left(A_i^2 + \frac{g}{n}A_i^4\right) + 2 \sum_{1 \leqq i < j \leqq p} C_{ij} \operatorname{tr} A_i A_j\right\} dA_1 \ldots dA_p \\ &= Z(0)\left\{1 - \frac{g}{n}\sum_{k=1}^p \int dA_k \operatorname{tr} A_k^4 \exp(-b_k \operatorname{tr} A_k^2) / \int dA_k \exp(-b_k \operatorname{tr} A_k^2) + \ldots\right\} \\ &= Z(0)\left\{1 - \frac{g}{4}(2n^2 + 1)\sum_{k=1}^p b_k^{-2} + 0(g^2)\right\}. \end{aligned}$$

So that

$$\frac{Z(g)}{Z(0)} \geqq \exp\left\{ -\frac{1}{4} g(2n^2+1) \sum_{k=1}^{p} (D^{-1})_{kk}^2 \right\}.$$

5.5. An obvious upper bound for $Z(g)/Z(0)$ is 1. Another of the form $k_1 g^{k_2}$ can be obtained by Schwartz's inequality.

5.6. Note that if $V(M)$ has a term in tr M^6(tr $M^8, \ldots$), then $xP_i(x)$ in Eq. (3.12) will also have a $P_{i-5}(x)(P_{i-7}(x), \ldots)$ present.

5.7. Let us represent the integral in Eq. (1.1) by a graph; the matrices $M^{(i)}$ are noted as points and the points i and j are joined by a line if $C_{ij} \neq 0$. If this graph contains no cycles, the angle variables can be integrated by using Eq. (2.24). The remaining integrations over the eigenvalues, even in the limit of large n, are not simple.

5.8. Examples. For a cyclic graph with p points,

$$Z(g) = \int \prod_{1}^{p} dA_i \exp\left\{ -\sum_{i=1}^{p} V(A_i) + 2c \sum_{i=1}^{p} \text{tr } A_i A_{i+1} \right\},$$

$A_{p+1} \equiv A_1$, with V as in Eq. (1.2).

$$(D^{-1})_{kk} = \frac{1}{p} \sum_{j=1}^{p} \left(1 - 2c \cos\frac{2\pi j}{p} \right)^{-1} = \left(th\theta th\frac{p\theta}{2} \right)^{-1},$$

$$Z(g) \geqq Z(0) \cdot \exp\left\{ -\frac{g}{4}(2n^2+1)p \left(th\theta th\frac{p\theta}{2} \right)^{-2} \right\},$$

where $2c\ ch\theta = 1$.

For a $p \times q$ square lattice graph with periodic boundary conditions,

$$Z(g) = \int \prod_{i=1}^{p} \prod_{j=1}^{q} dA_{ij} \exp \cdot \left\{ -\sum_{i=1}^{p} \sum_{j=1}^{q} [V(A_{ij}) - 2C_1 \text{ tr } A_{ij} A_{(i+1)j} - 2C_2 \text{ tr } A_{ij} A_{i(j+1)}] \right\},$$

$$A_{i(q+1)} \equiv A_{i1}, \quad A_{(p+1)j} \equiv A_{1j}, \quad V(A) = \text{Tr } A^2 + \frac{g}{n} \text{tr } A^4.$$

$$(D^{-1})_{ij,ij} = \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} \left(1 - 2C_1 \cos\frac{2\pi i}{p} - 2C_2 \cos\frac{2\pi j}{q} \right)^{-1}$$

$$\approx \frac{1}{4\pi^2} \int_0^{2\pi} \int (1 - 2C_1 \cos\theta - 2C_2 \cos\phi)^{-1} \, d\theta d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi} \{(1 - 2C_1 \cos\theta)^2 - 4C_2^2\}^{-1/2} d\theta$$

for p and q very large. Hence

$$\frac{Z(g)}{Z(0)} \geqq \exp\left[-\frac{g^2}{\pi^2} (2n^2+1)pq \{1 - 4(C_1 - C_2)^2\}^{-1} \left\{ \int_0^{\pi/2} (1 - \alpha^2 \sin^2\theta)^{-1/2} d\theta \right\}^2 \right],$$

with $\alpha^2 = 16C_1C_2\{1-4(C_1-C_2)^2\}^{-1}$.

For a star graph with m branches

$$Z(g) = \int dA \prod_1^m dB_i \exp\left\{ -V(A) - \sum_1^m V(B_i) + 2c \operatorname{tr} \sum_1^m AB_i \right\},$$

$$\frac{Z(g)}{Z(0)} = (1-mc^2)^{(1/2)n^2} (2c^{-m})^{(1/2)n(n-1)} \pi^{-(1/2)n} \left(\prod_1^n j! \right)^{-1} \cdot$$

$$\cdot \int \exp\{-V(X) + mc^2 X^2\} \{\det[F_{j-1}(x_k)]/\Delta(X)\}^m \Delta^2(X) dX$$

$$\geqq \exp\left[-\frac{g}{4}(2n^2+1)(1-mc^2)^{-2}\{1+m(1+c^2-mc^2)^2\} \right]$$

where

$$F_j(x) = \pi^{-1/2} \int \exp\left\{ -(y-x)^2 - \frac{g}{n} y^4 \right\} y^j dy$$

Appendix

Evaluation of the constant in Eq. (2.25).

We will need the

Lemma. *Let* $F(X) \equiv F(x_1, \ldots, x_n)$ *be a symmetric function of* $x_1, \ldots, x_n$ *and* $\Delta(X) = \prod_{1 \leqq i < j \leqq n} (x_i - x_j)$. *Then for arbitrary numbers* λ_{ij} *one has*

$$\int dX \Delta(X) F(X) \prod_{i<j} (x_i - x_j + \lambda_{ij}) = \int dX \Delta^2(X) F(X).$$

Proof. Expand the product $\prod_{i<j}(x_i - x_j + \lambda_{ij})$ in powers of $x_1, \ldots, x_n$ and note that

$$\int x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_2} \Delta(X) F(X) dX = 0$$

if any two of the α_i are equal; this is so because if $\alpha_i = \alpha_j$, then integrand is anti-symmetric in the variables x_i and x_j. Therefore the monomial in $x_1, \ldots, x_n$ which will give a non-zero contribution to the integral must have all α_i distinct, and its degree is at least

$$0 + 1 + 2 + \ldots + n - 1 = \tfrac{1}{2} n(n-1).$$

This is also the degree of $\prod_{i<j}(x_i - x_j) = \Delta(X)$. Hence terms containing any λ_{ij} drop out on integration. End of proof.

To calculate the constant in Eq. (2.25) we may choose

$$V(A) = \operatorname{tr} A^2 = \sum_{i=1}^n x_i^2.$$

Then

$$\int \exp\left\{ -\sum_1^n (x_i^2 + y_i^2 - 2c x_i y_i) \right\} \Delta(X) \Delta(Y) dX dY$$

$$= \int \exp\left\{ -\sum_1^n ((1-c^2)x_i^2 + (y_i - cx_i)^2 \right\} \Delta(X)\Delta(Y)dXdY$$

$$= \int \exp\left\{ -\sum_1^n ((1-c^2)x_i^2 + y_i^2) \right\} \Delta(X)\Delta(Y + cX)dXdY$$

$$= c^{(1/2)n(n-1)} \int \exp\left\{ -(1-c^2)\sum_1^n x_i^2 \right\} \Delta^2(X)dX \int e^{-\sum_1^n y_i^2} dY$$

by the lemma. The integration over the y_i is elementary. For that over x_i change variables to

$$x_i' = (1-c^2)^{1/2} x_i$$

so that

$$\int \exp\left\{ -(1-c^2)\sum_1^n x_i^2 \right\} \Delta^2(X)dX = (1-c^2)^{-(1/2)n^2} \int e^{-\sum_1^n x_i^2} \Delta^2(X)dX.$$

The last integral can be evaluated[2] by introducing Hermite polynomials which are orthogonal for the gaussian weight. The final result is

$$\int \exp\{ -\Sigma(x_i^2 + y_i^2 - 2cx_i y_i) \} \Delta(X)\Delta(Y)dXdY$$

$$= c^{(1/2)n(n-1)} \pi^{(1/2)n} (1-c^2)^{-(1/2)n^2} n! \prod_0^{n-1} (\pi^{1/2} 2^{-i} i!)$$

$$= \pi^n (\tfrac{1}{2}c)^{(1/2)n(n-1)} (1-c^2)^{-(1/2)n^2} \prod_1^n i!$$

For Eq. (5.6) we need to evaluate

$$\int \sum_1^n (x_i^4 + y_i^4) \exp\left\{ -\sum_1^n (x_i^2 + y_i^2 - 2x_i y_i) \right\} \Delta(X)\Delta(Y)dXdY,$$

$$= 2\int \sum_1^n x_i^4 \exp\left\{ -\sum_1^n ((1-c^2)x_i^2 + y_i^2) \right\} \Delta(X)\Delta(cX)dXdY$$

$$= 2nc^{(1/2)n(n-1)} (1-c^2)^{-(1/2)(n^2+4)} \pi^{n/2} \int \exp\left(-\sum_1^n x_i^2 \right) x_1^4 \Delta^2(X)dX$$

as above. Once more introducing Hermite polynomials, the last integral is seen to be[2]

$$\int \exp\left(-\sum_1^n x_i^2 \right) x_i^4 \Delta^2(X)dX$$

$$= (n-1)! \prod_0^{n-1} (2^{-i} i! \pi^{1/2}) \sum_{i=0}^{n-1} \frac{\int x^4 H_i^2(x) e^{-x^2} dx}{\int H_i^2(x) e^{-x^2} dx}$$

2 [4], Chap. 6

Finally, from the three term recurrence relation and orthogonality one gets

$$\int x^4 H_i^2(x) e^{-x^2} dx = \tfrac{3}{4}(2i^2 + 2i + 1) \int H_i^2(x) e^{-x^2} dx.$$

Putting everything together one sees that

$$\int \sum_1^n (x_i^4 + y_i^4) \exp\left\{ - \sum_1^n (x_i^2 + y_i^2 - 2c x_i y_i) \right\} \Delta(X) \Delta(Y) dX dY$$

$$= (1 - c^2)^{-2} (n^3 + \tfrac{1}{2} n) \int \exp\left\{ - \sum_1^n (x_i^2 + y_i^2 - 2 x_i y_i) \right\} \Delta(X) \Delta(Y) dX dY,$$

implying Eq. (5.6)

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