

## Gauge Dependence of World Lines and Invariance of the $S$ -Matrix in Relativistic Classical Mechanics\*

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**Abstract.** The notion of world lines is studied in the constraint Hamiltonian formulation of relativistic point particle dynamics. The particle world lines are shown to depend in general (in the presence of interaction) on the choice of the equal-time hyperplane (the only exception being the elastic scattering of rigid balls). However, the relative motion of a two-particle system and the (classical)  $S$ -matrix are independent of this choice.

### Introduction

We study the notion of particle world lines in the relativistic phase space formulation of classical point particle dynamics developed in [19] on the basis of Dirac's theory of constraint Hamiltonian systems<sup>1</sup> [4, 6, 7].

Aiming at a manifestly covariant picture we start with a  $8N$ -dimensional "large  $N$ -particle phase space"  $\Gamma^N$  equipped with a canonical Poisson bracket structure. The dynamics is specified by the introduction of a  $7N$ -dimensional Poincaré invariant submanifold  $\mathcal{M}$  of  $\Gamma^N$ , called the generalized ( $N$  particle) mass shell. It is

\* A preliminary version of this paper was circulated as ICTP, Trieste, Internal Report IC/79/59

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1 The constraint Hamiltonian approach to the relativistic point particle interaction was also adopted (in fact, rediscovered) in [9]. Recent work by Rohrlich [15], which proceeds on similar lines, differs from ours in that it abandons the notion of individual particle co-ordinates and trajectory (a generalized notion of "relative co-ordinates" – whose sum over all particles is not required to vanish – is used instead). As noted by Prof. Rohrlich (private communication of October 1978) this difference is not essential: a slight modification of his approach allows one to impose a linear relation among the relative co-ordinates  $\xi$  of [15] and hence to define single particle's co-ordinates. A Lagrangian approach to the problem of relativistic point particle interactions, which leads to similar constraint equations is being developed in the work of Takabayasi et al. (see [17] and further references cited therein). The work of Droz-Vincent [5], Bel, Martin [1] and others follows a similar path; their approach differs from ours by introducing from the outset non-canonical position variables (defined only implicitly in terms of the canonical four-dimensional co-ordinates and momenta used in this paper)

given by  $N$  equations

$$\varphi_k = \varphi_k(p_1, \dots, p_N; x_{12}, \dots, x_{N-1N}) = 0, \quad x_{\ell m} = x_\ell - x_m, \quad k, \ell, m = 1, \dots, N$$

which determine the particle energies as functions of the remaining variables. The submanifold  $\mathcal{M}$  is subject to some conditions recapitulated in Sect. I.A. We mention here the important requirement that  $\varphi_k$  are first-class constraints, which means that their Poisson brackets  $\{\varphi_k, \varphi_\ell\}$  vanish on  $\mathcal{M}$ . The functions  $\varphi_k$  not only determine the generalized  $N$ -particle mass shell  $\mathcal{M}$  but also generate  $N$  vector fields on which the restriction  $\omega|_{\mathcal{M}}$  of the symplectic form  $\omega = \sum_{k=1}^N dp_{k\mu} \wedge dx_k^\mu$  on  $\Gamma^N$  is degenerate. The relativistic Hamiltonian is defined as a linear combination of  $\varphi_k$  (with variable coefficients) that leaves invariant some  $(6N+1)$ -dimensional submanifold  $\mathcal{M}_H$  of  $\mathcal{M}$  which defines a one-parameter family of equal-time surfaces. (An example of such a surface is the plane  $nx_k = t$ ,  $k = 1, \dots, N$ , where  $n$  is a time-like vector which may depend on the momenta.) The selection of an equal-time surface, which excludes the unphysical relative time variables, is analogous to specifying a gauge condition and will also be referred to in the sequel in such terms.

In Sect. II.A we introduce a notion of equivalent dynamics which says, essentially, that two sets of constraints  $\varphi_k = 0$  and  $\bar{\varphi}_k = 0$  are equivalent, if they lead to the same particle world lines (for the same gauge and initial conditions) and to the same realization of the Poincaré group. (Equivalent dynamics correspond, in general, to different submanifolds  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  of  $\Gamma^N$ .) This notion is used in Sect. II.B to prove (for the case  $N=2$ ) that only piecewise straight world lines are independent of the choice of the equal-time surface (and hence, independent of the

Lagrange multipliers  $\lambda_k$  in the Hamiltonian  $H = \sum_k \lambda_k \varphi_k$ ). This is the price one pays for using a single-time Hamiltonian formalism for the description of directly interacting relativistic point particles (compare with recent results of Sokolov [16], obtained in an alternative approach to this problem). The relation of this result to the so-called “no-interaction theorem” of Currie et al. [3, 8, 10] is discussed in Sect. II.C (for a recent discussion see also [5, 11, 15]). It is demonstrated in Sect. II.D that in the two-particle case, the relative motion (expressed in terms of the variables  $x_\perp$  and  $p$

$$x_\perp = x + \frac{xP}{w^2} P, \quad x = x_1 - x_2, \quad P = p_1 + p_2, \quad w^2 = -P^2 (> 0), \quad (1)$$

$$p = \mu_2 p_1 - \mu_1 p_2, \quad \mu_1 + \mu_2 = 1, \quad \mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{w^2} \quad (2)$$

orthogonal to the total momentum  $P$  of the system) is gauge invariant and so is the two-particle  $S$ -matrix. The latter property is a consequence of the Birman-Kato invariance principle and remains true in the many-particle case.

## I. Constraint Hamiltonian Formulation of Relativistic $N$ -Particle Dynamics

For the reader's convenience, we start with a reformulation of the constraint Hamiltonian approach to the relativistic  $N$ -body problem developed in [19] (with

special attention to the case  $N=2$ ). In order to avoid unnecessary complications, we shall only deal with *spinless* point particles in this paper. (The general case of (massive) spinning particles is considered in [19].)

### I.A. Generalized $N$ -Particle Mass Shell

A manifestly covariant description of relativistic point particles requires the use of four-dimensional co-ordinates and momenta. We introduce the  $8N$ -dimensional “large phase space”  $\Gamma^N$  defined as the direct product of extended (eight-dimensional) single particle phase spaces:

$$\Gamma^N = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_N, \quad \Gamma_k = T^*M_k, \quad (1.1)$$

where  $T^*M_k$  is the cotangent bundle over the Minkowski space  $M_k$  of the  $k$ th particle (spanned by the co-ordinate and momentum 4-vectors  $x_k, p_k$ ). The Poincaré group is assumed to act linearly on  $\Gamma^N$

$$(a, A):(x_1, p_1; \dots; x_N, p_N) \rightarrow (Ax_1 + a, Ap_1; \dots; Ax_N + a, Ap_N). \quad (1.2)$$

Each  $\Gamma_k$  is equipped with a Poincaré invariant (contact) 1 form

$$\theta_k = p_k dx_k (= p_{k\mu} dx_k^\mu), \quad k = 1, \dots, N, \quad (1.3)$$

whose differential

$$\omega_k = d\theta_k = dp_k \wedge dx_k \quad (1.4)$$

is a symplectic form on  $\Gamma_k$ . The sum

$$\omega = \sum_{k=1}^N \omega_k = d \sum_{k=1}^N \theta_k \quad (1.5)$$

defines a Poincaré invariant symplectic structure on  $\Gamma^N$ , which incorporates the canonical Poisson bracket relations

$$\{p_{kv}, x_\ell^\mu\} = \delta_{k\ell} \delta_v^\mu, \quad \mu, v = 0, 1, 2, 3 \quad (1.6)$$

(all other brackets among the basic phase space co-ordinates vanishing).

The total momentum

$$P = p_1 + \dots + p_N \quad (1.7)$$

and the total angular momentum

$$M = p_1 \wedge x_1 + \dots + p_N \wedge x_N | (p \wedge x)_{\mu\nu} = p_\mu x_\nu - p_\nu x_\mu | \quad (1.8)$$

generate the (Poisson bracket) Lie algebra of the Poincaré group.

The *generalized  $N$  (interacting) particle mass shell* is defined as a  $7N$ -dimensional connected Poincaré invariant submanifold  $\mathcal{M}$  of  $\Gamma^N$  with the following properties:

(i) In any Lorentz frame  $\mathcal{M}$  can be locally defined by  $N$  equations of the following *canonical* (manifestly Euclidean and time translation invariant) *form*:

$$\varphi_k^{\text{can}} \equiv h_k(\mathbf{p}_1, \dots, \mathbf{p}_N; x_{12}, \dots, x_{N-1N}) - p_k^0 = 0, \quad k = 1, \dots, N, \quad x_{k\ell} = x_k - x_\ell, \quad (1.9)$$

where  $p_k$  are restricted by the assumption that the total momentum  $P$  is positive time-like

$$P^0 = \sum_{k=1}^N p_k^0 (= -P_0) > 0, \quad -P^2 = (P^0)^2 - \mathbf{P}^2 \equiv w^2 \geq 0; \quad (1.10)$$

here  $h_k$  are rotationally invariant functions whose further properties will be specified by the subsequent requirements.

(ii) *Compatibility* says, essentially, that the constraints  $\varphi_k (= \varphi_k^{\text{can}}) = 0$  are first class (in the terminology of Dirac [4, 7])

$$\{\varphi_k^{\text{can}}, \varphi_\ell^{\text{can}}\} = \frac{\partial h_\ell}{\partial x_k^0} - \frac{\partial h_k}{\partial x_\ell^0} + \{h_k, h_\ell\} = 0. \quad (1.11)$$

[The strong equation (1.11) is a consequence of the weak equality  $\{\varphi_k^{\text{can}}, \varphi_\ell^{\text{can}}\}|_{\mathcal{M}} = 0$  for  $\varphi_k^{\text{can}}$  given by (1.9).] More precisely, we shall adopt the following (stronger) mathematical requirement.

Consider the set  $\text{Ker}(\omega|_{\mathcal{M}})$  of all vectors tangent to  $\mathcal{M}$ , on which the restriction  $\omega|_{\mathcal{M}}$  of the symplectic form (1.1) vanishes. We assume that it is an integrable vector sub-bundle<sup>2</sup> of the tangent bundle  $T\mathcal{M}$ , and that the foliation

$$\mathcal{M} \rightarrow \Gamma_* = \mathcal{M} / \text{Ker}(\omega|_{\mathcal{M}}) \quad (1.12)$$

is a (locally trivial) fibre bundle (with an  $N$ -dimensional fibre) (cf. Appendix to [6]). The  $6N$ -dimensional base space  $\Gamma_*$  of this foliation plays the role of the physical phase space.

The existence of scattering states (a prerequisite for a scattering theory) requires the following additional assumption:

(iii) *Separability* (or *cluster decomposition property*): clusters of particles separated by large space-like intervals do not interact; in particular, whenever the functions  $h_k$  in (1.9) are defined globally, we should have

$$\lim_{x_k \rightarrow \infty} h_k = \sqrt{m_k^2 + \mathbf{p}_k^2},$$

where  $m_k$  is the mass of particle  $k$ .

*Comments.* 1. A relativistic Hamiltonian system is defined by the surface  $\mathcal{M}$  (and the form  $\omega$ ) and should not depend on the specific choice of (local) equations  $\varphi_k = 0$  describing  $\mathcal{M}$ . We shall exploit the independence of the physics on the particular  $\varphi_k$  by using different forms of the constraints depending on the problem under consideration. The canonical form (1.9) is useful for displaying some general properties of relativistic Hamiltonian systems (e.g. in establishing gauge dependence of canonical world lines in Sect. II.B). It has the drawback, however, of not being manifestly Lorentz invariant. The assumed Lorentz invariance implies<sup>3</sup>

2 A vector sub-bundle  $\mathcal{V}$  of a tangent bundle is *integrable* if the commutator  $[X, Y]$  of any two sections of  $\mathcal{V}$  is again a section of  $\mathcal{V}$ . If the submanifold  $\mathcal{M} \subset \Gamma$  is given in local co-ordinates  $\gamma = (\gamma^a) \in \Gamma$  by  $N$  equations  $\varphi_k(\gamma) = 0$ ,  $k = 1, \dots, N$ , then the points of  $\Gamma_*$  (1.12) can be identified with the  $N$ -dimensional integral surfaces  $\gamma = \gamma(\sigma_1, \dots, \sigma_N)$  of the system of partial differential equations

$$\frac{\partial \gamma}{\partial \sigma_k} = \{\varphi_k, \gamma\} \text{ on } \mathcal{M}$$

$\{M_{0i}, \varphi_k^{\text{can}}\} \approx 0$  which leads to a set of (strong) non-linear partial differential equation for  $h_k$ :

$$\sum_{\ell=1}^N \left( \mathbf{x}_\ell \frac{\partial}{\partial x_\ell^0} + x_\ell^0 \frac{\partial}{\partial \mathbf{x}_\ell} + h_\ell \frac{\partial}{\partial \mathbf{p}_\ell} \right) h_k - \mathbf{p}_k = 0. \quad (1.13)$$

In displaying a class of compatible two-particle constraints in Sect. I.B below (which includes realistic examples) we shall use manifestly covariant constraints  $\varphi_k = 0$  instead of (1.9).

2. Define the *canonical Hamiltonian constraint* as the sum of  $\varphi_k^{\text{can}}$  (1.9)

$$H^{\text{can}} \equiv \sum_{k=1}^N \varphi_k^{\text{can}} = h(\mathbf{p}_1, \dots, \mathbf{p}_N; x_{12}, \dots, x_{N-1 N}) - P^0 = 0$$

$$\left( P = \sum_k p_k \right), \quad (1.14)$$

where  $h = \sum_k h_k$  corresponds (e.g. in the non-relativistic limit) to the standard canonical Hamiltonian. We note that the evolution parameter  $t$  conjugate to the Hamiltonian  $H^{\text{can}}$  (in the sense that  $\dot{f} = \frac{df}{dt} = \{H^{\text{can}}, f\}$  for any smooth function  $f$  on  $\Gamma$ ) can be identified (within an additive constant) with each of the time components  $x_k^0$  ( $k=1, \dots, N$ ) in the reference frame under consideration. Indeed, we have

$$\frac{dx_k^0}{dt} = \{H^{\text{can}}, x_k^0\} = \frac{\partial H^{\text{can}}}{\partial p_{k0}} = - \frac{\partial H^{\text{can}}}{\partial p_k^0} = 1, \quad (1.15a)$$

so that

$$x_k^0 = t + c_k. \quad (1.15b)$$

Thus the relative time variables  $x_{jk}^0$  are constants of the motion with respect to the Hamiltonian (1.14)  $x_{jk}^0 = c_j - c_k$ . Their choice is a matter of convention (corresponding to the synchronization of clocks associated with different particles); we can set, in particular,  $c_k = 0$  without affecting the physics of the problem. The time evolution of the “physical variables”  $(\mathbf{p}_k, \mathbf{x}_k)$  is given by the standard Hamilton equations

$$\dot{\mathbf{p}}_k = \{H^{\text{can}}, \mathbf{p}_k\} = - \frac{\partial h}{\partial \mathbf{x}_k},$$

$$\dot{\mathbf{x}}_k = \{H^{\text{can}}, \mathbf{x}_k\} = \frac{\partial h}{\partial \mathbf{p}_k}, \quad k=1, \dots, N. \quad (1.16)$$

[Note that since  $(\mathbf{p}_j, \mathbf{x}_k)$  can, equivalently, serve as local co-ordinates on  $\Gamma_*$ , Eq. (1.16) can be regarded as an infinitesimal characterization of the action of the one-parameter group of time translations generated by  $P^0$  on  $\Gamma_*$ .]

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3 Following physicist’s tradition we use the weak equality sign  $\approx$  for equations valid on  $\mathcal{M}$

### I.B. Example: The Two-Particle Case

We shall look for a manifestly covariant pair of two-particle first class constraints

$$\varphi_1 = \frac{1}{2}(m_1^2 + p_1^2) + \phi_1 \approx 0, \quad \varphi_2 = \frac{1}{2}(m_2^2 + p_2^2) + \phi_2 \approx 0 \quad (1.17a)$$

or

$$\varphi_1 - \varphi_2 = \varphi + D \approx 0, \quad \varphi = \frac{1}{2}(m_1^2 + p_1^2 - m_2^2 - p_2^2) = pP, \quad D \equiv \phi_1 - \phi_2, \quad (1.17b)$$

$$H \equiv \mu_2 \varphi_1 + \mu_1 \varphi_2 = \frac{1}{2}(p^2 - b^2(w)) + \phi \approx 0, \quad \phi = \mu_2 \phi_1 + \mu_1 \phi_2, \quad (1.17c)$$

where  $\mu_{1,2}$  are defined by (2) and  $b^2$  is the free on shell value of the relative momentum square:

$$b^2(w) = \frac{1}{4} \left[ w^2 - 2(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{w^2} \right] \quad (1.18)$$

( $\phi_{1,2}$  or  $\phi$  and  $D$  are assumed to be functions of the scalar products of translation invariant vectors). In order to make the separation of the interaction terms  $\phi_k$  unique, we shall assume that they may depend on  $p$  through the scalar product  $x_{\perp} p = xp + \frac{1}{w^2}(xP)(pP)$  only<sup>4</sup>, and satisfy the separability condition:

$$\lim_{r \rightarrow \infty} \phi_k = 0 \quad \text{for} \quad r = \sqrt{x_{\perp}^2}, \quad x_{\perp} = x + \frac{1}{w^2}(xP)P. \quad (1.19)$$

In terms of the relative variables  $p$  and  $x$ , the total momentum  $P$ , and the dependent variables  $D$  and  $\phi$  (1.17) the compatibility condition (ii) assumes the form

$$\{\varphi_1, \varphi_2\} = \{\varphi + D, H\} = P \frac{\partial \phi}{\partial x} - p \frac{\partial D}{\partial x} + \{D, \phi\} \approx 0. \quad (1.20)$$

For a given  $D$  Eq. (1.20) can be regarded as a first order linear partial differential equation for  $\phi$  (whose solution involves a functional freedom). It was pointed out in [19] that the special solution

$$D = 0 = P \frac{\partial \phi}{\partial x}, \quad \text{or} \quad \phi_1 = \phi_2 = \phi(r, px_{\perp}; w) \quad (1.21)$$

involves enough freedom to accommodate (in its quantized version, including spin) the quasipotential equations [14] successfully applied so far. A solution with  $D \neq 0$  is considered in Sect. II.A, below.

The requirement (i) [asserting the existence of the local canonical form (1.9) of the constraints] implies

$$\det \left( \frac{\partial \varphi_k}{\partial p_{\ell}^0} \right) > 0 \quad \text{for} \quad P^0 = p_1^0 + p_2^0 > |\mathbf{P}|. \quad (1.22)$$

<sup>4</sup> Different sufficient conditions can be put forward for the uniqueness of the decomposition (1.17a) of  $\varphi_k$  into free mass shell (kinetic part) and interaction. [Separability (1.19) is not enough as it is demonstrated by the free mass shell constraints  $\varphi_1 = \frac{1}{2}(m_1^2 + p_1^2) + \varrho(r)(m_2^2 + p_2^2)$ ,  $\varphi_2 = \frac{1}{2}(m_2^2 + p_2^2)$  ( $\varrho \rightarrow 0$  for  $r \rightarrow \infty$ ,  $\varrho \neq 0$ .)] A  $(-1)^{\ell}$  dependence of the potential, where  $\ell$  is the total angular momentum [ $\ell^2 = x_{\perp}^2 p^2 - (x_{\perp} p)^2$ ], is also encountered in the quantum case (in which  $\ell = 0, 1, \dots$ ) – see [14]

That gives, in particular,  $\mu_k > 0$ , or  $w^2 > |m_1^2 - m_2^2|$  in the case of unbounded motion (in which  $\lim_{t=\pm\infty} \phi = 0$ ); we can actually derive the stronger inequality  $w > m_1 + m_2$  in that case.

We note that for  $\phi_1 = \phi_2$  (1.21) the free particle constraint

$$\varphi = \frac{1}{2}(m_1^2 + p_1^2 - m_2^2 - p_2^2) = pP \approx 0 \quad (1.23)$$

remains valid in the presence of interaction. It is conjugate to a gauge condition of the form

$$\chi^{(n)} \equiv \frac{nx}{nP} = 0, \quad \text{where } n^2 = \mathbf{n}^2 - n_0^2 < 0 \quad (1.24)$$

(in the sense that  $\{\varphi, \chi^{(n)}\} = 1$ ). Equation (1.24) which selects a family of equal-time surfaces, is left invariant by (all multiples of) the Hamiltonian constraint

$$\begin{aligned} H^{(n)} &\equiv \frac{1}{nP} \left[ \left( np_2 - n \frac{\partial \phi}{\partial p} \right) \varphi_1 + \left( np_1 + n \frac{\partial \phi}{\partial p} \right) \varphi_2 \right] \\ &= H - \frac{1}{nP} \left( n \frac{\partial \phi}{\partial p} + np \right) \varphi \approx 0, \end{aligned} \quad (1.25)$$

where  $H$  (1.17c) preserves [on  $\mathcal{M}$ , for  $\phi$  satisfying (1.21)] the manifestly covariant *Markov-Yukawa gauge*

$$\chi (= \chi^{(P)}) = - \frac{1}{w^2} P x \approx 0. \quad (1.26)$$

The practical usefulness of the Hamiltonian constraint (1.17c) in the local quasipotential approach [14] stems from a far reaching similarity of the centre-of-mass dynamics based on (1.23), (1.17c), and (1.26) and the corresponding non-relativistic Hamiltonian dynamics. Setting (in the centre-of-mass frame)

$$P = (w, \mathbf{0}), \quad x_{\perp} (\approx x) = (0, \mathbf{r}), \quad p \approx (0, \mathbf{p}), \quad (1.27)$$

we observe, in particular, that the three-dimensional angular momentum

$$\ell = \mathbf{p} \wedge \mathbf{r} \quad \left( \ell^2 \equiv \ell^2 = \frac{1}{w^2} W^2 = r^2 p^2 - (\mathbf{r}\mathbf{p})^2 \text{ for } W_{\kappa} = \frac{1}{2} \epsilon_{\kappa\lambda\mu\nu} P^{\lambda} M^{\mu\nu} \right) \quad (1.28)$$

is conserved and hence the (phase space) relative motion takes place in a plane orthogonal to  $\ell$  (just as in the non-relativistic case).

### *I.C. Two Examples: Electromagnetic Interaction and Elastic Scattering of Relativistic Balls*

The above framework allows us to interpret the relativistic two-particle interaction as a one-particle problem in an external field. Indeed, let us introduce, following [18, 20], an effective particle with *relativistic reduced mass*

$$m_w = \frac{m_1 m_2}{w} \quad (1.29a)$$

and *reduced energy*

$$E = \sqrt{m_w^2 + b_{(w)}^2} = \frac{w^2 - m_1^2 - m_2^2}{2w}. \quad (1.29b)$$

Then we can write the Hamiltonian constraint (1.17c) for the case of relativistic electromagnetic interaction of two oppositely charged particles in the form [18, 14]

$$\begin{aligned} H_{em} &= \frac{1}{2} \left[ m_w^2 + \mathbf{p}^2 - (E - V(r))^2 + \frac{\alpha^2}{4w^2 r^4} \right] \\ &= \frac{1}{2} (\mathbf{p}^2 - b_{(w)}^2) + \phi_{em}(r; w) \approx 0, \end{aligned} \quad (1.30a)$$

where

$$V(r) = -\frac{\alpha}{r}, \quad \alpha = -\frac{e_1 e_2}{4\pi} > 0, \quad \phi_{em} = -E \frac{\alpha}{r} - \frac{1}{2} \frac{\alpha^2}{r^2} + \frac{\alpha^2}{8w^2 r^4}. \quad (1.30b)$$

(The  $(E - V)^2$  term could have been guessed as a “minimal Coulomb coupling”; the whole expression  $\phi_{em}$  is derived from the one-photon exchange diagram of quantum electrodynamics. The corresponding Schrödinger equation  $H_{em}\psi = 0$  in the quantum framework gives the correct relativistic energy level including fine structure and recoil effects of order  $\alpha^4$  [18, 14].)

We shall use here this realistic example to argue that it is not reasonable to replace positivity condition (1.10) for the total energy by the stronger requirement that individual particle momenta are (positive) time-like. Assume indeed, for a moment, that

$$-p_2^2 = E_2^2 - \mathbf{p}^2 = m_2^2 + 2\phi_{em} \geq 0 \quad \text{for} \quad m_2 \leq m_1. \quad (1.31)$$

It is straightforward to verify, for example, in the case of  $ep$  scattering (in which  $m_2 = m_e$ ,  $m_1 = m_p = 1836m_e$ ), for the physical value of  $\alpha$  ( $= \frac{1}{137}$ ) that the equation

$$\frac{1}{E^2} (m_2^2 + 2\phi_{em}) = \frac{m_2^2}{E^2} - 2u - u^2 + \frac{E^2 u^4}{4w^2 \alpha^2} = 0$$

has two positive solutions  $u_1$  and  $u_2$  for  $u = \alpha/Er$  for any  $w \geq m_1 + m_2$ ; obviously the inequality (1.31) is violated for  $u_1 < u < u_2$ . Thus Eq. (1.31) implies for any fixed  $w$  in the scattering region a non-trivial restriction on the range of  $r$ . An analysis of the approximate solution of the Hamiltonian equations of motion obtained for kinematical configurations for which the last two terms in  $\phi_{em}$  (1.30b) can be neglected (see Sect. 4.B of [20]) shows that for sufficiently small total angular momentum  $\ell^2 = r^2 p^2 - (\mathbf{r}\mathbf{p})^2$  the two particles, originally (for  $t \rightarrow -\infty$ ) far apart, will, for finite times, come to the region ( $u_2 > u > u_1$ ), in which Eq. (1.31) is violated. If one postulates that Eq. (1.31) defines a wall (an infinite potential barrier) in phase space effectively forbidding small distances (of order  $r < 2\alpha E/m_2^2$ ) then the classical Rutherford scattering would be drastically changed for sufficiently small impact parameters in contradiction with experiment.

The limit in which the interaction term  $\phi$  tends to infinity in some domain of the invariant variables may be a meaningful idealization from a physical stand-



point and is perfectly admissible mathematically. A neat way to take care of such a limit is to restrict (in a Poincaré invariant manner) the range of phase space variables. We shall illustrate the idea on the example of the elastic scattering of two relativistic (spinless) balls of finite (energy dependent) radii.

The 14-dimensional “space of states”  $\mathcal{M}(R)$  of the system is defined as the portion of the free particle mass shell  $\mathcal{M}_0$  in which the invariant distance  $\sqrt{x_\perp^2}$  between the particle centres is larger than some positive number  $R(w)$ :

$$\mathcal{M}(R) = \{(p_1, p_2; x_1, x_2) \in \Gamma^2; p_k^0 = \sqrt{m_k^2 + \mathbf{p}_k^2}, r^2 = x_\perp^2 \geq R^2(w) > 0\}. \quad (1.32)$$

We assume the following continuity properties at the boundary of  $\mathcal{M}(R)$ : (a) particle trajectories are continuous everywhere (although particle velocities may have a jump on the boundary  $|x_\perp| = R$ ); (b) the generators  $P_\mu$  and  $M_{\mu\nu}$  of the Poincaré group are conserved.

The restriction of the range of the invariant distance together with the above continuity assumptions lead to a well-defined gauge invariant dynamics. It involves non-trivial scattering provided that the impact parameter

$$a = \frac{\ell}{b(w)} \quad (1.33)$$

does not exceed the total interaction radius  $R$ . Using the definition (1.32) of  $\mathcal{M}(R)$  as well as conditions (a) and (b), we recover the familiar reflection law: at the point of contact with the boundary  $|x_\perp| = R$  the tangent component of the relative momentum  $p = p(\tau)$  is continuous<sup>5</sup>, while its normal component changes sign. The differential cross-section is (just as in the non-relativistic case) independent of the scattering angle  $\theta$ , defined by

$$\cos\theta = \frac{1}{b^2} \mathbf{p}^{\text{in}} \mathbf{p}^{\text{out}} = 2 \frac{a^2}{R^2} - 1 \quad (1.34)$$

(where  $\mathbf{p}$  is the three-dimensional relative momentum in the centre-of-mass frame).

## II. World Lines and Scattering Matrix

### II.A. Equivalent Dynamics

As noted in Sect. I.B the evolution parameter  $t$ , conjugate to the canonical Hamiltonian (1.14), can be identified with the 0 components of the 4-vectors  $x_k$ . The gauge conditions

$$\chi_k \equiv x_k^0 - t = 0, \quad k = 1, \dots, N \quad (2.1)$$

are consistent with the Hamiltonian (1.14), since [due to (1.15)]

$$\frac{d\chi_k}{dt} = \{H^{\text{can}}, \chi_k\} + \frac{\partial \chi_k}{\partial t} = 0. \quad (2.2)$$

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<sup>5</sup> Note that this continuity property is, in general, an independent assumption (see e.g. Appendix II of [21]). In the two-particle case it comes out as a consequence of energy and angular momentum conservation

Given  $N$  points  $(\xi_1, \dots, \xi_N)$  on the plane  $x_1^0 = \dots = x_N^0 = t_0$  and initial velocities  $v_k = \dot{x}_k|_{t=t_0}$  we have a unique set of world lines<sup>6</sup>  $(x_1(t), \dots, x_N(t))$ , satisfying the initial conditions

$$x_k(t_0) = \xi_k, \quad \dot{x}_k(t_0) = v_k, \quad k = 1, \dots, N.$$

These world lines will depend, in general, on the Lorentz frame, which can be labelled by a unit time-like vector  $n$  ( $n_0 = -n^0 = \sqrt{1 + \mathbf{n}^2}$ ) such that  $x_k^0 = nx_k$ .

Two generalized mass shells  $\mathcal{M}$  and  $\bar{\mathcal{M}}$  will be considered as *physically equivalent* if for any fixed choice of the time-like vector  $n$  they lead to the same world lines for the same initial conditions, and if, in addition, they give rise to the same realization of the Poincaré group [the latter means that there is a one-to-one correspondence  $\gamma \rightarrow \bar{\gamma}$  of  $\mathcal{M}$  onto  $\bar{\mathcal{M}}$ , which leaves the world lines and the Poincaré group generators (1.7) and (1.8) invariant].

A necessary and sufficient condition for physical equivalence is the existence of a canonical (i.e. preserving the 2-form  $\omega$ ) isomorphic map  $f$  of some neighbourhood  $U \subset \Gamma^N$  of  $\mathcal{M}$  onto a neighbourhood  $\bar{U} \subset \Gamma^N$  of  $\bar{\mathcal{M}}$  satisfying the following conditions: (a)  $f(\mathcal{M}) = \bar{\mathcal{M}}$ ; (b)  $f$  weakly preserves the particle positions  $x_k$ ; in other words, if  $\pi$  is the projection of  $\Gamma^N$  onto the configuration space  $M^{4N}$  ( $M^4$  being the Minkowski space), then  $\pi|_{\mathcal{M}} = \pi|_{\bar{\mathcal{M}}} \circ f|_{\mathcal{M}}$ ; (c) the Poincaré group generators (1.7) and (1.8) are  $f$  invariant.

If the mass shell  $\mathcal{M}$  is separable [in the sense of (iii)] we demand (as a further condition for physical equivalence) that  $\bar{\mathcal{M}}$  is also separable and that the difference of the corresponding canonical constraints  $\varphi_k^{\text{can}} - \bar{\varphi}_k^{\text{can}}$  vanishes for  $x_{k\ell}^2 \rightarrow \infty$  ( $\ell \neq k$ ).

Note that this notion of physical equivalence singles out the Minkowski space trajectory along with the Poincaré group generators as a more fundamental object than the phase space picture. The notion of particle momentum (for fixed coordinates) is not determined by the canonical Poisson bracket relations. Indeed the transformations

$$\begin{aligned} x_k \rightarrow \bar{x}_k = x_k, \quad p_k \rightarrow \bar{p}_k = p_k + \partial_k F(\{\tfrac{1}{2}x_{\ell m}^2\}), \\ k, \ell, m = 1, \dots, N, \quad x_{\ell m} = x_\ell - x_m, \end{aligned} \quad (2.3)$$

are easily verified to be canonical (for any choice of the smooth function  $F$ ). Moreover, they leave the co-ordinates unaltered and because of the Poincaré invariance of  $F$ , the generators of the Poincaré group do not change either:

$$\sum_{k=1}^N \bar{p}_k = \sum p_k (=P), \quad \sum_{k=1}^N \bar{p}_k \wedge \bar{x}_k = \sum p_k \wedge x_k (=M). \quad (2.4)$$

In fact, it is not difficult to prove that locally the transformations of type (2.3) are the most general with all these properties. If, in addition,  $F \rightarrow 0$  for  $\mathbf{x}_k \rightarrow \infty$  then the

6 An alternative physical interpretation is proposed by Droz-Vincent [5] (see also [1]). Our  $x$ 's are regarded as auxiliary mathematical variables, while particle positions are defined as some functions  $q_k = q_k(x, p)$  such that  $\{\varphi_k, q_\ell\} \approx 0$  for  $k \neq \ell$  (say, for  $\varphi_k = \varphi_k^{\text{can}}$ ). As a consequence, the corresponding world lines  $\{q_k(t)\}$  are gauge invariant (in the sense to be specified below), but particle positions are no longer canonical. The difficulty in such an approach consists in actually finding the physical coordinates  $q_k(x, p)$  for arbitrary interactions

asymptotic states also coincide:  $\bar{p}_k^{as} = p_k^{as}$ . In the two-particle case the second Eq. (2.3) can be rewritten in terms of the single relative co-ordinate  $x = x_{12}$  as follows<sup>7</sup>:

$$\bar{p}_1 = p_1 + xB(\tfrac{1}{2}x^2), \quad \bar{p}_2 = p_2 - xB(\tfrac{1}{2}x^2), \quad (2.5)$$

[where  $B(u) = dF/du$ ]. Such transformations leave the  $x$ -space trajectories invariant and, therefore, relate physically equivalent theories (in the above terminology) to each other.

We can use the freedom in the choice of  $B$  in (2.5) [or  $F$  in (2.3)] to select a standard representative of the constraints (1.17). One way to do that is to assume that for the Markov-Yukawa gauge (1.26) and corresponding Hamiltonian the relative velocity vanishes weakly for  $p=0$ . We have

$$\begin{aligned} \dot{x} &= \left\{ \left( 1 - \frac{1}{s} P \frac{\partial D}{\partial p} \right) H + \frac{1}{s} \left( pP + P \frac{\partial \phi}{\partial p} \right) (\varphi + D), x \right\} \\ &= \left( 1 - \frac{1}{s} P \frac{\partial D}{\partial p} \right) \left( p + \frac{\partial \phi}{\partial p} \right) + \frac{1}{s} \left( pP + P \frac{\partial \phi}{\partial p} \right) \left( P + \frac{\partial D}{\partial p} \right) \\ &\rightarrow \left[ \left( 1 - \frac{1}{s} P \frac{\partial D}{\partial p} \right) \frac{\partial \phi}{\partial p} + \frac{1}{s} P \frac{\partial \phi}{\partial p} \left( P + \frac{\partial D}{\partial p} \right) \right]_{p=0} \approx 0. \end{aligned}$$

Hence, our standardization condition is

$$\left. \frac{\partial \phi}{\partial p} \right|_{p=0=x} \approx 0. \quad (2.7)$$

[Note that for

$$\begin{aligned} \varphi_1 &= \tfrac{1}{2} [m_1^2 + (p_1 + xB(\tfrac{1}{2}x^2))^2] + A(x_1^2, s), \\ \varphi_2 &= \tfrac{1}{2} \left[ m_2^2 + \left( p_2 - xB\left(\frac{x^2}{2}\right) \right)^2 \right] + A(x_2^2, s) \end{aligned}$$

we have

$$\phi = A(x_{1,2}^2, s) + pxB(\tfrac{1}{2}x^2) + \frac{x^2}{2} B^2(\tfrac{1}{2}x^2)$$

and (2.7) implies  $B=0$ .]

## II.B. Gauge Dependence of Interacting Particle's World Lines

We shall demonstrate in this section that particle world lines are not reparametrization invariant in the presence of non-trivial interaction (they depend on the vector  $n$  that specifies the equal-time plane). More precisely, we shall establish the following negative result.

<sup>7</sup> The fact that the functions  $\bar{\varphi}_{1,2} = \tfrac{1}{2} [m_{1,2}^2 + (p_{1,2} \pm xB(\tfrac{1}{2}x^2))^2]$  are in involution was first noticed in [22]

**Theorem 1.** *Let  $\mathcal{M}$  be a generalized two-particle mass shell, satisfying conditions (i) and (ii) [but not necessarily (iii)] of Sect. I.A. The projection  $\pi_k(\gamma_*)$  of each (two-dimensional) fibre  $\gamma_* \subset \mathcal{M}$  of the bundle  $\mathcal{M} \rightarrow \Gamma_*$  into the Minkowski spaces  $M_k$  of each particle ( $\pi_k(x_1, p_1; x_2, p_2) = x_k$ ,  $k=1, 2$ ) is a one-dimensional submanifold of  $M_k$  iff  $\mathcal{M}$  is (locally) physically equivalent to either a free particle mass shell or to a surface of type (2.14) (see Lemma 2, below). In both cases the space-time trajectories of the particles are straight lines.*

*Remark.* In a less technical language the theorem says that a two-particle system has gauge invariant world lines (in Minkowski space) only if we deal with a free motion or with the elastic scattering of rigid balls (described in Sect. I.C). Indeed, if the projections are two-dimensional, we need a (gauge dependent) subsidiary condition to define the one-dimensional world line of each particle.

*Proof.* In one direction the theorem is trivial. If the constraints are given by

$$\begin{aligned}\varphi_1^{\text{free}} &\equiv \frac{1}{2}[m_1^2 + (p_1 + xB(\frac{1}{2}x^2))^2] \approx 0, \\ \varphi_2^{\text{free}} &\equiv \frac{1}{2}[m_2^2 + (p_2 - xB(\frac{1}{2}x^2))^2] \approx 0\end{aligned}\quad (2.8)$$

(cf. Sect. II.A), or, more generally, if  $\frac{\partial \varphi_1}{\partial p_2} = 0 = \frac{\partial \varphi_2}{\partial p_1}$ , then obviously

$$\frac{\partial x_1}{\partial \sigma_2} = \{\varphi_2, x_1\} = 0 = \frac{\partial x_2}{\partial \sigma_1} (= \{\varphi_1, x_2\}), \quad (2.9)$$

and hence, the projections  $T_k = \pi_k \gamma_*$  of the fibre  $\gamma_*$  are one-dimensional.

The inverse statement is both more interesting and more difficult to establish: given that

$$\dim T_k = 1, \quad k=1, 2, \quad (2.10)$$

prove that the constraints  $\varphi_k$  can be chosen in the form (2.8). We shall proceed in two steps. First, we shall see that the assumption of the theorem leads to Eq. (2.9) for the canonical constraints (1.9). Second, we shall show that the general solution of (2.9) satisfying conditions (i), (ii) of Sect. I.A is given essentially by (2.8). These two steps form the content of the following two lemmas<sup>8</sup>.

**Lemma 1.** *If the world lines are one-dimensional then*

$$\frac{\partial}{\partial p_\ell} \varphi_k^{\text{can}} = \frac{\partial}{\partial p_\ell} h_k = 0 \quad \text{for } \ell \neq k \quad (2.11)$$

(where  $\varphi_k^{\text{can}}$  are given by (1.9)).

*Proof of Lemma 1.* Let  $\sigma_1, \dots, \sigma_N$  be some (time) parameters on the world lines  $T_1, \dots, T_N$ . Assumption (2.10) means that one can choose (in the neighbourhood of each point  $\gamma \in \mathcal{M}$ ) these parameters as local co-ordinates in the fibre (smoothly depending on the fibre). Then,

$$\{\varphi_k^{\text{can}}, x_\ell^\mu\} = B_{k\ell} \frac{dx_\ell^\mu}{d\sigma_\ell}, \quad k, \ell = 1, \dots, N \quad (\text{no sum!}), \quad (2.12)$$

<sup>8</sup> A complete proof is given in the appendix of ICTP, Trieste, Internal Report IC/79/59

where  $B_{k\ell}$  may depend on the point  $\gamma \in \mathcal{M}$  but not on the Lorentz index  $\mu$ . Since, according to (1.9)

$$\{\varphi_k^{\text{can}}, x_\ell^0\} = \delta_{k\ell}, \quad (2.13)$$

it follows that  $B_{k\ell} = \delta_{k\ell}$ . Lemma 1 is proven.

**Lemma 2.** *The constraints  $\varphi_k^{\text{can}} = 0$  (1.9), satisfying (1.11) and (2.11) can be replaced for  $N=2$  by equivalent constraints either of type (2.8) or of the form*

$$\varphi_1 = p_1 x + \frac{1}{2} C \sqrt{-x^2 + B(\frac{1}{2}x^2)}, \quad \varphi_2 = p_2 x + \frac{1}{2} C \sqrt{-x^2 - B(\frac{1}{2}x^2)}. \quad (2.14)$$

*Proof of Lemma 2.* Poincaré invariance, along with (2.12) tells us that each  $\varphi_k^{\text{can}}$  can be replaced by an equivalent constraint  $\bar{\varphi}_k$  depending on three scalar variables

$$s_k = \frac{1}{2} p_k^2, \quad u_k = x p_k, \quad k=1, 2, \quad v = \frac{1}{2} x^2. \quad (2.15)$$

Using the Poisson bracket relations

$$\begin{aligned} \{s_1, u_2\} = \{u_1, s_2\} = p_1 p_2, \quad \{s_1, v\} = u_1, \quad \{v, s_2\} = u_2, \\ \{u_1, v\} = 2v = \{v, u_2\}, \quad \{u_1, u_2\} = u_1 + u_2, \end{aligned} \quad (2.16)$$

we obtain

$$\begin{aligned} 0 \approx \{\bar{\varphi}_1, \bar{\varphi}_2\} &= \left( \frac{\partial \bar{\varphi}_1}{\partial s_1} \frac{\partial \bar{\varphi}_2}{\partial u_2} + \frac{\partial \bar{\varphi}_1}{\partial u_1} \frac{\partial \bar{\varphi}_2}{\partial s_2} \right) p_1 p_2 + \frac{\partial \bar{\varphi}_1}{\partial s_1} \frac{\partial \bar{\varphi}_2}{\partial v} u_1 \\ &+ \frac{\partial \bar{\varphi}_1}{\partial v} \frac{\partial \bar{\varphi}_2}{\partial s_2} u_2 + \frac{\partial \bar{\varphi}_1}{\partial u_1} \frac{\partial \bar{\varphi}_2}{\partial u_2} (u_1 + u_2) \\ &+ 2v \left( \frac{\partial \bar{\varphi}_1}{\partial u_1} \frac{\partial \bar{\varphi}_2}{\partial v} + \frac{\partial \bar{\varphi}_1}{\partial v} \frac{\partial \bar{\varphi}_2}{\partial u_2} \right). \end{aligned} \quad (2.17)$$

Since the variable  $p_1 p_2$  only appears in the first term, its coefficient should vanish.

There are three possibilities:

$$\text{a) } \frac{\partial \bar{\varphi}_1}{\partial s_1} \neq 0 \neq \frac{\partial \varphi_2}{\partial s_2}.$$

As a consequence of the implicit function theorem, in this case, the equations  $\bar{\varphi}_1 = 0 = \bar{\varphi}_2$  can be solved with respect to  $s_1$  and  $s_2$  (in some neighbourhood of the point under consideration) so that the equations defining the generalized mass shell  $\mathcal{M}$  (in that neighbourhood) can be written in the form

$$\varphi_k = s_k + F_k(u_k, v). \quad (2.18)$$

Then the vanishing of  $\frac{\partial \varphi_1}{\partial s_1} \frac{\partial \varphi_2}{\partial u_2} + \frac{\partial \varphi_1}{\partial u_1} \frac{\partial \varphi_2}{\partial s_2}$  gives the strong equation

$$\frac{\partial F_1}{\partial u_1} = - \frac{\partial F_2}{\partial u_2}. \quad (2.19)$$

Since the left-hand side of this equation is independent of  $u_2$ , while the right-hand side is independent of  $u_1$ , we deduce

$$F_1 = C_1(v) - u_1 B(v), \quad F_2 = C_2(v) + u_2 B(v), \quad (2.20)$$

or

$$\begin{aligned}\varphi_1 &= \frac{1}{2}[(p_1 - xB(\frac{1}{2}x^2))^2 + m_1^2(\frac{1}{2}x^2)] (\approx 0), \\ \varphi_2 &= \frac{1}{2}[(p_2 + xB(\frac{1}{2}x^2))^2 + m_2^2(\frac{1}{2}x^2)] (\approx 0).\end{aligned}$$

Inserting  $\varphi_1$  and  $\varphi_2$  into Eq. (2.17), we find that  $m_1^2$  and  $m_2^2$  are independent of  $x$ . This leads to (2.8).

$$\text{b) } \frac{\partial \bar{\varphi}_1}{\partial s_1} = 0 = \frac{\partial \bar{\varphi}_2}{\partial s_2} \quad (\text{locally}). \quad (2.21)$$

The canonical presentation of  $\mathcal{M}$  [consistent with (2.17)] in this case is (2.14). Note that  $\{\varphi_1, \varphi_2\} = \varphi_1 + \varphi_2 (\approx 0)$  for  $\varphi_k$  given by (2.14).

$$\text{c) } \frac{\partial \bar{\varphi}_1}{\partial s_1} = 0 \neq \frac{\partial \bar{\varphi}_2}{\partial s_2} \quad \text{or} \quad \frac{\partial \bar{\varphi}_1}{\partial s_1} \neq 0 = \frac{\partial \bar{\varphi}_2}{\partial s_2} \quad (\text{locally}).$$

There is no solution of the compatibility condition (2.17) in this case.

This proves Lemma 2.

To complete the proof of the theorem it remains to verify that the constraints (2.14) also lead to (locally) straight line trajectories. The simplest way to see it, is to use the Hamiltonian

$$H_{12} = \varphi_1 + \varphi_2 = Px + C\sqrt{-x^2}, \quad (2.22)$$

which gives  $\dot{x} = 0$ ,  $x_1(\tau) = x_1(0) + [x_1(0) - x_2(0)]\tau$ ,  $x_2(\tau) = x_2(0) + [x_1(0) - x_2(0)]\tau$ . The independence of the world lines of the choice of the Hamiltonian is implied by (2.9). Note that condition (1.22) implies that  $x = x_1 - x_2 (= x_1(0) - x_2(0))$  should be a (positive) time-like vector (for  $C > 0$ ).

### II.C. Comments: Relation to Earlier no Interaction Theorem

We proceed with a few remarks concerning Theorem 1 and its proof. We state, in particular, (in Comment 4), Theorem 2, which extends the result of Theorem 1 to the  $N$ -particle case under the additional assumption of non-degeneracy. The fact that the non-degeneracy assumption is essential is illustrated by the example of a gauge invariant dynamics in two space-time dimensions involving a 0 mass particle (Comment 3). The precise relation of Theorem 1 to the “no interaction theorem” [3, 10, 8] is analysed in Comment 6.

1) The constraints (2.14) violate the separability condition (iii) of Sect. I.A and seem globally unphysical. One could also exclude them locally by assuming that all coordinate differences are space-like on  $\mathcal{M}$ . We did not need, however, such an additional restriction, since these strange looking constraints appeared to be rather harmless (again leading to straight world lines).

2) Since our treatment (especially in Lemma 2) is local and uses smoothness in open neighbourhoods, it does not apply to boundary points of the generalized mass shell (if such exist as in the case of scattering of elastic balls, considered in Sect. I.C). Thus it leaves room for piecewise straight world lines, which are also frame independent.

3) The above proof can be extended to  $N$  particle interaction in  $D$ -dimensional space-time [of signature  $(D - 1, 1)$ ] provided that

$$D \geq 2N - 1. \quad (2.23)$$

Otherwise, the argument after Eq. (2.17) fails, since the scalar products of the translation invariant vectors are no longer independent. For example, for  $N = D = 2$  the determinant of the scalar products of the vectors  $p_1, p_2, x$  vanishes:

$$\begin{vmatrix} p_1^2 & p_1 p_2 & p_1 x \\ p_1 p_2 & p_2^2 & p_2 x \\ p_1 x & p_2 x & x^2 \end{vmatrix} = 0 \quad (\text{for } D=2).$$

Moreover, if  $D < 2N - 1$ , then the conclusion of Theorem 1 does not hold. This fact is illustrated by the example of  $D = 2$ , two-particle scattering with constraints

$$\varphi_1 = \frac{1}{2} p_1^2 \approx 0 (p_1^0 > 0), \quad \varphi_2 = \frac{m^2 + p_2^2}{2} + f(\xi) - f'(\xi) \frac{v p_1^2}{(p_1 x)^2} \approx 0, \quad (2.24a)$$

where

$$\xi = \frac{p_1 p_2}{p_1 x}, \quad v = \varepsilon(\mathbf{p}_1) (p_2^0 x - \mathbf{p}_2 \cdot \mathbf{x}^0), \quad \varepsilon(\mathbf{p}_1) \equiv \text{sign } \mathbf{p}_1 \quad (2.24b)$$

and the (smooth) function  $f$  satisfies  $f \leq 0$  (to make consistent  $p_1^0 > 0$  in the centre-of-mass frame),  $f(0) = 0$  (that ensures separability) and

$$\frac{\partial \varphi_2}{\partial p_2^0} \approx p_2^0 + f'(\xi) \frac{p_1^0}{p_1 x} > 0$$

whenever

$$x^2 > 0, \quad p_1^0 = |\mathbf{p}_1| > 0, \quad P^0 (= p_1^0 + p_2^0) > 0, \quad -P^2 = w^2 > m^2 \quad (2.24c)$$

[Eq. (2.24c) guarantees the existence of the canonical form (1.9) of the constraints.] The above conditions on  $f$  are automatically satisfied for

$$f(\xi) = -\frac{1}{2} \xi^2 F\left(\frac{1}{2} \xi^2\right), \quad F \geq 0. \quad (2.24d)$$

(Note that  $v$  is a Lorentz scalar on the forward cone  $p_1^0 = |\mathbf{p}_1|$ ; it is also invariant under space reflections.) The Poisson bracket  $\{\varphi_1, \varphi_2\}$  is proportional to  $\varphi_1$  and, therefore, vanishes weakly. In order to see that the constraints (2.24) generate a gauge invariant dynamics, it suffices to verify the equation

$$\{\varphi_2, x_1\} = \frac{\partial \varphi_2}{\partial p_1} \approx f'(\xi) \frac{(p_1 x) p_2 - (p_1 p_2) x - v p_1}{(p_1 x)^2} \approx 0. \quad (2.25)$$

[The vanishing of  $(p_1 x) p_2 - (p_1 p_2) x - p_1$  for  $p_1^0 = |\mathbf{p}_1|$  is indeed characteristic for two-dimensional configurations including a light-like vector.]

The simplest way to find the relative motion is to use the Markov-Yukawa gauge (1.26) which is preserved by the Hamiltonian

$$\begin{aligned} H &= \left[ \mu_2 + \frac{f'}{p_1 x} \left( \mu_1 - \mu_2 + \mu_1 \frac{v}{p_1 x} \right) \right] \varphi_1 + \mu_1 \varphi_2 \\ &= \left( \mu_2 + \frac{\mu_1 - \mu_2}{p_1 x} f' \right) \frac{1}{2} p_1^2 + \mu_1 \left( \frac{m_2^2 + p_2^2}{2} + f \right) \approx 0. \end{aligned} \quad (2.26)$$

For the centre-of-mass variables [ $P=(w, \mathbf{0})$ ,  $p=(0, \mathbf{p})$ ,  $x=(0, \mathbf{x})$ ,  $p_1=(|\mathbf{p}|, \mathbf{p})$  etc.] we have

$$\begin{aligned} \dot{\mathbf{x}} &= \{H, \mathbf{x}\} = \mathbf{p}, & \dot{\mathbf{p}} &= -f' \frac{\mathbf{p}}{x^2}, \\ \dot{t} (= \dot{x}_1^0 = \dot{x}_2^0) &= \left[ \varepsilon(p)w - \mathbf{p} - \frac{f'}{x} \right] \frac{\mathbf{p}}{w}. \end{aligned} \quad (2.27)$$

The energy integral, derived from the Hamiltonian constraint (2.26), gives

$$|\mathbf{p}| = \frac{w^2 - m^2}{2w} - \frac{1}{w} f \geq \frac{w^2 - m^2}{2w} > 0 \quad \text{for } f \leq 0, \quad w > m. \quad (2.28)$$

Thus  $\mathbf{p}$  never vanishes and, hence,  $\varepsilon(\mathbf{p})$  is a constant of the motion. From (2.27) and (2.28) we find the  $(t, \mathbf{x})$  trajectory

$$\varepsilon(\mathbf{p})(t - t_0) = \left[ \frac{1}{2}(w^2 + m^2) + f \left( -\varepsilon \frac{w}{x} \right) \right] \frac{\mathbf{x}}{w^2}. \quad (2.29)$$

If  $\mathbf{x}f$  is singular for  $\mathbf{x} \rightarrow 0$ , for instance, if  $f = -\frac{R^2 w^2}{2 x^2}$ , then  $\mathbf{x}$  never vanishes for finite times and  $\varepsilon(\mathbf{x}) = -\varepsilon(\mathbf{p})$ . If, on the contrary,  $f$  is regular and sufficiently small, say if

$$f = \frac{-\alpha wr}{R^4 w^2 + r^2}, \quad r = |\mathbf{x}|, \quad 0 < \alpha < \frac{4}{3\sqrt{3}} R^6 w^4 (w^2 + m^2),$$

then the correspondence  $t \leftrightarrow \mathbf{x}$  given by (2.29) is one to one and  $\mathbf{x}$  is an odd function of  $t - t_0$ , so that the scattering matrix is trivial.

4) A characteristic feature of the preceding example is the presence of a (free) zero mass particle which implies the degeneracy of the canonical Hamiltonian  $h(\mathbf{p}_1, \mathbf{p}_2; x) = |\mathbf{p}_1| + h_2(\mathbf{p}_2; x)$ . Independent of the precise expression for  $h_2$ , we have

$$\det \left( \frac{\partial^2 h}{\partial \mathbf{p}_k \partial \mathbf{p}_\ell} \right) = 0 \quad \text{for } \mathbf{p}_1 \neq \mathbf{0}.$$

It turns out that for a non-degenerate system Theorem 1 can be extended to the  $N$  particle case in four-dimensional space-time. Indeed, we have the following result.

**Theorem 2.** *Let the generalized  $N$ -particle mass shell satisfy the assumption*

$$\dim \pi_k(\gamma_*) = 1, \quad k = 1, \dots, N$$

of Theorem 1 and let in addition the canonical Hamiltonian  $h(\mathbf{p}_1, \dots, \mathbf{p}_N; x_{12}, \dots, x_{N-1N}) \left( \approx \sum_k p_k^0 \right)$  satisfy the non-degeneracy condition

$$\det \left( \frac{\partial^2 h}{\partial p_k^i \partial p_\ell^i} \right) \neq 0 \quad (i, j = 1, 2, 3; k, \ell = 1, \dots, N)$$

(the left-hand side standing for the determinant of the  $3N \times 3N$  matrix of second derivatives of the Hamiltonian). Then the Minkowski space trajectories of all particles are straight lines.

The rather technical proof of this theorem is a straightforward extension of a similar argument by Leutwyler [10] and will be omitted.



5) The definition of relativistic Hamiltonian dynamics, given in Sect. I.A, admits the following natural generalization. An  $N$  particle dynamics of the type  $k$  (an  $[N|k]$  dynamics, for short) ( $k=0, \dots, N-1$ ) is defined by a  $7N-k$  dimensional Poincaré invariant submanifold  $\mathcal{M}_{[N|k]}$  of  $\Gamma^N$  such that  $\text{Ker}(\omega|_{\mathcal{M}_{[N|k]}})$  is an  $N-k$  dimensional integrable sub-bundle of the tangent bundle  $T\mathcal{M}_{[N,k]}$ . If  $\mathcal{M}$  is given locally by  $N+k$  equations  $\varphi_a(\gamma)=0$ ,  $a=1, \dots, N+k$  then the skew symmetric matrix  $\{\varphi_a, \varphi_b\}$  should have rank  $2k$  on  $\mathcal{M}_{[N|k]}$ . We shall assume that  $N$  of these equations can be chosen in the canonical form (1.9) so that we can regard  $\mathcal{M}_{[N|k]} \subset \mathcal{M}_{[N|0]} = \mathcal{M}$ . The Hamiltonian is assumed to have zero Poisson brackets with all  $N+k$  constraints. For  $k=0$  we are back in the framework of this paper. In the opposite limiting case for  $k=N-1$ ,  $\text{Ker}(\omega|_{\mathcal{M}_{[N|N-1]}})$  is one-dimensional and so are the phase space trajectories, hence, particle world lines are gauge invariant for a trivial reason. An example of a two-particle dynamics of this type is given by the constraints (1.9), (1.17c) and the Markov-Yukawa gauge condition (1.26), also regarded as a constraint. It is actually this [2|1] formulation of two-particle dynamics which fits the quasipotential approach of [18, 14]. It provides a Lorentz invariant formulation of the relativistic two-body problem and thus gives a way out of the “no interaction theorem” of Currie, Jordan, Sudarshan (CJS) [3], Leutwyler [10], and Hill [8]. (It violates, of course, some of the assumptions of the CJS approach; most important, the three-dimensional co-ordinates  $\mathbf{x}_j$  and momenta  $\mathbf{p}_k$  ( $j, k=1, 2$ ) in an arbitrary Lorentz frame are not canonical on the generalized mass shell  $\mathcal{M}_{[2|1]}$ . This point is also discussed in [15].) Nevertheless, this definition of relativistic two-particle dynamics is not adopted either in [19] or in the present paper, because it does not fit the separability assumption (iii) (the expression  $xP(=0)$  having no limit for  $\mathbf{x} \rightarrow \infty$ ) and does not appear to be natural in the  $N$ -particle case (for  $N \geq 3$ ).

6) Theorem 1 is the counterpart (in the constraint Hamiltonian formulation of relativistic classical dynamics) of the CJS no interaction theorem [3, 10, 8]. The objective of this comment is to elucidate the precise relation between the two results. To do that, we start with a concise formulation of the CJS statement of the problem and main theorem (adapted to the language of the present paper).

A CJS  $N$ -particle system is defined by a Poisson bracket realization of the Lie algebra of the Poincaré group in the space  $\mathbb{R}^{6N}$ , spanned by the (three-dimensional) particle co-ordinates  $\mathbf{x}_j$  and momenta  $\mathbf{p}_k$  ( $j, k=1, \dots, N$ ) and equipped with the canonical symplectic form

$$\omega_{\text{CJS}} = \sum_{k=1}^N d\mathbf{p}_k \wedge d\mathbf{x}_k. \quad (2.30)$$

The Euclidean generators  $\mathbf{P}$  and  $\mathbf{M}$  ( $= \frac{1}{2}\varepsilon_{ijk}M^{jk}$ ) are assumed to have the standard form (1.7) and (1.8), while the Lorentz boosts  $M^{0j}$  and the Hamiltonian  $h = P^0$  are required to satisfy

$$\{M^{0i}, x_k^j\} = x_k^i \{h, x_k^j\} \quad (k=1, \dots, N; i, j=1, 2, 3). \quad (2.31)$$

(This assumption is referred to as the “world line condition” in [3].)

A CJS system is called non-degenerate if the equations  $\dot{\mathbf{x}}_k = [h, \mathbf{x}_k] \equiv \mathbf{y}_k(\mathbf{p}, \mathbf{x})$  can be solved with respect to the canonical momenta  $\mathbf{p}_k$  (or, equivalently, if the determinant of the second derivatives of  $h$  with respect to the momenta does not vanish – cf. Theorem 2).

**CJS Theorem** [3, 10]. *Every CJS two-particle system, and every non-degenerate  $N$ -particle system for  $N \geq 3$ , is canonically equivalent to a free CJS system (with Hamiltonian  $h = \sum_k \sqrt{m_k^2 + \mathbf{p}_k^2}$ ).*

*Note.* The CJS theorem was originally established in 4 space-time dimensions. It fails in 2 dimensions unless one adds extra assumptions [3, 8].

Given a generalized  $N$ -particle mass shell [satisfying conditions (i) and (ii) of Sect. I.A.] the question is whether it is possible to construct a CJS system. An obvious candidate for such a system is given (in some fixed Lorentz frame) by the set of gauge conditions  $x_k^0 = t$  which exclude the relative time variables from the canonical Hamiltonian  $h$  in (1.14). The sub-variety of  $\mathcal{M}$  thus obtained, however, is, in general, only a subset of  $\mathbb{R}^{6N}$  [which could be even empty, as it is the case for the two-particle mass shell given by Eq. (2.14) for time-like  $x_1 - x_2$ .] Assuming that it does coincide with  $\mathbb{R}^{6N}$  (or, more precisely, that  $\mathcal{M} \cap \{x_k^0 = t\} = \mathbb{R}^{6N}$  is a global section of the fibre bundle  $\mathcal{M} \rightarrow \Gamma_*$ ) we shall demonstrate that the property (2.31) of the Lorentz boosts is a consequence of the assumption of gauge invariance of world lines in our formulation.

Indeed, according to Lemma 1 of Sect. II.B, the condition  $\dim T_k = 1$  ( $k = 1, \dots, N$ ) implies that  $h_k$  is independent of  $\mathbf{p}_l$  for  $k \neq l$ . Then, the generators  $P_\mu$  (1.7),  $M_{ij}$  (1.8), and  $M^{0i} = \sum_k h_k x_k^i$  [for  $x_\ell^0 = t$  and  $h_k \approx p_k^0$  (1.9)] satisfy the Poisson bracket relations of the Lie algebra of the Poincaré group, and, moreover

$$\{M^{0i}, x_k^j\} = x_k^i \{h_k, x_k^j\} = x_k^i \{h, x_k^j\} = \left\{ \sum_\ell h_\ell x_\ell^i, x_k^j \right\}$$

in accord with (2.31).

To summarize, Theorem 1 has only a partial overlap with the CJS theorem. Our result can be obtained from the CJS statement (using Lemma 1 and the above argument) under the additional assumption that the intersection of  $\mathcal{M}$  with the equal time gauge is  $\mathbb{R}^{6N}$  (and that the Hamiltonian is non-degenerate for  $N \geq 3$ ). If we add to the CJS requirement the assumption that the system under consideration is obtained by restricting a generalized mass shell to the equal time gauge, then Theorem 1 provides an extension of the CJS result to non-degenerate systems (which may involve 0 mass particles) for  $D \geq 2N - 1$ .

We should also like to point out a difference in emphasis. Gauge dependence of world lines is the price one pays for enforcing the instantaneous canonical Hamiltonian framework upon the description of interacting relativistic point particles. As shown by Wigner and Van Dam (see, e.g. [23]) no such problem arises in a non-Hamiltonian approach involving retarded and advanced interactions. We contend, however, that the frame dependence of relativistic particle trajectories is *not* too high a price for preserving the Hamiltonian formalism, which, as we shall see in the next section, still provides reliable asymptotic results.

#### II.D. Gauge Invariance of the Two-Particle Relative Motion and of the Scattering Matrix

It turns out that a mathematical theorem – the Birman-Kato invariance principle – guarantees the gauge invariance of the  $S$  matrix and of the Moeller wave operators whenever they exist.

Before sketching the general argument, which will require some new apparatus, we shall present an elementary derivation of this result for the two-particle case.

For  $D(=\phi_1 - \phi_2)=0=P\frac{\partial\phi}{\partial x}$  the Hamiltonian (1.25), which preserves the gauge condition (1.24), is

$$H^{(m)} = H - \frac{1}{nP} \left( n \frac{\partial\phi}{\partial p} + np \right) \varphi (\approx 0), \quad (2.32)$$

where  $H$  is given by (1.17c). We observe that

$$\{\varphi, x_{\perp}\} = 0 = \{\varphi, p\}, \quad (2.33)$$

so that the time evolution of  $x_{\perp}$  and  $p$  is indeed independent of the choice of  $n$ :

$$\{H^{(m)}, x_{\perp}\} \approx \{H, x_{\perp}\}, \quad \{H^{(m)}, p\} \approx \{H, p\}. \quad (2.34)$$

The gauge dependence of the canonical centre-of-mass variable

$$\begin{aligned} X_c &= \frac{\partial}{\partial P} [(\mu_1 P + p)x_1 + (\mu_2 P - p)x_2] \\ &= X + \frac{\mu_1 - \mu_2}{w^2} (Px)P, \quad X = \mu_1 x_1 + \mu_2 x_2 \end{aligned} \quad (2.35)$$

can be found explicitly by solving the equation

$$\frac{\partial X_c}{\partial \sigma} = \{\varphi, X_c\} = \mathbf{p}, \quad \frac{\partial^2 X_c}{\partial \sigma^2} = 0. \quad (2.36)$$

If we denote the variable conjugate to  $H$  by  $\tau$ , so that

$$\frac{\partial X_c}{\partial \tau} = \{H, X_c\} = \frac{\partial H}{\partial P}, \quad (2.37)$$

then the  $\sigma$  dependence of  $X_c(\tau, \sigma)$  is given by

$$X_c(\tau, \sigma) = X_c(\tau, 0) + p(\tau)\sigma. \quad (2.38)$$

Assume now that we have an elastic scattering problem, for which the following limits exist:

$$\begin{aligned} \lim_{\tau \rightarrow \pm\infty} p_k(\tau) &= p_k^{\text{as}} = \lim_{\tau \rightarrow \pm\infty} \frac{1}{\tau} x_k(\tau, 0), \quad \text{as} = \begin{cases} \text{in for } \tau \rightarrow -\infty \\ \text{out for } \tau \rightarrow \infty \end{cases}, \\ (p_1^{\text{out}} + p_2^{\text{out}} = p_1^{\text{in}} + p_2^{\text{in}} = P) \end{aligned} \quad (2.39)$$

and

$$\begin{aligned} \lim_{\tau \rightarrow \pm\infty} [x_{\perp}(\tau) - p^{\text{as}}(\tau - \tau_{\text{as}})] &= a^{\text{as}} \\ p^{\text{as}} &= \mu_2 p_1^{\text{as}} - \mu_1 p_2^{\text{as}}, \quad a^{\text{as}} p^{\text{as}} = 0. \end{aligned} \quad (2.40)$$

(If the above limit exists, say, for  $\tau_{\text{as}}=0$ , then it exists for any  $\tau_{\text{as}}$ . Assuming  $p^{\text{as}} \neq 0$ , there is a unique value of  $\tau_{\text{as}}$  for which  $a^{\text{as}}$  and  $p^{\text{as}}$  are orthogonal.) For non-zero angular momentum,  $\ell > 0$ , the three 3-vectors  $\mathbf{p}^{\text{as}}$ ,  $\mathbf{a}^{\text{as}}$  and  $\ell = \mathbf{p}^{\text{as}} \wedge \mathbf{a}^{\text{as}}$  form an orthogonal frame in the centre-of-mass 3-space; moreover  $|\mathbf{a}^{\text{as}}| = \frac{l}{b}$  coincides with

the impact parameter (1.33) (since  $|\mathbf{p}^{\text{as}}|=b$ ). Noting that for  $\tau \rightarrow \pm\infty$ , we have  $r \rightarrow \infty$ , and  $\frac{\partial\phi}{\partial p_k} \rightarrow 0$ , we find

$$\frac{dx_1^{\text{as}}}{d\tau^{(n)}} = \left(\mu_2 - \frac{np}{nP}\right)p_1^{\text{as}}, \quad \frac{dx_2^{\text{as}}}{d\tau^{(n)}} = \left(\mu_\perp + \frac{np}{nP}\right)p_2^{\text{as}}. \quad (2.41)$$

Obviously, the corresponding normalized 4-velocities  $u_k^{\text{as}}$ ,  $k=1,2$  are independent of  $n$ . Since the scattering matrix transforms (by definition) the vectors  $p_k^{\text{in}}=m_k u_k^{\text{in}}$  and  $a^{\text{in}}$  into  $p_k^{\text{out}}=m_k u_k^{\text{out}}$  and  $a^{\text{out}}$  (all of which are gauge invariant), it is gauge invariant as well.

A deficiency of this argument is that it relies on the frame independence of the two-particle relative motion which cannot be expected to hold in the three (and more) particle case. Therefore, we shall sketch a more sophisticated argument, which has the virtue of being general. It is based on the notion of a (classical) wave operator, defined as follows.

Let  $L_f$  be the *Liouville operator* (Hamiltonian vector field) corresponding to a (smooth) function  $f(p, q)$  of the canonical co-ordinates in phase space

$$L_f = \frac{\partial f}{\partial p} \frac{\partial}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial}{\partial p}. \quad (2.42)$$

The (classical) wave operators  $w_\pm$  for the pair of Hamiltonians  $H_0$  and  $H$  are defined (whenever they exist) by the strong limits

$$w_\pm = w_\pm(H, H_0) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{tL_H} e^{-tL_{H_0}} \quad (2.43)$$

with respect to the  $L^1$  norm

$$\|f\| = \int_{I_*} |f(p, q)| \prod \frac{dpdq}{2\pi}.$$

The scattering operator is then given by

$$S = w_+^* w_- . \quad (2.44)$$

The Birman-Kato invariance principle (originally established in the framework of quantum theory and recently justified in the classical context – in the third Sokolov's paper [16]) says that for any smooth monotonously increasing function  $F(\xi)$  on the reals [such that  $F'(\xi) > 0$  everywhere]

$$w_\pm(H, H_0) = w_\pm(F(H), F(H_0)). \quad (2.45)$$

The gauge invariance of the two-particle  $S$  matrix is obtained as a consequence of (2.47) for  $H_0 = \frac{1}{2}(p^2 - b^2)$ ,  $H = H_0 + \phi$ ,  $F(H) = \lambda \left[ H - \frac{1}{nP} \left( n \frac{\partial\phi}{\partial p} + np \right) \phi \right]$  after noticing that  $\varphi \approx 0 = \{\varphi, H\} = \{\varphi, H_0\}$ . The Birman-Kato invariance principle also applies to the  $N$ -particle scattering for  $N \geq 3$  if the  $N$ -particle dynamics is defined in terms of the relativistic addition of interactions<sup>9</sup> (see [16, 20]).

<sup>9</sup> We do not discuss here the difficulties in constructing three (or more) particle separable interactions [13] which are overcome in Sokolov's work [16]. (If one neglects compatibility then the remaining conditions are fulfilled by a simple sum of two-particle interactions – see, e.g., [2])

We note finally that the quantum-mechanical bound state energy levels (evaluated in [18, 14] on the basis of a Schrödinger equation of the type  $H\psi=0$ ) also turn out to be gauge invariant.

*Acknowledgements.* It is a pleasure to thank F. Rohrlich and S. N. Sokolov as well as the participants of the quantum field theory seminar of the Steklov Mathematical Institute for useful discussions. The authors would like to thank Professor Abdus Salam and Professor P. Budinich, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, and the Scuola Internazionale Superiore di Studi Avanzati, Trieste, where the final version of this paper was completed.

## References

1. Bel, L., Martin, J.: *Ann. Inst. Henri Poincaré* **22A**, 173–195 (1975)
2. Crater, H.W.: *Phys. Rev.* **D18**, 2872–2881 (1978)
3. Currie, D.G., Jordan, T.F.: In: *Lectures in Theoretical Physics*. Barut, A.O., Brittin, W.E. (eds.), pp. 91–139; New York: Gordon and Breach 1968
4. Currie, D.G., Jordan, T.F., Sudarshan, E.C.G.: *Rev. Mod. Phys.* **35**, 350–375 (1963); *ibid.* 1032
5. Dirac, P.A.M.: *Proc. Roy. Soc. (London)* **A246**, 326–332 (1958); *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Yeshiva Univ., New York 1964
6. Droz-Vincent, Ph.: *Ann. Inst. Henri Poincaré* **27**, 407–424 (1977); *Phys. Rev.* **D19**, 702–706 (1979); *N-body relativistic systems*. Collège de France preprint, Paris (1979) (see also references to earlier work of this author cited therein)
6. Faddeev, L.D.: *Teor. Mat. Fiz.* **1**, 3–18 (1969); *Theor. Math. Phys. (USSR)* **1**, 1–13 (1969) (Engl. Transl.)
7. Hanson, A.J., Regge, T., Teitelboim, C.: *Constraint Hamiltonian Systems*. Roma: Accademia Nazionale dei Lincei 1976
8. Hill, R.N.: *J. Math. Phys. (N.Y.)* **8**, 1756–1773 (1967)
9. Komar, A.: *Phys. Rev.* **D18**, 1881–1886 (1978); *Phys. Rev.* **D18**, 1887–1893 (1978); *Phys. Rev.* **D18**, 3617–3623 (1978)
10. Leutwyler, H.: *Nuovo Cimento* **37**, 556–567 (1965)
11. Leutwyler, H., Stern, J.: *Ann. Phys. (N.Y.)* **112**, 94–164 (1978)
12. Logunov, A.A., Tavkheldize, A.N.: *Nuovo Cimento* **29**, 380–399 (1963)
13. Mutze, U.: *J. Math. Phys. (N.Y.)* **19**, 231–236 (1978)
14. Rizov, A.A., Todorov, I.T.: *Elem. Chastitsy Atom. Yad.* **6**, 669–742 (1975); *Sov. J. Part. Nucl.* **6**, 269–298 (1975) (Engl. Transl.)
14. Rizov, V.A., Todorov, T.I., Aneva, B.L.: *Nucl. Phys.* **B98**, 447–471 (1975)
15. Rohrlich, F.: *Ann. Phys. (N.Y.)* **117**, 292–322 (1979); *Phys. Lett.* **66A**, 268–270 (1978)
16. Sokolov, S.N.: Theory of relativistic direct interactions. Preprint IHEP OTF 78–125, Serpukhov (1978); *Teor. Mat. Fiz.* **36**, 193–207 (1978); *Theor. Math. Phys. (N.Y.)* **36**, 682–692 (1978) (Engl. Transl.); *Nuovo Cimento* **52A**, 1–22 (1979); see also earlier work by Sokolov, cited in these papers
17. Takabayasi, T.: *Progr. Theoret. Phys. (Kyoto)* **54**, 563–577 (1975); *Progr. Theoret. Phys. (Kyoto)* **57**, 331–332 (1977); *Relativistic mechanics of confined particles as extended model of hadrons – the bilocal case*. *Prog. Theor. Phys. Suppl.* **67**, 1–68 (1979) and references therein
18. Todorov, I.T.: *Phys. Rev.* **D3**, 2351–2356 (1971); *Quasipotential approach to the two-body problem in quantum field theory*. In: *Properties of Fundamental Interactions*, Vol. 9, Part C, pp. 951–979. Zichichi, A. (ed.). Bologna: Editrice Compositori 1973
19. Todorov, I.T.: *Communication of the JINR*, E2-10125, Dubna (1976)
20. Todorov, I.T.: *Constraint Hamiltonian dynamics of directly interacting relativistic point particles*. Lectures presented at the Summer School in Mathematical Physics, Bogazici University, Bebek, Istanbul, August 1979 and at the XVII<sup>th</sup> Winter School of Theoretical Physics, Karpacz, February–March 1980 (to be published)

21. Vinogradov, A.M., Kupersmidt, B.A.: *Usp. Mat. Nauk* **32**, No. 4, 175–236 (1977); *Russ. Math. Surv.* **32**:4, 177–244 (1977) (Engl. Transl.)
22. Vlaev, S.: The relativistic Coulomb problem for two classical spinless particles as a constraint dynamical problem. Diploma work, University of Sofia (1977)
23. Wigner, E.P.: Relativistic interaction of classical particles. In: Coral gables conference on fundamental interactions at high energy, Gudehus, T. et al. (ed.), pp. 344–355. New York: Gordon and Breach 1969

Communicated by A. Jaffe

Received July 23, 1979; in revised form August 12, 1980