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On the Spherically Symmetric Gauge Fields

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Abstract. The spherically symmetric gauge fields with a compact gauge group over 4-dimensional Minkowski space are determined completely. Expressions for the gauge potentials of these fields are obtained.

Part I. Construction of Spherically Symmetric Gauge Fields

1. Introduction

Spherical symmetry is one of the most important symmetry in nature. In a field theory the spherically symmetric fields are very useful for understanding the theory itself. For example, the important role of Schwarzschild solution in Einstein's gravitation theory is well-known. For the gauge theories, the fields with spherical symmetry are also very interesting. It is known that two gauge fields are equivalent if they are related by a gauge transformation. Consequently, in the construction of spherically symmetric gauge fields we have to consider the effect of gauge transformations. Spherically symmetric potentials of SU_2 gauge fields were firstly considered by Wu and Yang [1]. Now the complete classification of spherically symmetric SU_2 gauge fields is known [2]. Some spherically symmetric SU_3 gauge potentials were studied by several authors [3–5]. For the general gauge groups the problem of determining spherically symmetric field may be reduced to solve some system of partial differential equations by using Lie derivatives [6]. But, the results in [6] are local in character and no formulas for the gauge potential were obtained. The same problem was treated in a different way in a previous paper [7], an algebraic method for determining all spherically symmetric gauge fields was proposed. However, in this paper the expressions of the gauge potentials were given only for some special cases and the proof of the general theorem was not complete yet. It should be noted that some special form of spherically symmetric gauge fields were pointed in [8].

In the present paper we develop this method and give a complete determination of general spherically symmetric gauge fields for any compact gauge

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group. In Part I we state the method of construction and identify a special but important class of such fields. In Part II we prove that all possible spherically symmetric gauge fields can be obtained in this way. The proof is rather long, since we have to prove the completeness.

2. Definitions

Let G be a compact Lie group and $\mathscr{F}$ a gauge field with gauge group G over 4-dimensional Minkowski space-time R^{3+1}. We assume that the field $\mathscr{F}$ is regular except for $r=0$. Here $r=(x_1^2+x_2^2+x_3^2)^{1/2}$.

If the field $\mathscr{F}$ corresponds to a trivial bundle, then the gauge potential $b(x, dx) = b_\lambda(x)dx_\lambda$ ($\lambda=1,2,3,4$) is a g-valued 1-form defined on R^{3+1} except for $r=0$. Here g is the Lie algebra of G. For simplipifying the description we assume that G and g consist of $N \times N$ matrices.

Definition 1. If for each $A \in \mathrm{SO}_3$ there exists a G-valued function u_A such that

$$b(Ax, \mathrm{Ad}x) = (\mathrm{ad}\, u_A(x))b(x, dx) - (du_A(x))u_A^{-1}(x)\,, \tag{1}$$

then the field $\mathscr{F}$ is called spherically symmetric and the function u_A is defined as the complementary function associated with A.

If the field $\mathscr{F}$ corresponds a nontrivial bundle we have to cover R^{3+1} by two regions [9] i.e. $R^{3+1} = M^+ U M^-$

$$\begin{aligned} M^+ &= R^{3+1} - \{(0,0,r,t)\}\,, \\ M^- &= R^{3+1} - \{(0,0,-r,t)\}\,, \\ &(0 \leqq r < \infty, \quad -\infty < t < \infty)\,. \end{aligned} \tag{2}$$

Here $(0,0,r,t)$ and $(0,0,-r,t)$ are the north pole and south pole of the sphere $x_1^2+x_2^2+x_3^2=r^2$, $x_4=t$. The field $\mathscr{F}$ should be expressed as the combination of two gauge fields $\mathscr{F}^+$ and $\mathscr{F}^-$ which are defined by two gauge potentials b^+ and b^- over M^+ and M^- respectively. Moreover, b^+ and b^- are related by a gauge transformation

$$b^+(x, dx) = (\mathrm{ad}\, \zeta(x))b^-(x, dx) - (d\zeta(x))\zeta^{-1}(x) \tag{3}$$

in $M^+ \cap M^-$, where $\zeta(x)$ is a G-valued function.

Definition 2. $\mathscr{F}^+$ (or $\mathscr{F}^-$) is called almost spherically symmetric if the definition 1 holds true for all x and A, satisfying $x \in M^+$ (or M^-) and $Ax \in M^+$ (or M^-).

Definition 3. If $\mathscr{F}$ consists of $\mathscr{F}^+$ and $\mathscr{F}^-$, and $\mathscr{F}^+$, $\mathscr{F}^-$ are almost spherically symmetric, then $\mathscr{F}$ is called a spherically symmetric field.

Remark. If $\mathscr{F}$ corresponds to a trivial bundle it can also be represented as a combination of $\mathscr{F}^+$ and $\mathscr{F}^-$. In this case it is ready seen that definition 3 agrees with Definition 1.

3. Construction of Spherically Symmetric Gauge Fields

We give here a description of the method of constructing spherically symmetric gauge fields. The main procedure is to construct $\mathscr{F}^+$ and $\mathscr{F}^-$, then connect them by a gauge transformation. In order to obtain $\mathscr{F}^+$ (and $\mathscr{F}^-$), we construct the complementary function u_A corresponding to each rotation A satisfying $Ax_0 \neq x_0'$ where $x_0 = (0, 0, r, t)$, $x_0' = (0, 0, -r, t)$. Then, we determine the possible gauge potential of x_0, and further we obtain the gauge potential b^+ at any arbitrary points of $\mathscr{F}^+$, the gauge potential b^- is constructed similarly. Moreover, we need a quantization condition to combine the b^+ and b^-.

Let (ψ, θ, ϕ) be the Eulerian angles for the rotation A, i.e. $A = C_3(\psi)C_1(\theta)C_3(\phi)$, where $C_a(\alpha)$ is the rotation around x_a-axis through an angle α.

(a) The first step is to take an element Y of g such that the quantization condition

$$\exp(4\pi Y) = e \tag{4}$$

be satisfied, where e is the unit element of G. Let the value of a complementary function $u_{C_3(\phi)}$ at x_0 be

$$u_{C_3(\phi)}(x_0) = \exp(\phi Y). \tag{5}$$

The condition (4) is a crucial one for combining $\mathscr{F}^+$ and $\mathscr{F}^-$ in a global gauge field $\mathscr{F}$.

(b) In order to obtain $u_A(x_0)$ for general A we set

$$u_{C_3(\psi)C_1(\theta)}(x_0) = u_{C_3(\psi)}(x_0), \qquad 0 \leqq \theta < \pi. \tag{6}$$

In Part II we shall prove that any other possible choices of $u_{C_3(\psi)C_1(\theta)}(x_0)$ will be equivalent to (6) via a gauge transformation. Then, the value of the complementary function $u_A(x)$ at $x = x_0$ is

$$u_A(x_0) = u_{C_3(\psi)C_1(\theta)C_3(\phi)}(x_0) = u_{C_3(\psi)}(x_0)u_{C_3(\phi)}(x_0) = u_{C_3(\psi+\phi)}(x_0), \quad (\theta < \pi). \tag{7}$$

(c) Let $x = Bx_0$, $B \in SO_3$, $x \neq (0, 0, -r, t)$. It is seen in Part II that the formula

$$\begin{aligned} u_A(x) &= u_{C_3(\psi_1)C_1(\theta_1)C_3(\phi_1)}(Bx_0) \\ &= u_{C_3(\psi_1)C_1(\theta_1)C_3(\phi_1)B}(x_0)u_B^{-1}(x_0) = u_{AB}(x_0)u_B^{-1}(x_0) \end{aligned}$$

defines the value of a complementary function $u_{C_3(\psi_1)C_1(\theta_1)C_3(\phi_1)}$ at the point x. It is easy to verify that this value is independent upon the choice of B. Thus, we obtain the complementary function $u_A(x)$.

(d) To obtain the possible gauge potential at the point x_0, we put $x = x_0$ and $A = C_3(\phi)$ in Eq. (1) and obtain

$$b(x_0, C_3(\phi)dx) = (\operatorname{ad} \exp(\phi Y))b(x_0, dx). \tag{8}$$

For simplicity one can choose a gauge such that $x_i b_i(x) = 0$, then

$$b_3(x_0) = 0. \tag{9}$$

By differentiating Eq. (8) with respect to ϕ, we obtain a system of linear equations

$$-b_2(x_0) + [Y, b_1(x_0)] = 0, \qquad b_1(x_0) + [Y, b_2(x_0)] = 0, \tag{10}$$

$$[Y, b_4(x_0)] = 0. \tag{11}$$

For given G and Y, it is quite easy to solve these equations. Thus, (9)–(11) give $b_\lambda(x_0)$ or

$$b(x_0, dx) = b_\lambda(x_0)dx_\lambda = b_\lambda(0,0,r,t)dx_\lambda \qquad (r>0). \tag{12}$$

For abbreviation, we write

$$b_\lambda(r,t) = b_\lambda(0,0,r,t) = b_\lambda(x_0), \qquad b(x_0, dx) = b(r,t,dx). \tag{12'}$$

(e) Using (1) and (6), after some calculation we obtain the expression of b^+ at any point of M^+:

$$b^+(C_3(\psi)C_1(\theta)x_0, dx) = (\mathrm{ad}\exp(\psi Y))b(r,t,C_1(-\theta)C_3(-\psi)dx) - \frac{r-x_3}{r}Yd\psi, \tag{13}$$

where $C_3(\psi)C_1(\theta)x_0$ $(\pi > \theta \geqq 0)$ is a general point of M^+.

This is the formula for gauge potentials of the field $\mathscr{F}^+$. In Part II, we shall verify that it is almost spherically symmetric.

(f) Similarly, the field $\mathscr{F}^-$ is defined by the potential

$$b^-(C_3(\psi)C_1(\theta-\pi)x_0', dx) = (\mathrm{ad}\exp(-\psi Y))b(r,t,C_1(\pi-\theta)C_3(-\psi)dx) + \frac{r+x_3}{r}Yd\psi, \tag{14}$$

where $x_0' = (0,0,-r,t)$ and $b(x_0', dx) = b(x_0, dx) = b(r,t,dx)$.

(g) It is readily verified that

$$b^+(x,dx) = \mathrm{ad}(\exp(2\psi Y))b^-(x,dx) - (d\exp(2\psi Y))\exp(-2\psi Y). \tag{15}$$

Hence, $\mathscr{F}^+$ and $\mathscr{F}^-$ can be combined together as a spherically symmetric gauge field $\mathscr{F}$ if (4) holds.

Thus the construction of spherically symmetric gauge fields is accomplished. In short, we have the theorem

Main Theorem. *All the spherically symmetric gauge fields of a compact gauge group are the combination of two fields $\mathscr{F}^+$ and $\mathscr{F}^-$ which are defined over M^+ and M^- with gauge potentials* (13) *and* (14) *respectively. In* (13) *and* (14) *Y is an arbitrary element of g satisfying* $\exp(4\pi Y) = e$ *and $b(r,t,dx)$ is determined by* (9)–(11).

4. Simple Examples

We shall give some examples to illustrate the general construction in Sect. 3.

(a) U_1 gauge fields. When G is U_1, the Lie algebra g is generated by i, thus we have to take $Y = \frac{m}{2}i$, where m is an integer. From Eq. (9)–(11), the general solution for $b_\lambda(x_0)$ is

$$b_1(x_0) = b_2(x_0) = b_3(x_0) = 0, \qquad b_4(x_0) = \sigma(r,t)i.$$

Consequently, from (13), (14) we have

$$\begin{aligned} b^+(x,dx) &= -\frac{r-x_3}{r}Yd\psi + i\sigma(r,t)dt = -\frac{1-\cos\theta}{2}mid\psi \\ &\quad + \sigma(r,t)idt\,, \\ b^-(x,dx) &= \frac{r+x_3}{r}Yd\psi + i\sigma(r,t)dt = \frac{1+\cos\theta}{2}mid\psi \\ &\quad + \sigma(r,t)idt\,. \end{aligned} \tag{16}$$

So the field consists of a standard m-monopole [9] and a spherically symmetric electromagnetic field with scalar potential $\sigma(r,t)$.

(b) SU_2 gauge fields. Without loss of generality we may take

$$Y = i\begin{bmatrix} \frac{m}{2} & 0 \\ 0 & -\frac{m}{2} \end{bmatrix} \quad (m \text{ is an integer}). \tag{17}$$

From (10) we see that

$$b_1(x_0) + [Y,[Y,b_1(x_0)]] = 0\,. \tag{18}$$

If $m \neq 0, 1$, we have

$$b_1(x_0) = b_2(x_0) = b_3(x_0) = 0$$

$$b_4(x_0) = \sigma\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The field is reduced to a U_1 field and the potential has the form (15), the only change is that i should be replaced by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

If $m=0$ or 1, we obtain the strictly spherically symmetric SU_2 gauge field and the synchro-spherically symmetric SU_2 gauge fields [2]. We shall discuss them in the following section.

5. Spherically Symmetric Gauge Fields of Proper Type

Definition 4. Let $\mathscr{F}$ be a spherically symmetric gauge field corresponding to a trivial bundle. If there exists a gauge such that the complementary function u_A does not depend on x, then it is called a spherically symmetric gauge field of proper type [7]. It has been shown that if a spherically symmetric field $\mathscr{F}$ is regular everywhere, then it is of proper type. In fact, in this case the bundle is trivial. Let $0, 0', x$ be the points $(0,0,0,0)$, $(0,0,0,t)$, (x_1,x_2,x_3,t) respectively. Choose a gauge such that the phase factors for each segment $00'$ and $0'x$ are all equal to e. The phase factor for the arc $x\,x+dx$ is $e-b(x,dx)$ where $b(x,dx)$ is the gauge potential [10]. This quantity is the phase factor for the infinitessimal loop $00'xx'0''0$, where $x'=x+dx$, $0''=(0,0,0,t+dt)$ [11]. The spherical symmetry implies the existence of u_A such that

$$b(Ax, dAx) = (\operatorname{ad} u_A) b(x,dx)\,. \tag{19}$$

Here u_A is independent of x.

Evidently, a spherically symmetric gauge field $\mathscr{F}$ of proper type can be singular at $r=0$. The construction of spherically symmetric gauge fields of proper type can be accomplished in the following way. Let $\tau: A \to u_A$ be an homomorphism of SO_3 to G and τ_1 the corresponding homomorphism of the Lie algebra so_3 to g, X_1, X_2, X_3 be a set of standard base of so_3 and $Y_i = \tau_1 X_i$ $(i=1,2,3)$. It is easily seen that

$$u_{C_3(\phi)} = \exp(\phi Y_3) . \tag{20}$$

Moreover, $u_{C_3(\psi)C_1(\theta)}$ is given by the homorphism τ, i.e. $u_{C_3(\psi)C_1(\theta)} = \tau C_3(\psi) C_1(\theta)$, $b_\lambda(x_0)$ are still determined by (9)–(11) with $Y = Y_3$ and the gauge potential is given by

$$b(Ax_0, \operatorname{Ad} x) = (\operatorname{ad} u_A) b(x_0, dx) . \tag{21}$$

The gauge potential obtained this way is equivalent to the gauge potential obtained through the method in Sect. 3 with the same $u_{C_3(\phi)(x_0)} = \exp(\phi Y_3)$ and $b(x_0, dx)$. This is a consequence of Lemma 7. However, in the present case the potential corresponds to a trivial bundle and admits a simpler explicit expression. Besides, we can conclude that a spherically symmetric gauge field $\mathscr{F}$ is of proper type if and only if the element Y of the Lie algebra, described in the main theorem, be $\tau_1 X_3$.

For SU_2 case, we know that the representations of SO_3 in a two dimensional space are either a trivial representation $A \to e$ or the spin representation $A \to u_A$ with $(\operatorname{ad} u_A) = A$, if a suitable base for SU_2 is chosen. Then Y_3 is 0 or $\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The SU_2 gauge field obtained are strictly spherically symmetric and synchro-spherically symmetric (see the end of Appendix II) respectively.

In the appendix we shall list all possible gauge potential for SU_3 gauge field. A class of SU_N gauge potentials is also presented.

Part II. Proof of the Main Theorem

6. Sets of Complementary Functions

At first we shall analyze in detail the properties of complementary functions defined in Definition 1. The analysis is made for the case that $\mathscr{F}$ corresponds to a trivial bundle, but all results are valid for the fields $\mathscr{F}^+$ and $\mathscr{F}^-$ with some obvious modifications.

Because there may be a set of complementary functions $\{u_A\}$ associated with each $A \in SO_3$, we use the following notations:

$U_A = \{u_A\}$ – the set of complementary functions associated with A.

$u_A(x)$ – the value of u_A at x.

$U_A(x) = \{u_A(x)\}$ – the set of values at x of complementary functions associated with A.

Lemma 1. *Let $A, B \in SO_3$. Then*

$$\{u \mid u = u_A(Bx) u_B(x)\} = U_{AB}(x) . \tag{22}$$

Proof. Using (1), we see that

$$\begin{aligned} b(ABx, ABdx) &= \operatorname{ad} u_A(Bx) b(Bx, Bdx) - du_A(Bx) u_A^{-1}(Bx) \\ &= \operatorname{ad} u_A(Bx) \operatorname{ad} u_B(x) b(x, dx) - \operatorname{ad} u_A(Bx) du_B(x) u_B^{-1}(x) \\ &\quad - du_A(Bx) u_A^{-1}(Bx) \\ &= \operatorname{ad}(u_A(Bx) u_B(x)) b(x, dx) - d(u_A(Bx) u_B(x)) (u_A(Bx) u_B(x))^{-1} . \end{aligned}$$

Hence $u_A(Bx)u_B(x) \in U_{AB}(x)$. Similarly we have

$$u_B^{-1}(x) \in U_{B^{-1}}(Bx) . \tag{23}$$

Consequently, for each $u_{AB}(x) \in U_{AB}(x)$ and $u_B(x) \in U_B(x)$ we have

$$u_{AB}(x) u_B^{-1}(x) = u_{AB}(x) u_{B^{-1}}(Bx) \in U_A(Bx) . \tag{24}$$

Thus (22) is proved.

For simplicity we write (22) as

$$U_A(Bx) U_B(x) = U_{AB}(x) . \tag{25}$$

Lemma 2. *Under the gauge transformation*

$$b'(x, dx) = (\operatorname{ad} \zeta(x)) b(x, dx) - (d\zeta(x)) \zeta^{-1}(x) , \tag{26}$$

the complementary function u_A becomes u'_A defined by

$$u'_A(x) = \zeta(Ax) u_A(x) \zeta^{-1}(x) . \tag{27}$$

Proof. Using (1), (26), and (27), it is easily seen that

$$b'(Ax, \operatorname{Ad} x) = (\operatorname{ad} u'_A(x)) b'(x, dx) - (du'_A(x)) u'^{-1}_A(x) .$$

This is the conclusion of Lemma 2.

Let x_0 be a fixed point, say $(0, 0, r, t)$, and E the unit element of SO_3.

Lemma 3. *$U_E(x_0)$ is a closed subgroup of G.*

Proof. From Lemma 1 we see that $U_E(x_0)$ is a subgroup. If $\alpha \in U_E(x_0)$, then there is a G-valued function $u \in U_E$ such that $u(x_0) = \alpha$. The function u is the solution of the differential equation

$$du = ub - bu \tag{28}$$

with the initial condition $u(x_0) = \alpha$. Suppose that $\{\alpha_n\}$ is a sequence of elements in $U_E(x_0)$ and $\alpha_n \to \alpha_0$ as $n \to \infty$. For each α_n the differential equation (28) has a solution $u_n(x)$ and $u_n(x_0) = \alpha_n$. The value $u_n(x)$ can be obtained by solving a system of ordinary equation

$$\frac{du}{d\sigma} = u(\sigma) b\left(x(\sigma), \frac{dx(\sigma)}{d\tau}\right) - b\left(x(\sigma), \frac{dx(\sigma)}{d\sigma}\right) u(\sigma) \tag{29}$$

with initial condition $u(0) = \alpha_n$, where $x(\sigma)$ $0 \leqq \sigma \leqq 1$ is a smooth arc connecting x_0 and x. Moreover, the value of $u_n(x)$ is independent of the choice of the arc $x(\sigma)$. Consequently, $u_n(x)$ converges to a solution $u(x)$ of (29) with initial condition

$u(x_0)=\alpha_0$ and $u(x)$ is independent of the choice of $x(\sigma)$, so it is a solution of (28). Hence $\alpha_0 \in U_E(x_0)$. This proves that the group $U_E(x_0)$ is closed.

Lemma 4. *Suppose that the Lie algebra g does not contain a nontrivial center. If $U_E(x_0)$ is a nondiscrete proper subgroup of G, then the field is reducible to a gauge field whose gauge group has Lie algebra $g_1 \neq g$.*

Proof. Suppose that $u \in U_E$. Then u satisfies (28). The integrability condition of (28) is

$$(\mathrm{ad}\, u(x)) f_{\lambda\mu}(x) = f_{\lambda\mu}(x) \,. \tag{30.0}$$

Here $f_{\lambda\mu}(x)$ is the field strength. The successive gauge derivative $f_{\lambda\mu/\nu}(x), \dots$ satisfy

$$(\mathrm{ad}\, u(x)) f_{\lambda\mu/\nu}(x) = f_{\lambda\mu/\nu}(x) \,, \tag{30.1}$$

$$(\mathrm{ad}\, u(x)) f_{\lambda\mu/\nu\sigma}(x) = f_{\lambda\mu/\nu\sigma}(x) \,. \tag{30.2}$$

Let the subalgebra generated by $f_{\lambda\mu}(x)$, $f_{\lambda\mu/\nu}(x), \dots$ be $\Sigma(x)$. From (30.0), (30.1), ..., it is seen that each element of $\Sigma(x)$ remains unchanged under ad $u(x)$ with $u(x) \in U_E(x)$. Moreover, by their construction $\Sigma(x)$ are parallel along any path with respect to the gauge potential b. Consequently, the loop phase factors at point x_0 (or holonomy group at that point) keep $\Sigma(x_0)$ unchanged. From the hypothesis on G and $U_E(x)$ we see that $\Sigma(x_0) \neq g$ and the loop phase factors at the point x_0 belong to a subgroup G_1 of lower dimension. Consequently, the field is reducible to a G_1 gauge field. Lemma 4 is proved.

If g contains a nontrivial center, then G is decomposed to the direct product $U_1 \times U_1 \times \dots \times U_1 \times G'$, where the Lie algebra of G' does not contain nontrivial center. The gauge potential is also decomposed. Further, spherically symmetric U_1 gauge fields are readily constructed (see Sect. 4). Hence it remains to consider the fields with gauge group G'.

Consequently, without loss of generality we can assume that $U_E(x_0)$ is discrete and g does not contain nontrivial center.

Lemma 5. *The mapping $A \to U_A(x_0)$ is a smooth mapping from SO_3 into the coset space $G/U_E(x_0)$ where $x_0=(0,0,r,t)$.*

Proof. The integrability condition of (1) is

$$\begin{gathered} f_{\lambda_1\lambda_2}(Ax) a^{\lambda_1}_{\mu_1} a^{\lambda_2}_{\mu_2} u_A(x) - u_A(x) f_{\mu_1\mu_2}(x) = 0 \,, \\ A = (a^{\lambda}_{\mu}) \in SO_3 \,. \end{gathered} \tag{31.0}$$

Differentiation gives

$$f_{\lambda_1\lambda_2/\lambda_3}(Ax) a^{\lambda_1}_{\mu_1} a^{\lambda_2}_{\mu_2} a^{\lambda_3}_{\mu_3} u_A(x) - u_A(x) f_{\mu_1\mu_2/\mu_3}(x) = 0 \,. \tag{31.1}$$

Let $x=x_0$ and consider $u(x_0)$ as unknowns. We solve these equations firstly. The independent equations in (31.0), (31.1), ... must be finite in number, for otherwise, the solution $u_A(x_0)$ does not exist. Evidently, when $A=E$, $u_A=e$ satisfies these equations.

We use the implicit function theorem to prove the smoothness of $u_A(x_0)$ near E. Let $u=u(\alpha_1, \dots, \alpha_r)$ be a parametric representation of the group G near e and $u(0, \dots, 0)=e$. The system (31.0), (31.1), ... with $x=x_0$ can be considered as equations of $\alpha_1, \ldots, \alpha_r$. Differentiating these equations with respect to $\alpha_a (a=1, \ldots, r)$ and letting $A=E$, $\alpha_a=0$, we get

$$f_{\lambda_1\lambda_2}(x_0)u_a - u_a f_{\lambda_1\lambda_2}(x_0) = 0 ,$$
$$f_{\lambda_1\lambda_2/\lambda_3}(x_0)u_a - u_a f_{\lambda_1\lambda_2/\lambda_3}(x_0) = 0 ,$$

where $u_a = \left.\frac{\partial u}{\partial \alpha_a}\right|_{\alpha_a=0} \in g$. From these equations we should have $u_a=0$. Otherwise, $\Sigma(x_0)$ would not be equal to g. The implicit function theorem implies that $u_A(x_0)$ with $u_E(x_0)=e$ is smooth with respect to A near E.

Consider the solution of (31.0), (31.1), ... near C. Let $A=CB$, $w=u_C^{-1}(Bx_0)u_A(x_0)$. Noting

$$f_{\lambda_1\lambda_2/\lambda_3\ldots\lambda_s}(Ax_0)a^{\lambda_1}_{\mu 1}\ldots a^{\lambda_s}_{\mu_s} = u_C(Bx_0) f_{\nu_1\nu_2/\nu_3\ldots\nu_s}(Bx_0) b^{\nu_1}_{\mu_1}\ldots b^{\nu_s}_{\mu_s} u_C^{-1}(Bx_0) ,$$
$$B=(b^\lambda_\mu) ,$$

we obtain

$$f_{\lambda_1\lambda_2}(Bx_0)b^{\lambda_1}_{\mu_1}b^{\lambda_2}_{\mu_2}w(x_0) - w(x_0)f_{\mu_1\mu_2}(x_0) = 0 ,$$
$$f_{\lambda_1\lambda_2/\lambda_3}(Bx_0)b^{\lambda_1}_{\mu_1}b^{\lambda_2}_{\mu_2}b^{\lambda_3}_{\mu_3}w(x_0) - w(x_0)f_{\mu_1\mu_2/\mu_3}(x_0) = 0 ,$$

and hence w is smooth with respect to B near E. Consequently $u_A(x_0)$ is smooth with respect to A near arbitrary C. The Eq. (1) is integrable for $u_A(x)$ if we take the obtained $u_A(x_0)$ as initial condition.

7. Determination of the Potentials

Let (ψ, θ, ϕ) be the Eulerian angles for rotation A, i.e. $A=C_3(\psi)C_1(\theta)C_3(\phi)$. From Lemmas 1 and 5 we may write

$$u_{C_3(\psi)C_1(\theta)C_3(\phi)}(x_0) = u_{C_3(\psi)C_1(\theta)}(x_0)u_{C_3(\phi)}(x_0) \tag{32}$$

with

$$u_{C_3(\phi)}(x_0) = \exp(\phi Y) , \qquad u_{C_3(2\pi)}(x_0) \in U_E(x_0) , \tag{33}$$

where $Y=Y(r,t)$ is a g-valued smooth function. Moreover, $u_{C_3(\psi)C_1(\theta)}(x_0)$ is smooth with respect to (ψ, θ) and

$$u_{C_3(\psi)C_1(0)}(x_0) = u_{C_3(\psi)}(x_0) , \tag{34}$$
$$u_{C_3(2\pi)C_1(\theta)}(x_0) = u_{C_3(0)C_1(\theta)}(x_0)u_{C_3(2\pi)}(x_0) . \tag{35}$$

When $\mathscr{F}$ corresponds to a trivial bundle we have also

$$u_{C_3(\psi)C_1(\pi)}(x_0)(x_0) = u_{C_1(\pi)}(x_0)u_{C_3(-\psi)}(x_0) . \tag{36}$$

From Lemma 1 it is seen that

$$u_A(Bx_0) = u_{AB}(x_0)u_{B^{-1}}(Bx_0) , \tag{37}$$

so $u_A(x)$ can be determined by the whole set of $u_A(x_0)$.

Lemma 6.

$$\left.\frac{\partial u_{C_3(\psi)C_1(\theta)}(x_0)}{\partial\theta}\right|_{\theta=0} u^{-1}_{C_3(\psi)}(x_0)=\alpha\cos\psi+\beta\sin\psi\ , \tag{38}$$

where α, β take values in g and are independent of ψ.

Proof. From Lemma 5 and (37) we have

$$u_{C_3(\psi_1)C_1(\theta_1)}(x)=u_{C_3(\psi_1)C_1(\theta_1)C_3(\psi)C_1(\theta)}(x_0)u^{-1}_{C_3(\psi)C_1(\theta)}(x_0)\ , \tag{39}$$

where x denotes the point $C_3(\psi)C_1(\theta)x_0$. For simplicity we neglect the symbols r and t. Let $C_3(\psi_1)C_1(\theta_1)C_3(\psi)C_1(\theta)=C_3(\psi')C_1(\theta')C_3(\phi')$. From the matrices in both sides it is easily seen that

$$\cos\theta'=-\sin\theta_1\cos\psi\sin\theta+\cos\theta_1\cos\theta\ ,$$

$$\operatorname{ctg}\phi'=\operatorname{ctg}\psi\cos\theta+\operatorname{ctg}\theta_1\frac{\sin\theta}{\sin\psi}\ ,$$

$$\operatorname{ctg}\psi'=-\frac{\sin\psi_1\sin\psi\sin\theta-\cos\psi_1\cos\theta_1\cos\psi\sin\theta-\cos\psi_1\sin\theta_1\cos\theta}{\cos\psi_1\sin\psi\sin\theta+\sin\psi_1\cos\theta_1\cos\psi\sin\theta+\sin\psi_1\sin\theta_1\cos\theta}\ .$$

From these equations and

$$\theta'|_{\theta=0}=\theta_1\ ,\qquad \phi'|_{\theta=0}=\psi\ ,\qquad \psi'|_{\theta=0}=\psi_1\ , \tag{40}$$

we have

$$\left.\frac{\partial\theta'}{\partial\theta}\right|_{\theta=0}=\cos\psi\ ,\qquad \left.\frac{\partial\phi'}{\partial\theta}\right|_{\theta=0}=-\sin\psi\operatorname{ctg}\theta_1\ ,\qquad \left.\frac{\partial\psi'}{\partial\theta}\right|_{\theta=0}=\frac{\sin\psi}{\sin\theta_1}\ . \tag{41}$$

Consequently,

$$\begin{aligned}\left.\frac{\partial u_{C_3(\psi_1)C_1(\theta_1)}(x)}{\partial\theta}\right|_{\theta=0}=\sin\psi\Bigg(&\frac{1}{\sin\theta_1}\frac{\partial u_{C_3(\psi_1)C_1(\theta_1)}(x_0)}{\partial\psi_1}\\ &-\frac{\cos\theta_1}{\sin\theta_1}u_{C_3(\psi_1)C_1(\theta_1)}(x_0)\frac{\partial u_{C_3(\psi)}(x_0)}{\partial\psi}u^{-1}_{C_3(\psi)}(x_0)\\ &+\cos\psi\frac{\partial u_{C_3(\psi_1)C_1(\theta_1)}(x_0)}{\partial\theta_1}\\ &-u_{C_3(\psi_1)C_1(\theta_1)}(x_0)\left.\frac{\partial u_{C_3(\psi)C_1(\theta)}(x_0)}{\partial\theta}\right|_{\theta=0}u^{-1}_{C_3(\psi)}(x_0)\Bigg)\end{aligned} \tag{42}$$

On the other hand, if $f(x)$ is any differentiable function of x at $x=x_0$, we have $dx_1=\sin\psi d\theta$, $dx_2=-\cos\psi d\theta$ at x_0. Thus

$$\begin{aligned}\left.\frac{\partial f}{\partial\theta}\right|_{\substack{\psi=\text{const}\\ \theta=0}}d\theta&=\left.\frac{\partial f}{\partial x_1}\right|_{x=x_0}dx_1+\left.\frac{\partial f}{\partial x_2}\right|_{x=x_0}dx_2\\ &=\left(\left.\frac{\partial f}{\partial x_1}\right|_{x=x_0}\sin\psi-\left.\frac{\partial f}{\partial x_2}\right|_{x=x_0}\cos\psi\right)d\theta\ .\end{aligned} \tag{43}$$

Comparing to (42), we obtain the conclusion of Lemma 6.

If $u_{C_3(\phi)}(x_0)$ is given, we may use a special $u_{C_3(\psi)C_1(\theta)}(x_0)$ satisfying (34), (35), and (39) to construct a set of complementary functions $u_A(x)$. Of course we may use another $u'_{C_3(\psi)C_1(\theta)}(x_0)$ satisfying the same condition to construct another set of complementary functions $u'_A(x)$. However, we have

Lemma 7. *$u'_A(x)$ and $u_A(x)$ are related by* (26) *with a suitable $\zeta(x)$.*

Proof. Let

$$\zeta(x)=u'_{C_3(\psi)C_1(\theta)}(x_0)u^{-1}_{C_3(\psi)C_1(\theta)}(x_0)\,. \tag{44}$$

It is easily verified that $\zeta(x)$ is a G-valued function on M^+ and is regular at x_0.

Let $A=C_3(\psi_1)C_1(\theta_1)C_3(\phi_1)$, $AC_3(\psi)C_1(\theta)=C_3(\psi')C_1(\theta')C_3(\phi')$, then $C_3(\psi')C_1(\theta')x_0=AC_3(\psi)C_1(\theta)x$, and

$$\begin{aligned}u'_A(x)&=u'_{C_3(\psi')C_1(\theta')}(x_0)u_{C_3(\phi')}(x_0)u'^{-1}_{C_3(\psi)C_1(\theta)}(x_0)\\&=\zeta(Ax)u_{C_3(\psi')C_1(\theta')}(x_0)u_{C_3(\phi')}(x_0)u^{-1}_{C_3(\psi)C_1(\theta)}(x_0)\zeta^{-1}(x)\\&=\zeta(Ax)u_A(x)\zeta^{-1}(x)\,.\end{aligned}$$

Lemma 7 follows from Lemma 2.

From Lemma 7 it is seen that without loss of generality we may take

$$u_{C_3(\psi)C_1(\theta)}(x_0)=u_{C_3(\psi)}(x_0)\,,\qquad(\theta \neq \pi)\,. \tag{45}$$

Lemma 8. *If $b(x_0,dx)$ satisfies*

$$b(x_0,u_{C_3(\phi)}(x_0)dx)=(\mathrm{ad}\,u_{C_3(\phi)}(x_0))b(x_0,dx)-du_{C_3(\phi)}(x)|_{x=x_0}u^{-1}_{C_3(\phi)}(x_0)\,, \tag{46}$$

then

$$b(x,dx)=(\mathrm{ad}\,u_A(x_0))b(x_0,A^{-1}dx)-du_A(x)\Big|_{\substack{x=x_0\\dx=A^{-1}dx}}u_A^{-1}(x_0)$$

is a spherically symmetric gauge potential, where $x=Ax_0$.

Proof. We first prove that $b(x,dx)$ does not depend on the choice of A. Let $Ax_0=Bx_0=x$

$$\underset{1}{b}(Ax_0,dAx)=(\mathrm{ad}\,u_A(x_0))\,(x_0,dx)-du_A(x)|_{x=x_0}u_A^{-1}(x)\,,$$

$$\underset{2}{b}(Bx_0,dBx)=(\mathrm{ad}\,u_B(x_0))b(x_0,dx)-du_B(x)|_{x=x_0}u_B^{-1}(x)\,.$$

we want to prove that $\underset{1}{b}(Ax_0,dx)=\underset{2}{b}(Bx_0,dx)$.

By a direct calculation this is equivalent to

$$\begin{aligned}b(x_0,dB^{-1}Ax)&=(\mathrm{ad}\,u_B^{-1}(x_0)u_A(x_0))b(x_0,dx)\\&\quad+u_B^{-1}(x_0)du_B(x)\Big|_{\substack{x=x_0\\dx=B^{-1}Adx}}\\&\quad-u_B^{-1}(x_0)du_A(x)|_{x=x_0}u_A^{-1}(x_0)u_B(x_0)\,.\end{aligned} \tag{47}$$

It is easily seen that $B^{-1}A=C_3(\phi)$ for some ϕ and

$$u_B^{-1}(x_0)u_A(x_0)=u_{B^{-1}}(Bx_0)u_A(x_0)=u_{B^{-1}}(Ax_0)u_A(x_0)=u_{B^{-1}A}(x_0)\,.$$

Moreover, differentiating $u_{B^{-1}A}(x)=u_{B^{-1}}(Ax)u_A(x)$ and setting $x=x_0$, we have

$$\begin{aligned} du_{B^{-1}A}(x)|_{x=x_0} &= du_{B^{-1}}(Ax)|_{x=x_0}u_B(x_0)+u_{B^{-1}}(Ax_0)du_A(x)|_{x=x_0} \\ &= du_B^{-1}(B^{-1}Ax)|_{x=x_0}u_B(x_0)+u_B^{-1}(x_0)du_A(x)|_{x=x_0} . \end{aligned}$$

Thus we see that (47) is equivalent to (46). So we proved that $b(x,dx)$ is independent of the choice of A.

Similarly, a direct calculation gives that

$$b(ABx_0,dABx_0)=(\operatorname{ad} u_A(Bx_0))b(Bx_0,dBx)-du_A(Bx)|_{x=x_0}u_A(Bx_0) .$$

Lemma 8 is proved.

Thus the problem is reduced to construct $b(x_0,dx)$ such that (46) is satisfied.

It is easily seen that (46) is equivalent to (10), (11) and

$$[Y,b_3(x_0)]=0 , \tag{48}$$

if we take $u_{C_3(\phi)}(x_0)=\exp(\phi Y)$, since (10), (11), and (48) are the equivalent formulas of (46) in the Lie algebra. For the time being Y can depend on x_0.

Now we are going to construct the potential. Let $b(x_0,dx)$ be a solution of (10), (11), and (48). From (42) and (45) it follows that

$$\left.\frac{\partial u_{C_3(\psi_1)C_1(\theta_1)}(\psi,\theta)}{\partial\theta}\right|_{\theta=0}=\sin\psi\,\frac{1-\cos\theta_1}{\sin\theta_1}\,Y. \tag{49}$$

Noting that

$$r\sin\psi d\theta=dx_1 ,$$

we have

$$b^+(Ax_0,dAx)=(\operatorname{ad} u_3(\psi_1))b(x_0,dx)-\frac{1}{r}\left(\frac{1-\cos\theta_1}{\sin\theta_1}\right)Ydx_1 , \tag{50}$$

where $A=C_3(\psi_1)C_1(\theta_1)$. Replacing (ψ_1,θ_1) by (ψ,θ) and noticing that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=A\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}=\begin{bmatrix} r\sin\psi\sin\theta \\ -r\cos\psi\sin\theta \\ \cos\theta \end{bmatrix} .$$

$$A^{-1}=\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi\cos\theta & \sin\psi\cos\theta & \sin\theta \\ \cos\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta \end{bmatrix} ,$$

we obtain (13). Here we use (48) instead of $b_3(x_0)=0$. Thus all possible almost spherically symmetric fields $\mathscr{F}^+$ are constructed.

By the symmetry we can choose a suitable gauge such that the potential of $\mathscr{F}^-$ is (14). Since x_0 and x_0' lie in opposite direction, we have to replace ψ by $-\psi$ in the construction of $u_{C_3(x)(x_0')}$. Moreover, it is noted that

$$C_3(\psi)C_1(\theta)x_0=C_3(\psi)C_1(\theta-\pi)x_0' , \tag{51}$$

We prove formula (4). Let $\zeta(x)=\exp(2\psi Y)$. It is easily seen that

$$b^+(x,dx)=(\operatorname{ad}\zeta(x))b^-(x,dx)-(d\zeta(x))\zeta^{-1}(x) , \tag{52}$$

so $\mathscr{F}^+$ and $\mathscr{F}^-$ are equivalent at each point of $M_+ \cap M_-$. It is required that $\zeta(x)$ be a single valued function of $M_+ \cap M_-$. Hence we must have (4).

In order to simplify $Y(r,t)$ we shall use the following lemma.

Lemma 9. *If* $Y(r,t)$ *is a g-valued smooth function, satisfying* $\exp(4\pi Y(r,t)) = e$, *then there exists a G-valued smooth function* $\kappa(r,t)$ *and an element* $Y_0 \in g$ *such that*

$$Y(r,t) = (\mathrm{ad}\,(\kappa(r,t)))Y_0 . \tag{53}$$

Proof. The local existence of a continuous $\kappa(r,t)$ is known [12]. In addition we shall prove that $\kappa(r,t)$ can be a smooth function of r and t. Let H be the subgroup of G

$$H = \{\alpha \in G | (\mathrm{ad}\,\alpha)Y_0 = Y_0\} ,$$

h the Lie algebra of H and $h^\perp$ the orthogonal complementary of h. For any given r_1, t_1, we have

$$Y(r_1, t_1) = (\mathrm{ad}\,\lambda)Y_0 .$$

Consider the equation

$$Y(r,t) = \mathrm{ad}\,(\lambda \exp(k(r,t)))Y_0$$

with $k(r,t)$ *as unknown function taking values in* $h^\perp$. The equation is satisfied by $r = r_1$, $t = t_1$, and $k = 0$. Differentiating the equation and set $r = r_1$, $t = t_1$, and $k = 0$ we obtain

$$[dk, Y_0] = 0 .$$

Since $dk \in h^\perp$ we have $dk = 0$. From the implicit function theory we obtain the existence of a smooth solution $\kappa(r,t)$ near an arbitrary point (r_1, t_1). So the set $K: 0 < r < \infty$, $-\infty < t < \infty$ may be covered by a system of neighborhoods $\{U_\alpha\}$ such that

$$Y(r,t) = (\mathrm{ad}\,\kappa_\alpha(r,t))Y_0$$

in U_α. In $U_\alpha \cap U_\beta$ define $g_{\beta\alpha} = \kappa_\beta \kappa_\alpha^{-1} \in H$. We have H bundle over K. Since K is homeomorphic to R^2 the H bundle is a trivial bundle. Hence for each U_α there exists $\psi_\alpha \in H$ such that $g_{\beta\alpha} = \psi_\beta \psi_\alpha^{-1}$. Define

$$\kappa = \kappa_\alpha \psi_\alpha^{-1}$$

which is equal to $\kappa_\beta \psi_\beta^{-1}$ in $U_\alpha \cap U_\beta$. We obtain the conclusion of Lemma 9.

Consequently, in the construction of gauge potentials we may assume $Y(r,t)$ be independent of (r,t).

Further, we define a G-valued function $\zeta(r,t)$ by

$$\frac{d\zeta}{dr} = \zeta(r,t)b_3(r,t) , \qquad r(r_0, t) = e , \qquad (r_0 \neq 0) .$$

By the gauge transformation via $\zeta(r,t)$ we obtain $b_3(r,t) = 0$ instead of (48).

Thus the proof of the main theorem is accomplished.

Appendix I

Spherically Symmetric SU_3 Gauge Fields

For the SU_3 case we may assume

$$Y=\frac{i}{2}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$

where m_1, m_2, m_3 are integers with $m_1+m_2+m_3=0$. The spherically symmetric gauge potentials are classified as follows.

I. Proper Type

(a) $m_1=2, m_2=0, m_3=-2$. Y is the generator Y_3 of the 3-dimensional irreducible representation of SO_3. This is the case considered in [5] (see also Appendix II).

(b) $m_1=1, m_2=0, m_3=-1$. Y is the generator Y_3 of the reducible representation which is the 2-dimensional irreducible representation of SO_3 acturely. We have

$$b(x,dx)=\begin{bmatrix} b'(x,dx) & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{bmatrix}+i\begin{bmatrix} \lambda(r,t) & 0 & 0 \\ 0 & \lambda(r,t) & 0 \\ 0 & 0 & -2\lambda(r,t) \end{bmatrix} dx_\psi ,$$

where $b'(x,dx)$ is a syncro-spherically symmetric potential and $\lambda(r,t)$ is an arbitrary function.

(c) $m_1=m_2=m_3=0$. We have

$$b_i=0 ,$$

if we take the gauge $x^ib_i=0$ and

$$b_4=(a_{ij}(r,t)) ,$$

where $(a_{ij}(r,t))$ is an arbitrary function valued in su_3. In this case the "magnetic part" of the potential vanishes

II. Improper Type

(a) $m_i \neq m_j$, $m_i \neq m_j \pm 2$ $(i,j=1,2,3,\ i\neq j)$. We have

$$b_i^+dx^i=-\frac{r-x_3}{r}\,Yd\psi , \qquad b_i^-dx^i=\frac{r+x_3}{r}\,Yd\psi$$

$$b_4^-=b_4^+=i\begin{bmatrix} \lambda(r,t) & 0 & 0 \\ 0 & \mu(r,t) & 0 \\ 0 & 0 & -\lambda(r,t)-\mu(r,t) \end{bmatrix} .$$

(b) $m_1=m_2 \neq m_3$, $m_3 \neq m_1 \pm 2$.

b_i^+ and b_i^- are same as II(a).

$$b_4^+ = b_4^- = i \begin{bmatrix} \lambda(r,t) & z(r,t) & 0 \\ \bar{z}(r,t) & \mu(r,t) & 0 \\ 0 & 0 & -\lambda(r,t)-\mu(r,t) \end{bmatrix}.$$

(c) $m_2 = m_1 - 2$, $m_3 \neq m_1, m_2$, $m_1 + 2, m_2 - 2$.

$b_4^\pm$ are same as II(a). $b_i^\pm$ are equal to those of II(a) with the additional terms which are gauge potentials of a syncro-spherically symmetric field in the gauge considered in Sect. 3.

Remark. Except the cases I(a) and I(c) the fields are reducible.

Appendix II

A Special Class of Spherically Symmetric SU_N Gauge Fields

Let

$$Y = i \begin{bmatrix} l & & & & & \\ & l-1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & -l+1 & \\ & & & & & -l \end{bmatrix} \quad (N = 2l+1)$$

be a diagonal matrix. Y is a generator Y_3 of the irreducible representation of SO_3. Let

$$D^l_{mn}(\psi, \theta, 0) = T^l_{mn}(\phi, \theta),$$

where D^l_{mn} are the generalized spherical functions and $\phi = \psi - \frac{\pi}{2}$. So (θ, ϕ) are the spherical coordinates of a unit sphere. Using the method of Sect. 3 we obtain the matrix expressions of the gauge potentials

$$\begin{aligned}
b_1(x) &= \left[\sum_{s=-l+1}^{l} \{(T^l_{ms}\bar{T}^l_{ns-1} - T^l_{ms-1}\bar{T}^l_{ns})(-d_s \sin\phi + e_s \cos\theta \cos\phi) \right. \\
&\quad \left. - i(T^l_{ms}\bar{T}^l_{ns-1} + T^l_{ms-1}\bar{T}^l_{ns})(d_s \cos\theta \cos\phi + e_s \sin\phi)\} \right], \\
b_2(x) &= \left[\sum_{s=-l+1}^{l} \{(T^l_{ms}\bar{T}^l_{ns-1} - T^l_{ms-1}\bar{T}^l_{ns})(d_s \cos\phi + e_s \cos\theta \sin\phi) \right. \\
&\quad \left. + i(T^l_{ms}\bar{T}^l_{ns-1} + T^l_{ms-1}\bar{T}^l_{ns})(-d_s \cos\theta \sin\phi + e_s \cos\phi)\} \right], \\
b_3(x) &= \left[\sum_{s=-l+1}^{l} \{(T^l_{ms}\bar{T}^l_{ns-1} - T^l_{ms-1}\bar{T}^l_{ns})(-e_s \sin\theta) + i(T^l_{ms}\bar{T}^l_{ns-1} + T^l_{ms-1}\bar{T}^l_{ns} d_s \sin\theta)\} \right], \\
b_4(x) &= \left[i \sum_{s=-l+1}^{l} c_s T_{ms}\bar{T}_{ns} \right],
\end{aligned} \tag{A}$$

where m, n are the indices for the elements of $N \times N$ matrices, c_s, d_s, e_s are arbitrary functions of (r, t).

In particular, for $N=2$ (A) are equivalent to the general form of the synchro-spherically symmetric potentials

$$b_i^a=\varepsilon_{iak}x_k V(r,t)+\delta_{ia}S(r,t)+x_i x_a T(r,t)\ ,$$
$$b_4^a=x_a U(r,t)$$

with the gauge condition

$$S+r^2T=0\ .$$

For $N=3$ the expression (A) are consistent with the expression obtained in [15].

Note. From a letter from Prof. R. Jackiw dated March 31, 1980, we became aware that he solved the same problem, using the method in [6]. He stated the results and the method in his lectures in February 1980 at Schladming, Austria. The results are consistent with ours. Our proof is more complicated, but we do not assume a priori that the complementary function $U_A(x)$ is single-valued in the local sense and differentiable with respect to A. We are grateful to Prof. Jackiw for his letter and for telling us some negligence in the Appendix I of our preprint ITP-SB-100-79.

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