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Vlasov Hydrodynamics of a Quantum Mechanical Model

Heide Narnhofer*[1] and Geoffrey L. Sewell[2]

1 Institut für Theoretische Physik, Universität Wien, Wien, Osterreich
2 Department of Physics, Queen Mary College, London, England

Abstract. We derive the Vlasov hydrodynamics from the microscopic equations of a quantum mechanical model, which simulates that of an assembly of gravitating particles. In addition we show that the local microscopic dynamics of the model corresponds, on a suitable time-scale, to that of an ideal Fermi gas.

1. Introduction

It is only in certain limited contexts (e.g. [1–3]) that the extraction of macroscopic dynamical laws from the microscopic equations of motion of 'large' quantum systems has been rigorously effected. To the best of our knowledge, there are, as yet, no rigorous derivations of hydrodynamics from quantum statistical mechanics.

The present article is devoted to the passage from the microscopic equations of motion to a form of hydrodynamics, i.e. that due to Vlasov, for a quantum mechanical model which, though relatively simple, does have some physical significance, as we shall presently explain. The model, $\Sigma^{(N)}$, is that of an assembly of N particles of the same species, in $\mathbb{R}^3$, with Hamiltonian

$$H^{(N)} = \frac{1}{2} N^{-2/3} \sum_{j=1}^{N} p_j^2 + N^{-1} \sum_{\substack{j,k=1 \\ j<k}}^{N} V(x_j - x_k), \tag{1.1}$$

where

$$[x_j, p_k] = i\delta_{jk}, [x_j, x_k] = [p_j, p_k] = 0, \tag{1.2}$$

and where V satisfies certain regularity conditions. As we are concerned with the properties of the model in the limit $N \to \infty$, we formulate the dynamics of a sequence of systems $\{\Sigma^{(N)}\}$. Furthermore we restrict our considerations, for

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reasons which will presently become clear, to evolutions from initial states for which

$$\langle x_j^2 \rangle^{(N)} \leqq A, \quad \text{and } N^{-2/3} \langle p_j^2 \rangle^{(N)} \leqq B, \tag{1.3}$$

where A, B are finite constants, independent of N. Although the model appears to have a more natural physical significance when the constituent particles are fermions (cf. the next paragraph), our analysis is equally applicable to the case where they are bosons. The principal result we obtain is that the model obeys the laws of Vlasov hydrodynamics in the limit $N \to \infty$. This result thus parallels the classical one of Ref. [4].

In the case where its constituent particles are fermions, $\Sigma^{(N)}$ is closely related to the gravitational system, $\Sigma_G^{(N)}$, whose equilibrium states we obtained in an earlier article [5], and which we shall now briefly describe. $\Sigma_G^{(N)}$ consists of N fermions of one species, that interact via Newtonian gravitational forces. It satisfies certain simple scaling laws, and in one particular scaling, the region Ω that it occupies is N-independent, while the gravitational constant and particle mass are proportional to N^{-1} and $N^{2/3}$, respectively. In this scaling, the Hamiltonian for $\Sigma_G^{(N)}$ is given by (1.1), provided that $V(x) = -|x|^{-1}$ and Dirichlet conditions are imposed at the boundary, $\partial\Omega$, of Ω. Furthermore, the mean square momentum per particle of $\Sigma_G^{(N)}$ scales as $N^{2/3}$. Thus, we see that, if $V(x)$ is chosen to be a suitably regularized approximant to $-|x|^{-1}$, then $\Sigma^{(N)}$ simulates $\Sigma_G^{(N)}$, with the bounds given by (1.3) replacing the boundary condition at $\partial\Omega$. Likewise, $\Sigma^{(N)}$ may be made to simulate the atomic model $\Sigma_A^{(N)}$, whose equilibrium properties were obtained in Ref. [6], provided that additional terms are introduced in $H^{(N)}$ to represent the presence of the nuclei.

We formulate the dynamics of $\Sigma^{(N)}$ in terms of a hierarchy of n-particle characteristic functions, corresponding to reduced density matrices. The formulation is effected on two distinct time-scales, which we term 'microscopic' and 'macroscopic' (or 'hydrodynamic'). In view of the condition (1.3) and the assumed regularity properties of V, it turns out (Lemma 5) that $\langle x_j^2 \rangle_t^{(N)}$ and $N^{-2/3} \langle p_j^2 \rangle_t^{(N)}$ remain uniformly bounded w.r.t. N over any finite time interval, whether measured on the microscopic or macroscopic scale. From this it follows that the interparticle spacing scales as $N^{-1/3}$, and thus, in the limit $N \to \infty$, any neighbourhood of a point $X (\in \mathbb{R}^3)$ will generally carry with it an infinity of particles. Thus it emerges that, as in the cases of $\Sigma_G^{(N)}$ [5] and $\Sigma_A^{(N)}$ [6], each point X carries a system whose algebra of observables is isomorphic, via a scale transformation, to that of an infinitely extended assembly of particles.

The microscopic description is concerned with the evolution of the states induced on the observables attached, in this way, to different spatial points X. In this description, we re-scale displacement from X, momentum and time relative to $N^{-1/3}$, $N^{1/3}$ and 1, respectively. The units of distance and time thus correspond essentially to the mean interparticle spacing and to the mean time required to traverse that spacing. The results we obtain on the microdynamics are that, in the limit $N \to \infty$, the evolution of the local states corresponds to that of an ideal quantum gas, and that, if the observables attached to different points X are initially uncorrelated, then they subsequently remain so. These results arise because, in the scaling concerned, the forces on the particles vanish in the limit $N \to \infty$.

In view of the fact that the interparticle spacing $\sim N^{-1/3}$, the second result may naturally be interpreted as signifying that the purity of a phase is preserved.

In the macroscopic description, we scale distance, momentum and time in ratio to 1, $N^{1/3}$ and $N^{1/3}$, respectively: thus, in this description the unit of distance is essentially the mean radius of the system and that of time is the interval required for a particle to traverse that distance. The result that we obtain is that, in the limit $N \to \infty$, the macroscopic evolution of the system is given by the *classical* Vlasov hydrodynamics: the quantum properties of the system, that serve to generate this dynamics are buried in the structure of the initial state.

The essential reasons why such a result could be anticipated may be seen from the following heuristic argument. If we define $P_j := N^{-1/3} p_j$, $t := N^{1/3} T$ and $\hbar_N := N^{-1/3}$, the Hamiltonian $H^{(N)}$ takes the form

$$H^{(N)} = \frac{1}{2} \sum_{j=1}^{N} P_j^2 + N^{-1} \sum_{\substack{j,k=1 \\ j<k}}^{N} V(x_j - x_k), \tag{1.4}$$

with

$$[x_j, P_k] = i\hbar_N \delta_{jk}; \tag{1.5}$$

and the unitary operator governing the evolution of $\Sigma^{(N)}$ is

$$U^{(N)}(T) = \exp(iH^{(N)}T/\hbar_N). \tag{1.6}$$

Thus, by (1.5) and (1.6), the quantum properties of the dynamics become governed, in the macroscopic description, by an effective 'Planck's constant' $\hbar_N$, which tends to zero as $N \to \infty$. Hence, one could expect that, in this limit, the system becomes classical; and further, a classical system with Hamiltonian given by (1.4) is known [4] to evolve according to a mean field theory corresponding to Vlasov hydrodynamics. Our result may therefore be regarded as a natural combination of a classical limit and mean field theory.

The material of this article will be organised as follows. Section 2 will constitute a self-contained formulation of the theory of the microscopic evolution. In Sect. 3, we shall provide a description of the model at the hydrodynamical level and state our main Theorems, the proof of which will be constituted by the results of Sects. 4 and 5. In fact, Sect. 4 will be devoted to the derivation of the classical Vlasov hierarchy for the limit $N \to \infty$; while Sect. 5 will provide a proof of the existence and uniqueness of the global solution of this hierarchy, together with a proof that, in cases where the initial state satisfies a factorisation condition, representing molecular chaos, then so too do the subsequent states, and their evolution is governed entirely by the one-particle Vlasov equation. Thus, in this case we have a 'propagation of molecular chaos', as in the classical model of Ref. [4].

Through this article we shall denote $\mathbb{R}^3$ by Γ and the Hilbert space $L^2(\Gamma^N)$ by $\mathscr{H}^{(N)}$.

2. Microscopic Dynamics

We shall now formulate the states and dynamics of the above-described N-particle system $\Sigma^{(N)}$ in a scaling that corresponds to the microscopic description discussed in Sect. 1. Thus, if $\rho^{(N)}$ is a density matrix in $\mathscr{H}^{(N)}$, representing a state of $\Sigma^{(N)}$, we define the evolute of $\rho^{(N)}$ at time t, on the microscopic time-scale, to be

$$\tilde{\rho}_t^{(N)} = e^{-iH^{(N)}t}\rho^{(N)}e^{iH^{(N)}t}, \tag{2.1}$$

where the Hamiltonian $H^{(N)}$ is given by Eq. (1.1). For $X \in \Gamma$, $t \in \mathbb{R}$ and $n(\leq N) \in \mathbb{N}$, we define the local characteristic function $\tilde{\mu}_{X,t}^{(N,n)}: \Gamma^n \times \Gamma^n \to C$, by the formula

$$\tilde{\mu}_{X,t}^{(N,n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) = \mathrm{Tr}\left[\tilde{\rho}_t^{(N)} \exp i \sum_{j=1}^{n} (N^{-1/3}\xi_j p_j + N^{1/3}\eta_j(x_j - X))\right]$$
$$\forall \xi_j, \eta_j \in \Gamma; j = 1, \ldots, n, \tag{2.2}$$

where $\xi_j p_j$ and $\eta_j(x_j - X)$ are inner products in Γ.

Thus, $\tilde{\mu}_{X,t}^{(N,n)}$ yields the n-particle correlation functions at time t, in the scaling where distance $\sim N^{-1/3}$ and particle momenta $\sim N^{1/3}$. Suppose now that $\{\rho^{(N)}\}$ is a sequence of density matrices corresponding to states of the systems $\{\Sigma^{(N)}\}$, such that $\tilde{\mu}_{X,t}^{(N,n)}$ converges to $\tilde{\mu}_{X,t}^{(n)}$, as $N \to \infty$, in the weak topology dual† to $L_1(\Gamma^{2n})$. In this case, the family of characteristic functions $\tilde{\mu}_{X,t} := \{\tilde{\mu}_{X,t}^{(n)} | n \in \mathbb{N}\}$ yields the local correlations at X, for time t. In fact, $\tilde{\mu}_{X,t}$ corresponds to the state of an infinite system, attached to X, whose observables are the self-adjoint elements of the C^*-algebra of the CAR or CCR over $L^2(\mathbb{R}^3)$.

We now extend our description of the microscopic properties of $\{\Sigma^{(N)}\}$ by defining characteristic functions, that serve to specify not only the local states but also the correlation functions for observables attached to different points of Γ. Thus, for each finite point set $(X_1, \ldots, X_l) \in \Gamma$ and corresponding set of integers $(n_1, \ldots, n_l)$, we define

$$\tilde{\mu}_{X_1,\ldots,X_l;t}^{(N;n_1,\ldots,n_l)}: (\Gamma^{n_1} \times \ldots \times \Gamma^{n_l})^2 \to C$$

by the formula

$$\tilde{\mu}_{X_1,\ldots,X_l;t}^{(N;n_1,\ldots,n_l)}(\xi^{(1)}, \ldots, \xi^{(l)}; \eta^{(1)}, \ldots, \eta^{(l)})$$
$$= \mathrm{Tr}\left[\rho_t^{(N)} \exp i \sum_{k=1}^{l} \sum_{j=1}^{n_k} (N^{-1/3}\xi_j^{(k)} p_{n_1+\ldots+n_{k-1}+j} + N^{1/3}\eta_j^{(k)}(x_{n_1+\ldots+n_{k-1}+j} - X_k)\right],$$
$$\forall \xi^{(k)} = (\xi_1^{(k)}, \ldots, \xi_{n_k}^{(k)}), \quad \eta^{(k)} = (\eta_1^{(k)}, \ldots, \eta_{n_k}^{(k)}); \quad \xi_j^{(k)}, n_j^{(k)} \in \Gamma. \tag{2.3}$$

Suppose now that $\{\rho^{(N)}\}$ is a sequence of density matrices such that, for all values of $(X_1, \ldots, X_l)$, $(n_1, \ldots, n_l)$ and t, the function $\tilde{\mu}_{X_1,\ldots,X_l;t}^{(N;n_1,\ldots,n_l)}$ converges to $\tilde{\mu}_{X_1,\ldots,X_l;t}^{(n_1,\ldots,n_l)}$, say, as $N \to \infty$, in the weak topology dual to $L_1((\Gamma^{n_1+\ldots+n_l})^2)$. Then, in this case $\tilde{\mu}_t$, defined as the family of functions $\{\tilde{\mu}_{X_1,\ldots,X_l;t}^{(n_1,\ldots,n_l)} | (X_1, \ldots, X_l) \in \Gamma; n_1, \ldots, n_k \in \mathbb{N}\}$, specifies not only the local states of the system, but also the correlations between observables attached to different points of Γ. We note, in passing, that $\tilde{\mu}_t$ corres-

† It is rather easy to verify that sequences of states $\{\rho^{(N)}\}$ satisfying this convergence condition can be found: the same may be said for the corresponding condition specified in the following paragraph. On the other hand, one can easily show that a pointwise convergence of the characteristic functions $\tilde{\mu}_{X,t}^{(N,n)}$ is out of the question, because of their X-dependence

ponds to a state on the algebra of observables that we termed '*hydrolocal*' in Ref. [5]. The following Theorem shows that the microdynamics of the system, as specified by the time-dependence of $\tilde{\mu}_t$, corresponds to local free evolutions.

Theorem 1. *If the gradient of the potential V is bounded, then the temporal evolution of* $\tilde{\mu}_t$ *is given by the formula*

$$\tilde{\mu}^{(n_1,\dots,n_l)}_{X_1,\dots,X_l;t}(\xi^{(1)},\dots,\xi^{(l)};\eta^{(1)},\dots,\eta^{(l)}) \\ = \tilde{\mu}^{(n_1,\dots,n_l)}_{X_1,\dots,X_l;0}(\xi^{(1)}+\eta^{(1)}t,\dots,\xi^{(l)}+\eta^{(l)}t;\eta^{(1)},\dots,\eta^{(l)}). \tag{2.4}$$

Corollary 2. *If* $\tilde{\mu}_0$ *has the property that, for arbitrary* $(X_1,\dots,X_l)\in\Gamma$ *and* $n_1,\dots,n_l\in\mathbb{N}$,

$$\tilde{\mu}^{(n_1,\dots,n_l)}_{X_1,\dots,X_l;0}(\xi^{(1)},\dots,\xi^{(l)};n^{(1)},\dots,\eta^{(l)}) = \prod_{k=1}^{l}\tilde{\mu}^{(n_k)}_{X_k,0}(\xi^{(k)},\eta^{(k)}), \tag{2.5}$$

then

$$\tilde{\mu}^{(n_1,\dots,n_l)}_{X_1,\dots,X_l;t}(\xi^{(1)},\dots,\xi^{(l)};\eta^{(1)},\dots,\eta^{(l)}) = \prod_{k=1}^{l}\tilde{\mu}^{(n_k)}_{X_k,t}(\xi^{(k)},\eta^{(k)}). \tag{2.6}$$

Comment. Condition (2.5) signifies that observables attached to different points of Γ are uncorrelated at $t=0$. Since the mean interparticle spacing for $\Sigma^{(N)}\sim N^{-1/3}$, this may be interpreted as signifying that the initial limit state, represented by $\tilde{\mu}_0$, corresponds to a pure phase. Corollary 2 tells us then that the purity of a phase is preserved under microscopic time evolution.

Proof of Theorem 1. We shall prove the Theorem for the case $l=1$: generalisation to arbitrary l is trivial. For this purpose, we start by defining

$$\tilde{x}^{(N)}_j(t) = e^{iH^{(N)}t}x_j e^{-iH^{(N)}t},\quad \tilde{p}^{(N)}_j(t) = e^{iH^{(N)}t}p_j e^{-iH^{(N)}t}, \\ \tilde{F}^{(N)}_j(t) = -N^{-1}\sum_{k\neq j}\nabla V(\tilde{x}^{(N)}_j(t)-\tilde{x}^{(N)}_k(t)), \tag{2.7}$$

and noting that, by (2.1) and (2.2),

$$\tilde{\mu}^{(N,n)}_{X,t}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n) \\ = \mathrm{Tr}\left[\rho^{(N)}\exp \mathrm{i}\sum_{j=1}^{n}(N^{-1/3}\xi_j\tilde{p}^{(N)}_j(t)+N^{1/3}\eta_j(\tilde{x}^{(N)}_j(t)-X))\right]. \tag{2.8}$$

Further, it follows from Eqs. (1.1) and (2.7) that the Heisenberg Eqs. of motion for $\tilde{x}^{(N)}_j(t)$ and $\tilde{p}^{(N)}_j(t)$ imply that

$$\tilde{p}^{(N)}_j(t) = p_j + \int_0^t dt' F^{(N)}_j(t'); \\ \tilde{x}^{(N)}_j(t) = x_j + N^{-2/3}p_j t + N^{-2/3}\int_0^t dt'\int_0^{t'} dt''\,F^{(N)}_j(t''). \tag{2.9}$$

Hence, by (2.8) and (2.9),

$$\tilde{\mu}^{(N,n)}_{X,t}(\xi_1,\dots,\xi_k;\eta_1,\dots,\eta_k) \\ = \mathrm{Tr}\left[\rho^{(N)}\exp i\sum_{j=1}^{n}(N^{-1/3}(\xi_j+\eta_j t)p_j + N^{1/3}\eta_j(x_j-X)+N^{-1/3}\tilde{Q}^{(N)}_j(t))\right], \tag{2.10}$$

where

$$\tilde{Q}_j^{(N)}(t) = \xi_j \int_0^t dt' \tilde{F}_j^{(N)}(t') + \eta_j \int_0^t dt' \int_0^{t'} dt'' \tilde{F}_j^{(N)}(t''). \tag{2.11}$$

In view of our assumption that ∇V is bounded, it follows from (2.7) and (2.11) that $\| \tilde{Q}_j^{(N)}(t) \|$ is uniformly bounded w.r.t. N. Further, as

$$\| e^{i(A+B)} - e^{iA} \| \leqq \| B \|$$

for any self-adjoint operator A, it follows easily from Eqs. (2.10) that

$$\begin{aligned} &|\tilde{\mu}_{X,t}^{(N,n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) - \tilde{\mu}_{X,0}^{(N,n)}(\xi_1 + \eta_1 t, \dots, \xi_n + \eta_n t; \eta_1, \dots, \eta_n)| \\ &\quad \leqq N^{-1/3} \| \tilde{Q}_j^{(N)}(t) \| . \end{aligned}$$

Hence, in view of the uniform boundedness of $\| \tilde{Q}_j^{(N)}(t) \|$ and the definition of $\tilde{\mu}_{X,t}^{(n)}$ as the limit of $\tilde{\mu}_{X,t}^{(N,n)}$, in a specified topology, it follows that

$$\tilde{\mu}_{X,t}^{(n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) = \tilde{\mu}_{X,0}^{(n)}(\xi_1 + \eta_1 t, \dots, \xi_n + \eta_n t; \eta_1, \dots, \eta_n), \qquad \square$$

which is the required result.

3. Hydrodynamical Description

The state of $\Sigma^{(N)}$ at time t, in the scaling for the macroscopic description, specified in Sect. 1, is given by the density matrix

$$\rho_t^{(N)} = \exp(-iN^{1/3} H^{(N)} t) \rho^{(N)} \exp(iN^{1/3} H^{(N)} t); \tag{3.1}$$

and the n-particle characteristic function ($n \leqq N$) in this scaling is $\mu_t^{(N,n)}: \Gamma^n \times \Gamma^n \to C$, where

$$\mu_t^{(N,n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) = \mathrm{Tr}\left[\rho_t^{(N)} \exp i \sum_{j=1}^{n} (N^{-1/3} \xi_j p_j + \eta_j x_j) \right], \tag{3.2}$$

where $\xi_j p_j$ and $\eta_j x_j$ are inner products in Γ.

We are now in a position to state our two main Theorems on the macroscopic time evolution of $\Sigma^{(N)}$ in the limit $N \to \infty$.

Theorem 3. *Assume that the initial states* $\{\rho^{(N)}\}$ of the systems $\{\Sigma^{(N)}\}$ *satisfy the condition (1.3) and that, further, the Fourier transform* $\hat{V}$, *of* V, *is a* C_0*-class function on* $\mathbb{R}^3$. *Then, under these conditions* $\mu_t^{(N,n)}$ *converges pointwise, for each* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$, *to a characteristic function* $\mu_t^{(n)}$ *for a classical probability measure* $m_t^{(n)}$ *on* $\Gamma^n \times \Gamma^n$, *as* $N \to \infty$ *over some subsequences of integers; and further,*

(1) *the sequence* $\{m_t^{(n)}\}$ *defines a probability measure* m_t *on* $\Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$; *and*

(2) *the sequence* $\mu_t := \{\mu_t^{(n)}\}$ *satisfies the Vlasov hierarchy, namely,*

$$\frac{d\mu_t}{dt} = (L + K)\mu_t, \tag{3.3}$$

where

$$(L\mu_t)^{(n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) = \sum_{j=1}^{m} \eta_j \frac{\partial \mu_t^{(n)}}{\partial \xi_j}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n), \tag{3.4}$$

$$(K\mu_t)^{(n)}(\xi_1,\ldots,\xi;\eta_1,\ldots,\eta_n)=\sum_{j=1}^{n}\int d^3q\,\hat{V}(q)(q\xi_j)$$
$$\cdot\mu_t^{(n+1)}(\xi_1,\ldots,\xi_n,0;\eta_1,\ldots,\eta_j+q,\ldots,\eta_n,-q), \tag{3.5}$$

and where we define the Fourier transform $\hat{V}$ of V according to the convention that $V(x)=\int d^3q\hat{V}(q)e^{iqx}$.

Theorem 4. *Under the same conditions on V as in Theorem 3, the Vlasov hierarchy (3.3) admits a unique global solution for the evolution $\mu_0\to\mu_t$. Furthermore if, at $t=0$, μ_t satisfies the factorization condition*

$$\mu_t^{(n)}(\xi_1,\ldots,\xi_n;\eta_1,\ldots,\eta_n)=\prod_{j=1}^{n}\mu_t^{(1)}(\xi_j,\eta_j), \tag{3.6}$$

then it fulfils this condition for all $t\in\mathbb{R}$. In this case, $\mu_t^{(1)}$ is the unique solution of the Vlasov equation, as expressed in terms of the one-particle characteristic function, i.e.

$$\frac{d\mu_t^{(1)}}{dt}(\xi,\eta)=\eta\frac{\partial\mu_t^{(1)}}{\partial\xi}(\xi,\eta)+\int d^3q\,\hat{V}(q)(q\xi)\mu_t^{(1)}(\xi,\eta+q)\mu_t^{(1)}(0,-q). \tag{3.7}$$

Comments. (1) Theorem 4 tells us that the dynamics governed by the Vlasov hierarchy preserves 'molecular chaos', as represented by the factorization condition (3.6). As $\mu_t^{(n)}$ is the limit of $\mu_t^{(N,n)}$ (Theorem 3) and as the interparticle spacing for $\Sigma^{(N)}\sim N^{-1/3}$, it is natural, according to standard arguments [7; *p.* 295], to interpret condition (3.6) as one that is satisfied by a system in a pure phase. Furthermore, it is not too difficult to construct a sequence $\{\rho^{(N)}\}$ of initial states (e.g. suitable quasi-free ones) for which the limit characteristic functions $\mu_t^{(n)}$ satisfy (3.6) at $t=0$ and hence, by Theorem 4, for all t.

(2) Although it is a fairly simple matter to weaken the condition on $\hat{V}$ for Theorem 3, it seems to us to be difficult to do the same for Theorem 4—at least within the present framework.

4. Derivation of the Vlasov Hierarchy

This Section will be devoted to a proof of Theorem 3.

Lemma 5. *Assume that $|\nabla V|$ is bounded and that condition (1.3) is satisfied. Then, for each $T\in\mathbb{R}_+$, $\exists$ finite constants A_T, B_T such that*

$$\mathrm{Tr}(\rho_t^{(N)}x_j^2)\leqq A_T \text{ and } N^{-2/3}\,\mathrm{Tr}(\rho_t^{(N)}p_j^2)\leqq B_T\,\forall t\in[-T,T]. \tag{4.1}$$

Proof. Defining

$$x_j^{(N)}(t)=\exp(iN^{1/3}H^{(N)}t)x_j\exp(-iN^{1/3}H^{(N)}t),$$
$$p_j^{(N)}(t)=\exp(iN^{1/3}H^{(N)}t)p_j\exp(-iN^{1/3}H^{(N)}t), \tag{4.2}$$

and

$$F_j^{(N)}(t)=-N^{-1}\sum_{k\neq j}\nabla V(x_j^{(N)}(t)-x_k^{(N)}(t)), \tag{4.3}$$

it follows from (3.1), (4.2) and (4.3) that

$$\mathrm{Tr}(\rho_t^{(N)}x_j^2)=\mathrm{Tr}(\rho^{(N)}(x_j^{(N)}(t))^2)\ \textit{and}\ \mathrm{Tr}(\rho_t^{(N)}p_j^2)=\mathrm{Tr}(\rho^{(N)}(p_j^{(N)}(t))^2), \tag{4.4}$$

while it follows from (1.1), (4.2) and (4.3) that the Heisenberg equations of motion for $x_j^{(N)}(t)$ and $p_j^{(N)}(t)$ yield the formulae

$$x_j^{(N)}(t)=x_j+N^{-1/3}p_jt+\int_0^t dt_1\int_0^{t_1}dt_2F_j^{(N)}(t_2) \tag{4.5a}$$

and

$$p_j^{(N)}(t)=p_j+N^{1/3}\int_0^t dt_1F_j^{(N)}(t_1). \tag{4.5b}$$

In view of our assumption of the boundedness of ∇V, it follows from (4.3) that $\| F_j^{(N)}(t)\|$ is uniformly bounded w.r.t. N and t. It is a simple matter to show that this uniform boundedness of $\| F_j^{(N)}(t)\|$, together with the assumed condition (1.3) and Eqs. (4.4) and (4.5), imply the required result. □

Lemma 6. *Assume again the condition of Lemma 5. Then, for all* $y:=(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n,t)$ *in any compact region* K *of* $\Gamma^n\times\Gamma^n\times\mathbb{R}$, *the first and second derivatives of* $\mu_t^{(N,n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)$ w.r.t. *the* ξ's *and* η's *are all uniformly bounded* w.r.t. *N and y.*

Proof. It follows from the commutation relations (1.2) that

$$\begin{aligned}\exp i\sum_{j=1}^n(N^{-1/3}\xi_jp_j+\eta_jx_j)&=\exp\left(i\sum_{j=1}^nN^{-1/3}\xi_jp_j\right)\exp\left(i\sum_{j=1}^n\eta_jx_j\right)\\&\quad\cdot\exp\left(-\frac{i}{2}\sum_{j=1}^nN^{-1/3}\xi_j\eta_j\right)\\&=\exp\left(i\sum_{j=1}^n\eta_jx_j\right)\left(\exp i\sum_{j=1}^nN^{-1/3}\xi_jp_j\right)\\&\quad\cdot\exp\left(\frac{i}{2}\sum_{j=1}^nN^{-1/3}\xi_j\eta_j\right).\end{aligned} \tag{4.6}$$

Hence, by Eqs. (3.2) and (4.6), the first and second derivatives of $\mu_t^{(N,n)}$ w.r.t. the ξ's and η's are given by the following formulae, the right-hand sides of which are well-defined, in view of Lemma 5,

$$\begin{aligned}&\frac{\partial\mu_t^{(N,n)}}{\partial\xi_j}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)\\&=iN^{-1/3}\,\mathrm{Tr}\left[\rho_t^{(N)}(p_j-\tfrac{1}{2}\eta_j)\exp i\sum_{j=1}^n(N^{-1/3}\xi_jp_j+\eta_jx_j)\right]\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\frac{\partial\mu_t^{(N,n)}}{\partial\eta_j}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)\\&=i\,\mathrm{Tr}\left[\rho_t^{(N)}(x_j+\tfrac{1}{2}N^{-1/3}\xi_j)\exp i\sum_{j=1}^n(N^{-1/3}\xi_jp_j+\eta_jx_j)\right]\end{aligned} \tag{4.8}$$

$$\frac{\partial^2 \mu_t^{(N,n)}}{\partial \xi_j \partial \xi_k}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)$$
$$= -N^{-2/3}\,\mathrm{Tr}\left[(p_j + \tfrac{1}{2}\eta_j)\rho_t^{(N)}(p_j - \tfrac{1}{2}\eta_j)\exp i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right] \tag{4.9}$$

$$\frac{\partial^2 \mu_t^{(N,n)}}{\partial \xi_j \partial \eta_k}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)$$
$$= -N^{-1/3}\,\mathrm{Tr}\left[(x_j - \tfrac{1}{2}N^{-1/3}\xi_j)\rho_t^{(N)}(p_j - \tfrac{1}{2}\eta_j)\exp i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right] \tag{4.10}$$

and

$$\frac{\partial^2 \mu_t^{(N,n)}}{\partial \eta_j \partial \eta_k}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)$$
$$= -\mathrm{Tr}\left[(x_j - \tfrac{1}{2}N^{-1/3}\xi_j)\rho_t^{(N)}(x_j + \tfrac{1}{2}N^{-1/3}\xi_j)\exp i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right]. \tag{4.11}$$

The required result now follows, in view of Lemma 5, from a simple application to Eqs. (4.7)–(4.11) of the general inequality

$$|\mathrm{Tr}(A\rho BC)| \leqq [\mathrm{Tr}(\rho A^*A)]^{1/2}[\mathrm{Tr}(\rho BB^*)]^{1/2}\|C\|. \qquad \square$$

Proposition 7. *Assume again the conditions of Lemma 5, together with the additional one that $V \in L_1(\mathbb{R}^3)$. Then the functions $\mu_t^{(N,n)}$ are differentiable* w.r.t. *t and satisfy the BBGKY hierarchy*

$$\frac{d\mu_t^{(N,n)}}{dt} = L_n \mu_t^{(N,n)} + K_{n,n+1}^{(N)}\mu_t^{(N,n)}, \quad \text{for } n = 1, \ldots, N, \tag{4.12}$$

where

$$L_n \mu_t^{(N,n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) = \sum_{j=1}^{n} \eta_j \frac{\partial \mu_t^{(N,n)}}{\partial \xi_j}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) \tag{4.13}$$

and

$$K_{n,n+1}\mu_t^{(N,n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)$$
$$= N^{-1}\sum_{\substack{j,k=1\\ j<k}}^{n} \int d^3q\,\hat{V}(q)(2N^{1/3}\sin\tfrac{1}{2}N^{-1/3}q(\xi_j - \xi_k))$$
$$\cdot \mu_t^{(N,n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_j + q, \ldots, \eta_k - q, \ldots, \eta_n)$$
$$+ \left(1 - \frac{n}{N}\right)\sum_{j=1}^{n} \int d^3q\,\hat{V}(q)(2N^{1/3}\sin\tfrac{1}{2}N^{-1/3}q\xi_j).$$
$$\cdot \mu_t^{(N,n+1)}(\xi_1, \ldots, \xi_n, 0; \eta_1, \ldots, \eta_j + q, \ldots, \eta_n, -q), \tag{4.14}$$

$\mu_t^{(N,N+1)} = 0$, *and* $\hat{V}$ *is the Fourier transform of* V, *defined by the same convention as in Theorem 3.*

Proof. It follows from (1.1) and (1.2), together with the definition $V(x) := \int d^3q\,\hat{V}(q)e^{iqx}$, that

$$\exp\left(i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right)H^{(N)}\exp\left(-i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right)$$
$$= H^{(N)} - N^{-2/3}\sum_{j=1}\eta_j(p_j - \tfrac{1}{2}\eta_j) + N^{-1}\sum_{\substack{j,k=1\\ j<k}}^{n}\int d^3q\,\hat{V}(q)e^{iq(x_j-x_k)}\left[e^{iN^{-1/3}q(\xi_j-\xi_k)} - 1\right]$$
$$+ N^{-1}\sum_{j=1}^{n}\sum_{k=n+1}^{N}\int d^3q\,\hat{V}(q)e^{iq(x_j-x_k)}\left[e^{iN^{-1/3}q(\xi_j-\xi_k)} - 1\right]. \tag{4.15}$$

Further, it follows from (3.1) and (3.2) that

$$\frac{d\mu_t^{(N,n)}}{dt}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)$$
$$= iN^{1/3}\,\mathrm{Tr}\left[\rho_t^{(N)}\left[H^{(N)}, \exp i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)\right]\right]. \tag{4.16}$$

The commutator in this last formula is obtained by multiplying equation (4.15), from the right, by $\exp i\sum_{j=1}^{n}(N^{-1/3}\xi_j p_j + \eta_j x_j)$. On formulating the commutator in this way, the required result may be obtained as a simple consequence of Eqs. (4.6), (4.7), (4.15) and (4.16). □

Proposition 8. *Under the assumptions of Proposition 7,* $\mu_t^{(N,n)}$ *converges pointwise to a function* $\mu_t^{(n)}$ *and, correspondingly,* $\partial\mu_t^{(N,n)}/\partial\xi_j$, $\partial\mu_t^{(N,n)}/\partial\eta_j$ *converge pointwise to* $\partial\mu_t^{(n)}/\partial\xi_j$, $\partial\mu_t^{(n)}/\partial\eta_j$, *respectively, as* $N\to\infty$ *over a subsequence of integers that is independent of t. Here,* $\mu_t^{(n)}$ *is the characteristic function for a measure* $m_t^{(n)}$ *on* $\Gamma^n\times\Gamma^n$, *such that* $m_t := \{m_t^{(n)}\}$ *canonically defines a probability measure on* $\Gamma^{\mathbb{N}}\times\Gamma^{\mathbb{N}}$. *Further, the convergence of* $\mu_t^{(N,n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)$ *to* $\mu_t^{(n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)$, *and likewise of the first derivatives of* $\mu_t^{(N,n)}$ w.r.t. ξ_j,η_j *to the corresponding ones of* $\mu_t^{(n)}$, *is uniform in* $(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n;t)$ *over any compact region in* $\Gamma^n\times\Gamma^n\times\mathbb{R}$.

Proof. Let K be a compact subset of $\Gamma^n\times\Gamma^n\times\mathbb{R}$. Then, it follows from Lemma 6 and Proposition 7 that we may employ the Arzela–Ascoli theorem to show that $\mu_t^{(N,n)}$, $\partial\mu_t^{(N,n)}/\partial\xi_j$, $\partial\mu_t^{(N,n)}/\partial\eta_j$ converge pointwise, and uniformly w.r.t. their arguments $(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)$ and t, for $(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n;t)\in K$, as $\mathrm{N}\to\infty$ over some subsequence of integers. Further, the uniformity of this convergence ensures that the limits of $\partial\mu_t^{(N,n)}/\partial\xi_j$, $\partial\mu_t^{(N,n)}/\partial\eta_j$ are $\partial\mu_t^{(n)}/\partial\xi_j$ and $\partial\mu_t^{(n)}/\partial\eta_j$, respectively, where $\mu_t^{(n)} = \lim_{N\to\infty}\mu_t^{(N,n)}$. Now let $\{K_n\}$ be an increasing absorbing sequence of compacts in $\Gamma^n\times\Gamma^n\times\mathbb{R}$. Then it follows from the above result, applied to each K_n in turn, that we may use the diagonalization procedure to prove that $\mu_t^{(N,n)}$, $\partial\mu_t^{(N,n)}/\partial\xi_j$ and $\partial\mu_t^{(N,n)}/\partial\eta_j$ converge pointwise to $\mu_t^{(n)}$, $\partial\mu_t^{(n)}/\partial\xi_j$ and $\partial\mu_t^{(n)}/\partial\eta_j$, for all $t\in\mathbb{R}$, as $N\to\infty$ over some sequence in $\mathbb{N}$.

It now remains for us to prove our statements concerning $\mu_t^{(n)}$ as the characteristic function of a probability measure. For this purpose we first infer from the definition (3.2) and the commutation relations (1.2) that

$$\sum_{j,k=1}^{l} c_j\bar{c}_k\mu_t^{(N,n)}(\xi^{(j)}-\xi^{(k)},\eta^{(j)}-\eta^{(k)})\exp\frac{-i}{2}N^{-1/3}(\xi^{(j)}\eta^{(k)}-\xi^{(k)}\eta^{(j)})\geqq 0$$
$$\forall c_1,\dots,c_l\in C;\, l\in\mathbb{N};\,\xi^{(1)},\dots,\xi^{(l)},\eta^{(1)},\dots,\eta^{(l)}\in\Gamma^n, \tag{4.17}$$

and that

$$\mu_t^{(N,n)}(0;0)=1. \tag{4.18}$$

Hence, as $\mu_t^{(n)}$ is the pointwise limit of $\mu_t^{(N,n)}$, it follows from (4.17) and (4.18) that

$$\sum_{j,k=l}^{l} c_j\bar{c}_k\mu_t^{(n)}(\xi^{(j)}-\xi^{(k)},\eta^{(j)}-\eta^{(k)})\geqq 0\,\forall c_1,\dots,c_l\in C;\, l\in\mathbb{N};$$
$$\xi^{(1)},\dots,\xi^{(l)};\eta^{(1)},\dots,\eta^{(l)}\in\Gamma^n, \tag{4.19}$$

and

$$\mu_t^{(n)}(0;0)=1. \tag{4.20}$$

In view of the continuity of $\mu_t^{(n)}$ (Lemma 6), it follows from (4.19) and (4.20) that $\mu_t^{(n)}$ is indeed the characteristic function for a probability measure $m_t^{(n)}$ on $\Gamma^n\times\Gamma^n$, i.e.

$$\mu_t^{(n)}(\xi;\eta)=\int dm_t^{(n)}(x;p)e^{i(\xi p+\eta x)}\quad\forall\xi,\eta\in\Gamma^n. \tag{4.21}$$

Finally, it follows immediately from (3.2) that, for $n<N$,

$$\mu_t^{(N,n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)\equiv\mu_t^{(N,n+1)}(\xi_1,\dots,\xi_n,0;\eta_1,\dots,\eta_n,0)$$

and hence, as $\mu_t^{(n)}$ is the limit of $\mu_t^{(N,n)}$,

$$\mu_t^{(n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)\equiv\mu_t^{(n+1)}(\xi_1,\dots,\xi_n,0;\eta_1,\dots,\eta_n,0). \tag{4.22}$$

One infers readily from (4.21) and (4.22) that $\{m_t^{(n)}\}$ correspond to a unique probability measure m_t on $\Gamma^{\mathbb{N}}\times\Gamma^{\mathbb{N}}$, such that if Δ_n is a Borel subset of $\Gamma^n\times\Gamma^n$ and $\hat{\Delta}_n$ is the cylinder set $\Delta_n\times\Gamma^{\mathbb{N}'}\times\Gamma^{\mathbb{N}'}$, with $\mathbb{N}':=\mathbb{N}\backslash\{1,\dots,n\}$, then $m_t(\hat{\Delta}_n)\equiv m_t^{(n)}(\Delta_n)$. □

Proof of Theorem 3. Proof of the statement concerning the convergence of $\mu_t^{(N,n)}$, together with that of the further property (1), is provided by Proposition 8, for which the basic assumptions are weaker than those of Theorem 3.

In order to establish the Vlasov hierarchy (3.3), we rewrite Eq. (4.12) in the form

$$\mu_t^{(N,n)}=\mu_0^{(N,n)}+\int_0^t dt'(L_n\mu_{t'}^{(N,n)}+K_{n,n+1}^{(N)}\mu_{t'}^{(N,n)}). \tag{4.23}$$

By Proposition 8 and the definitions of L_n, $K^{(N)}_{n,n+1}$, L and K (Eqs. (3.4), (3.5), (4.13) and (4.14)), together with the assumed condition for $\hat{V}$, we may pass to the limiting form of (4.23), as $N \to \infty$, in a straightforward way, and thereby obtain the equation.

$$\mu_t^{(n)} = \mu_0^{(n)} + \int_0^t dt' ((L+K)\mu_{t'})^{(n)},$$

which is equivalent to the required result. □

5. The Vlasov Dynamics

In this Section we shall establish two propositions concerning the Vlasov dynamics; and it will be seen that Theorem 4 is an immediate consequence of these propositions. It will be assumed throughout this Section that $\hat{V}$ is a C_0-class function.

Proposition 9. *The Vlasov hierarchy (3.3) has a unique global solution.*

Proof. We reformulate the Vlasov hierarchy in interaction representation. Thus, we start by defining $\bar{\mu}_t := \{\bar{\mu}_t^{(n)} | n \in \mathbb{N}\}$ to be the sequence of characteristic functions, related to μ_t by the formula

$$\bar{\mu}_t^{(n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) = \mu_t(\xi_1 - \eta_1 t, \dots, \xi_n - \eta_n t; \eta_1, \dots, \eta_n); \tag{5.1}$$

and then note that the Vlasov hierarchy (3.3) is equivalent to the equation

$$\bar{\mu}_t = \mu_0 + \int_0^t dt_1 \bar{K}(t_1) \bar{\mu}_{t_1}, \tag{5.2}$$

where

$$(\bar{K}(t)\bar{\mu}_t)^{(n)}(\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n) \equiv \sum_{k=1}^{n} \int d^3\eta_{n+1} \hat{V}(\eta_{n+1})\, \eta_{n+1}(\xi_k - \eta_k t)$$
$$\cdot \bar{\mu}_t^{(n+1)}(\xi_1, \dots, \xi_k + \eta_{n+1} t, \dots, q_n, -\eta_{n+1} t; \eta_1, \dots, \eta_k + \eta_{n+1}, \dots, \eta_n, -\eta_{n+1}) \tag{5.3}$$

A straightforward iteration of (5.2) yields the formula

$$\bar{\mu}_t = \mu_0 + \sum_{k=1}^{l-1} \int_0^t dt_1 \dots \int_0^{t_{k-1}} dt_k \bar{K}(t_1) \dots \bar{K}(t_k)\mu_0 + \Delta_l \bar{\mu}_t \tag{5.4}$$

where

$$\Delta_l \bar{\mu}_t = \int_0^t dt_1 \dots \int_0^{t_{l-1}} dt_l \bar{K}(t_1) \dots \bar{K}(t_l) \bar{\mu}_{t_l}. \tag{5.5}$$

An estimate of $\Delta_l \bar{\mu}_t$ will be provided in the Appendix, and will there be shown to imply that, under the specified assumptions on $\hat{V}$, there exists a positive number τ, which does not depend on μ_0, such that

$$(\Delta_l \bar{\mu}_t)^{(n)} \to 0, \text{ pointwise, as } l \to \infty, \forall n \in \mathbb{N}, \text{ and } |t| \leqq \tau. \tag{5.6}$$

Hence, it follows from (5.4) and (5.6) that, for $|t| \leqq \tau$, (5.2) possesses a unique solution, namely

$$\bar{\mu}_t = \mu_0 + \sum_{k=1}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{k-1}} dt_k \bar{K}(t_1) \ldots \bar{K}(t_k)\mu_0 . \quad (5.7)$$

As the value of τ, that governs (5.6), is independent of μ_0, we may similarly prove the uniqueness of $\bar{\mu}_t$, and hence of μ_t, over the successive time intervals $[n\tau, (n+1)\tau]$ (and $[-(n+1)\tau, -n\tau]$) for all $n \in \mathbb{N}$, and thereby establish that the hierarchy (3.3) has a unique global solution. □

Definition 10. Let $m_0^{(1)}$ be a probability measure on Γ^2, and let $(x, p) \to (X_t(x, p), P_t(x, p))$ be the one-parameter group of canonical transformations of Γ^2, uniquely specified by the equations of motion (cf. [4, 8]) □

$$\frac{dX_t(x,p)}{dt} = P_t(x,p), \quad \frac{dP_t(x,p)}{dt} = -\int dm_0^{(1)}(x', p')\nabla V(X_t(x,p) - X_t(x', p')) \quad (5.8)$$

with

$$X_0(x, p) = x, P_0(x, p) = p.$$

We define $m_t^{(1)}$ to be the probability measure on Γ^2, whose characteristic function $\mu_t^{(1)}$ is given by the formula

$$\mu_t^{(1)}(\xi, \eta) = \int dm_0^{(1)}(x, p) \exp i(\xi P_t(x, p) + \eta X_t(x, p)). \quad (5.9)$$

Thus, $m_t^{(1)}$ is the weak solution of the Vlasov equation for measures, as formulated in [8].

Proposition 11. *The characteristic function for $m_t^{(1)}$, as specified in Def. 10, is the unique solution of the Vlasov equation (3.7). Furthermore, if m_0 (respectively m_t) is the probability measure on $\Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$, given by the product of copies $m_0^{(1)}$ (respectively $m_t^{(1)}$) on the components $\Gamma \times \Gamma$, and μ_0 (respectively μ_t) is the corresponding family of characteristic functions on $\{\Gamma^n \times \Gamma^n | n \in \mathbb{N}\}$, then the evolution $m_0 \to m_t$ is given by the unique solution $\mu_0 \to \mu_t$ of the Vlasov hierarchy (3.3).*

Proof. In order to show that $\mu_t^{(1)}$, as defined by (5.9), satisfies Eq. (3.7), we pass to the interaction representation and define

$$\bar{\mu}_t^{(1)}(\xi, \eta) = \mu_t^{(1)}(\xi - \eta t, \eta). \quad (5.10)$$

Thus, by (5.9) and (5.10),

$$\bar{\mu}_t^{(1)}(\xi, \eta) = \int dm_0^{(1)}(x, p) \exp i(\xi(p + \bar{P}_t(x, p)) + \eta(x + \bar{X}_t(x, p) - t\bar{P}_t(x, p)), \quad (5.11)$$

where

$$\bar{X}_t(x, p) = X_t(x, p) - x - pt, \quad \text{and } \bar{P}_t(x, p) = P_t(x, p) - p; \quad (5.12)$$

i.e., by (5.8) and (5.12),

$$\bar{X}_t(x, p) = -\int_0^t dt_1 \int_0^{t_1} dt_2 \int dm_0^{(1)}(x', p')\nabla V(X_{t_2}(x, p) - X_{t_2}(x', p')) \quad (5.13)$$

and

$$\bar{P}_t(x,p) = -\int_0^t dt_1 \int dm_0^{(1)}(x',p')\nabla V(X_{t_1}(x,p) - X_{t_1}(x',p')). \tag{5.14}$$

In view of the boundedness of ∇V, it follows from (5.13) and (5.14) that, for t in any finite time interval, $d\bar{X}_t/dt$ and $d\bar{P}_t/dt$ are both bounded. Therefore, we may obtain $d\bar{\mu}_t^{(1)}/dt$ from (5.11) by differentiating under the integral sign on the r.h.s. Thus, using (5.8) and (5.12), we find that

$$\frac{d\bar{\mu}_t^{(1)}(\xi,\eta)}{dt} = i(\xi - \eta t)\int dm_0^{(1)}(x,p)\frac{d\bar{P}_t(x,p)}{dt}\exp i((\xi - \eta t)P_t(x,p) + \eta X_t(x,p)). \tag{5.15}$$

On using (5.10) and (5.14) to express $d\bar{\mu}_t^{(1)}/dt$ and $d\bar{P}_t/dt$ in terms of $\mu_t^{(1)}$ and X_t, respectively, we see that (5.15) may be rewritten in the following form.

$$\frac{d\mu_t^{(1)}}{dt}(\xi,\eta) = \eta\frac{\partial\mu_t^{(1)}}{\partial\xi}(\xi,\eta) - i\xi\int dm_0^{(1)}(x,p)dm_0^{(1)}(x',p')\nabla V(X_t(x,p) - X_t(x',p'))\exp i(\xi P_t(x,p) + \eta X_t(x,p)). \tag{5.16}$$

Thus, as $\nabla V(X_t(x,p) - X_t(x',p')) = i\int d^3q\, q\,\hat{V}(q)\exp iq(X_t(x,p) - X_t(x',p'))$, it follows from (5.16) that

$$\frac{d\mu_t^{(1)}}{dt}(\xi,\eta) = \eta\frac{\partial\mu_t^{(1)}}{\partial\xi}(\xi,\eta) + \int d^3q(q\xi)\hat{V}(q)\int dm_0^{(1)}(x,p)\exp i(\xi P_t(x,p) + (\eta+q)X_t(x,p))\int dm_0^{(1)}(x,p)\cdot\exp(-iqX_t(x,p)).$$

This equation is equivalent to (3.7) since, by (5.9) the last two integrals in the product on its r.h.s. are $\mu_t^{(1)}(\xi,\eta+q)$ and $\mu_t^{(1)}(0,-q)$. Thus, $\mu_t^{(1)}$ satisfies (3.7).

Let μ_0 (respectively μ_t) be the family $\{\mu_0^{(n)}$ (respectively $\mu_t^{(n)})\,|\,n\in\mathbb{N}\}$ of characteristic functions corresponding to the probability measure m_0 (respectively m_t) on $\Gamma^{\mathbb{N}}\times\Gamma^{\mathbb{N}}$, given by the product of copies of $m_0^{(1)}$ (respectively $m_t^{(1)}$) on the components $\Gamma\times\Gamma$. Thus $\mu_\sigma^{(n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n) = \prod_{j=1}^n \mu_\sigma^{(1)}(\xi_j,\eta_j)$ for $\sigma = 0$ or t, from which it follows that, as $\mu_t^{(1)}$ satisfies (3.7), then μ_t is a solution of (3.3). Therefore, by Prop. 9, μ_t is the unique solution of the hierarchy (3.3). This result also implies that $\mu_t^{(1)}$ is the unique solution of (3.7). For, if $\tilde{\mu}_t^{(1)}$ were another solution of that equation then one could construct a second solution $\tilde{\mu}_t$ of the hierarchy (3.3), with initial value μ_0, by the prescription $\tilde{\mu}_t^{(n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n) = \prod_{j=1}^n \tilde{\mu}_t^{(1)}(\xi_j,\eta_j)$; and this would contradict Prop. 9.

Appendix

We shall now establish (5.6) for some $r > 0$. Thus, we start by defining the operator $\theta_k(\xi,\eta)$ on the functions on $\Gamma^m\times\Gamma^m$, with $(\xi,\eta)\in\Gamma^2$ and $m, k(\leq m)\in\mathbb{N}$, by the formula

$$(\theta_k(\xi,\eta)f)(\xi_1,\dots,\xi_m;\eta_1,\dots,\eta_m):=f(\xi_1,\dots,\xi_k-\xi,\dots,\xi_m;\eta_1,\dots,\eta_k-\eta,\dots,\eta_m). \tag{A.1}$$

With this definition, it follows from (5.3) and (5.5) that $(\Delta_l\bar{\mu}_t)^{(n)}$ may be expressed in the following form

$$\begin{aligned}&(\Delta_l\bar{\mu}_t)^{(n)}(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n)\\&=(-1)^l\sum_{k_1=1}^{n}\dots\sum_{k_l=1}^{n+l-1}\int_0^t dt_1\dots\int_0^{t_{l-1}}dt_l\prod_{m=1}^{l}(d\xi_{n+m}d\eta_{n+m}\delta(\xi_{n+m}-\eta_{n+m}t_m))\cdot\\&\cdot\{\hat{V}(-\eta_{n+1})\eta_{n+1}(\xi_{k_1}-\eta_{k_1}t_1)\theta_{k_2}(\xi_{n+1},\eta_{n+1})\hat{V}(-\eta_{n+2})\eta_{n+2}(\xi_{k_2}-\eta_{k_2}t_2)\\&\cdot\theta_{k2}(\xi_{n+2},\eta_{n+2})\hat{V}(-\eta_{n+3})\dots\hat{V}(-\eta_{n+l})\eta_{n+l}(\xi_{k_l}-\eta_{k_l}t_l)\theta_{k_l}(\xi_{n+l},\eta_{n+l})\\&\cdot\bar{\mu}_{t_l}^{(n+l)}(\xi_1,\dots,\xi_{n+l};\eta_1,\dots,\eta_{n+l})\}.\end{aligned} \tag{A.2}$$

It follows from (A.1) that the quantity in curly brackets in (A.2) is equal to

$$\begin{aligned}&[\hat{V}(-\eta_{n+1})\eta_{n+1}(\xi_{k_1}-\eta_{k_1}t_1)][\hat{V}(-\eta_{n+2})\eta_{n+2}(\xi_{k_2}-\eta_{k_2}t_2-\delta_{k_1k_2}\\&\cdot(\xi_{n+1}-\eta_{n+1}t_2))]\dots[\hat{V}(-\eta_{n+l})\eta_{n+l}(\xi_{k_l}-\eta_{k_l}t_l-\sum_{m=1}^{l-1}(\xi_{n+m}-\eta_{n+m}t_m)\delta_{k_mk_l})]\\&\cdot[\theta_{k_1}(\xi_{n+1},\eta_{n+1})\dots\theta_{k_l}(\xi_{n+l},\eta_{n+l})\bar{\mu}_{t_l}^{(n+l)}(\xi_1,\dots,\xi_{n+l};\eta_1,\dots,\eta_{n+l})].\end{aligned} \tag{A.3}$$

Since $\tilde{\mu}_{t_l}^{(n+l)}$ is a characteristic function, it follows from (A.1) that the modulus of the term in the last square brackets in (A.3) cannot exceed unity. Furthermore, the presence of the δ-functions in (A.2) permits us to replace ξ_{n+m} by $\eta_{n+m}t_m$, for $m\leqq l$, in (A.3). Hence, the modulus of the term in curly brackets in (A.2) can be majorized by

$$\begin{aligned}&|\hat{V}(-\eta_{n+1})\eta_{n+1}(\xi_{k_1}-\eta_{k_1}t_1)|\cdot|\hat{V}(-\eta_{n+2})\eta_{n+2}(\xi_{k_2}-\eta_{k_2}t_2-\delta_{k_1k_2}\eta_{n+1}(t_2-t_1)|\dots\\&\cdot\left|\hat{V}(-\eta_{n+l})\eta_{n+l}(\xi_{k_l}-\eta_{k_l}t_l-\sum_{m=1}^{l-1}\delta_{k_mk_l}(t_l-t_m))\right|.\end{aligned} \tag{A.4}$$

We note now that the constants $A:=\sup\{|\hat{V}(\eta)\eta|;\eta\in\Gamma\}$ and $B:=\sup\{|\eta||\eta\in\operatorname{supp}\hat{V}\}$ are both finite, in view of our assumptions on $\hat{V}$. Thus, defining $C:=\max\{|\xi_1|,\dots,|\xi_n|\}$ and $D:=\max\{|\eta_1|,\dots,|\eta_n|\}$, and taking account both of the ordering of the t_m's and of the ranges of values of the k_m's, as specified in (A.2), we see that the expression (A.4) may be majorized by

$$A^l(C+D|t|)[(C+D|t|)\varepsilon_{k_2n}+B|t|\delta_{k_1k_2}]\dots\left[(C+D|t|)\varepsilon_{k_ln}+\sum_{m=1}^{l-1}B|t|\delta_{k_lk_m}\right], \tag{A.5}$$

where

$$\varepsilon_{kn}=\begin{matrix}1\text{ for }k\leqq n\\0\text{ for }k>n\end{matrix} \tag{A.6}$$

Since the volume of supp $\hat{V}$ is majorized by $\frac{4}{3}\pi B^3$, and since (A.6) provides an upper bound to the term in curly brackets in (A.2), it follows from the latter Eq., after a little manipulation, that

$$|\Delta_l \bar{\mu}_t^{(n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n)| \\ \leqq (\tfrac{4}{3}\pi B^3 A)^l ((C + D|t|)n)((C + D|t|)n + (n-1)B|t|) \ldots \\ \cdot ((C + D|t|)n + (n + l - 1)B|t|)\frac{|t|^l}{l!}.$$

From this estimate, it follows that $\Delta_l \bar{\mu}_t^{(n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) \to 0$ as $l \to \infty$, for all $(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) \in \Gamma^n \times \Gamma^n$ and $|t| \leqq \tau$, with τ chosen to be $(\frac{8}{3}\pi AB^4)^{-1/2}$.

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