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Hyperfunction Solutions of the Zero-Rest-Mass Field Equations

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Abstract. In this paper it is shown how the Penrose transform maps tangential hyperfunction Dolbeault groups with coefficients in a power of the hyperplane section bundle on the hyperquadric of null twistors in projective twistor space isomorphically to all hyperfunction solutions of the massless field equations of nonnegative helicity on compactified Minkowski space. This is an extension of the Penrose transform which generated real-analytic solutions of the same field equations on the same space (cf. Eastwood, M., Penrose, R., Wells, R.O., [10]). In additions, one obtains the result that each hyperfunction solution of the massless field equations of nonnegative helicity is the sum of massless fields of positive and negative frequency, a generalization of the usual Fourier decomposition for solutions with appropriate growth conditions.

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0. Introduction

It was shown some time ago by Penrose that solutions of the zero-rest-mass field equations could be generated by transforming holomorphic functions defined on open subsets of 3-dimensional complex projective space $\mathbb{P}_3(\mathbb{C})$ to spinor fields on

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Minkowski space by a contour integration process [24]. This original result, now called the *Penrose transform*, depends on *twistor geometry* (originated by Penrose in [25]), has been refined by various authors, and a more complete theory has emerged (see [35, 19] for recent surveys of this subject). In twistor geometry it is compactified Minkowski space, denoted by M, which appears naturally. This is, in fact, the *conformal compactification*, with respect to the conformal group in physics. Conformally invariant field equations (essentially describing particles with zero mass, e.g. Maxwell's equations) are naturally defined on this compact manifold M.

In [10] it was shown that the zero-rest-mass field equations, originally introduced by Dirac [9] (including the massless Dirac equation, the homogeneous Maxwell equations, and the linearized Einstein equations), could be derived from twistor geometry. Moreover, all of the *holomorphic* solutions of the differential equations on a given open subset (of a specified type) of the natural *complexification* $\mathbb{M}$ of M are the Penrose transform of (smooth) cohomology classes in corresponding open sets in $\mathbb{P}_3(\mathbb{C})$. As an example, one had [35, 10] an *isomorphism*

$$\mathscr{P}: H^1(\mathbb{P}^{\pm}, \mathcal{O}_{\mathbb{P}}(-n-2) \xrightarrow{\cong} \mathscr{Z}_n(\mathbb{M}^{\pm}), \tag{0.1}$$

where

$$\mathbb{P}^+ = \{z \in \mathbb{P}_3(\mathbb{C}) : |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 > 0\}$$
$$\mathbb{P}^- = \{z \in \mathbb{P}_3(\mathbb{C}) : |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 < 0\},$$

and

$$\mathbb{M}^+ = \{z = x - iy \in \mathbb{C}^4 : y_0^2 - y_1^2 - y_2^2 - y_3^2 > 0\}$$

which is a tube domain (and $\mathbb{M}^-$ is defined similarly) with distinguished boundary

$$M_0 = \{z = x - iy \in \mathbb{C}^4 : y = 0\} \cong \mathbb{R}^4$$

which can be identified with affine Minkowski space. Here

$$\mathscr{Z}_{2s}(\mathbb{M}^+) = \{\varphi_{A' \dots D'}(z) \text{ holomorphic in } \mathbb{M}^+ : \nabla^{AA'} \underbrace{\varphi_{A' \dots D'}}_{2s} = 0\}$$

are the positive helicity holomorphic zero-rest-mass fields ($s = 1$ corresponds to the self-dual Maxwell's equations, for instance), while $H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2))$ denotes the sheaf cohomology groups of $\mathbb{P}^+$ with coefficients in the sheaf of sections of the $(-n-2)$-power of the hyperplane section bundle $H \to \mathbb{P} := \mathbb{P}_3(\mathbb{C})$ (i.e. local holomorphic functions which are homogeneous of degree $-n-2$ in homogeneous coordinates). The power of the hyperplane section bundle $H^m \to \mathbb{P}$ determines the helicity of the field equations whose solutions are generated by these cohomology classes. In [10] both positive and negative helicity cases were considered, but in this paper, where we study generalized functions as solutions, we shall restrict ourself to the simpler case of nonnegative helicity (the other case still involves unresolved problems at this time).

Consider now the same field equations on real (compactified) Minkowski space M (well-defined since the field equations are conformally invariant). Since these equations are hyperbolic in nature, one has many solutions which are not smooth. We recall that hyperfunctions are a generalization of distributions originally studied by Martineau and Sato which are defined on any real-analytic manifold, and which have been utilized in various contexts since [29, 22, 16]. Locally hyperfunctions can be thought of as elements of a dual space to the space of real-analytic functions endowed with an appropriate topology. Globally this is not valid; hyperfunctions are not elements of a dual space in general. Sato showed that they could be characterized in a very useful manner as local sums of jumps in boundary values of holomorphic functions defined on a complexification of the given real-analytic manifold. In Schwartz' theory of distributions, one differentiates by simply differentiating the test functions and invoking the usual formula for integration by parts. In the theory of hyperfunctions, one represents the hyperfunctions as the (("algebraic sum of jumps of") boundary values of holomorphic functions, then differentiates the holomorphic functions appearing in this representation, and then takes boundary values of the differentiated holomorphic functions to get the required derivatives of the generalized functions. There is a well-defined calculus of such generalized functions, and its utilization depends on detailed results in the theory of holomorphic functions of several complex variables, just as Schwartz' theory depends on detailed results concerning locally convex topological vector spaces. In addition, one uses homological algebra as it appears in algebraic geometry and complex analysis as a tool for keeping track of the "algebraic sums" which appears in the description of hyperfunctions given above. These generalized functions have been applied extensively to various problems involving classification of differential equations and their solutions, in particular their solutions with specified types of singularities (see [16], which traces recent developments concerning hyperfunctions and more generally microfunctions in this area).

The purpose of this paper is to generalize the Penrose transform acting on smooth data (as developed in [10]) to act on hyperfunction data in such a way that: a) it agrees with [10] when restricted to smooth data and b) it generates all hyperfunction solutions of the field equations. This can be carried out quite completely. The mathematical tools necessary for this development are described in more detail below. First we will formulate the principal results in the paper.

The Dolbeault groups $H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2))$ which appear in (0.1) have a natural generalization to the boundary of $\mathbb{P}^+$, denoted by P. Namely, one can consider *tangential Dolbeault groups* on P

$$'H^{0,q}_{\mathscr{E}}(P)=\frac{\operatorname{Ker}\bar{\partial}_P:\mathscr{E}^{0,q}(P)\to\mathscr{E}^{0,q+1}(P)}{\operatorname{Im}\bar{\partial}_P:\mathscr{E}^{0,q-1}(P)\to\mathscr{E}^{0,q}(P)},$$

where $\bar{\partial}_P$ is the tangential $\bar{\partial}$-operator (denoted by $\bar{\partial}_b$ by many authors). These groups were first studied by Kohn and Rossi [20] and have been studied extensively since [33, 12]. One can consider differential forms with various smoothness classes, and for our purposes it is useful to consider hyperfunction coefficients, denoted by $\mathscr{B}^{0,q}(P)$, and to consider differential forms with coefficients in a vector bundle, just as in the case of smooth coefficients. Thus we let

$$'H^{0,q}_{\mathscr{B}}(P,V)$$

denote the tangential Dolbeault groups with hyperfunction coefficients, and with coefficients in the holomorphic vector bundle V on $\mathbb{P}$ restricted to P. This is *intrinsic* to the CR-structure of P, and does not depend on the ambient space [27].

In a different direction we recall that a solution φ of field equations (not necessarily the massless field equations) on Minkowski space (compact or affine) is of *positive frequency* if φ is the boundary value of a solution of the same field equations on $\mathbb{M}^+$. A solution has *negative frequency* if it is the boundary value of a solution in $\mathbb{M}^-$. Here boundary value is taken in whatever manner makes sense in the given context, i.e., continuous, C^∞, distribution, L^2, or hyperfunction, etc. This concept is important in quantum field theory [8, 31], but we will not elaborate this point here. Any solution admitting a Fourier decomposition in any reasonable sense will split into two Fourier components corresponding to positive and negative frequencies in the Fourier exponentials which is where this concept arises.

We can now formulate the principal results of this paper:

A) The Penrose transform $\mathscr{P}$ is well-defined on tangential Dolbeault cohomology of hyperfunction type and there is an isomorphism

$$\mathscr{P}: {}'H^{0,1}_{\mathscr{B}}(P, H^{-n-2}) \xrightarrow{\cong} \mathscr{Z}_{\mathscr{B},n}(M),$$

where $\mathscr{Z}_{\mathscr{B},n}(M)$ is the vector space of all hyperfunction solutions on M of the zero-rest-mass field equations of nonnegative helicity $n/2$.

B) Every hyperfunction solution of the massless field equations of nonnegative helicity on M is the sum of hyperfunction solutions of positive and negative frequency, i.e., the natural boundary value mapping (in the sense of Martineau [23])

$$\mathscr{Z}_n(\mathbb{M}^+) \oplus \mathscr{Z}_n(\mathbb{M}^-) \to \mathscr{Z}_{\mathscr{B},n}(M)$$

is an isomorphism.

The proof of B) depends on A) and the fact that there is a natural mapping

$$b: H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2)) \oplus H^1(\mathbb{P}^-, \mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow{\cong} {}'H^{0,1}_{\mathscr{B}}(M, H^{-n-2})$$

whicj is an isomorphism. This last isomorphism is a natural generalization of the decomposition of a function on the real axis into the jump of holomorphic functions from the upper and lower half plane derived from Cauchy's formula. Thus this generalized Cauchy decomposition yields the Fourier decomposition in B) for the solutions of the field equations, and the Penrose transform is the mechanism which transforms the one decomposition into the other.

There are several problems which remain open at this time. Can the Penrose transform be defined as a mapping to affine Minkowski space M_0 (i.e., it must be defined on $P-I$, where I is a certain complex projective line P [10])? This is not clear. Also does any hyperfunction solution of the field equations on M_0 extend as a hyperfunction solution on all of M? For distribution solutions one can see that all solutions do extend and are the sum of positive and negative frequency solutions (see [11]). In that paper it is shown by general Fourier analysis that all such solutions are of positive and negative frequency, and then that *because of this* they have a hyperfunction extension to M, but part of this analysis breaks down for hyperfunction solutions on M_0. It is likely that all hyperfunction solutions of the massless field equations on M_0 extend to the compactification M, but this is not yet proved.

It is our hope that this reinterpretation of positive and negative frequency and weak solutions in terms of holomorphic data on projective twistor space will be useful in the many unresolved problems concerning "weak solutions" of the much more difficult nonlinear differential equations of field theory which arise when one considers self-interactions as well as non-abelian gauge theories.

We now give a survey of the contents of the paper. The language developed in [10] for describing invariantly twistor geometry and the associated field equations is adopted in its entirety in this paper. In particular, there is a fundamental diagram

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P} & & \mathbb{M} \end{tikzcd}$$

representing holomorphic surjections of compact complex manifolds, where the ν fibres are projective lines and the μ fibres are projective planes. The manifolds $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$ are flag manifolds of subspaces of $\mathbb{T}$, the 4-dimensional complex vector space of twistors. The null twistors, [satisfying $\Phi(Z)=0$, where $Z=(Z^0, \ldots, Z^3)$, and Φ is the twistor Hermitian form of type $(++--)$] define real submanifolds of $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$

$$\begin{tikzcd} & F \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ P & & M \end{tikzcd}$$

where μ and ν are the restrictions of the mappings above. This is a subfibration in a natural sense, where the fibres of ν are projective lines as before, but the fibres of μ are real circles (the light rays in M which can be identified with real compactified Minkowski space, see [35]). The form Φ determines also in a natural manner open subsets

$$\begin{tikzcd} & \mathbb{F}^+ \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P}^+ & & \mathbb{M}^+ \end{tikzcd}$$

(essentially the subspaces where Φ is positive definite).

In [10] we decomposed the mapping

$$\mathscr{P}: H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2)) \to \mathscr{Z}_n(\mathbb{M}^+) \tag{0.2}$$

into a composition of mappings:

$$H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow{\mu^*} H^1(\mathbb{F}^+, \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n-2)), \tag{0.3a}$$

$$H^1(\mathbb{F}^+, \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow{\sigma} H^1(\mathbb{F}^+, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)), \tag{0.3b}$$

$$\begin{aligned} H^1(\mathbb{F}^+, \mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) &\xrightarrow{L} H^0(\mathbb{M}^+, \nu^1_*\mu^*\mathcal{O}_{\mathbb{P}}(-n-2)) \\ &\cong H^0(\mathbb{M}^+, \mathcal{O}_{(A'\ldots D')}[-1]'), \end{aligned} \tag{0.3c}$$

and we showed that:

1) μ^* is an isomorphism,

2) $\operatorname{Im}\sigma = \ker d_\mu$, where d_μ is a differential operator acting on

$$H^1(\mathbb{P}^+, \mu^* \mathcal{O}_{\mathbb{P}}(-n-2)),$$

(a spectral sequence argument),

3) L is an isomorphism (Leray spectral sequence),

4) $L(\ker d_\mu) = \ker \nabla$, where ∇ is the massless field operator of the appropriate type.

In these four steps, the last three present no great difficulty in generalizing to this setting, and this is carried out in Sect. 6 (this presupposes that we use relative cohomology as discussed below to represent hyperfunction data however). The difficult point is to generalize μ^* to: a) pullback hyperfunction data from $\mathbb{P}$ to $\mathbb{F}$, and b) to characterize its image. This is done in Sects. 2–5, and is discussed in more detail below.

In Sect. 1 we look at open sets defined near P to define $\mathscr{P}$ (using the results in [10]) on real-analytic data on the hyperquadratic P. This gives a mapping

$$\mathscr{P}: H^1(P, \mathcal{O}_{\mathbb{P}}(-n-2)) \to \mathscr{Z}_{\mathscr{A},n}(M) \tag{0.4}$$

into real-analytic solutions of the field equations, but it is not clear at this point that it is surjective. We then show that

$$H^1(P, \mathcal{O}_{\mathbb{P}}(-n-2)) \hookrightarrow {}'H^{0,1}_{\mathscr{B}}(P, H^{-n-2}) \tag{0.5}$$

and moreover that

$$\begin{aligned} {}'H^{0,1}_{\mathscr{B}}(P, H^{-n-2}) &\cong H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-n-2)) \oplus H^1(\mathbb{P}^-, \mathcal{O}_{\mathbb{P}}(-n-2)) \\ &\cong H^2_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n-2)), \end{aligned} \tag{0.6}$$

where this last group is *relative sheaf cohomology* (see Sect. 2). We have defined $\mathscr{P}$ on $H^1(P, \mathcal{O}_{\mathbb{P}}(-n-2))$ and we want to extend it to ${}'H^{0,1}_{\mathscr{B}}(P, H^{-n-2})$ in (0.5). We see that in (0.6) we have three different representations of this hyperfunction data. We use the relative cohomology representation to effect our transform. In this case we are able to use sheaf-theoretic techniques to characterize the image as the solutions of the equations we are interested in. In particular, steps 2–4 above are simpler in terms of relative cohomology also. We then show how the transform can be computed in terms of the other representations, and that the different methods of transforming data are equivalent.

More specifically, we formulate in Sect. 2 the appropriate generalization of the pullback mapping μ^* for relative sheaf cohomology:

$$\tilde{\mu}^*: H^2_P(\mathbb{P}, \mathscr{F}) \to H^5_F(\mathbb{F}, \mu^{-1}\mathscr{F})$$

for appropriate sheaves $\mathscr{F}$. For this purpose we give in Sect. 3 a generalization of the Leray spectral sequence for a fibration, generalized to relative cohomology and involving relative direct image sheaves. We compute the relative direct images sheaves which occur in our geometric context explicitly. Then using standard vanishing theorems of complex geometry we are able to conclude that the spectral sequence degenerates. Taking the limit of the spectral sequence gives the desired pullback mapping (Theorem 2.2). The mapping given by the pullback which generalizes (0.3a) is an *isomorphism* in the case of a negative line bundle. More generally there are obstructions which have been isolated, but not computed.

In Sect. 4 we show that $\tilde{\mu}^*$ can be identified with the dual of a fibre-integral mapping, essentially integrating real-analytic forms over the fibres of $F \xrightarrow{\mu} P$. Here we have to take the CR-structure induced on F and P from $\mathbb{F}$ and $\mathbb{P}$ into account [33]. We show in particular that the fibre-integral mapping (Stoll [30])

$$I : \mathscr{A}^r(F) \to \mathscr{A}^{r-1}(P)$$

has the property that

$$I(\mathscr{A}^{5,0}(F)) \hookrightarrow \mathscr{A}^{3,1}(P) \tag{0.7}$$

a *shifting* of the bidegree (a well-defined concept on CR-manifolds). This is proved by generalizing Stoll's arguments for the case of a complex-analytic fibration, and choosing adapted frames for the CR-structures on F and P (in the manner initiated by E. Cartan). This gives an appropriate splitting of the cotangent bundle exact sequence which is defined by the fibration, and yields (0.7).

In Sect. 5 we compare the abstract (spectral sequence) and the fibre-integral pullbacks by choosing appropriate hyperfunction resolutions of the various analytic sheaves involved (this uses results of Komatsu [21]). We also obtain concrete maps of resolutions involving the fibre-integration map, which allows us to make the comparisons. This is important because it allows us to show that the restriction of the mapping defined on hyperfunction data in (0.4) *restricts* to the Penrose transform defined earlier. This uses the fact that the dual to the fibre-integral mapping has this restriction property by a version of Fubini's theorem.

In Sect. 6 we consider, just as in [10], the spectral sequence of the differential sheaf

$$0 \to \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n-2) \to \Omega_\mu^0(-n-2) \to \Omega_\mu^1(-n-2) \to \Omega_\mu^2(-n-2)$$

but in a relative cohomology context which causes no difficulty. The arguments concerning this spectral sequence and the corresponding relative Leray spectral sequence for the holomorphic fibration $\mathbb{F} \xrightarrow{\nu} \mathbb{M}$ go over just as in [10]. This then yields the principal results which include the fact that (0.4) is surjective (in Sect. 6) as well as the principal results A) and B) listed above, which are elaborated in Sects. 6 and 7.

Some of these results were announced in a rough form in [34]. In [14] there is a different description of the Penrose transform for smooth data on $\mathbb{M}^+$ for the case of self-dual Maxwell's equations. The theorems in [14] are a special case of the general results in [10]. They includes, however, an explicit inverse, and an extension to the boundary for L^2 boundary data, which then is a special case of the weak solutions studied in this paper. Their methods involve explicit computations involving fibre-integrals, and are special cases of more general results concerning Radon transforms on q-concave spaces [13].

1. Real-Analytic and Hyperfunction Forms on the Null-Twistor Hyperquadric

We shall recall the definitions of the basic geometric spaces on which the Penrose transform operates. We refer to [35] and [10] for more details concerning the geometry of twistors. The vector space of *twistors* $\mathbb{T}$ is by definition a four-dimensional complex vector space endowed with an Hermitian form Φ of type $(++--)$. We have the fundamental diagram of complex flag manifolds

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P} & & \mathbb{M} \end{tikzcd} \tag{1.1}$$

where $\mathbb{P}$ is the space of 1-dimensional subspaces of $\mathbb{T}$, $\mathbb{M}$ is the space of 2-dimensional subspaces of $\mathbb{T}$, $\mathbb{F}$ is the space of pairs of nested 1- and 2-dimensional subspaces of $\mathbb{T}$, and where μ and ν are the natural holomorphic mappings. If $L \subset \mathbb{T}$ is a subspace, then one defines

$$\Phi(L) > 0, \ \Phi(L) = 0, \ \Phi(L) < 0$$

to mean that

$$\Phi(v) > 0, \ \Phi(v) = 0, \ \Phi(v) < 0 \quad \text{for all non-zero} \quad v \in L,$$

respectively. We then define

$$P = \{x \in \mathbb{P} : \Phi(x) = 0\}$$

$$F = \{x \in \mathbb{F} : \Phi(x) = 0\}$$

$$M = \{x \in \mathbb{M} : \Phi(x) = 0\}$$

while $\mathbb{F}^+$, $\mathbb{P}^+$, $\mathbb{M}^+$ are the open subsets of $\mathbb{F}$, $\mathbb{P}$, $\mathbb{M}$ where $\Phi > 0$, and $\mathbb{F}^-$, $\mathbb{P}^-$, $\mathbb{M}^-$ are the open subsets of $\mathbb{F}$, $\mathbb{P}$, $\mathbb{M}$ where $\Phi < 0$, respectively [35]. We see that

$$F = \overline{\mathbb{F}^+} \cap \overline{\mathbb{F}^-} = \partial \mathbb{F}^+ \cap \partial \mathbb{F}^-$$

and similarly, P is the common boundary of $\mathbb{P}^+$ and $\mathbb{P}^-$, while M is the common boundary of $\mathbb{M}^+$ and $\mathbb{M}^-$. Note that $P = \partial \mathbb{P}^+ = \partial \mathbb{P}^-$, which is not true in the other two cases, where F and M are of lower dimension than the dimension of the full topological boundary. We recall that M is 4-real-dimensional *compactified Minkowski space*, while P is the 5-real-dimensional *hyperquadric of null twistors* which parametrizes all of the light rays in M.

We recall from [10] the basic *Penrose transform*

$$\mathscr{P} : H^1(\mathbb{P}^+, \mathcal{O}(-n-2)) \xrightarrow{\cong} \mathscr{Z}_n'(\mathbb{M}^+),$$

where $\mathscr{Z}_n'(\mathbb{M}^+)$ is the vector space of holomorphic right-handed massless fields of helicity $n/2$, for $n \in \mathbb{Z}$. One of the principal purposes of this paper is to extend this transform to map intrinsic data on P to solutions of the same differential equations on M.

We will use the notation and terminology of [27] concerning the complex of $(0,q)$-forms on $P \subset \mathbb{P}$, since P is a real-analytic hypersurface in a complex manifold. In particular we have the complex of real-analytic forms intrinsically[1] defined on P

$$\longrightarrow {}'\mathscr{A}^{0,q-1}(P) \xrightarrow{\bar\partial_P} {}'\mathscr{A}^{0,q}(P) \xrightarrow{\bar\partial_P} {}'\mathscr{A}^{0,q+1}(P) \longrightarrow \dots, \tag{1.2}$$

1 We use the notation ${}'\mathscr{A}, {}'H, \dots$, etc. to denote intrinsic data on the real submanifolds P, F, and M. These would be well-defined notions even without the ambient complex manifolds

where $\bar{\partial}_P$ is the tangential $\bar{\partial}$-operator on the hypersurface P. Similarly, we have the complex of *hyperfunction forms* on P

$$\longrightarrow {}'\mathscr{B}^{0,q-1}(P) \xrightarrow{\bar{\partial}_P} {}'\mathscr{B}^{0,q}(P) \xrightarrow{\bar{\partial}_P} {}'\mathscr{B}^{0,q+1}(P) \longrightarrow \dots \tag{1.3}$$

which can be thought of as differential forms on P with hyperfunction coefficients (recalling that hyperfunctions are a generalization of distributions, well-defined on any real-analytic manifold, and hence hyperfunction forms are natural generalizations of the *currents* of de Rham [7, 27]). Now let ${}'H^{0,q}_{\mathscr{A}}(P)$ and ${}'H^{0,q}_{\mathscr{B}}(P)$ denote the cohomology of the complexes (1.2) and (1.3), respectively [i.e., the space $\ker\bar{\partial}_P/\operatorname{im}\bar{\partial}_P$ at the $(0,q)$-term of the complex]. This is the basic data which we want to transform by a generalization of $\mathscr{P}$ to give weak solutions of the zero-mass-field equations on compactified real Minkowski space M. First we will see how to transform the intrinsic real-analytic data. To do this we will represent real-analytic cohomology classes on P as boundary values of holomorphic data on $\mathbb{P}^+$ and $\mathbb{P}^-$. We will need to consider the complexes (1.2) and (1.3) for differential forms with coefficients in the restriction to P of a holomorphic vector bundle V defined on all of $\mathbb{P}$. This is done in a standard fashion, and we denote by

$${}'H^{0,q}_{\mathscr{A}}(P,V) \quad \text{and} \quad {}'H^{0,q}_{\mathscr{B}}(P,V)$$

the (intrinsic) real-analytic and hyperfunction cohomology groups on P with coefficients in $V(=V|_P)$. These are the *tangential Dolbeault groups.* If now V is a holomorphic vector bundle on a complex manifold X, we will denote by $H^{0,q}(X,V)$ the Dolbeault group (of smooth forms) on X, and we have the fundamental theorem of Dolbeault

$$H^{0,q}(X,V) \cong H^q(X,\mathscr{O}_X(V)),$$

where the second cohomology group is the sheaf cohomology of the space X with coefficients in the sheaf of holomorphic sections of the holomorphic vector bundle V [36]. Let us note that the tangential Dolbeault groups do *not* have, in the same manner, a sheaf cohomology representation on the hypersurface P as the Poincaré lemma is not true for the complex of sheaves involved (in fact, the obstruction to the Poincaré lemma being true for this complex is sometimes infinite dimensional, in particular at the $(0,1)$-level for P [2]). We can however relate these intrinsic tangential Dolbeault groups on P to sheaf cohomology groups on neighboring open subsets of $\mathbb{P}$. First we recall that if S is a closed subset of a paracompact topological space X, and $\mathscr{F}\to X$ is a sheaf of abelian groups on X, then $H^q(S,\mathscr{F}) := H^q(S,\mathscr{F}|_S)$, and there is an isomorphoism $H^q(S,\mathscr{F}) \cong \liminf_{U\supset S} H^q(U,\mathscr{F})$ [15]. For Dolbeault groups on a complex manifold X we define

$$H^{0,q}(S,V) = \lim_{\substack{\longrightarrow \\ U\supset S}} \operatorname{ind} H^{0,q}(U,V)$$

for S a closed subset of X, and V a holomorphic vector bundle on X.

Lemma 1.1. *Suppose* $H^1(\mathbb{P},\mathscr{O}_{\mathbb{P}}(V)) = H^2(\mathbb{P},\mathscr{O}_{\mathbb{P}}(V)) = 0$, *then*
a) ${}'H^{0,1}_{\mathscr{A}}(P,V) \cong H^{0,1}(\overline{\mathbb{P}^+},V) \oplus H^{0,1}(\overline{\mathbb{P}^-},V) \cong H^1(\overline{\mathbb{P}^+},\mathscr{O}_{\mathbb{P}}(V)) \oplus H^1(\overline{\mathbb{P}^-},\mathscr{O}_{\mathbb{P}}(V))$,
b) ${}'H^{0,1}_{\mathscr{B}}(P,V) \cong H^{0,1}(\mathbb{P}^+,V) \oplus H^{0,1}(\mathbb{P}^-,V) \cong H^1(\mathbb{P}^+,\mathscr{O}_{\mathbb{P}}(V)) \oplus H^1(\mathbb{P}^-,\mathscr{O}_{\mathbb{P}}(V))$.

Proof. We consider the exact sequences

$$\dots \longrightarrow H^{0,1}(\mathbb{P}, V) \longrightarrow H^{0,1}(\overline{\mathbb{P}^+}, V) \oplus H^{0,1}(\overline{\mathbb{P}^-}, V) \xrightarrow{b_{\mathscr{A}}} {}'H^{0,1}_{\mathscr{A}}(P, V)$$
$$\longrightarrow H^{0,2}(\mathbb{P}, V) \longrightarrow \dots$$
$$\dots \longrightarrow H^{0,1}(\mathbb{P}, V) \longrightarrow H^{0,1}(\mathbb{P}^+, V) \oplus H^{0,1}(\mathbb{P}^-, V) \xrightarrow{b_{\mathscr{B}}} {}'H^{0,1}_{\mathscr{B}}(P, V)$$
$$\longrightarrow H^{0,2}(\mathbb{P}, V) \longrightarrow \dots$$

which follow directly from Theorem 6.6 and Theorem 5.6 in [27], extended to vector bundle coefficients [2, 17]. The mappings $b_{\mathscr{A}}$ and $b_{\mathscr{B}}$ represent the "jumps" in the boundary values of the cohomology classes on $\mathbb{P}^{\pm}$. Using the hypotheses, the conclusion now follows immediately, where the second isomorphism in each case is just an application of Dolbeault's theorem representing Dolbeault groups in terms of sheaf cohomology. $\square$

Let H be the hyperplane section bundle on $\mathbb{P}(c_1(H)=1)$, then we have the following corollary to the lemma above.

Corollary 1.2. *If* $V=H^n$, $n\in\mathbb{Z}$, *then*

a) ${}'H^{0,1}_{\mathscr{A}}(P, H^n) \cong {}'H^1(\overline{\mathbb{P}^+}, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(\overline{\mathbb{P}^-}, \mathcal{O}_{\mathbb{P}}(n))$,

b) ${}'H^{0,1}_{\mathscr{B}}(P, H^n) \cong H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(\mathbb{P}^-, \mathcal{O}_{\mathbb{P}}(n))$.

Proof. If K is the canonical bundle on $\mathbb{P}$, i.e., $K=\wedge^3 T^*\mathbb{P}$, then one has $K\cong H^{-4}$ on $\mathbb{P}\cong\mathbb{P}_3$. Since $H^n\otimes K^*=H^n\otimes H^4=H^{n+4}$ is a positive line bundle for $n\geqq 0$, we see by Kodaira's vanishing theorem [36, p. 226] that $H^q(\mathbb{P}, \mathcal{O}(H^n))=0$, $q\geqq 1$, $n\geqq 0$. On the other hand, H^n is negative for $n<0$, so $H^q(\mathbb{P}, \mathcal{O}(H^n))=0$, for $n<0$, $q<3$, by the dual version of the same theorem. $\square$

There is another extrinsic way of looking at real-analytic data on P, F, and M. Namely, one can consider sheaf cohomology $H^q(P, \mathscr{F})$, $H^q(F, \mathscr{F}')$, and $H^q(M, \mathscr{F}'')$ for sheaves $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ defined on neighborhoods of P, F, and M in $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$, respectively.

Lemma 1.3. $H^1(P, \mathcal{O}_{\mathbb{P}}(n)) \cong {}'H^{0,1}_{\mathscr{A}}(P, H^n), \quad n\in\mathbb{Z}$.

Proof. We will show that

$$H^1(P, \mathcal{O}_{\mathbb{P}}(n)) \cong H^1(\overline{\mathbb{P}^+}, \mathcal{O}(n)) \oplus H^1(\overline{\mathbb{P}^-}, \mathcal{O}(n))$$

and use the previous results. Let U_j be a sequence of neighborhoods of $\overline{\mathbb{P}^+}$ and V_j be a sequence of neighborhoods of $\overline{\mathbb{P}^-}$ and then $U_j\cap V_j$ is a fundamental neighborhood system for P. Then we have the Mayer-Vietoris sequence, for a fixed j,

$$\dots \to H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to H^1(U_j, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(V_j, \mathcal{O}_{\mathbb{P}}(n))$$
$$\to H^1(U_j\cap V_j, \mathcal{O}_{\mathbb{P}}(n)) \to H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to \dots .$$

Using the vanishing theorems of Kodaira as in Corollary 1.2, we see that

$$\begin{array}{ccc}
H^1(U_j, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(V_j, \mathcal{O}_{\mathbb{P}}(n)) & \xrightarrow{\cong} & H^1(U_j\cap V_j, \mathcal{O}_{\mathbb{P}}(n)) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \\
H^1(\overline{\mathbb{P}^+}, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(\overline{\mathbb{P}^-}, \mathcal{O}_{\mathbb{P}}(n)) & \xrightarrow{\cong} & H^1(P, \mathcal{O}_{\mathbb{P}}(n))
\end{array}$$

where the natural isomorphism is induced in the limit. $\square$

Now the pullback mapping

$$\mu^* : H^1(P, \mathcal{O}_{\mathbb{P}}(n)) \to H^1(F, \mu^{-1}\mathcal{O}_{\mathbb{P}}(n)) \tag{1.4}$$

is well-defined by the direct limit process, since

$$\mu^* : H^1(U'', \mathcal{O}_{\mathbb{P}}(n)) \to H^1(U', \mu^{-1}\mathcal{O}_{\mathbb{P}}(n))$$

is well-defined and injective for suitable neighborhoods U of M such that the fibres of $\mu : U' \to U''$ are connected [2]. Note that it is not possible to choose a fundamental neighborhood system of M, $\{U\}$, so that the fibres of $U' \xrightarrow{\mu} U''$ are 1-connected, since M is diffeomorphic to $S^1 \times S^3$.

We now recall one of the fundamental results from [10] mentioned earlier concerning the Penrose transform acting on holomorphic data:

$$\mathscr{P} : H^1(\overline{\mathbb{P}^+}, \mathcal{O}_{\mathbb{P}}(-n-2)) \to \mathscr{Z}_n'(\overline{\mathbb{M}^+}), \qquad n \in \mathbb{Z}, \tag{1.5}$$

$$\mathscr{P} : H^1(\overline{\mathbb{P}^-}, \mathcal{O}_{\mathbb{P}}(-n-2)) \to \mathscr{Z}_n'(\overline{\mathbb{M}^-}), \qquad n \in \mathbb{Z}. \tag{1.6}$$

We then use Lemmas 1.1 and 1.3 to define

$$\mathscr{P} : {}'H^{0,1}_{\mathscr{A}}(P, H^{-n-2}) \to \mathscr{Z}'_{\mathscr{A},n}(M) \tag{1.7}$$

by letting f correspond to $f^+ + f^-$ under the direct sum decomposition, where $f^+ \in H^1(\overline{\mathbb{P}^+}, \mathcal{O}_{\mathbb{P}}(-n-2))$ and $f^- \in H^1(\overline{\mathbb{P}^-}, \mathcal{O}_{\mathbb{P}}(-n-2))$, and we let

$$\mathscr{P}(f) = \mathscr{P}(f_1) + \mathscr{P}(f_2).$$

Here $\mathscr{Z}'_{\mathscr{A},n}(M)$ is the vector space of global real-analytic solutions of the positive helicity zero-rest-mass equations on M [10]. The mapping is clearly injective, but is not obviously surjective, although this will be a consequence of results derived later in this paper. We now relate this real-analytic data on P to the hyperfunction data on P.

Lemma 1.4. *The natural homomorphism*

$$'H^{0,1}_{\mathscr{A}}(P, H^{-n-2}) \to {}'H^{0,1}_{\mathscr{B}}(P, H^{-n-2})$$

is an injection.

Proof. The natural homomorphism of these cohomology groups is given by the injection of the complex (1.2) into the complex (1.3). But this does not imply, in general, injectivity of the corresponding cohomology groups. However, we can use the Penrose transform as given by (1.5) and (1.6) and Lemma 1.1 to deduce the desired result from the injectivity of the mapping $\mathscr{Z}_n'(\overline{M^+}) \hookrightarrow \mathscr{Z}_n'(M^+)$, since this mapping is simply the restriction of holomorphic fields from the closure of $\mathbb{M}^+$ to its interior. □

Now we have the Penrose transform defined on $'H^{0,1}_{\mathscr{A}}(P, H^{-n-2})$, and the object is to extend it to $'H^{0,0}_{\mathscr{B}}(P, H^{-n-2})$, which we will do in the next section by reinterpreting the hyperfunction cohomology on P in terms of relative sheaf cohomology on P. We note that we could simply use Corollary 1.2b) to map

2 As in [10], we let $U' = \nu^{-1}(U)$, $U'' = \mu \circ \nu^{-1}(U)$

hyperfunction data to direct sums of holomorphic solutions on $\mathbb{M}^{\pm}$, and then we could take hyperfunction boundary values in the sense of Martineau [23] on M, obtaining a mapping of the desired sort. This turns out to be quite appropriate, but it is difficult to characterize the image as being *all* hyperfunction solutions if we proceed this way. It will turn out that all hyperfunction solutions are in the image of the mapping just described, but that is a consequence of the major results developed in the next several sections.

2. Relative Cohomology, Hyperfunction Forms, and the Pullback Mapping

If X is a topological space, S is a closed subset of X, and $\mathscr{F}$ is a sheaf of abelian groups on X, then we denote by $H^q_S(X, \mathscr{F})$, $q \geqq 0$, the *relative cohomology of X with respect to S and the sheaf $\mathscr{F}$*. If $\mathscr{F} \to \mathscr{C}^*(\mathscr{F})$ is the canonical flabby resolution of X, then $H^q_S(X, \mathscr{F})$ is defined to be $H^q(\Gamma_S(X, \mathscr{C}^*(\mathscr{F})))$, where $H^q(K^*)$ is the q-th cohomology group of the given complex K^* and $\Gamma_S(X, \mathscr{S})$ denotes sections of a sheaf $\mathscr{S}$ over X with support in S [4, 15]. In the case where $S = X$, then we have the usual definition of sheaf cohomology. Relative cohomology can be computed in terms of coverings or in terms of specific resolutions, just as in the case for ordinary cohomology, and we will see examples of such computations in this section.

It is a fundamental result of Sato that hyperfunctions can be represented in terms of relative cohomology [29, 22]. In this direction, we have, in particular, the following proposition.

Proposition 2.1. *Let $n \in \mathbb{Z}$, then:*

$$H^2_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \cong H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(\mathbb{P}^-, \mathcal{O}_{\mathbb{P}}(n)) \cong {}'H^{0,1}_{\mathscr{B}}(P, H^n).$$

Proof. We use the long exact sequence for relative cohomology [4]

$$\ldots \to H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to H^1(\mathbb{P} - P, \mathcal{O}_{\mathbb{P}}(n)) \to H^2_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \to \ldots.$$

Now noting that

$$H^1(\mathbb{P} - P, \mathcal{O}_{\mathbb{P}}(n)) \cong H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(n)) \oplus H^1(\mathbb{P}^-, \mathcal{O}_{\mathbb{P}}(n))$$

and using the fact that $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = 0$, as before, as well as Lemma 1.1, the result follows. □

On a complex manifold X, let $V \to X$ be a holomorphic vector bundle, and let

$$0 \longrightarrow \Omega^p_X(V) \longrightarrow \mathscr{B}^{p,0}(V) \xrightarrow{\bar\partial} \mathscr{B}^{p,1}(V) \xrightarrow{\bar\partial} \ldots \longrightarrow \mathscr{B}^{p,q}(V) \longrightarrow \ldots \tag{2.1}$$

be the resolution of $\Omega^p_X(V)$ by hyperfunction (p, q)-forms with coefficients in V (Komatsu [21]). A basic generalization of Dolbeault's theorem due to Komatsu is

$$\begin{aligned} H^q_S(X, \Omega^p_X(V)) &\cong \frac{\operatorname{Ker} \bar\partial : \Gamma_S(X, \mathscr{B}^{p,q}(V)) \to \Gamma_S(X, \mathscr{B}^{p,q+1}(V))}{\operatorname{Im} \bar\partial : \Gamma_S(X, \mathscr{B}^{p,q-1}(V)) \to \Gamma_S(X, \mathscr{B}^{p,q}(V))} \\ &= H^q(\Gamma_S(X, \mathscr{B}^{p*}(V)) \end{aligned} \tag{2.2}$$

i.e., the relative cohomology can be represented by hyperfunction-forms with supports on S. This is a straightforward consequence of the fact that (2.1) is a flabby resolution of $\Omega^p_X(V)$, which is not true for C^∞ or distribution-valued forms.

Now if $Y \subset X$ is a C^∞ real oriented submanifold of the complex manifold X of real codimension r, then let $[Y]$ denote the current of integration over Y [7]. The current $[Y]$ is a current (differential form with distribution coefficients) of degree r. Any current is also a hyperfunction form, since the distributions on X inject into the hyperfunctions on X. Any hyperfunction form φ on the complex manifold X of degree r can be decomposed uniquely into the sum of hyperfunction forms of specified bidegree or type

$$\varphi = \varphi^{r,0} + \varphi^{r-1,1} + \ldots + \varphi^{0,r}$$

just as in the case for differential forms with smooth coefficients. Thus if Y is as above, then

$$[Y] = [Y]^{r,0} + [Y]^{r-1,1} + \ldots + [Y]^{0,r}$$

and $[Y]^{r-s,s}$ is a hyperfunction form on X of type $(r-s,s)$ with support on Y.

We now give an explicit realization in terms of differential forms of the mapping given in Lemma 1.4

$$'H^{0,1}_{\mathscr{A}}(P, H^n) \to 'H^{0,1}_{\mathscr{B}}(P, H^n)$$

of real-analytic data into hyperfunction data on P with corresponding mappings on F and M. We assume that P, F, and M are equipped with an orientation compatible with the orientation of the ambient complex manifolds $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$.

Proposition 2.3. *Let $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ be locally free analytic sheaves on $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$, respectively, then the following mappings determined by Dolbeault representatives of the cohomology groups are well-defined:*

a) $$\begin{array}{ccc} H^1(P,\mathscr{F}) & \longrightarrow & H^2_P(\mathbb{P},\mathscr{F}) \\ \Psi & & \Psi \\ \varphi & \longmapsto & \varphi \wedge [P]^{0,1} \end{array}$$

b) $$\begin{array}{ccc} H^1(F,\mathscr{F}) & \longrightarrow & H^5_F(\mathbb{F},\mathscr{F}') \\ \Psi & & \Psi \\ \varphi' & \longmapsto & \varphi' \wedge [F]^{0,4} \end{array}$$

c) $$\begin{array}{ccc} H^1(M,\mathscr{F}'') & \longrightarrow & H^4_M(\mathbb{M},\mathscr{F}'') \\ \Psi & & \Psi \\ \varphi'' & \longmapsto & \varphi'' \wedge [M]^{0,4} \end{array}$$

Proof. First we note that $P \subset \mathbb{P}$, $F \subset \mathbb{F}$, and $M \subset \mathbb{M}$ have real codimensions 1, 4, and 4, respectively. Second, we see that φ, φ', φ'' are smooth differential forms and hence the wedge product with a differential form with singular coefficients (in this case measure coefficients) is well-defined, and gives an element of the appropriate cohomology class by the representation of relative cohomology given by (2.2). $\square$

Remark. It follows from Lemma 1.4 that a) is injective, and it follows from a standard result of Sato that c) is injective [29, 22]. The homomorphism b) is also injective, but this is more subtle and is a consequence of some general results in [28]. This point will be discussed in more detail at a later point in this paper.

We now come to one of our principal results, namely that the mapping μ^* in (1.4) extends to relative cohomology. We will first study the pullback in terms of abstract sheaf cohomology. Then we will show how it can be computed in terms of differential forms, and that it is compatible with the usual notion of pullback of smooth data.

Theorem 2.2. *Let $\mathscr{F}$ be a locally free analytic sheaf on $\mathbb{P}$, then there exists canonically an exact sequence*

$$0 \longrightarrow H_P^2(\mathbb{P}, \mathscr{F}) \xrightarrow{\tilde{\mu}^*} H_F^5(\mathbb{F}, \mu^{-1}\mathscr{F}) \longrightarrow H_P^1(\mathbb{P}, \mathscr{F}) \longrightarrow 0 .$$

We will prove later (Lemma 3.4) that $H_P^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n)) = 0$, $n > 0$, and thus we obtain the corollary to the above theorem:

Corollary 2.3. *There exists canonically an isomorphism, for $n \geqq 0$;*

$$H_P^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n)) \xrightarrow[\cong]{\tilde{\mu}^*} H_F^5(\mathbb{F}, \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n)) .$$

The proof of this theorem and its corollary will be presented in detail in Sect. 3 using a spectral sequence argument.

3. A Generalized Leray Spectral Sequence for Relative Cohomology

The Leray spectral sequence for a proper mapping is well known [15], and has had many applications in topology, algebraic geometry, and complex analysis. We need for our purposes a slight generalization of this standard result which is very useful in the proof of Theorem 2.2.

Let $X \xrightarrow{\pi} Y$ be a proper continuous mapping of topological spaces, and let $\mathscr{S} \to X$ be a sheaf of abelian groups over X. Let $S \subset X$ be a closed subset, and let $T \subset Y$ be a closed subset satisfying $\pi(S) \subset T$. The *relative direct image sheaves*

$$R_S^q \pi(\mathscr{S})$$

(relative to the closed subset $S \subset X$) are defined as the sheaves (indexed by the integer $q \geqq 0$) generated by the presheaves

$$U \mapsto H_S^q(\pi^{-1}(U), \mathscr{S}),$$

where U is open in Y.

Theorem 3.1. *There exists a spectral sequence*

$$E_2^{pq} = H_T^p(Y, R_S^q \pi(\mathscr{S})) \Rightarrow H_S^r(X, \mathscr{S}) .$$

Proof. The proof of this theorem is a straightforward generalization of the proof of Théorème 4.17.1 in [15], and won't be repeated here. One simply replaces the direct images of the canonical flabby resolution of $\mathscr{S}$ on X by the 0-th relative direct images of the same resolution, noting that all of the arguments still make sense. The usual Leray theorem is the case where $S = X$, and $T = Y$. $\square$

We want to apply Theorem 3.1 to our geometric setting, where basically $F \subset \mathbb{F}$, and $P \subset \mathbb{P}$ with $\mu : \mathbb{F} \to \mathbb{P}$ will be the geometric objects to be considered. However, we need to be careful in our choice of closed subset on $\mathbb{F}$, for reasons which will become apparent later. We know that $F \xrightarrow{\mu} P$ is a fibre-bundle with S^1 as fibres, and the group SU(2, 2) acts naturally on this fibration [35]. We want to consider an extension of this fibration to a neighborhood of P in $\mathbb{P}$. This extension fibration can be realized group-theoretically in terms of a 1-parameter subgroup $\{g_t\}$ of $SL(4, \mathbb{C})$ which has the property that $\{g_t(P)\}$ is a foliation of a neighborhood of P in $\mathbb{P}$ by "translates" of P, for $|t| < \varepsilon$, and $g_0 = \{\mathrm{id}\}$. This is constructed as follows. Let

$$C : D \to \mathbb{C}^+$$

be the Cayley transform mapping the unit disc D to the upper half plane $\mathbb{C}^+$, and let $g_t = C^{-1} T_t C$, where $T_t : \mathbb{C} \to \mathbb{C}$ is given by $z \to z + it$, considered as an element of $SL(2, \mathbb{C})$. Then considering $\{g_t\}$ as a subgroup of $SL(4, \mathbb{C})$ by the embedding of $SL(2, \mathbb{C})$ into $SL(4, \mathbb{C})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & 0 & b \\ 0 & I_2 & 0 \\ c & 0 & d \end{bmatrix}$$

gives the desired subgroup of $SL(4, \mathbb{C})$. Here we have chosen homogeneous coordinates on $\mathbb{T}$ so that the fundamental Hermitian form Φ has the matrix representation

$$\Phi = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

and where $\mathbb{P}^+$, P, and F are considered as SU(2, 2)-orbits in this realization. Then applying g_t to F in $\mathbb{F}$ will yield

$$F_t := g_t(F) \xrightarrow{\mu_t} P_t := g_t(P),$$

a 1-parameter family of fibrations equivalent to $F \xrightarrow{\mu} P$, and we let

$$\tilde{F} := \bigcup_{|t| < \varepsilon} F_t \to N := \bigcup_{|t| < \varepsilon} P_t$$

be the desired extension of F to the neighborhood N of P. We now have the following fundamental lemma.

Lemma 3.2. *Let $\mathscr{F} \to N$ be a sheaf of abelian groups, then*

$$R^p_{\tilde{F}} \mu(\mu^{-1} \mathscr{F}) \cong \begin{cases} \mathscr{F}, & p = 3, 4 \\ 0, & p \neq 3, 4. \end{cases}$$

Proof. Let $\mathbb{F}_x = \mu^{-1}(x)$, $\tilde{F}_x = \mu^{-1}(x) \cap \tilde{F}$, and $F_x = \tilde{F}_x$, if $x \in P$. Since μ is proper, we see that, for $x \in N$,

$$\begin{aligned} R^p_{\tilde{F}} \mu(\mu^{-1} \mathscr{F})_x &\cong H^p_{\tilde{F}_x}(\mathbb{F}_x, \mu^{-1} \mathscr{F}) \\ &\cong H^p_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathscr{F}_x, \end{aligned} \tag{3.1}$$

where the last isomorphism is valid because $\mu^{-1}(\mathscr{F})$ is constant on the fibre $\mathbb{F}_x$. Now

$$H^p_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z}) \cong H^p_{S^1}(\mathbb{P}_2, \mathbb{Z}),$$

where $S^1 \hookrightarrow \mathbb{P}_2$ is some embedding of S^1 in $\mathbb{P}_2$. We then have the relative cohomology exact sequence

$$\ldots \to H^p(\mathbb{P}_2 - S^1, \mathbb{Z}) \to H^p(\mathbb{P}_2, \mathbb{Z}) \to H^{p+1}_{S^1}(\mathbb{P}_2, \mathbb{Z}) \to \ldots. \tag{3.2}$$

We know from standard results the cohomology groups

$$H^p(\mathbb{P}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p \quad \text{even} \\ 0, & p \quad \text{odd} \end{cases} \tag{3.3}$$

and we can compute $H^p(\mathbb{P}_2 - S^1, \mathbb{Z})$ easily. Namely, let T be a small tubular neighborhood of $S^1 \hookrightarrow \mathbb{P}_2$. Then we have the Mayer-Vietoris sequence, where we let $\mathbb{P}'_2 = \mathbb{P}_2 - S^1$,

$$\ldots \to H^p(\mathbb{P}_2, \mathbb{Z}) \to H^p(\mathbb{P}'_2, \mathbb{Z}) \oplus H^p(T, \mathbb{Z}) \to H^p(\mathbb{P}'_2 \cap T, \mathbb{Z}) \to \ldots. \tag{3.4}$$

We note that

$$H^p(\mathbb{P}'_2 \cap T, \mathbb{Z}) \cong H^p(S^1 \times S^2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & p = 0, 1, 2, 3 \\ 0, & p \geqq 4, \end{cases} \tag{3.5}$$

$$H^p(T, \mathbb{Z}) \cong H^p(S^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & p = 0, 1 \\ 0, & p \geqq 2. \end{cases} \tag{3.6}$$

Here we have used the fact that $\pi_1(\mathbb{P}_2) = 0$, so that the embedded circle can be considered (by a homotopy equivalence) as embedded in a coordinate neighborhood and the normal bundle tubular neighborhood T is diffeomorphic to $S^1 \times D^3$. Using (3.3), (3.5), and (3.6) in the exact sequence (3.4) we obtain readily

$$H^p(\mathbb{P}_2 - S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & p = 2 \\ 0, & p \neq 0, 2. \end{cases} \tag{3.7}$$

It then follows from (3.3) and (3.7) and the sequence (3.2) that

$$H^p_{S^1}(\mathbb{P}_2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & p = 3, 4 \\ 0, & p \neq 3, 4. \end{cases} \tag{3.8}$$

In general we see that we have

$$\begin{array}{ccccccc} \ldots \to H^2(\mathbb{F}_x, \mathbb{Z}) & \to & H^2(\mathbb{F}_x - F_x, \mathbb{Z}) & \to & H^3_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z}) \to \ldots \\ \wr\| & & \wr\| & & \wr\| \\ \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \end{array}$$

The manifold $\tilde{F}$ has a natural orientation, and so does N, thus inducing an orientation on the fibres $\{\tilde{F}_x\}$. One can choose a generator for $H^3_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z})$ in a smooth manner. We will consider $H^1_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^1_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{C})$, and get a convenient choice for this generator in terms of currents. Let $[\tilde{F}_x]$ denote the current of integration over $\tilde{F}_x$ as a submanifold of $\mathbb{F}_x \cong \mathbb{P}_2$. Then $[\tilde{F}_x]$ is a closed current of degree 3 on $\mathbb{F}_x$ which is supported on $\tilde{F}_x$ and which represents a generator for $H^3_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{C})$. This current represents in the de Rham-current representation the Poincaré dual of the cycle $F_x \subset \mathbb{F}_x$. We want to do this in a smooth manner, although it's intuitively clear that the above construction varies smoothly. Consider the current of integration defined by $[\tilde{F}]$, where $\tilde{F}$ is considered as a codimension-3 submanifold of $\mu^{-1}(N)$. Then $[\tilde{F}]$ is a 3-form in $\mu^{-1}(N)$ which will correspond in some sense to the parametrized family of 1-cycles. Let $\mathscr{L}^3_{\mathbb{F}}$ be the germs of d-closed hyperfunction 3-forms on $\mathbb{F}$. Then we see that $[\tilde{F}]$ restricted to small neighborhoods of F_x in $\mu^{-1}(N)$, gives a well-defined element

$$[\tilde{F}]_x \in \Gamma(\tilde{F}_x, \mathscr{L}^3_{\mathbb{F}})$$

since

$$\Gamma(\tilde{F}_x, \mathscr{L}^3_{\mathbb{F}}) = \lim_{U \ni x} \operatorname{ind} \Gamma(\tilde{F} \cap \mu^{-1}(U), \mathscr{L}^3_{\mathbb{F}}).$$

We now claim that $[\tilde{F}]_x$ defines a generator of $H^1_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{C})$ for each $x \in N$. To see this, we let

$$0 \to \mathbb{C} \to \mathscr{B}^*_{\mathbb{F}}$$

be the resolution of the constant sheaf $\mathbb{C}$ by the flabby sheaves of hyperfunction forms on $\mathbb{F}$ [21]. Then for any open set $W \subset \mathbb{F}$, and for any closed subset $S \subset W$, we have

$$H^q_S(W, \mathbb{C}) \cong H^q(\Gamma_S(W, \mathscr{B}^*_{\mathbb{F}})).$$

Thus, by using this representation we see that, for U open in N, $[\tilde{F}]|_{\mu^{-1}(U)}$ defines an element

$$\{[\tilde{F}]|_{\mu^{-1}(U)}\} \in H^1_{\tilde{F} \cap \mu^{-1}(U)}(\mu^{-1}(U, \mathbb{C}).$$

Thus taking the inductive limit over $U \ni x$, we find

$$\lim_{U \ni x} \operatorname{ind} \{[\tilde{F}]|_{\mu^{-1}(U)}\} = \{[\tilde{F}]_x\},$$

where $\{[\tilde{F}]_x\}$ is the cohomology class in $H^1(\Gamma_{\tilde{F}_x}(\mathbb{F}_x, \mathscr{B}^*_{\mathbb{F}}))$ represented by $[\tilde{F}]_x$. Thus we have

$$\{[\tilde{F}]_x\} \in H^1(\Gamma_{\tilde{F}_x}(\mathbb{F}_x, \mathscr{B}^*_{\mathbb{F}})) \cong H^1_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{C}),$$

and hence $\{[\tilde{F}]_x\}$ is a well-defined generator for $H^1_{\tilde{F}}(\mathbb{F}_x, \mathbb{C})$.

Returning to the main assertion of the lemma we recall that [see (3.1)]

$$R^p_{\tilde{F}}\mu(\mu^{-1}\mathscr{F})_x \cong H^p_{\tilde{F}}(\mathbb{F}_x, \mathbb{Z}) \otimes \mathscr{F}_x.$$

Now $[\tilde{F}]_x$ is a generator for $H^3_{\tilde{F}_x}(\mathbb{F}_x, \mathbb{Z}) \cong \mathbb{Z}$, since there is no torsion here, and we thus have

$$R^3_{\tilde{F}}\mu(\mu^{-1}\mathscr{F})_x \cong \mathbb{Z}[\tilde{F}]_x \otimes \mathscr{F}_x \cong \mathscr{F}_x. \tag{3.9}$$

This is a sheaf homomorphism since the mapping is effected by a globally defined current $[\tilde{F}]$. Thus

$$R^3_{\tilde{F}}\mu(\mu^{-1}\mathscr{F}) \cong \mathscr{F}.$$

Similarly one sees that $R_F^4\mu(\mu^{-1}\mathscr{F})\cong\mathscr{F}$, although we haven't written down an explicit generator. The remainder of the lemma follows from (3.8). $\square$

We can now use Lemma 3.2 to compute the terms in the spectral sequence given by Theorem 3.1 for this geometric setting. Namely, consider

$$E_2^{pq}=H_P^p(\mathbb{P},R_F^q\mu(\mu^{-1}\mathscr{F}))\Rightarrow H_F^r(\mathbb{F},\mu^{-1}\mathscr{F}), \tag{3.10}$$

where $\mathscr{F}$ is a locally free analytic sheaf on $\mathbb{P}$. Then we see that

$$R_F^q\mu(\mu^{-1}\mathscr{F})_x=\begin{cases}R_F^q\mu(\mu^{-1}\mathscr{F})_x, & x\in P\\ 0, & x\notin P.\end{cases}$$

This follows from the fact that, for $\mathscr{S}$ any sheaf on $\mathbb{F}$,

$$\begin{aligned}R_F^q\mu(\mathscr{S})_x&=\lim_{U\ni x}\operatorname{ind} H^q_{F\cap\mu^{-1}(U)}(\mu^{-1}(U),\mathscr{S})\\ &\cong H^q_{F_x}(\mathbb{F}_x,\mathscr{S}),\\ &\cong H^q_{F_x}(\mathbb{F}_x,\mathscr{S}),\quad \text{if}\quad x\in P\end{aligned}$$

since μ is proper. But we also have

$$\begin{aligned}R_F^q\mu(\mathscr{S})_x&=\lim_{U\ni x}\operatorname{ind} H^q_{F\cap\mu^{-1}(U)}(\mu^{-1}(U),\mathscr{S})\\ &=H^q_F(\mathbb{F}_x,\mathscr{S}),\quad \text{for}\quad x\in P,\end{aligned}$$

and $R_F^q\mu(\mathscr{S})$ clearly vanishes for $x\notin P$. Thus we find that

$$\begin{aligned}H_P^p(\mathbb{P},R_F^q\mu(\mu^{-1}\mathscr{F}))&=H_P^p(\mathbb{P},R_F^q\mu(\mu^{-1}\mathscr{F}))\\ &=\begin{cases}H_P^p(\mathbb{P},\mathscr{F}), & q=3,4\\ 0, & q\neq 3,4\end{cases}\end{aligned} \tag{3.11}$$

by Lemma 3.2.

Now we want to calculate $H_P^p(\mathbb{P},\mathscr{F})$, for suitable $\mathscr{F}$. First we note that we have the long exact sequence for relative cohomology for any sheaf $\mathscr{F}$ on $\mathbb{P}$:

$$\begin{aligned}0\to H_P^0(\mathbb{P},\mathscr{F})\quad&\to H^0(\mathbb{P},\mathscr{F})\quad\to H^0(\mathbb{P}-P,\mathscr{F})\to H_P^1(\mathbb{P},\mathscr{F})\\ \to H^1(\mathbb{P},\mathscr{F})\quad&\to H^1(\mathbb{P}-P,\mathscr{F})\to H_P^2(\mathbb{P},\mathscr{F})\quad\to H^2(\mathbb{P},\mathscr{F})\\ \to H^2(\mathbb{P}-P,\mathscr{F})&\to H_P^3(\mathbb{P},\mathscr{F})\quad\to H^3(\mathbb{P},\mathscr{F})\quad\to H^3(\mathbb{P}-P,\mathscr{F})\\ \to H_P^4(\mathbb{P},\mathscr{F})\quad&\to H^4(\mathbb{P},\mathscr{F})\quad\to\dots.\end{aligned} \tag{3.12}$$

Lemma 3.3. *If $\mathscr{F}$ is a locally free analytic sheaf on $\mathbb{P}$, then*

$$H_P^0(\mathbb{P},\mathscr{F})=0$$

$$H_P^q(\mathbb{P},\mathscr{F})=0,\qquad q\geqq 4.$$

Proof. It's clear that holomorphic sections of a vector bundle on $\mathbb{P}$ with supports in the real hypersurface P must vanish identically. On the other hand, we see that $H^q(\mathbb{P}-P,\mathscr{F})=0$, $q\geqq 3$, since $\mathbb{P}-P$ is an open 3-dimensional complex manifold, and similarly $H^q(\mathbb{P},\mathscr{F})=0$, $q\geqq 4$, since $\mathbb{P}$ is a 3-dimensional complex manifold (cf. [18]). Using this in (3.12) given the desired result. $\square$

We now have a more specialized vanishing of relative cohomology.

Lemma 3.4. $H^1_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n))=0$, $n \geqq 0$.

Proof. Letting $\mathscr{F}=\mathcal{O}_{\mathbb{P}}(-n)$, then by the Kodaira vanishing theorem we have that, since $\mathscr{F}$ is a negative line bundle, $H^q(\mathbb{P}, \mathscr{F})=0$, $q=0,1,2$ [36]. Let $\tau^{-1}(x) = \mu \circ \nu^{-1}(x)$, $x \in \mathbb{M}$, then $\tau^{-1}(x) \cong \mathbb{P}_1(\mathbb{C})$, for $x \in \mathbb{M}$ [35]. If $f \in H^0(\mathbb{P}-P, \mathscr{F})$, then

$$f|_{\tau^{-1}(x)} \in H^0(\tau^{-1}(x), \mathcal{O}_{\tau^{-1}(x)}(-n)) \cong H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(-n))=0, \quad \text{for} \quad x \in \mathbb{M}^+ \cup \mathbb{M}^- .$$

Thus f vanishes on $\tau^{-1}(x)$, but since $\mathbb{P}-P$ is covered by such projective lines, it follows that $f \equiv 0$. Thus we conclude that $H^0(\mathbb{P}-P, \mathscr{F})=H^1_P(\mathbb{P}, \mathscr{F})=0$. □

Remark. We can calculate $H^3_P(\mathbb{P}, \mathscr{F})$ for $\mathscr{F}$ a negative line bundle, and we see that

$$\begin{aligned} &H^3_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n))=0, && n=1,2,3, \\ &H^3_P(\mathbb{P}, \mathcal{O}(-4+n)) \cong \odot^n(\mathbb{C}^4), && n \geqq 0 . \end{aligned}$$

This uses the fact that $\mathbb{P}-P=\mathbb{P}^+ \cup \mathbb{P}^-$ and that $\mathbb{P}^+$ and $\mathbb{P}^-$ are 2-complete in the sense of Andreotti-Grauert [1] and hence $H^q(\mathbb{P}^\pm, \mathscr{F})=0$, $q \geqq 2$, yielding $H^3_P(\mathbb{P}, \mathscr{F}) \cong H^3(\mathbb{P}, \mathscr{F})$ in this case. The above result is then a standard calculation.

Thus using (3.11) and Lemma 3.3 we find that the spectral sequence (3.9) becomes, for any locally free analytic sheaf

$$\begin{array}{c|cccccccc} p\uparrow & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ & 0 & 0 & 0 & H^3_P(\mathbb{P}, \mathscr{F}) & H^3_P(\mathbb{P}, \mathscr{F}) & 0 & \dots \\ & 0 & 0 & 0 & H^2_P(\mathbb{P}, \mathscr{F}) & H^2_P(\mathbb{P}, \mathscr{F}) & 0 & \dots \\ & 0 & 0 & 0 & H^1_P(\mathbb{P}, \mathscr{F}) & H^1_P(\mathbb{P}, \mathscr{F}) & 0 & \dots \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline & & & & & & & \rightarrow q \end{array} \tag{3.13}$$

all other terms being zero. Since this is the E_2 term of the sequence, we see that

$$\begin{aligned} &E_2^{23} \cong E_3^{23} \cong \dots \cong E_\infty^{23} \cong H^2_P(\mathbb{P}, \mathscr{F}) \\ &E_2^{32} \cong E_3^{32} \cong \dots \cong E_\infty^{32} \cong H^1_P(\mathbb{P}, \mathscr{F}) \\ &E_2^{pq} = E_3^{pq} = \quad = E_\infty^{pq} = 0, \quad p+q=5, p \neq 2,3 . \end{aligned}$$

Because the spectral sequence (3.9) vanishes except in the two columns in (3.13) above, we obtain canonically the exact sequence

$$0 \rightarrow E_\infty^{32} \rightarrow H^5_F(\mathbb{F}, \mu^{-1}\mathscr{F}) \rightarrow E_\infty^{23} \rightarrow 0$$

(see [5, Chap. XV, Proposition 5.5]). This yields

$$0 \rightarrow H^2_P(\mathbb{P}, \mathscr{F}) \rightarrow H^5_F(\mathbb{F}, \mu^{-1}\mathscr{F}) \rightarrow H^1_P(\mathbb{P}, \mathscr{F}) \rightarrow 0$$

and Theorem 2.2 is proved. Lemma 3.4 above completes the proof of Corollary 2.3.

4. Pulling Back Hyperfunction Forms by Fibre-Integration

In this section we want to give an explicit construction of a pullback mapping

$$H_P^2(\mathbb{P}, \mathscr{F}) \xrightarrow{\tilde{\mu}^*} H_P^5(\mathbb{P}, \mu^*\mathscr{F})$$

in terms of hyperfunction differential forms which will be seen to be comparable with the usual pullback of smooth forms. In the following section we will show that this pullback and the one given in the preceding section agree. First we need the notion of "integration over the fibre" for real-analytic forms, and then we will dualize to get our desired mapping. We refer the reader to Stoll [30] for a thorough discussion of fibre-integration and we will use and adapt some of his main results for our purposes.

Let M and N be real-analytic manifolds and suppose that $M \xrightarrow{f} N$ is a proper, real-analytic, surjective, maximal rank mapping with fibre dimension$=q$. The *fibre integral* is a mapping

$$f_* : \mathscr{A}^p(M) \to \mathscr{A}^{p-q}(M)$$

for $p \geqq q$. This mapping has various nice functorial properties, e.g. $f_*(d\alpha) = df_*(\alpha)$, etc., and is defined by an explicit integration process which won't be elaborated on here. We will use certain properties as we proceed, referring to Stoll for details. If M and N are both complex manifolds and f is holomorphic, then one finds that f_* preserves bidegree. In fact, if the fibre has complex dimension q, then, for a form φ of type (r,s), one finds that $f_*(\varphi)$ is a form of type $(r-q, s-q)$ on N, and moreover f_* commutes with $\bar{\partial}$.

We need a generalization of this last result to CR-manifolds, that is, manifolds whose real tangent bundle has a real subbundle which admits a complex structure (see [33] for a survey of this topic). First we need the notion of (p,q)-forms on a CR-manifold. Suppose that $M \subset X$ is a CR-submanifold where X is an n-dimensional complex manifold. Suppose that the CR-dimension of M is d. Then we can find real 1-forms $\{\theta_1, \ldots, \theta_l\}$ which annihilate the CR-tangent space $H(M) \subset T(M)$. Moreover, we can find, in addition, forms $\{\omega_1, \ldots, \omega_d, \bar{\omega}_1, \ldots, \bar{\omega}_d\}$ so that

$$\{\theta_1, \ldots, \theta_l, \omega_1, \ldots, \omega_d, \bar{\omega}_1, \ldots, \bar{\omega}_d\}$$

span the cotangent space to $T^*(M)$ near a point $p \in M$, where $l + 2d = \dim M$. There is a natural group of changes of frame just as in the case of hypersurfaces (cf. [32, 6]), and we see that the group consists of matrices of the form

$$\left[\begin{array}{c|c} \mathrm{GL}(l, \mathbb{R}) & 0 \\ \hline * & \mathrm{GL}(d, \mathbb{C}) \end{array}\right]$$

acting on $\{\theta_1, \ldots, \theta_l, \omega_1, \ldots, \omega_d\}$ considered as a column vector. We define the sheaf of C^∞ (p,q)-forms on M as germs of $(p+q)$-forms on M of the form

$$\mathscr{E}_M^{p,q} = \left\{ \sum_{\substack{i_1 < \ldots < i_r \\ j_1 < \ldots < j_s \\ k_1 < \ldots < k_q \\ r+s=p}} a_{i_1 \ldots i_r j_1 \ldots j_s k_1 \ldots k_q} \theta_{i_1} \wedge \ldots \wedge \theta_{i_r} \wedge \omega_{j_1} \wedge \ldots \wedge \omega_{j_s} \wedge \bar{\omega}_{k_1} \wedge \ldots \wedge \bar{\omega}_{k_q} \right\}.$$

One can check easily that the concept of (p, q)-form is independent of the choice of frame. Moreover, one finds that if

$$j: M \hookrightarrow X$$

is the embedding, then

$$j^*(\mathscr{E}_X^{p,q}) \hookrightarrow \mathscr{E}_M^{p,q}, \quad 0 \leqq p \leqq n, \quad 0 \leqq q \leqq d, \tag{4.1a}$$

$$j^*(\mathscr{E}_X^{p-l,q+l}) \hookrightarrow \mathscr{E}_M^{p,q}, \quad 0 \leqq p \leqq n, \quad q+l>d, \quad 0 \leqq l \leqq p. \tag{4.1b}$$

This can be verified by using specific frames for the ambient space adapted to the CR-submanifold M near a given point, but the argument will be omitted here (cf. [27], where this is discussed in detail for hypersurfaces).

Let X and Y be complex manifolds, and consider CR-submanifolds $M \subset X$, $N \subset Y$ and a real-analytic CR-mapping f

$$\begin{tikzcd} M \arrow[r, hook, "\sigma"'] \arrow[d, "f"] & X \\ N \arrow[r, hook, "\tau"'] & Y \end{tikzcd} \tag{4.2}$$

with f surjective, and where f is the restriction of a holomorphic mapping of maximal rank mapping a neighborhood of M onto a neighborhood of N. Now we have the fact that if σ and τ denote the embeddings in (4.2), then [letting $\mathscr{A}_M^{p,q}$ denote the sheaf of real-analytic (p, q)-forms on M, etc.]

$$\sigma^*(\mathscr{A}_X^{p,q}) \subset \mathscr{A}_M^{p,q}$$

$$\tau^*(\mathscr{A}_Y^{p,q}) \subset \mathscr{A}_N^{p,q}$$

as above, and moreover, there are tangential $\bar{\partial}$-operators $\bar{\partial}_M$, $\bar{\partial}_N$ satisfying the commutative diagram

$$\begin{tikzcd} \mathscr{A}_X^{p,q} \arrow[r, "\bar{\partial}"] \arrow[d, "\sigma^*"] & \mathscr{A}_X^{p,q+1} \arrow[d, "\sigma^*"] \\ \mathscr{A}_M^{p,q} \arrow[r, "\bar{\partial}_M"] & \mathscr{A}_M^{p,q+1} \end{tikzcd}$$

and similarly for $N \hookrightarrow Y$ (cf. [27]).

Proposition 4.1. *Suppose that* $\dim_{\mathbb{R}} M = 7, \dim_{\mathbb{C}} X = 5,$ *and* $N = Y$, *with* $\dim_{\mathbb{C}} Y = 3$, *then there is a commutative diagram*

$$\begin{tikzcd} \mathscr{A}_M^{5,0}(M) \arrow[r, "\bar{\partial}_M"] \arrow[d, "f_*"] & \mathscr{A}_M^{5,1}(M) \arrow[r, "\bar{\partial}_M"] \arrow[d, "f_*"] & \mathscr{A}_M^{5,2}(M) \arrow[r] \arrow[d, "f_*"] & 0 \\ \mathscr{A}_N^{3,1}(N) \arrow[r, "\bar{\partial}_N"] & \mathscr{A}_N^{3,2}(N) \arrow[r, "\bar{\partial}_N"] & \mathscr{A}_N^{3,3}(N) \arrow[r] & 0 \end{tikzcd}$$

where f_* *is the fibre integral mapping.*

Remark. We see that in this proposition the fibre integration preserves the bigrading of the differential forms on the CR-manifolds involved. This is true in general for CR-mappings. Exactly which bidegrees map to which bidegrees depends on the circumstance, and, as we see from the proposition, it is not necessarily obvious, given our previous intuition concerning fibre integrals for real or complex manifolds. We won't work out the general case in this paper, as we need only the special result given above.

Proof. The basic idea in the fibre-integral is to find a splitting of the exact sequence

$$0 \to f^*(T_x^*(N)) \to T^*(M)|_{F_x} \to T(F_x) \to 0,$$

where F_x is the fibre over $x \in N$. The fibre integral is defined in terms of a splitting, and is then shown to be independent of the splitting (see [30]). Moreover, the nature of the splitting determines the preservation of properties such as type, commutation with $\bar{\partial}$, etc., as is worked out by Stoll in the complex-analytic case. For our case it will then suffice to show that we can always find suitable frames for $T^*(M)$ and $T^*(N)$ so that we have an appropriate splitting. Namely, if $\{\theta_1, \theta_2, \theta_3, \omega_1, \bar{\omega}_1, \omega_2, \bar{\omega}_2\}$ is a typical CR-frame for $T^*(M)$, and if $\{\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2, \zeta_3, \bar{\zeta}_3\}$ is a typical frame for $T^*(N)$ (complex manifold), then we claim that for the given mapping f, and near any point $x \in N$, we can always choose frames so that

$$\begin{aligned} f^*(\zeta_i) &= \omega_i, \quad f^*(\bar{\zeta}_i) = \bar{\omega}_i, \quad i = 1, 2, \\ f^*(\zeta_3 + \bar{\zeta}_3) &= \theta_1, \\ f^*(i(\zeta_3 - \bar{\zeta}_3)) &= \theta_2. \end{aligned} \tag{4.3}$$

These will be *frames adapted to the* CR-*mapping f.*

To see that we can find frames satisfying (4.2) we note that the CR-tangent bundle to M denoted by $H(M) \underset{i}{\to} T(M)$, has a real dual $H^*(M)$, and one has a natural surjective projection

$$T^*(M) \xrightarrow{i^*} H^*(M).$$

We note that $H^*(M)$ has a complex structure, and the pullback mapping

$$f^*: T^*(N) \to T^*(M)$$

composed with the projection i^*, gives

$$i^* \circ f^*: T^*(N) \to H^*(N)$$

which is complex-linear. This mapping is necessarily of maximal rank, and it follows that, since $\{\omega_1, \omega_2\}$ span $H^*(M)$ considered as a complex vector space,

$$\begin{aligned} \omega_1 &= a_1 f^*(\zeta_1) + a_2 f^*(\zeta_2) + a_3 f^*(\zeta_3), \\ \omega_2 &= b_1 f^*(\zeta_1) + b_2 f^*(\zeta_2) + b_3 f^*(\zeta_3). \end{aligned}$$

By choosing a new choice of frame $\{\zeta_1', \zeta_2', \zeta_3'\}$ for $T^*(N)$, we can require that $a_1 = b_2 = 1$, and all other coefficients vanish. Thus for this new frame we have

$$\begin{aligned} f^*(\zeta_1') &= \omega_1, \\ f^*(\zeta_2') &= \omega_2. \end{aligned}$$

Now, we have that $\{f^*(\operatorname{Re}\zeta_3'), f^*(\operatorname{Im}\zeta_3')\}$ span over $\mathbb{R}$ a 2-dimensional subspace of $T^*(M)$, and hence

$$f^*(\operatorname{Re}\zeta_3')=\sum\alpha_i\theta_i+\sum a_i\omega_i+\sum\bar{a}_i\bar{\omega}_i$$
$$f^*(\operatorname{Im}\zeta_3')=\sum\beta_i\theta_i+\sum b_i\omega_i+\sum\bar{b}_i\bar{\omega}_i.$$

Letting

$$\zeta_3''=\zeta_3'+c_1\zeta_1'+c_2\zeta_2'$$
$$\zeta_1''=\zeta_1'$$
$$\zeta_2''=\zeta_2'$$

we can choose c_1 and c_2 so that $a_i=b_i=0$ in a similar expansion for $f^*(\operatorname{Re}\zeta_3'')$, $f^*(\operatorname{Im}\zeta_3'')$. Thus these real vectors are real-linear combinations of $\{\theta_1,\theta_2,\theta_3\}$. We can choose appropriate linear combinations of $\{\theta_i\}$, call them $\{\theta_i'\}$ so that

$$f^*(\operatorname{Re}\zeta_3'')=\theta_1'$$
$$f^*(\operatorname{Im}\zeta_3'')=\theta_2'.$$

Thus we have frames satisfying (4.3).

Using such adapted frames we see that if

$$\varphi=a\theta_1\wedge\theta_2\wedge\theta_3\wedge\omega_1\wedge\omega_2$$

is a form of type $(5,0)$ on M, then

$$\varphi=af^*(\zeta_3+\bar{\zeta}_3)\wedge f^*(i(\zeta_3-\bar{\zeta}_3))\wedge\theta_3\wedge f^*(\zeta_1)\wedge f^*(\zeta_2).$$

We see also that if F_x is the fibre, then $\theta_3|_{F_x}\neq 0$. Thus we can express our form φ as

$$\varphi=-ia\theta_3\wedge f^*(\zeta_3\wedge\bar{\zeta}_3\wedge\zeta_1\wedge\zeta_2).$$

This shows that the fibre-integral of this form of type $(5,0)$ is of type $(3,1)$ (since $\zeta_3\wedge\bar{\zeta}_3\wedge\zeta_1\wedge\zeta_2$ is of type $(3,1)$, cf. [30]. The action of f_* on higher order forms is determined in a similar manner and we omit further details. That $\bar{\partial}_M f_*=f_*\bar{\partial}_N$ follows from simple type considerations and the fact that d commutes with f_* [30]. □

Remark. The same ideas would show that if $M\subset N$ is a generic 6-dimensional CR-submanifold of a 5-complex-dimensional manifold fibered over a 5-dimensional CR-hypersurface in a 3-complex-dimensional manifold, then we would have

$$\begin{array}{ccccc}
\mathscr{A}_M^{5,0}(M) & \xrightarrow{\bar{\partial}_M} & \mathscr{A}_M^{5,1}(M) & \longrightarrow & 0\\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\mathscr{A}_N^{3,1}(N) & \xrightarrow{\bar{\partial}_N} & \mathscr{A}_N^{3,2} & \longrightarrow & 0\,.
\end{array}\tag{4.4}$$

This is the situation we have in our fibration $F\xrightarrow{\mu}P$. But we need to use the fibre integration in the context of $\tilde{F}\xrightarrow{\mu}N$, where $\tilde{F}$ is the extension of the fibration

$F \xrightarrow{\mu} P$ to a neighborhood N of P as discussed in Sect. 3. We will have occasion to refer to (4.4) later however.

Let $W = \mu^{-1}(N)$ and consider a complex of real-analytic forms defined in fixed neighborhoods W of F and N of P:

$$\begin{array}{ccccccc}
\mathscr{A}^{5,0}(W) & \xrightarrow{\bar\partial} & \mathscr{A}^{5,1}(W) & \xrightarrow{\bar\partial} & \mathscr{A}^{5,2}(W) & \longrightarrow \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & \\
\mathscr{A}^{3,1}(N) & \xrightarrow{\bar\partial} & \mathscr{A}^{3,2}(N) & \xrightarrow{\bar\partial} & \mathscr{A}^{3,3}(N) &
\end{array}$$

where I is the mapping induced by the fibre integration over the fibres of $\tilde{F} \to N$, which satisfies the hypotheses of Proposition 4.1. Now the hyperfunction forms with compact supports are dual to the above spaces, and we obtain by duality

$$\begin{array}{ccccccc}
0 & \longleftarrow & \mathscr{B}_c^{0,5}(W) & \xleftarrow{\bar\partial} & \mathscr{B}_c^{0,4}(W) & \xleftarrow{\bar\partial} & \mathscr{B}_c^{0,4}(W) \\
\uparrow & & \uparrow{\scriptstyle I^*} & & \uparrow{\scriptstyle I^*} & & \uparrow{\scriptstyle I^*} \\
\mathscr{B}_c^{0,3}(N) & \longleftarrow & \mathscr{B}_c^{0,2}(N) & \longleftarrow & \mathscr{B}_c^{0,1}(N) & \longleftarrow & \mathscr{B}_c^{0,0}(N)
\end{array}$$

where $\mathscr{B}_c^{**}(\ )$ denotes hyperfunctions with compact support. Thus we obtain a mapping

$$H_c^{0,2}(N) \xrightarrow{I^*} H_c^{0,5}(W)$$

and if we restrict our attention to hyperfunction forms with support in P, we obtain readily

$$H_P^{0,2}(N) \xrightarrow{I^*} H_F^{0,5}(W).$$

This construction can be extended to vector bundle coefficients with no difficulty, and we obtain the natural mapping

$$H_P^{0,2}(\mathbb{P}, V) \xrightarrow{I^*} H_F^{0,5}(\mathbb{F}, \mu^* V) \tag{4.5}$$

where $V \to \mathbb{P}$ is a holomorphic vector bundle. We have used the fact that

$$H_P^{0,2}(\mathbb{P}, V) \cong H_P^{0,2}(N, V)$$

and similarly on $\mathbb{F}$. We also used the dual bundle $V^* \to \mathbb{P}$ for coefficients of the real-analytic forms used in defining I whose dual gives (4.5).

Theorem 4.2. *The diagram*

$$\begin{array}{ccc}
H_P^{0,2}(\mathbb{P}, V) & \xrightarrow{I^*} & H_F^{0,5}(\mathbb{F}, \mu^* V) \\
\downarrow & & \downarrow \\
H^{0,1}(P, V) & \xrightarrow{\mu^*} & H^{0,1}(F, \mu^* V)
\end{array}$$

where the vertical arrows are given by Proposition 2.3, is commutative.

Proof. Let $\langle\ ,\ \rangle$ denote the pairing between the compactly supported hyperfunction forms supported on P or F and the real-analytic forms defined near P or

F. We need to show that if

$$\varphi \in \mathscr{A}^{0,1}(P, V) \quad \text{and} \quad \bar{\partial}\varphi = 0$$

then

$$I^*(\varphi \wedge [P]^{0,1}) = -\mu^*(\varphi) \wedge [F]^{0,4}$$

Thus we compute, for $\psi \in \mathscr{A}^{5,0}(F, V^*)$,

$$\begin{aligned}\langle \mu^*\varphi \wedge [F]^{0,4}, \psi\rangle &= -\langle [F]^{0,4}, \psi \wedge \mu^*\varphi\rangle \\ &= -\int_F \psi \wedge \mu^*\varphi \\ &= -\int_P I(\psi \wedge \mu^*\varphi),\end{aligned}$$

where I is the fibre integral over $\tilde{F}$, and the last step is a global version of Fubini's theorem [30]. But one of the properties of the fibre integral is that

$$I(\psi \wedge \mu^*\varphi) = \varphi \wedge I(\psi)$$

[30], and thus we obtain

$$\begin{aligned}\langle \mu^*\varphi \wedge [F]^{0,4}, \psi\rangle &= -\int_P \varphi \wedge I(\psi) \\ &= -\varphi \wedge [P]^{0,1}, I(\psi)\rangle \\ &= -\langle I^*(\varphi \wedge [P]^{0,1}), \psi\rangle.\end{aligned}$$

Since $\bar{\partial}$ commutes with I^* we obtain the desired result. □

5. Comparison of the Abstract and Fibre-Integral Pullbacks

Let $\mathscr{F}$ be a locally-free analytic sheaf on $\mathbb{P}$. Then from Theorem 2.2 and (4.5) we have the diagram

$$\begin{tikzcd} H^2_P(\mathbb{P}, \mathscr{F}) \arrow[r, "\tilde{\mu}^*"] \arrow[dr, "I^*"'] & H^5_F(\mathbb{F}, \mu^{-1}\mathscr{F}) \arrow[d, "i"] \\ & H^2_F(\mathbb{F}, \mu^*\mathscr{F}) \end{tikzcd}$$

where i is the homomorphism indicated by the natural sheaf homomorphism

$$\mu^{-1}\mathscr{F} \to \mu^*\mathscr{F} = \mu^{-1}\mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{F}} \quad \text{on } \mathbb{F}.$$

We want to show that this diagram is commutative. We have the following mappings of resolutions of sheaves on $\mathbb{P}$ and $\mathbb{F}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "g_0"] & \mathscr{B}^{0,*}_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{F} \arrow[d, "g"] \\ 0 \arrow[r] & R^3_{\tilde{F}}\mu(\mu^{-1}\mathscr{F}) \arrow[r] \arrow[d, "h_0"] & \{\tilde{F}\} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{B}^{0,*}_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{F} \arrow[d, "h"] \\ 0 \arrow[r] & \mathscr{Z}^{0,3}_{\mathbb{F}} \otimes \mu^*\mathscr{F} \arrow[r] & \mathscr{B}^{0,*+3}_{\mathbb{F}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mu^*\mathscr{F}. \end{tikzcd}$$

Here we have let

$$\mathscr{Z}_X^{0,q} = \mathrm{Ker}(\mathscr{B}_X^{0,q} \to \mathscr{B}_X^{0,q+1})$$

be the $\bar{\partial}$-closed hyperfunction forms of type $(0,q)$ on a complex manifold X. We have set

$$g_0(f_x) = [\tilde{F}]_x \otimes f_x \in R_{\tilde{F}}^3 \mu(\mu^{-1}\mathscr{F})_x$$

which is well-defined by (3.9). We let $\{\tilde{F}\}$ denote the subsheaf of $\mathscr{B}_{\mathbb{F}}^3$ generated by $\{\tilde{F}\}_x$ at each $x \in \tilde{F}$ as an $\mathcal{O}_{\mathbb{P}}$-module. We let g be the natural extension of g_0 to the differential sheaves indicated. We define h by

$$h([\tilde{F}]_x \otimes \varphi^{0,q} \otimes f_x) = I^*(\varphi^{0,q}) \otimes f_x \in \mathscr{B}_{\mathbb{F}}^{0,q+3} \otimes \mu^* \mathscr{F}\,.$$

The mapping h on the resolutions induces the mapping h_0. Then we see that each of the resolutions involved is a specific resolution of the sheaf in question. Such mappings induce mappings on cohomology. In particular, then we have natural mappings, by the "abstract de Rham theorem" [15]:

$$\begin{tikzcd}
H_P^2(\mathbb{P}, \mathscr{F}) \arrow[d, "\cong"', "g_0"] & H^2(\Gamma_P(\mathbb{P}, \mathscr{B}_{\mathbb{P}}^{0,*} \otimes \mathscr{F})) \arrow[l, "\cong"'] \arrow[d, "\cong"', "g"] \\
H_P^2(\mathbb{P}, R_{\tilde{F}}^3 \mu(\mu^{-1}\mathscr{F})) \arrow[d, "h_0"] & H^2(\Gamma_P(\mathbb{P}, \{F\} \otimes \mathscr{B}_{\mathbb{P}}^{0,*} \otimes \mathscr{F}) \arrow[l, "\alpha"', "\cong"] \arrow[d, "h"] \\
H_{\tilde{F}}^2(\mathbb{F}, \mathscr{Z}_{\mathbb{F}}^{0,3} \otimes \mu^* \mathscr{F}) & H^2(\Gamma_{\tilde{F}}(\mathbb{F}, \mathscr{B}_{\mathbb{F}}^{0,*+3} \otimes \mu^* \mathscr{F})\,. \arrow[l, "\beta"', "\cong"]
\end{tikzcd} \tag{5.1}$$

In addition we have,

$$\begin{tikzcd}
H_{\tilde{F}}^5(\mathbb{F}, \mu^{-1}\mathscr{F}) \arrow[d, "i"'] & H_P^2(\mathbb{P}, \mathscr{F}) \arrow[l, "\tilde{\mu}^*"'] \arrow[r, "g_0"] \arrow[d, "\tilde{h}_0"] & H_P^2(\mathbb{P}, R_{\tilde{F}}^3 \mu(\mu^{-1}\mathscr{F})) \arrow[d, "h_0"] \\
H_{\tilde{F}}^5(\mathbb{F}, \mu^* \mathscr{F}) & H^2(\Gamma_{\tilde{F}}(\mathbb{F}, \mathscr{B}_{\mathbb{F}}^{0,*+3} \otimes \mu^* \mathscr{F}) \arrow[l, "\delta"', "\cong"] \arrow[r, "\beta"] & H_{\tilde{F}}^2(\mathbb{F}, \mathscr{Z}_{\mathbb{F}}^{0,3} \otimes \mu^* \mathscr{F})
\end{tikzcd} \tag{5.2}$$

where $\tilde{h}_0$ is the induced mapping, using the fact that α is an isomorphism. Moreover, δ is the isomorphism given by the Dolbeault representation of $H_{\tilde{F}}^5(\mathbb{F}, \mu^* \mathscr{F})$ in terms of hyperfunction forms. We note that

$$\begin{tikzcd}
R_{\tilde{F}}^3 \mu(\mu^{-1}\mathscr{F}) \arrow[dd, "h_0"] & \mathscr{F} \arrow[l, "\alpha_0"'] \arrow[dr] & \\
 & & \mu^{-1}\mathscr{F} \arrow[dl] \\
\mathscr{Z}_{\mathbb{F}}^{0,3} \otimes \mu^* \mathscr{F} & \mu^* \mathscr{F} \arrow[l] &
\end{tikzcd}$$

is commutative, where

$$\alpha_0(f_x) = [\tilde{F}]_x \otimes f_x$$

and

$$h_0([\tilde{F}]_x \otimes f_x) = [\tilde{F}]_x^{0,3} \otimes f_x\,.$$

This implies the commutativity of (5.2).

Thus we see that the combined diagrams (5.1) and (5.2) are compatible, and that we have the following theorem.

Theorem 5.1. *The diagram*

$$\begin{tikzcd} H^2_P(\mathbb{P},\mathscr{F}) \arrow[r, "\tilde{\mu}^*"] \arrow[dr, "I^*"'] & H^5_P(\mathbb{F},\mu^{-1}\mathscr{F}) \arrow[d, "i"] \\ & H^5_F(\mathbb{F},\mu^*\mathscr{F}) \end{tikzcd}$$

is commutative.

6. The Penrose Transform Acting on Hyperfunction Data

We now consider the relative de Rham sequence on $\mathbb{F}$ used in [10],

$$0 \longrightarrow \mu^{-1}\mathcal{O}_{\mathbb{P}}(n) \longrightarrow \Omega^0_\mu(n) \xrightarrow{d_\mu} \Omega^1_\mu(n) \longrightarrow \Omega^2_\mu(n) \longrightarrow 0$$

and consider the spectral sequence of relative cohomology groups

$$E_1^{pq} = H^p_F(\mathbb{F},\Omega^q_\mu(n)) \Rightarrow H^r_F(\mathbb{F},\mu^{-1}\mathcal{O}_{\mathbb{P}}(n)) \tag{6.1}$$

which is the natural generalization of the spectral sequence used in [10] (cf. [4, 15]). To see when this degenerates we will, as before, compute the direct images of the sheaves $\Omega^q_\mu(n)$, with respect to the mapping ν, and then use a version of the relative Leray spectral sequence (Sect. 2) to compute the terms in (6.1). The degeneration of the spectral sequence (6.1) will then yield the differential equations, as before. The direct image sheaves are the same as in [10]; in this case we don't need a relative version of them.

We will consider, for simplicity, only the case of $\mathscr{F}=\mathcal{O}_{\mathbb{P}}(-n-2)$, $n>0$ (which will correspond to positive helicity solutions of the field equations). Thus we have from [10, Eq. (2.11)] that

$$\begin{aligned} &\nu^1_*\Omega^0_\mu(-n-2) \cong \mathcal{O}_{(A'B'\dots D')}[-1]' \quad (n \text{ indices}), \qquad n \geqq 0 \\ &\nu^1_*\Omega^1_\mu(-n-2) \cong \mathcal{O}_{A(B'\dots D')}[-2]' \quad (n-1 \text{ indices}), \qquad n \geqq 1 \\ &\nu^1_*\Omega^2_\mu(-n-2) \cong \mathcal{O}_{(C'\dots D')}[-1][-3]' \quad (n-2 \text{ indices}), \qquad n \geqq 2 \end{aligned} \tag{6.2}$$

and all other direct images vanish. Thus the relative Leray spectral sequence, for fixed $j=0,1,2$, is

$$H^p_M(\mathbb{M},\nu^q_*\Omega^j_\mu(-n-2)) \Rightarrow H^r_F(\mathbb{F},\Omega^j_\mu(-n-2)) \tag{6.3}$$

and because (6.2) are the only nonvanishing direct image sheaves this is totally degenerate. Thus we have the Leray isomorphism

$$H^r_F(\mathbb{F},\Omega^j_\mu(-n-2)) \underset{\cong}{\xleftarrow{L}} H^{r-1}_M(\mathbb{M},\nu^1_*\Omega^j_\mu(-n-2)). \tag{6.4}$$

It follows from Sato's fundamental theorem that

$$H^s_M(\mathbb{M},\mathscr{L})=0, \qquad s \neq 4, \tag{6.5}$$

where $\mathscr{L}$ is any locally free analytic sheaf on $\mathbb{M}$, since M is a real-analytic submanifold of its complexification [29, 22]. Thus we see that

$$H_F^r(\mathbb{F}, \Omega_\mu^j(-n-2))=0\,, \qquad r \neq 5\,.$$

Therefore, we are left with the only nonzero terms in the spectral sequence (6.1) at the E_1 level is given by $E_1^{5,q}$, $q=0,1,2$,

$$\begin{array}{l|ccccc}
p\uparrow & 0 & & 0 & & 0 \\
& H_F^5(\mathbb{F}, \Omega_\mu^0(-n-2)) & \longrightarrow & H_F^5(\mathbb{F}, \Omega_\mu^1(-n-2)) & \longrightarrow & H_F^5(\mathbb{F}, \Omega_\mu^2(-n-2)) \\
& 0 & & 0 & & 0 \\
& \vdots & & \vdots & & \vdots \\
& 0 & & 0 & & 0 \\
\hline
& & & & & \rightarrow q
\end{array}$$

Also the mapping d_μ induces the spectral sequence mapping d_1, and we obtain

$$\begin{tikzcd}
H_F^5(\mathbb{F}, \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n-2)) \arrow[r, tail, "\sigma", "\cong"'] \arrow[dr, "\cong"'] & \mathrm{Ker}[H_F^5(\mathbb{F}, \Omega_\mu^0(-n-2)) \xrightarrow{d_\mu} H_F^5(\mathbb{F}, \Omega_\mu^1(-n-2))] \\
& \mathrm{Ker}[H_M^4(\mathbb{M}, \mathcal{O}_{(A'\dots D')}[-1]') \xrightarrow{\nabla} H_M^4(\mathbb{M}, \mathcal{O}_{A(B'\dots D')}[-2]')] \arrow[u]
\end{tikzcd} \tag{6.6}$$

where ∇ is the induced differential operator.

We see that the spectral sequence (6.4) also degenerates locally. Namely, for U open in $\mathbb{M}$, and letting $U'=\mu^{-1}(U)$, we have

$$\begin{tikzcd}
H^5_{F\cap U'}(U', \Omega_\mu^0(-n-2)) \arrow[r, "d_\mu"] \arrow[d, "L", "\cong"'] & H^5_{F\cap U'}(U', \Omega_\mu^1(-n-2)) \arrow[d, "L", "\cong"'] \\
H^4_{M\cap U}(U, \mathcal{O}_{(A'\dots D')}[-1]') \arrow[r, "\nabla"] & H^4_{M\cap U}(U, \mathcal{O}_{A(B'\dots D')}[-2]')\,.
\end{tikzcd} \tag{6.7}$$

This uses the fact that (6.5) holds locally also [29, 22]. If U_0 is open in M, let U be open in $\mathbb{M}$ so that $U_0=U\cap M$. We then let $\mathscr{Z}'_{\mathscr{B},n}$ be the sheaf on M generated by the presheaf

$$U_0 \mapsto \mathrm{Ker}[H^4_{M\cap U}(U, \mathcal{O}_{(A'\dots D')}[-1']) \xrightarrow{\nabla} H^4_{M\cap U}(U, \mathcal{O}_{A(B'\dots D')}[-2]')]\,. \tag{6.8}$$

This turns out to be the sheaf of hyperfunction solutions to the massless field equations of positive helicity on M. We'll come back to this point later, but for the time being we let $\mathscr{Z}'_{\mathscr{B},n}$ be defined simply as the local solutions of (6.8) as indicated.

Therefore, at the global level we have now the mappings [Theorem 2.2, (6.6), (6.8)]

$$\begin{aligned}
&H_P^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow[\cong]{\tilde{\mu}^*} H_F^5(\mathbb{F}, \mu^{-1}\mathcal{O}_{\mathbb{P}}(-n-2)) \\
&\xrightarrow[\cong]{\sigma} \mathrm{Ker}\,(H_F^5(\mathbb{F}, \Omega_\mu^0(-n-2)) \to H_F^5(\mathbb{F}, \Omega_\mu^1(-n-2)) \xrightarrow[\cong]{L^{-1}} \mathscr{Z}'_{\mathscr{B},n}(M)
\end{aligned}$$

and we denote the composition by $\mathscr{P}$, and call it the *Penrose transform of hyperfunction data*. Thus we obtain the fundamental result

$$\mathscr{P}: H_P^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n-2)) \xrightarrow{\cong} \mathscr{Z}'_{\mathscr{B},n}(M)\,, \qquad n>0\,, \tag{6.9}$$

and we have the following theorem.

Theorem 6.1. *The Penrose transform acting on hyperfunction data is compatible with the Penrose transform acting on real-analytic data* (1.7), *and moreover one has the following commutative diagram, where the vertical maps are given by Proposition 2.3, and* $n>0$,

$$\begin{tikzcd} H^2_P(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-n-2)) \arrow[r, "\mathscr{P}", "\cong"'] & \mathscr{Z}'_{\mathscr{B},n}(M) \\ H^1(P, \mathcal{O}_{\mathbb{P}}(-n-2)) \arrow[u] \arrow[r, "\mathscr{P}", "\cong"'] & \mathscr{Z}'_{\mathscr{A},n}(M) \arrow[u] \end{tikzcd}.$$

In particular, $\mathscr{P}$ *acting on real-analytic data is mapped onto the set of all real-analytic solutions of the positive helicity massless field equations on* M.

Remark. This theorem justifies using the same notation $\mathscr{P}$ to denote the Penrose transform. We can also consider transforming data of an intermediate sort, such as distribution, C^∞, Sobolev classes, etc. all of which are encompassed between real-analytic and hyperfunction smoothness, and we would have a well-defined mapping. The nature of the image will depend on the smoothness class considered, and this will bear further investigation.

Proof. It suffices to show that the diagram

$$\begin{tikzcd} H^1(F, \Omega^j_\mu(-n-2) \arrow[r] \arrow[d, "L_{\mathscr{A}}"', "\cong"] & H^5_F(\mathbb{F}, \Omega^j_\mu(-n-2)) \\ H^0(M, v^1_*\Omega^j_\mu(-n-2)) \arrow[r] & H^4_M(\mathbb{M}, v^1_*\Omega^j_\mu(-n-2)) \arrow[u, "L_{\mathscr{B}}", "\cong"'] \end{tikzcd}$$

is commutative, where $L_{\mathscr{A}}$ is an inductive limit of Leray isomorphisms on open neighborhoods of $M \subset \mathbb{M}$, and $L_{\mathscr{B}}$ is given in (6.6), and the horizontal maps are given by Proposition 2.3. What we need is an explicit representation for $L_{\mathscr{B}}$. We can find such a representation in the neighborhood of fibres of v, and this will suffice.

Let $\mathscr{F} = \Omega^j_\mu(-n-2)$, which will be fixed in this argument. Then we have that

$$H^5_F(\mathbb{F}, \mathscr{F}) \cong H^1_F(\mathbb{F}, \mathscr{F} \otimes \mathscr{Z}^{0,4}_{\mathbb{F}})$$
$$H^4_M(\mathbb{M}, v^1_*\mathscr{F}) \cong H^0_M(\mathbb{M}, \mathscr{B}^{0,4}_{\mathbb{M}} \otimes v^1_*\mathscr{F})$$

(noting that $\mathscr{B}^{0,4}_{\mathbb{M}} = \mathscr{Z}^{0,4}_{\mathbb{M}}$ by degree considerations). We also have the fibre-integration mapping for the holomorphic fibration $\mathbb{F} \xrightarrow{v} \mathbb{M}$, the fibres being compact 1-complex-dimensional submanifolds:

$$\mathscr{A}^{5,1}(F) \xrightarrow{I} \mathscr{A}^{4,0}(M).$$

This fibre integral is well-defined on neighborhoods $v^{-1}(U)$ of F fibred over $U \supset M$, and thus well defined in the inductive limit. Therefore we have by duality

$$\Gamma_F(\mathbb{F}, \mathscr{B}^{0,4}_{\mathbb{F}}) \xleftarrow{I^*} \Gamma_M(\mathbb{M}, \mathscr{B}^{0,4}_{\mathbb{M}}).$$

We note that I^* commutes with $\bar\partial$, and thus

$$\Gamma_M(\mathbb{F}, \mathscr{Z}^{0,4}_{\mathbb{M}}) = \Gamma_M(\mathbb{F}, \mathscr{B}^{0,4}_{\mathbb{M}}) \xrightarrow{I^*} \Gamma_{\mathbb{F}}(\mathbb{F}, \mathscr{Z}^{0,4}_{\mathbb{F}}).$$

We then have the following representation for $L_{\mathscr{B}}$ localized near a fibre $v^{-1}(x)$, where $x \in M$:

$$\begin{tikzcd}
H^1(F,\mathscr{F}) \arrow[r] \arrow[d, "L_{\mathscr{A}}"] & H^1(v^{-1}(x),\mathscr{F}) \arrow[r] \arrow[d, "\mathrm{id}"] & H^1(v^{-1}(x),\mathscr{F}) \otimes \Gamma(v^{-1}(x),\mathscr{L}_{\mathbb{F}}^{0,4}) \arrow[r] & {} \\
\mathrm{H}^0(\mathrm{M}, v^1_*\mathscr{F}) \arrow[r] & \mathrm{H}^0(\mathrm{x}, v^1_*\mathscr{F}) \arrow[r] & \mathrm{H}^0(\mathrm{x}, v^1_*\mathscr{F}) \otimes \mathscr{B}^{0,4}_{\mathbb{M},x} \arrow[r] \arrow[u, "\mathrm{id}\otimes I^*"'] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^1(v^{-1}(x),\mathscr{L}_{\mathbb{F}}^{0,4}\otimes\mathscr{F}) & H^1_F(\mathbb{F},\mathscr{L}_{\mathbb{F}}^{0,4}\otimes\mathscr{F}) \arrow[l] \arrow[r, "\cong"'] & H^5_F(\mathbb{F},\mathscr{F}) \\
{} \arrow[r] & H^0(x,\mathscr{B}_{\mathbb{M}}^{0,4}\otimes v^1_*\mathscr{F}) \arrow[u] & H^0_M(\mathbb{M},\mathscr{B}_{\mathbb{M}}^{0,4}\otimes v^1_*\mathscr{F}) \arrow[l] \arrow[r, "\cong"] \arrow[u] & H^4_M(\mathbb{M}, v^1_*\mathscr{F}) \arrow[u, "L_{\mathscr{B}}"']
\end{tikzcd}$$

Now we note that

$$I^*([P]^{0,4}) = [F]^{0,4}$$

since, for ψ a $(5,1)$-form defined near F,

$$\begin{aligned}
\langle I^*([P]^{0,4}), \psi\rangle &= \langle [P]^{0,4}, I(\psi)\rangle \\
&= \int_P I(\psi) = \int_F \psi = \langle [F]^{0,4}, \psi\rangle .
\end{aligned}$$

It follows that the above diagram is commutative, and hence that $L_{\mathscr{A}}$ is compatible with $L_{\mathscr{B}}$.

Let $\mathscr{P}_{\mathscr{A}}$ denote $\mathscr{P}$ acting on real-analytic data. We know that $\mathscr{P}_{\mathscr{A}}$ is injective (see Sect. 1). To see that it is surjective, we let $\varphi \in \mathscr{Z}'_{\mathscr{A},n}(M)$, then we can consider φ as a hyperfunction solution, and we will see in the next section that the boundary-value mapping

$$\begin{aligned}
\mathscr{Z}'_n(\mathbb{M}^+) \oplus \mathscr{Z}'_n(\mathbb{M}^-) &\xrightarrow{b_M} \mathscr{Z}'_{\mathscr{B},n}(M) \\
(f^+, f^-) &\mapsto b(f^+, f^-)
\end{aligned} \tag{6.10}$$

is an *isomorphism*, where the hyperfunction boundary values are taken in the distinguished boundary of the tube domains $\mathbb{M}^+$ and $\mathbb{M}^-$ in the sense of Martineau [23] (restricting to an affine coordinate system, and covering M by such coordinate systems). For smooth boundary values, one has that $b(\varphi^+, \varphi^-)$ is the difference $(\varphi^+ - \varphi^-)|_M$ of the boundary values of φ^+ and φ^- on M. It follows from the Martineau construction that $\varphi \in \mathscr{Z}'_{\mathscr{B},n}(M)$ is real-analytic on M if and only if φ^+ and φ^- given by the isomorphism (6.10) have real-analytic boundary values on M (this is essentially the edge-of-the-wedge theorem). Assuming the isomorphism (6.11) and using the above remarks it follows that φ^+ and φ^- are both holomorphic in $\mathbb{M}^+ \cup N(M)$ and $\mathbb{M}^- \cup N(M)$ where $N(M)$ is a neighborhood of M. But we know that $\tau^{-1}(\mathbb{M}^+ \cup N(M))$ and $\tau^{-1}(\mathbb{M}^- \cup N(M))$ are neighborhoods of $\overline{\mathbb{P}^+}$ and $\overline{\mathbb{P}^-}$ respectively ($\tau^{-1} = \mu \circ v^{-1}$). Therefore we see that by (1.4) and (1.5) there exist $f^\pm \in H^1(\overline{\mathbb{P}^\pm}, \mathcal{O}_{\mathbb{P}}(-n-2))$ such that $\mathscr{P}(f^\pm) = \varphi^\pm$. Therefore $f^+ - f^-$ defines a well-defined element $f \in H^1(P, \mathcal{O}_{\mathbb{P}}(-n-2))$ such that $\mathscr{P}_{\mathscr{A}}(f) = \varphi$. Thus $\mathscr{P}_{\mathscr{A}}$ is surjective. $\square$

Remark. One can formulate tangential cohomology groups on P with C^∞ and distribution coefficients as in [2, 27]. In these two cases one obtains in the same manner as in the above proof that $\mathscr{P}$ maps distributional data to all distribution solutions of the field equations on M, and C^∞ data to all C^∞ solutions of the field equations on M. Again, the basic tool is the work of Martineau in understanding the smoothness classes of the boundary values of holomorphic functions from the tube domains $\mathbb{M}^+$ and $\mathbb{M}^-$. We won't bother to formulate this explicitly in this paper, as there are no new ideas involved.

7. Penrose Transform Representations of Hyperfunction Massless Fields

We will now combine conceptually the ideas described separately in the previous sections. We have the following commutative diagram, where we have let $\mathscr{F}=\mathcal{O}_{\mathbb{P}}(-n-2)$, for $n>0$, and we let $\mathscr{F}'=\mu^*\mathscr{F}$, and $\mathscr{F}''=\nu^1_*\mu^*\mathscr{F}$ be the natural derived sheaves in this context. The mapping in the diagram will be discussed in more detail below:

$$\begin{array}{ccccc}
\sum\limits_{\pm} H^1(\mathbb{P}^{\pm},\mathscr{F}) & \xrightarrow{\mu^*} & \sum\limits_{\pm} H^1(\mathbb{F}^{\pm},\mathscr{F}') & \xrightarrow[\cong]{L} & \sum\limits_{\pm} H^0(\mathbb{M}^{\pm},\mathscr{F}'') \\
\cong\downarrow b_P & & \downarrow b_F & & \downarrow b_M \\
{}'H^{0,1}_{\mathscr{B}}(P,\mathscr{F}) & \xrightarrow{I^*} & {}'H^{0,1}_{\mathscr{B}}(F,\mathscr{F}') & \xrightarrow[\cong]{L''} & {}'H^{0,0}_{\mathscr{B}}(M,\mathscr{F}'') \\
\cong\downarrow s_P & & \cong\downarrow s_F & & \cong\downarrow s_M \\
H^2_P(\mathbb{P},\mathscr{F}) & \xrightarrow{\tilde{\mu}^*} & H^5_F(\mathbb{F},\mathscr{F}') & \xrightarrow[\cong]{} & H^4_M(\mathbb{M},\mathscr{F}'') .
\end{array} \tag{7.1}$$

The mappings b_M, b_F, and b_M denote boundary value maps; they are the "jumps at the boundary of the boundary values taken from each side". More precisely, b_M is the boundary value mapping of Martineau [23], where we take boundary values in some open affine $M_0\subset M$, $M_0\cong\mathbb{R}^4$, and there is a finite covering of M by such coordinate systems on M (take $M_0=M\cap\mathbb{M}^I$, where $\mathbb{M}^I$ is described in [10, Sect. 1]). Then we have $\mathbb{M}^{\pm}$ with distinguished boundary $M_0\cong\mathbb{R}^4$, and the vector bundles become trivial spinor bundles, so we just use the Martineau construction of boundary values of holomorphic functions in tube domains for scalar functions. The mapping s_M is Sato's characterization of intrinsic hyperfunctions on M, generalized to having vector bundle coefficients. The mappings s_P and s_F are generalizations of the mapping s_M. The mapping s_P is described in [27, 17]. The mapping s_F is a further generalization of this to higher codimension, and is a special case of a general result [28, Chap. II, Corollary 3.5.8] due to Sato et al. These mappings depend on the Cauchy-Kowaleski theorem among other things. As we saw in Sect. 1, b_P is an isomorphism. We define b_F to be the induced mapping. An intrinsic description of b_F would involve a generalization of Martineau's work which involves more generalized tube domains (à la $\mathbb{F}^{\pm}$), and where the holomorphic functions are replaced by cohomology classes. This hasn't been carried out, as far as we know, but in our case it is not necessary, as the mapping is induced from the others. For the horizontal maps we have that μ^* and L are the usual pullback and the Leray direct image mapping as developed in [10]. The maps $\tilde{\mu}^*$ and L' are the generalizations of μ^* and L described in the earlier sections of this paper. The mapping I^* can be described as the dual of the fibre integral mapping

$$I:{}'\mathscr{A}^{5,0}(F)\to{}'\mathscr{A}^{3,1}(P)$$

with suitable bundle coefficients, and this will be compatible with $\tilde{\mu}^*$ (see the Remark following Proposition 4.1). The mapping L'' has not been described intrinsically, and is induced from the others. An intrinsic description would involve a more delicate Leray theorem for tangential Dolbeault groups on the CR-fibration $F \xrightarrow{v} M$.

Now it's well known that not all hyperfunctions on M are boundary values from $\mathbb{M}^{\pm}$. This is true, in general, only if the hyperfunctions satisfy a differential equation whose characteristics are appropriately related to the boundary of $\mathbb{M}^{\pm}$ [3]. But we *do* have solutions of differential equations, and we can obtain such a characterization for the massless fields of positive helicity.

Let us restrict our attention in (7.1) to the kernels of the natural differential operators we have been dealing with. Recall that $\mathscr{F}' = \Omega^0_\mu(-n-2)$. Then we let

$$Z^1(\mathbb{F}^+, \mu^*\mathscr{F}) = \operatorname{Ker}[H^1(\mathbb{F}^+, \Omega^0_\mu(-n-2) \xrightarrow{d_\mu} H^1(\mathbb{F}^{\pm}, \Omega^1_\mu(-n-2)]$$

$$'Z^{0,1}_{\mathscr{B}}(F, \mu^*\mathscr{F}) = \operatorname{Ker}['H^{0,1}_{\mathscr{B}}(F, \Omega^0_\mu(-n-2)) \xrightarrow{d_\mu} {'H^{0,1}_{\mathscr{B}}}(F, \Omega^1_\mu(-n-2)]$$

etc., replacing H by Z in the various 6 function spaces in the right-hand side of diagram (7.1). Using (6.6) and the corresponding result in [10] we then obtain the commutative diagram

$$\begin{array}{ccccc}
\sum\limits_{\pm} H^1(\mathbb{P}^{\pm}, \mathscr{F}) & \xrightarrow{\mu^*} & \sum\limits_{\pm} Z^1(\mathbb{F}^{\pm}, \mathscr{F}') & \xrightarrow{L} & \sum\limits_{\pm} Z^0(\mathbb{M}^{\pm}, \mathscr{F}'') \\
\cong \downarrow b_P & & \downarrow b_F & & \downarrow b_M \\
'H^{0,1}_{\mathscr{B}}(P, \mathscr{F}) & \xrightarrow{I^*} & 'Z^{0,1}_{\mathscr{B}}(F, \mathscr{F}') & \xrightarrow{L''} & 'Z^{0,0}_{\mathscr{B}}(M, \mathscr{F}'') \\
\cong \downarrow s_P & & \cong \downarrow s_F & & \cong \downarrow s_M \\
H^2_P(\mathbb{P}, \mathscr{F}) & \xrightarrow{\tilde{\mu}^*} & Z^5_F(\mathbb{F}, \mathscr{F}') & \xrightarrow{L'} & Z^4_M(\mathbb{M}, \mathscr{F}'') .
\end{array} \tag{7.2}$$

Theorem 7.1. *All mappings in* (7.2) *are isomorphisms.*

Proof. This is now a trivial consequence of the fact that $\tilde{\mu}^*$ and L' are isomorphisms [Theorem 2.2 and (6.6)], the corresponding isomorphisms for μ^* and L (see [10]), and the vertical isomorphisms on the left. □

This theorem gives a fairly complete description of global hyperfunction solutions of the massless field equations of positive helicity on M.

We note that it was verified in [10] that the abstract differential operator derived from the spectral sequences corresponded to the classical zero-rest-mass equations (as described in [26, 35], for instance) as follows:

$$\begin{array}{ccc}
H^0(\mathbb{M}^+, v^1_*\Omega^0_\mu(-n-2)) & \xrightarrow{\nabla} & H^0(\mathbb{M}^+, v^1_*\Omega^1_\mu(-n-2)) \\
\wr\Vert & & \wr\Vert \\
H^0(\mathbb{M}^+, \mathcal{O}_{(A'\ldots D')}[-1]') & \xrightarrow{\nabla} & H^0(\mathbb{M}^+, \mathcal{O}_{B(C'\ldots D')}[-2]') \\
\Cup & & \Cup \\
\varphi_{A'\ldots D'} & \longmapsto & \nabla^{A'}_B \varphi_{A'B'\ldots D'}
\end{array} \tag{7.3}$$

where the spinor sheaves over $\mathbb{M}^+$ correspond to the classical spinor fields on $\mathbb{M}^+$. To differentiate a hyperfunction f means to differentiate the holomorphic functions whose boundary values determine f, and thus the differential operators which define $'Z^{0,0}_{\mathscr{B}}(M, \mathscr{F}'')$ and $Z^4_M(\mathbb{M}, \mathscr{F}'')$, which are compatible with the operator ∇ in (7.3) above, must correspond to the classical differential operators, when we restrict attention to affine Minkowski space $M_0 \subset M$, which has a coordinate system compatible with the coordinates on $\mathbb{M}^+$ and $\mathbb{M}^-$ (the coordinate chart $\mathbb{M}^I$ in [10]). This justifies our claims made at various times in the paper that $\mathscr{L}'_{\mathscr{A},n}$, $\mathscr{L}'_{\mathscr{B},n}$, etc., are solutions of the massless field equations of positive helicity.

We have the following corollary to Theorem 7.1. We first recall the concept of positive and negative frequency. A solution f of a field equation (such as the massless field equations) on affine Minkowski space M is said to be of *positive frequency* if f is the boundary value (in the hyperfunction sense, for instance) of a holomorphic solution to the same equation in $\mathbb{M}^+$. A solution is said to have *negative frequency* if f is the boundary value of a holomorphic solution defined on $\mathbb{M}^-$ [8, 26]. We can equally well speak of solutions to conformally invariant equations on M being of positive or negative frequency, provided the differential equations on M have holomorphic extensions to $\mathbb{M}^+$ and $\mathbb{M}^-$, which is certainly the case for the massless field equations. We thus have the following corollary to Theorem 7.1.

Corollary 7.2. *Any hyperfunction solution of the positive helicity massless field equations on M is the sum of solutions of positive and negative frequency.*

Proof. This is a direct consequence of the isomorphism b_M in 7.2. □

The formalism described in this paper goes through also for helicity 0, just as in [10], where the spectral sequence (6.1) degenerates at second order and the induced differential operator becomes the wave operator acting on scalar densities. With this modification, the results are otherwise the same, and we won't write it out explicitly. It has been shown (see [11]) that all distribution solutions of the zero-rest-mass field equations on affine Minkowski space have hyperfunction extensions to compactified Minkowski space which satisfy the corresponding field equations. Thus such solutions can always be represented as the Penrose transform of holomorphic data (or boundary values of such).

We have not attempted at this time to carry over the developments in [10] concerning left-hand fields, potentials, background coupled fields, etc. to this hyperfunction context, although it's clear that much of the development will work. There are some points however which would need clarification, as we used the negativity of the coefficient bundle to get the precise isomorphism we obtained. The obstruction, in general [e.g., cokernel $\tilde{\mu}^* \cong H^1_P(\mathbb{P}, \mathscr{F})$] would have to be incorporated into the potential theory or gauge freedom.

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Note added in proof. Toby Baily has recently communicated a proof of the conjecture made in the Introduction that hyperfunction massless fields on affine Minkowski space extend as hyperfunction solutions of the same field equations to compactified Minkowski space.