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The Classical Field Limit of $P(\varphi)_2$ Quantum Field Theory

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Abstract. It will be shown that, for a convex polynomial P, the $P(\varphi)_2$ quantum field theory without cutoff has a classical field limit as Planck's constant h tends to zero. This extends work of Hepp [1], who considered theories with a space cutoff.

0. Introduction

The purpose of this paper is to show that, for suitable interactions P, the two-dimensional $P(\varphi)_2$ models have a limit, as Planck's constant h tends to zero, which describes a classical field theory. The framework in which this result is proved was first formulated by Hepp [1], who proved a similar result for models with a space cutoff. In the present case, without any cutoff, the technical details are much harder, in particular because the physical Hilbert space Ξ_{ph} of the interacting theory turns out to depend on h.

The central idea is to define a vector $\Psi(h, u^0, v^0) \in \Xi_{ph}(h)$, which depends on initial conditions u^0, v^0 for the classical field theory (see Theorem 1), and then to prove that the expectation value at any time t, of a bounded function of the time zero field tends as h tends to zero, to the same function of the classical field at time t with the initial conditions u^0, v^0. The method, which is also the underlying method in Hepp's paper, is to make a change of variable in the physical Hilbert space, a space of fields, to centre around the classical field, instead of around the zero field. This change takes Ψ to the vacuum vector, and transforms the Hamiltonian into a time dependent Hamiltonian (see Lemma 2). To make this formalism rigorous, a space cutoff ℓ is introduced, and then the problem is to define and control the time dependent Hamiltonian uniformly in h and ℓ. The definition of the time dependent Hamiltonian is by the method of time dependent quadratic forms [9] and the control is by cluster expansion techniques [8], the latter requiring an analytic continuation in an appropriate, but quite unphysical, parameter to a Euclidean region.

It should be noted, that just as there are classical field states, the vectors $\Psi(h, u^0, v^0)$, so also are there classical particle states in the $P(\phi)_2$ models [12].

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1. Statement of Theorem

Let $P(x)=\sum_{n=3}^{N}\lambda_n x^n$ be a polynomial which is bounded below and convex. In particular, N is even, $\lambda_N>0$, and P and the second derivative of P are everywhere non-negative.

Let μ_h be the Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance $h(-\Delta+m_0^2)^{-1}$, where Δ is the Laplacian on $\mathbb{R}^2$, m_0 is the bare mass, and h is Planck's constant.

The cutoff Schwinger functions for the interaction determined by P are given by

$$S_\Lambda(f_1,\dots,f_n)=Z^{-1}\int\phi(f_1)\dots\phi(f_n)e^{-h^{-1}\int_\Lambda :P(\phi):d^2x}d\mu_h(\phi) \tag{1}$$

where

$$Z=\int e^{-h^{-1}\int_\Lambda :P(\phi):d^2x}d\mu_h(\phi).$$

Eckmann [2] has shown that, for h sufficiently small, the limit $\lim_{\Lambda\to\mathbb{R}^2} S_\Lambda$ can be constructed using cluster expansion techniques and defines a family of Schwinger functions for a Wightman theory with unique vacuum. Let Ξ_{ph} denote the physical Hilbert space of this theory, and let H_{ph} be the Hamiltonian. Note that $\Xi_{ph}=\Xi_{ph}(h)$ does depend on h. Eckmann's Theorem is the special case of the results to be proved below when the classical field vanishes.

The classical field equation corresponding to the interaction P is

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+m_0^2u+P'(u)=0, \tag{2}$$

which is equivalent to the Hamilton equations

$$\frac{\partial u}{\partial t}=v,\quad \frac{\partial v}{\partial t}=\frac{\partial^2 u}{\partial x^2}-m_0^2u-P'(u). \tag{3}$$

In this paper it is impossible to take units with $h=1$, so (2) implies that m_0 has units of inverse length, rather than of mass.

A proof of the following existence theorem is given in [3, 4 Sect. X.13].

Theorem 1. *Let $u^0, v^0\in C_0^\infty(\mathbb{R}^1)$ be real valued. Then, for P as above, there are unique real valued functions $u(t,x), v(t,x)\in C^\infty(\mathbb{R}^2)$ satisfying the equations (3) with the initial conditions $u(0,x)=u^0(x)$, $v(0,x)=v^0(x)$. Write $u(t)=u(t,\cdot)$, $v(t)=v(t,\cdot)$. Then $u(t), v(t)\in C_0^\infty(\mathbb{R}^1)$, and* $\operatorname{supp} u(t)\subset\{x: d(x,\operatorname{supp} u^0)\leqq t\}$.

Throughout this paper we take a fixed time T and fixed initial conditions $u^0, v^0\in C_0^\infty(\mathbb{R})$. Without further mention, all space cutoffs ℓ will be chosen to satisfy $\operatorname{supp} u(t)\subset[-\ell,\ell]$ for $0\leqq t\leqq T$.

Let φ^0 and π be the canonical time zero fields on Ξ_{ph}, as defined by the C^* algebra approach to $P(\varphi)_2$ [5, 6].

The main result of this paper, Theorem 5, shows that, for h sufficiently small, there is a vector $\Psi(h,u^0,v^0)\in\Xi_{ph}(h)$, which, for any t with $0\leqq t\leqq T$ and for any $f\in\mathscr{S}(\mathbb{R}^1)$, has expectation value for $e^{i\varphi^0(f)}$ approximating the corresponding classical value $e^{iu(t)(f)}$, and that the expectation value for $e^{ih^{-1/2}(\varphi^0-u(t))(f)}$ is approximated by an h independent matrix element in a Gaussian field theory.

Here

$$u(t)(f) = \int u(t, x) f(x) d^1 x.$$

An explicit expression for $\Psi(h, u^0, v^0)$ is

$$\Psi(h, u^0, v^0) = \exp i(\varphi^0(v^0) - \pi(u^0))/h \; \Omega, \tag{4}$$

where Ω is the vacuum vector in Ξ_{ph}.

In order to simplify the h dependence in Eq. (1), let μ be the Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance $(-\Delta + m_0^2)^{-1}$, and let $C: L^2(d\mu_h) \to L^2(d\mu)$ be the unitary map defined by $C 1 C^* = 1$, $C\varphi(f)C^* = h^{1/2}\varphi(f)$. Since C is unitary, it follows that,

$$S_\Lambda(f_1, \ldots, f_n) = Z^{-1} h^{(1/2)n} \int \varphi(f_1) \ldots \varphi(f_n) e^{-h^{-1}\int_\Lambda :P(h^{1/2}\varphi):d^2x} d\mu(\varphi) \tag{5}$$

and that

$$Z = \int e^{-h^{-1}\int_\Lambda :P(h^{1/2}\varphi):d^2x} d\mu(\varphi).$$

The factor $h^{(1/2)n}$ in (5) demonstrates that, if Ξ is the Hilbert space of the field theory with Schwinger functions equal to $h^{-(1/2)n} S_{\mathbb{R}^2}$, then C induces a unitary map $C: \Xi_{ph} \to \Xi$, such that $C\Omega = \Omega$, $C\varphi(f)C^* = h^{1/2}\varphi(f)$. The existence of Ξ_{ph} is equivalent to that of Ξ, but note that the equivalence breaks down at $h = 0$, and that Theorem 5 shows that it is Ξ_{ph} to which the usual physical interpretation applies, while Ξ is a mathematical convenience, giving a precise expression of the deviation of quantum fields from classical fields in the limit $h \to 0$.

Introduce a space cutoff ℓ. Let H_0 be the free Hamiltonian corresponding to the Euclidean measure μ.

Let
$$H_\ell = H_0 + h^{-1} \int_{-\ell}^{\ell} :P(h^{1/2}\varphi^0):d^1x.$$

The following time dependent Hamiltonians are also needed;

$$H_\ell(s) = H_0 + h^{-1} \int_{-\ell}^{\ell} \{ :P(h^{1/2}\varphi^0 + u(s)): - P(u(s)) - h^{1/2}\varphi^0 P'(u(s)) \} d^1x.$$

Since $u(s) \in C_0^\infty(\mathbb{R}^1)$, it is a standard result that H_ℓ and $H_\ell(s)$ define self-adjoint semi-bounded operators on Fock space $\mathscr{F} = L^2(d\mu)$ [7]. Let $E_\ell = E_\ell(h)$ be the lower bound of H_ℓ, and, more generally, if H is any semi-bounded operator, let $E(H)$ denote its lower bound.

Let
$$Q_\ell(s) = \sum_{n=3}^{N} \sum_{r=2}^{n-1} \lambda_n \binom{n}{r} h^{(1/2)r-1} \int_{-\ell}^{\ell} :(\varphi^0)^r: u(s)^{n-r} d^1x$$

so that
$$H_\ell(s) = H_\ell + Q_\ell(s).$$

Let
$$\hat{H}_\ell = H_\ell - E_\ell.$$

Choose $\alpha > 0$. Then, because all the operators involved have a common core, the following inequalities between quadratic forms are immediate consequences of the definitions of the $E(H)$.

$$0 \leqq \frac{\alpha}{1+\alpha}\hat{H}_\ell \leqq \hat{H}_\ell + Q_\ell(s) + \frac{1}{1+\alpha}(E_\ell - E(H_\ell + (1+\alpha)Q_\ell(s)))$$
$$\leqq \frac{1+\alpha}{\alpha}\hat{H}_\ell + \frac{1}{1+\alpha}(E_\ell - E(H_\ell + (1+\alpha)Q_\ell(s)))$$
$$+ \frac{1}{\alpha}(E_\ell - E(H_\ell - \alpha Q_\ell(s))) \tag{6}$$

Define $K_\ell(s) = \hat{H}_\ell + Q_\ell(s) + \varepsilon_1$, where ε_1 is a constant, independent of h and ℓ, which will be chosen in Lemma 6, and proved to bound $(E_\ell - E(H_\ell + (1+\alpha)Q_\ell(s)))$, so that $K_\ell(s) \geqq 0$.

Let $\mathscr{H}_{+1} \subset \mathscr{F} \subset \mathscr{H}_{-1}$ be the scale of spaces associated with $\hat{H}_\ell$. $\mathscr{H}_{\pm 1}$ depend on h and ℓ. It will be shown that there is an operator $U(t,0)$ on $\mathscr{F}$, such that $t \to U(t,0)\Omega_\ell$ is the unique continuous function $x(t)$ from $\mathbb{R}$ to $\mathscr{H}_{+1}$ such that $x(0) = \Omega_\ell$, and the equation $\frac{d}{dt}(x(t)) = -iK_\ell(t)x(t)$ holds in $\mathscr{H}_{-1}$. Formally, $U(t,0) = T\exp\left(-i\int_0^t K_\ell(s)ds\right)$ where T is the "time ordering symbol".

Using this uniqueness result, we prove

Lemma 2. *Set* $\Psi_\ell(h,u^0,v^0) = C^* e^{ih^{-1/2}(\varphi^0(v^0) - \pi(u^0))}\Omega_\ell$. *Then*

$$(\Psi_\ell(h,u^0,v^0), C^* e^{iH_\ell t} C e^{ih^{-1/2}(\varphi^0 - u(t))(f)} C^* e^{-iH_\ell t} C \Psi_\ell(h,u^0,v^0))$$
$$= (\Omega_\ell, U(t,0)^* e^{i\varphi^0(f)} U(t,0)\Omega_\ell). \tag{7}$$

Proof. Set $x(t) = e^{ia(t)} e^{-ih^{-1/2}(\varphi^0(v(t)) - \pi(u(t)))} e^{-iH_\ell t} C \Psi_\ell(h,u^0,v^0)$ where

$$a(t) = h^{-1}\int_0^t \int \{P(u) - \tfrac{1}{2}uP'(u)\} d^1x\, ds + (E_\ell - \varepsilon_1)t$$

Then, by explicit differentiation and use of the canonical commutation relations, $x(t)$ satisfies the hypotheses given above, and so equals $U(t,0)\Omega_\ell$. It is at this point that it is necessary that $\operatorname{supp} u(t) \subset [-\ell, \ell]$. The analytic details are dealt with by noting that $\Omega_\ell \in \mathscr{H}_{+1}$, and that, by explicit calculation, $e^{i\varphi^0(f)}$, $e^{i\pi(f)} \in \mathscr{L}(\mathscr{H}_{\pm 1}, \mathscr{H}_{\pm 1})$. □

By the C^* algebra approach to the limit $\ell \to \infty$ [5, 6], the left hand side of Eq. (7) converges to

$$(\Psi(h,u^0,v^0), e^{iH_{Ph}t/h} e^{ih^{-1/2}(\varphi^0 - u(t))(f)} e^{-iH_{Ph}t/h} \psi(h,u^0,v^0))$$

if it converges at all. It will, in fact, be proved to converge uniformly in h.

Letting $h \to 0$ while holding ℓ fixed leads to the classical field theory results of Hepp [1]. In this paper these results will be extended by letting $\ell \to \infty$ before taking $h \to 0$. In order to do this, the powerful techniques of the cluster expansion [8] are needed. The cluster expansion is defined in the "Euclidean time region", that is, an analytic continuation is made in the time variable from real to imaginary time. In the present case, the continuation is made, not in time, but in a completely unphysical variable z. Formally, we set $U(z,t,s) = T\exp\left(-iz\int_s^t K_\ell(r)dr\right)$.

$U(z, t, s)$ will be defined as a bounded operator on $\mathscr{F}$ for $\operatorname{Im} z \leqq 0$, and the cluster expansion will be used in the region $\operatorname{Re} z = 0$. Of course, $U(t, s) = U(1, t, s)$.

In order to make the analytic continuation, the following extension of Vitali's theorem will be used,

Lemma 3. Let $\Gamma_1 = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, $\Gamma_2 = \{z \in \mathbb{C} : |z| > 0, \operatorname{Im} z \leqq 0\}$. For $0 \leqq h \leqq h_0$ *suppose that a)* $f_n(z, h)$ *is a sequence of functions which are analytic in* z *on* Γ_1 *and continuous and differentiable in* z *on* Γ_2.

b) On any compact subset of Γ_2, $|f_n(z, h)|$ *and* $|\frac{df_n}{dz}(z, h)|$ *are bounded uniformly in* h *and* z.

c) $f_n(z, h)$ *converges for* $z \in \Gamma_1$ *with* $\operatorname{Re} z = 0$ *to a function continuous in* h.

Then, there exists a unique function $f(z, h)$ *which is analytic in* z *on* Γ_1, *continuous in* z *on* Γ_2, *and continuous in* h *for* $z \in \Gamma_2$, *such that* $f_n(z, h)$ *converges to* $f(z, h)$ *for* $z \in \Gamma_2$ *and* $0 \leqq h \leqq h_0$.

Proof. 1. Fix h. By Vitali's theorem $f(z, h)$ exists and is analytic in z on Γ_1. Thus $f_n(z, h)$ converges on a dense subset of Γ_2. But, on any compact subset of Γ_2, the sequence is uniformly bounded, and also, because the derivatives are uniformly bounded, is uniformly continuous. Thus $f_n(z, h)$ converges to a unique function $f(z, h)$ which is continuous in z. Further, the continuity in z is uniform in h.

2) Let $(h_n)_{n=1}^{\infty}$ be a sequence between 0 and h_0 converging to h. Then, we can apply the same method to show that $f(z, h_n) \to f(z, h)$ for all $z \in \Gamma_2$, because

a′) $f(z, h_n)$ is analytic in z on Γ_1 and continuous in z on Γ_2.

b′) On any compact subset of Γ_2, $|f(z, h_n)|$ is bounded uniformly in h and n, and the sequence $f(z, h_n)$ is equicontinuous in z. □

c′) $f(z, h_n)$ converges for $z \in \Gamma_1$ with $\operatorname{Re} z = 0$.

Theorem 4. *Given a time* $T > 0$ *and a pair of classical initial conditions* $u^0, v^0 \in C_0^\infty(\mathbb{R}^1)$, *there exists* $h_0 > 0$, *such that, for* $0 \leqq h \leqq h_0$ *and for* $0 \leqq t \leqq T$, $\Xi_{ph}(h)$ *can be defined by the cluster expansion, and* $U(z, t, 0)$ *can be defined by Theorem 7. Let* $F(\ell, h, t, z) = (\Omega_\ell, U(\bar{z}^{-1}, t, 0)^* e^{i\varphi^0(f)} U(z, t, 0)\Omega_\ell)$ *and for any sequence* $l_n \to \infty$, *set* $f_n(z, h) = F(\ell_n, h, t, z)$. *Then* $f_n(z, h)$ *satisfies the hypotheses of Lemma 3.*

Theorem 4 is the central technical result of this paper. Hypotheses a and b of Lemma 3 will be verified in Sect. 2, where $U(z, t, s)$ will be constructed, and hypothesis c will be verified in Sect. 3.

An immediate consequence of Theorem 4 is the main physical result of this paper:

Theorem 5. (i) $\lim_{h \to 0} (\Psi(h, u^0, v^0), e^{itH_{ph}/h} e^{i\varphi^0(f)} e^{-itH_{ph}/h} \Psi(h, u^0, v^0)) = e^{iu(t)(f)}$

(ii) $\lim_{h \to 0} (\Psi(h, u^0, v^0), e^{itH_{ph}/h} e^{ih^{-1/2}(\varphi^0 - u(t))(f)} e^{-itH_{ph}/h} \Psi(h, u^0, v^0))$
$= (\Omega_0, T(e^{-i\int_0^t H^0(s)ds})^* e^{i\varphi^0(f)} T(e^{-i\int_0^t H^0(s)ds})\Omega_0)$.

Here Ω_0 *is the free vacuum in* $\mathscr{F}$,

and

$$H^0(s) = H_0 + \tfrac{1}{2} \int_{\mathbb{R}} :(\varphi^0)^2(x) : P''(u(s, x)) d^1 x .$$

Note that, since T is chosen arbitrarily in Theorem 4, Theorem 5 holds for all time t.

2. Time Ordered Quadratic Forms

The inequalities (6) show that $K_\ell(s)\in\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{-1})$ (the space of bounded operators from $\mathscr{H}_{+1}$ to $\mathscr{H}_{-1}$). Lemma 6, which will be proved in Sect. 3, shows that $\|K_\ell(s)\|_{+1,-1}$ is bounded independently of ℓ and h. In addition, define

$$\begin{aligned}\Delta K_\ell(t,s)&=(t-s)^{-1}(K_\ell(t)-K_\ell(s))\\&=\sum_{n=3}^{N}\sum_{r=2}^{n-1}\lambda_n\binom{n}{r}h^{(1/2)r-1}\int_{-\ell}^{\ell}:(\varphi^0)^r(x):\Delta u^{n-r}(t,s,x)d^1x\end{aligned}$$

where

$$\Delta u^{n-r}(t,s,x)=(t-s)^{-1}(u^{n-r}(t,x)-u^{n-r}(s,x)).$$

Now

$$\begin{aligned}0&\leqq\hat{H}_\ell+\alpha\Delta K_\ell+(E_\ell-E(H_\ell+\alpha\Delta K_\ell))\\&\leqq 2\hat{H}_\ell+(E_\ell-E(H_\ell+\alpha\Delta K_\ell))+(E_\ell-E(H_\ell-\alpha\Delta K_\ell)),\end{aligned}$$

so Lemma 6 also shows that $\Delta K_\ell\in\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{-1})$ with $\|\Delta K_\ell\|_{+1,-1}$ bounded independently of ℓ, h, s, and t.

By a similar bound, $\lim_{t\to s}\Delta K_\ell(t,s)$ exists in norm in $\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{-1})$, because $u\in C^\infty(\mathbb{R}^2)$.

Lemma 6. *Given a bare mass m_0 and a time $T>0$, there exists $\alpha_0>0$ such that, for $0\leqq s<t\leqq T$, for $0\leqq\alpha\leqq\alpha_0$, for all bare masses $m\geqq m_0$, and for $0<h\leqq 1$, there are constants $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ independent of s, t, α, m, h, and of ℓ, such that*

$$\begin{gathered}E_\ell-E(H_\ell+(1+\alpha)Q_\ell(s))\leqq\varepsilon_1,\ E_\ell-E(H_\ell-E(H_\ell-\alpha)Q_\ell(s))\leqq\varepsilon_2,\\E_\ell-E(H_\ell+\alpha\Delta K_\ell(t,s))\leqq\varepsilon_3,\ E_\ell-E(H_\ell-\alpha\Delta K_\ell(t,s))\leqq\varepsilon_4.\end{gathered}$$

Now we make an abstraction of this situation. Lemma 6 shows that the hypotheses of Theorem 7, which are modelled closely on those of the theorems quoted in the proof below, are satisfied. Thus Theorem 7 can be applied to the proof of Theorem 4.

Theorem 7. *Let H be a positive self-adjoint operator on a Hilbert space $\mathscr{H}$ with norm $\|\ \|$, and let $\mathscr{H}_{+1}\subset\mathscr{H}\subset\mathscr{H}_{-1}$ be the corresponding scale of spaces, so that $\|x\|_{\pm1}=\|(H+1)^{\pm1/2}x\|$. For $0\leqq t\leqq T$, let $H(t)$ be a self-adjoint operator on $\mathscr{H}$ satisfying $0\leqq C^{-1}H\leqq H(t)\leqq CH$, where C is positive constant independent of t. Suppose further, that $\Delta H(t,s)=(t-s)^{-1}(H(t)-H(s))$ belongs to $\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{-1})$ for $0\leqq s<t\leqq T$, is strongly uniformly continuous in s and t, and that $\|\Delta H(t,s)\|_{+1,-1}\leqq D$, where D is a constant independent of s and t. Finally, suppose that $\lim_{s\to t}\Delta H(t,s)$ exists uniformly in t for $0\leqq t\leqq T$, belongs to $\mathscr{L}(\mathscr{H}_{+1},\mathscr{H}_{-1})$, and is strongly uniformly continuous in t.*

Then, for all $z\in\mathbb{C}$ with $\operatorname{Im}z\leqq 0$, and for $0\leqq s\leqq t\leqq T$,

(i) *An operator $U(z,t,s)\in\mathscr{L}(\mathscr{H}_{-1},\mathscr{H}_{-1})$ exists with $\|U(z,t,s)\|_{-1,-1}\leqq 1$.*

(ii) For all $\psi \in \mathscr{H}_{+1}$, $x(t) = U(z,t,s)\psi \in \mathscr{H}_{+1}$, *is continuous in* t *in* $\mathscr{H}_{+1}$, *and is the unique solution in* $\mathscr{H}_{-1}$ *to* $\frac{d}{dt}x(t) = -izH(t)x(t)$ *with* $x(s) = \psi$.

(iii) $U(z,t,s) \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{+1})$ *with* $\| U(z,t,s) \|_{+1,+1} \leqq C^2 e^{D(t-s)}$

(iv) $U'(z,t,s) = \frac{d}{dz} U(z,t,s) \in \mathscr{L}(\mathscr{H}_{+1}, \mathscr{H}_{-1})$ *exists and satisfies* $\| U'(z,t,s) \|_{+1,-1} \leqq (t-s) C^3 e^{(t-s)D}$.

Proof. Parts (i) and (ii) are due to Kisyński[9]. A brief reformulation of Kisyński's method is given here in a simple and accessible form in order to prove parts (iii) and (iv).

Let $H_1(s)$ be the closure of $H(s)$ considered as an operator in $\mathscr{H}_{-1}$. Then $H_1(s)$ is self-adjoint with domain $\mathscr{H}_{+1}$ because it is closed and symmetric, and $H(s) \pm i$ have dense image in $\mathscr{H}_{-1}$. The hypotheses of Theorem 7 have been chosen so that the operators $H_1(s)$ satisfy the hypotheses of Kato and Yosida's theorems on time dependent operators with common domains, in particular Theorem X.70 of [4], which is a replication of Theorem 1, Sect. X.14 of [10].

It follows that $U(z,t,s)$ can be constructed on $\mathscr{H}_{-1}$ as a limit of piecewise constant propagators. For $k = 1,2,\ldots$ set

$$U_k(z,t,s) = \exp(-iz(t-s)H_1((i-1)T/k)) \text{ if } (i-1)T/k \leqq s \leqq t \leqq iT/k$$

and

$$U_k(z,t,r) = U_k(z,t,s)\,U_k(z,s,r) \text{ if } 0 \leqq r \leqq s \leqq t \leqq T.$$

Then, by the quoted theorem, $U(z,t,s)\varphi = \lim_{k\to\infty} U_k(z,t,s)\varphi$ exists for $\varphi \in \mathscr{H}_{-1}$, and if $\psi \in \mathscr{H}_{+1}$, then $x(t) = U(z,t,s)\psi$ satisfies (ii) and $\| x(t) \|_{-1} \leqq \| \psi \|_{-1}$.

This proves (i) and (ii), except for uniqueness, which is proved by Kisyński ([9] Theorem 2.4).

(iii) follows from the construction of [4], Theorem X. 70. In fact, in all essentials, it is given as the lemma to that theorem, which shows further that,

$$\| U_k(z,t,s) \|_{+1,+1} \leqq C^2 (1 + (D/k))^2 e^{(t-s)D}. \tag{8}$$

(iv) Given $t \geqq s$ and k, set $j = [kt/T]$, $\ell = [ks/T]$. Then, if $j = \ell$, $U_k'(z,t,s) = -i(t-s)U_k(z,t,s)H_1(jT/k)$ if $j > \ell$,

$$\begin{aligned} U_k'(z,t,s) = & -i(t - j(T/k))H_1(jT/k)\,U_k(z,t,s) \\ & -i(T/k) \sum_{n=\ell+1}^{j-1} U_k(z,t,nT/k)H_1(nT/k)U_k(z,nT/k,s) \\ & -i((\ell+1)(T/k) - s)U_k(z,t,s)H_1(\ell T/k) \end{aligned}$$

Using (i) and (8), it follows that

$$\| U_k'(z,t,s) \|_{+1,-1} \leqq (T/k)(j - \ell + 1)C^3 (1 + (D/k))^2 e^{(t-s)D}$$

and that $U_k'(z,t,s)$ converges strongly on $\mathscr{H}_{+1}$, uniformly in z, t, and s, to $-i\int_s^t U(z,t,r)H_1(r)\,U(z,r,s)\,dr$, so this equals $\frac{d}{dz}U(z,t,s)$, and (iv) is satisfied.

3. The Euclidean Region

The Euclidean region is the set of $z = -i\zeta$ for $\zeta \in \mathbb{R}$, $\zeta > 0$. This section proves that $f_n(z, h)$ defined in Theorem 4 satisfies hypothesis c of Lemma 3. Lemma 6 will also be proved here.

With the notation of Theorem 4,

$$F(\ell, h, t, -i\zeta) = (\Omega_\ell, T(e^{-\zeta^{-1}\int_0^t K_\ell(s)ds})^* e^{i\varphi^0(f)} T(e^{-\zeta\int_0^t K_\ell(s)ds})\Omega_\ell).$$

The Feynman–Kac formula and the definition of Ω_ℓ (see [7], section V. 4) show that this is equal to the limit as $S \to \infty$ of

$$\frac{(\Omega_0, e^{-SH_\ell} T(e^{-\zeta^{-1}\int_0^t (K_\ell(s)+E_\ell)ds})^* e^{i\varphi^0(f)} T(e^{-\zeta\int_0^t (K_\ell(s)+E_\ell)ds}) e^{-SH_\ell}\Omega_0)}{(\Omega_0, e^{-(2S+(\zeta^{-1}+\zeta)t)H_\ell}\Omega_0)}$$

$$= \frac{\Theta(\Omega_0\, e^{-SH_\ell} T(e^{-\int_{-t/\zeta}^0 H\,(\zeta s+t)ds})^* e^{i\varphi^0(f)} T(e^{-\int_0^{\zeta t} H_\ell(\zeta^{-1}s)ds}) e^{-SH_\ell}\Omega_0)}{(\Omega_0, e^{-(2S+(\zeta^{-1}+\zeta)t)H_\ell}\Omega_0)}$$

$$= \frac{\Theta \int e^{i\varphi^0(f)} e^{-h^{-1}\int_\Lambda \{:P(h^{0/2}\varphi+g): - P(g) - h^{1/2}\varphi P'(g)\}d^2x} d\mu(\varphi)}{\int e^{-h^{-1}\int_\Lambda :P(h^{1/2}\varphi):d^2x} d\mu(\varphi)}$$

where

$$\Theta = e^{-(\zeta^{-1}+\zeta)t\varepsilon_1},$$

Λ is the subset $[-S - \zeta^{-1}t, S+\zeta t] \times [-\ell, \ell]$ of $\mathbb{R}^2$, g is the function on $\mathbb{R}^2$ defined by

$$\begin{aligned} g(s,x) &= 0 \text{ for } s < -t/\zeta \text{ and } s > \zeta t \\ &= u(-\zeta s, x) \text{ for } -t/\zeta \leqq s \leqq 0 \\ &= u(s/\zeta, x) \text{ for } 0 \leqq s \leqq \zeta t, \end{aligned}$$

and

$$\varphi^0(f) = \varphi(\delta_0 \otimes f).$$

Note that the adjoint of $U(z, t, s)$ has the time ordering reversed.

The required results, (convergence in ℓ and uniform continuity in h of $F(\ell, h, t, -i\zeta)$), now follow by standard methods of constructive quantum field theory [8], on proving that the measure

$$d\nu_\Lambda(\varphi) = Z^{-1} e^{-h^{-1}\int_\Lambda (:P(h^{1/2}\varphi+g): - P(g) - h^{1/2}\varphi P'(g))d^2x} d\mu(\varphi)$$

has a convergent cluster expansion:

Theorem 8. *There exists $\varepsilon > 0$, independent of m_0, such that, if $0 \leqq h^{1/2} \leqq \varepsilon m_0^2$ then, for any Wick ordered polynomials A, B of the form $\int :\varphi(x_1)^{n_1}: \ldots :\varphi(x_j)^{n_j}: w(x_1, \ldots, x_j)\, dx$ with* supp A *and* supp B *separated by a strip of width d, there is a constant $M_{A,B}$ and a positive constant m independent of A, B, and h such that*

$$|\int AB\, d\nu_\Lambda - \int A\, d\nu_\Lambda \int B\, d\nu_\Lambda| \leqq M_{A,B} e^{-md}.$$

Proof. By Theorem 1, $\sup_{(s,x)\varepsilon\Lambda} |g(s,x) < \infty$, and also there is a fixed interval $[a, b] \subset$

$[-\ell,\ell]$ such that $\operatorname{supp} u(s) \subset [a,b]$ for $0 \leqq s \leqq T$. Define $\Lambda_0 = [-t/\zeta, \zeta t] \times [a,b]$. Then $\operatorname{supp} g \subset \Lambda_0$.

Now, following [8], and adopting the notation of that paper, we expand $S_\Lambda(x) = \int \prod_{i=1}^{n} \varphi(x_i) dv_\Lambda$ into a sum of terms labelled by unions of lattice squares X and lattice lines Γ, with the modification that in the present case, the bounded set X_0 outside which the expansion is made, is chosen to be $X_0 = \Lambda_0 \cup \{x_1, \dots, x_n\}$. Thus

$$S_\Lambda(x) = \sum_{X,\Gamma} \int \partial^\Gamma \int \prod_{i=1}^{n} \varphi(x_i) e^{-W(\Lambda_0, g)} e^{-V(\Lambda \cap X)} d\varphi_{s(\Gamma)} ds(\Gamma) \times Z_{\partial X}(\Lambda \sim X)/Z(\Lambda) \quad (9)$$

where the notation is that of [8] Sect. 3, except that

$$V(\Lambda) = h^{-1} \int_\Lambda :P(h^{1/2}\varphi): d^2x$$

and

$$W(\Lambda_0, g) = h^{-1} \int_{\Lambda_0} \{ :P(h^{1/2}\varphi + g): - P(g) - h^{1/2}\varphi P'(g)\} d^2x - V(\Lambda_0)$$

$$= \sum_{n=3}^{N} \sum_{r=2}^{n-1} \lambda_n \binom{n}{r} h^{(1/2)r-1} \int_{\Lambda_0} :\varphi^r: g^{n-r} d^2x .$$

By choice of X_0, $Z_{\partial X}(\Lambda \sim X)$ is independent of g.

The central result of Eckmann's letter [2] is that the cluster expansion (9) converges for $g = 0$. This follows straightforwardly from the techniques of [8], given Lemma 9 part (i). The proof that (9) converges when g is non-zero also follows from the proof of [8], using both parts of Lemma 9 and Hölder's inequality to replace corollary 9.6 of [8].

Lemma 9. *For any $m_0 > 0$, there are constants K_1 and K_2 such that, for all h with $0 \leqq h \leqq 1$ and for all Λ*

(i) (*Eckmann*) $1 \leqq \int e^{-4V(\Lambda)} d\mu(\varphi) \leqq e^{K_1|\Lambda|}$.

(ii) $\int e^{-4h^{-1}\int_\Lambda \{:P(h^{1/2}\varphi + g): - P(g) - h^{1/2}\varphi P'(g)\} d^2x} d\mu(\varphi) \leqq e^{K_2|\Lambda|}$.

Lemma 9 will be proved as part of Theorem 10, but first we reduce the proof of Lemma 6 to a similar form.

Let $f_i \in C_0^\infty(\mathbb{R}^1)$ $i = 1, 2, \dots, N-2$ with $\operatorname{supp} f_i \subset [a,b]$. Define

$$L = \sum_{n=3}^{N} \sum_{r=2}^{n-1} \lambda_n \binom{n}{r} h^{(1/2)r-1} \int_a^b :(\varphi^0)^r: f_{n-r} d^1x .$$

Now follow [7] Lemma VI. 14 for the proof that $E_\ell - E(H_\ell + L)$ is bounded above. Lemma 6 of this paper states that for suitable f_i in the definition of L, this upper bound is independent of ℓ and h, while Simon [7] is only concerned about independence from ℓ. However, allowing for differences in notation, there are no problems with the dependence on h, and corresponding to inequality (VI. 21) of [7], we obtain

$$-E(H_\ell + L) \leqq -\tfrac{1}{2} E_{\ell + (1/2)a} - \tfrac{1}{2} E_{\ell - (1/2)b} - E(H(a,b,\lambda_0) + \lambda_0 L)/\lambda_0 \quad (10)$$

where

$$\lambda_0 = 2/(1 - e^{-m_0}) \text{ and } H(a, b, \lambda) = H_0 + \lambda \int_a^b h^{-1} :P(h^{1/2} \varphi^0) d^1 x.$$

Now, there are constants α_∞ and β_∞ (possibly depending on h) such that

$$-\alpha_\infty \ell \leqq E_\ell \leqq -\alpha_\infty \ell - \beta_\infty,$$

so

$$-\tfrac{1}{2}E_{\ell+(1/2)a} - \tfrac{1}{2}E_{\ell-(1/2)b} \leqq -E_\ell - \tfrac{1}{4}\alpha_\infty(b-a) - \beta_\infty. \tag{11}$$

The next step is to verify that $-\frac{1}{4}\alpha_\infty(b-a) - \beta_\infty$ is bounded above independently of h.

$$-\tfrac{1}{4}\alpha_\infty(b-a) < 0 \tag{12}$$

because $b > a$ and $\alpha_\infty > 0$ ([7] Theorem VI.2a).

A proof that β_∞ is bounded below is also given by Simon ([7] Theorems VI.10, VI.7, and VI.2), and reduces to

$$B_\infty \geqq CE(H(0, t, 2r(1 - e^{-m_0 t})^{-1}))/t \tag{13}$$

where C is a constant independent of h, $t > 0$ is arbitrary, and $1 < r < 2$. (t and r are also independent of h).

Combining inequalities (10) – (13) gives

$$E_\ell - E(H_\ell + L) \leqq -CE(H(0, t, 2r(1 - e^{-m_0 t})^{-1}))/t - E(H(a, b, \lambda_0) + \lambda_0 L)/\lambda_0,$$

so the proof of Lemma 6 has been reduced to that of Theorem 10.

Theorem 10. *Suppose given finite positive constants m_0, ρ, and $\bar{\lambda}$. Take any set of $N-1$ bounded functions $\{f, f_1, \dots, f_{N-2}\}$ in $C_0^\infty(\mathbb{R}^2)$ such that $\|f\|_\infty < \rho$, $\|f_i\|_\infty < \rho$ $i = 1, 2, \dots, N-2$, and define, for $\Lambda \subset \mathbb{R}^2$,*

$$U(\Lambda) = h^{-1} \int_\Lambda \{ :P(h^{1/2}\varphi + f): - P(f) - h^{1/2}\varphi P'(f)\} d^2 x,$$

$$L(\Lambda) = \sum_{n=3}^{N} \sum_{r=2}^{n-1} \lambda_n \binom{n}{r} h^{1/2r-1} \int_\Lambda :\varphi^r(x): f_{n-r}(x) d^2 x.$$

Then, there is a constant $\alpha_0 > 0$, depending only on m_0, ρ, and $\bar{\lambda}$, and a constant $K = K(\alpha_0, m_0, \rho, \bar{\lambda})$, such that, for all α and λ with $0 \leqq \lambda \leqq \bar{\lambda}$ and $|\alpha| \leqq \alpha_0$,

$$\int e^{-\lambda U(\Lambda) - \alpha\lambda L(\Lambda)} d\mu \leqq e^{K|\Lambda|}.$$

Proof. The proof is in five steps.

(i) Choose ε with $0 < \varepsilon m_0$. Then there is a constant K_3 such that

$$\int e^{(1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2 x} d\mu(\varphi) \leqq e^{K_3|\Lambda|}. \tag{14}$$

Proof. Let σ be the Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance $C(\varepsilon) = (-\Delta + m_0^2 - \varepsilon^2 \chi_\Lambda)^{-1}$ (χ_Λ is the characteristic function of Λ).

Then

$$d\mu(\varphi) = M^{-1} e^{-(1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2 x} d\sigma(\varphi) \tag{15}$$

where

$$M = \int e^{-(1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2x} d\sigma(\varphi).$$

(15) can be proved quite easily, by calculating the generating functional of the measure defined by the right hand side.

It follows that $\int e^{+(1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2x} d\mu(\varphi) = M^{-1}$. Now, by Jensen's inequality,

$$M \leqq \exp(-\tfrac{1}{2}\varepsilon^2 \int \int_\Lambda :\varphi^2: d^2x d\sigma(\varphi)) = \exp(-\tfrac{1}{2}\varepsilon^2 \int_\Lambda c(\varepsilon)(x) d^2x)$$

where

$$c(\varepsilon)(x) = \lim_{y \to x} (C(\varepsilon)(x, y) - C(0)(x, y)).$$

$$C(\varepsilon) - C(0) = \varepsilon^2(-\Delta + m_0^2 - \varepsilon^2 \chi^\Lambda)^{-1} \chi^\Lambda (-\Delta + m_0^2)^{-1},$$

so

$$\int_\Lambda c(\varepsilon)(x) d^2x = \varepsilon^2 \int_\Lambda \chi_\Lambda(x) C(\varepsilon)(x, y) \chi_\Lambda(y) C(0)(y, x) d^2x d^2y. \tag{16}$$

But

$$0 \leqq -\Delta + m_0^2 - \varepsilon^2 \leqq -\Delta + m_0^2 - \varepsilon^2 \chi_\Lambda$$

so it follows by the proof of proposition 7.3 of [8], that $C(\varepsilon) \in \mathscr{B}^{2,\delta}_{\text{loc}}(\mathbb{R}^4)$ for $0 \leqq \delta < \frac{1}{2}$.

Hence $C(\varepsilon) L^2_{\text{loc}}(\mathbb{R}^4)$, so the proof of (i) is concluded by applying Hölder's inequality to (16).

(ii) By (i) and Hölder's inequality

$$\int e^{-\lambda U(\Lambda) - \lambda \alpha L(\Lambda)} d\mu \leqq e^{K_3|\Lambda|} \int e^{-(2\lambda U(\Lambda) + 2\lambda\alpha L(\Lambda) + (1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2x)} d\mu.$$

(iii) The proof of the Theorem is now completed by showing that there is a constant $K_4 = K_4(\alpha_0, m_0, \rho, \bar{\lambda})$ such that

$$\int e^{-(2\lambda U(\Lambda) + 2\lambda\alpha L(\Lambda) + (1/2)\varepsilon^2 \int_\Lambda :\varphi^2: d^2x)} d\mu \leqq e^{K_4|\Lambda|}.$$

The demonstration of this is based on the proof in [11] Sect. 2.3 that the exponential of a semi-bounded Wick polynomial in φ is in $L^p(d\mu(\varphi))$ for all $p \geqq 1$.

Introduce a momentum cutoff κ. Let $\varphi_\kappa(x)$ be the cutoff field, and $C_\kappa(x) = \int \varphi_\kappa^2(x) d\mu$ be the cutoff covariance.

For $a, b \in \mathbb{R}$ and $\mathbf{b} = (b_1, \ldots, b_{N-2}) \in \mathbb{R}^{N-2}$, define

$$p(a, b) = P(a + b) - P(b) - aP'(b)$$

and

$$q(a, \mathbf{b}) = \sum_{n=3}^{N} \sum_{r=2}^{n-1} \lambda_n \binom{n}{r} a^r b_{n-r}.$$

Then, by the method of [11] Sect. 2.3, it is sufficient to show that there is a constant $K_5 = K_5(\alpha_0, m_0, \rho, \bar{\lambda})$, which is independent of x, κ, and h, such that for $a = h^{1/2} \varphi_\kappa(x), b = f(x), b_i = f_i(x) \quad i = 1, \ldots, N-2$,

(iv) $\quad 2\lambda h^{-1} :p(a, b): + 2\lambda\alpha h^{-1} :q(a, \mathbf{b}): + \frac{1}{2}\varepsilon^2 :\varphi_\kappa^2(x): \geqq -K_5(C_\kappa(x) + 1)^{(1/2)N}$

First we control the polynomials p and q.

(v) For all $a, b, \mathbf{b}$ such that $|b| \leqq \rho, |b_i| \leqq \rho$ $i = 1, \ldots, N-2$, there is a positive constant $\alpha_0 = \alpha_0(\bar{\lambda}, \rho, m_0)$ and a positive constant $K_6 = K_6(\alpha_0, m_0, \rho, \bar{\lambda})$ such that for all α and λ with $0 \leqq \lambda \leqq \bar{\lambda}$ and $|\alpha| \leqq \alpha_0$

$$2\lambda p(a,b) + 2\lambda\alpha q(a,\mathbf{b}) + \tfrac{1}{2}\varepsilon^2 a^2 \geqq K_6 \sum_{r=2}^{N} |a|^r.$$

Proof. As a function of a, $p(a, b)$ is a convex polynomial with $p(0, b) = 0, p'(0, b) = 0$, $P''(a, b) = P''(a + b) \geqq 0$. Thus $p(a, b) \geqq 0$.

For

$$|\alpha| \leqq 1, \quad |b| \leqq \rho, \quad |b_i| \leqq \rho$$

$a^{-N}(p(a, b) + \alpha q(a,\mathbf{b})) \to \lambda_N$ uniformly in $\alpha, b, \mathbf{b}$, as $a \to \infty$ so there exists $\delta \geqq 1$ such that $|a| \geqq \delta$ implies that

$$P(a,b) + \alpha q(a,\mathbf{b}) \geqq \tfrac{1}{2}\lambda_N a^N \geqq \lambda_N (2N)^{-1} \sum_{r=2}^{N} |a|^r.$$

For $|a| \leqq \delta, |b_i| \leqq \rho, a^{-2}q(a,\mathbf{b})$ is bounded, so there exists $\alpha_0 \leqq 1$ such that

$$2\lambda|\alpha q(a,\mathbf{b})| \leqq \tfrac{1}{4}\varepsilon^2 a^2 \text{ for } 0 \leqq \lambda \leqq \bar{\lambda}, |\alpha| \leqq \alpha_0.$$

Thus, for

$$|a| \leqq \delta, \quad 2\lambda p(a,b) + 2\lambda\alpha q(a,\mathbf{b}) + \tfrac{1}{2}\varepsilon^2 a^2 \geqq \tfrac{1}{4}\varepsilon^2 a$$
$$\geqq \tfrac{1}{4}\varepsilon^2 (N\delta^{N-2})^{-1} \sum_{r=2}^{N} |a|^r.$$

Hence (v), with $K_6 = \min\{\tfrac{1}{4}\varepsilon^2(N\delta^{N-2})^{-1}, \lambda_N/4N\}$.

Proof of (iv). Set $b = f(x), b_i = f_i(x), a = h^{1/2}\varphi_\kappa(x)$.
Then, undoing the Wick ordering,

$$2\lambda h^{-1} :p(a,b): + 2\lambda\alpha h^{-1} :q(a,\mathbf{b}): + \tfrac{1}{2}\varepsilon^2 :\varphi_\kappa^2(x):$$
$$= h^{-1}\left\{2\lambda p(a,b) + 2\lambda\alpha q(a,\mathbf{b}) + \tfrac{1}{2}\varepsilon^2 a^2 - K_6 \sum_{r=2}^{N} |a|^r\right\}$$
$$+ \left(\sum_{n=3}^{N}\sum_{r=2}^{n} h^{(1/2)r-1}\Big(K_6|a|^r/d_r - \sum_{j=1}^{[(1/2)r]} k_{n,r,j} C_\kappa(x)^j a^{r-2j}\Big)\right)$$

where

$$d_r = N - r + 1 \text{ for } r \neq 2, \quad d_2 = N-2;$$

and $k_{n,r,j}$ is independent of h and bounded.

The term in { } is positive by (v), and the second term can be minimised as a function of a to give the required result (iv). □

This completes the proof of Theorem 10.

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