Commun. Math. Phys. 78, 409–428 (1981)

Communications in
Mathematical
Physics

The Group with Grassmann Structure $UOSP(1.2)$

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Abstract. The finite-dimensional representations of the Lie superalgebra $osp(1.2)$ and the group with Grassmann structure $OSP(1.2)$ have been studied. The explicit expression of the projection operator of the superalgebra $osp(1.2)$ has been found. The operator permits an arbitrary finite-dimensional representation to be expanded in the components multiple to the irreducible ones. The Clebsch-Gordan coefficients for the tensor product of two arbitrary irreducible representations have been obtained. The matrix elements of the irreducible representations of the group $UOSP(1.2)$ [the analoque of the compact form of the group $OSP(1.2)$] are studied. The explicit form of these matrix elements, the differential equations satisfied by them, and the integral of their product have been found.

1. Introduction

The Lie superalgebras and the Lie groups with Grassmann structure[1] have been extensively used since recently in physics. These objects appeared first in the problems relevant to the secondary quantization of the fermion systems [5], then in the dual model, and finally in the supergravity and the supersymmetric field theory (see the review in [6]). The natural problem arises, therefore, to develop a formalism of the theory of representation of the Lie superalgebras and groups with Grassmann structure up to the extent as was achieved for some of the semisimple Lie groups [14].

The present work studies in detail the finite-dimensional representations of the Lie superalgebra $osp(1.2)$ and the generated group with Grassmann structure $OSP(1.2)$. The representations of the Lie superalgebra $osp(1.2)$ were studied earlier in [8, 10, 11].

In the first part of the present work, the projection operator method developed earlier for the usual semisimple Lie algebras [1, 2] is applied to the superalgebra

1 Other names for these objects may be found elsewhere, namely supergroups or graded groups. The terms used here seems to us to reflect better the essence of these mathematical objects

0010-3616/81/0078/0409/$04.00

$osp(1.2)$. This technique has been used to obtain the explicit expressions of the Clebsch-Gordan coefficients for the tensor product of the arbitrary irreducible representations of the superalgebra $osp(1.2)$.

Studied in the second part are the matrix elements of the irreducible representations of the group $UOSP(1.2)$, i.e. the analogue of the compact form of the group $OSP(1.2)$. We shall find the explicit form of these matrix elements, the differential equations satisfied by them, and the integrals of their product.

2. The Basis of Finite-Dimensional Representation with the Highest Weight of the Superalgebra *osp*(1.2)

Let some of the known definitions be reminded [7]. A complex (or real) linear space V is called Z_2-graded if it is presented in the form of the direct sum of two subspaces, i.e. $V = V_0 \oplus V_1$. The elements V_0 are called even, and those of V_1 odd. The elements that are either even or odd are called homogeneous. There exists a parity function α defined on homogeneous elements by the formula:

$$\alpha(x) = \begin{cases} 0, & \text{if} \quad x \in V_0 \\ 1, & \text{if} \quad x \in V_1 . \end{cases} \tag{2.1}$$

The Lie superalgebra (or, which is the same, the Z_2-graded Lie algebra) is the Z_2-graded space $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ with bilinear operation (called commutator) satisfying the following axioms

$$\begin{gathered} [x, y] = (-1)^{\alpha(x)\alpha(y)+1}[y, x], \\ (-1)^{\alpha(x)\alpha(z)}[x, [y, z]] + (-1)^{\alpha(z)\alpha(y)}[z, [x, y]] + (-1)^{\alpha(y)\alpha(x)}[y, [z, x]] = 0 . \end{gathered} \tag{2.2}$$

for all the homogeneous elements x, y, z. The commutator $[x, y]$ will also be designated $[x, y]_-$ if $\alpha(x) = 0$ or $\alpha(y) = 0$ and $[x, y]_+$ if $\alpha(x) = \alpha(y) = 1$.

A representation of a superalgebra $\mathfrak{A}$ in a finite-dimensional graded vector space $V = V_0 \oplus V_1$ is the realization of the algebra $\mathfrak{A}$ by the operators T_x in V subject to the condition: if $x \in \mathfrak{A}_0$ then $T_x V_0 \subset V_0$, $T_x V_1 \subset V_1$ and if $x \in \mathfrak{A}_1$ then $T_x V_0 \subset V_1$, $T_x V_1 \subset V_0$. We assume that there exists on V such a nondegenerate bilinear Hermitian form (denoted by brackets $\langle \mid \rangle$) that V_0 and V_1 are orthogonal with respect to this form, i.e.,

$$\langle V_0 | V_1 \rangle = 0 . \tag{2.3}$$

The elements $L_\pm$, L_0, $R_\pm$ satisfying the conditions

$$[L_0, L_\pm]_- = \pm L_\pm , \qquad [L_+, L_-]_- = 2L_0 , \tag{2.4a}$$

$$[L_0, R_\pm]_- = \pm \tfrac{1}{2} R_\pm , \qquad [L_\mp, R_\pm]_- = R_\mp , \qquad [L_\pm, R_\pm]_- = 0 , \tag{2.4b}$$

$$[R_\pm, R_\pm]_+ = \pm \tfrac{1}{2} L_\pm , \qquad [R_+, R_-]_+ = -\tfrac{1}{2} L_0 \tag{2.4c}$$

form the Cartan-Weyl basis of the superalgebra $osp(1.2)$ [8]. The elements $L_\pm$, L_0 form the basis in the even subspace $osp_0(1.2)$[2], while $R_\pm$ form the basis in the odd subspace $osp_1(1.2)$:

$$osp(1.2) = osp_0(1.2) \oplus osp_1(1.2) . \tag{2.5}$$

2 $osp_0(1.2)$ is the simple Lie algebra A_1

The operators of the representation of the superalgebra $osp(1.2)$ which correspond to the basis elements $L_{\pm}$, L_0, $R_{\pm}$, will be denoted with the same letters. These operators satisfy the conditions (2.4a, b, c), where $[x,y]_-$ means the usual commutator, and $[x,y]_+$ anticommutator.

Let $V(J)$ be the space of the finite-dimensional representation with the highest weight J of the superalgebra $osp(1.2)$. The basis in the space $V(J)$ is constructed in the reduction $osp(1.2)\supset A_1=osp_0(1.2)$. The highest vector will be denoted by the symbol $|J\lambda\rangle$. The following equalities are valid for this vector

$$L_0|J\lambda\rangle=J|J\lambda\rangle\,, \tag{2.6a}$$

$$L_+|J\lambda\rangle=R_+|J\lambda\rangle=0\,, \tag{2.6b}$$

where λ means the parity of the highest vector $|J\lambda\rangle$. It follows from (2.6b) that the highest vector exhibits a definite parity, i.e. it belongs to either even $V_0(J)$ or odd subspace $V_1(J)$.

The basis in the space $V(J)$ will be constructed using the projection operator of the algebra A_1. This operator is of the form [2, 12]

$$P=\sum_{r=0}^{\infty} C_r(L_0)L_-^r L_+^r\,, \tag{2.7a}$$

$$C_r(L_0)=\frac{(-1)^r\Gamma(2L_0+2)}{r!\,\Gamma(2L_0+r+2)} \tag{2.7b}$$

where $\Gamma(\ )$ is the operator gamma-function. The main properties of the operator P are

$$[L_0,P]=0\,,\qquad P^2=P\,, \tag{2.8a}$$

$$L_+P=PL_-=0\,. \tag{2.8b}$$

Obviously, if the operator P acts on the vector with weight $M=L$ (with respect to the operator L_0), it will cut off a component of the vector which is highest relative to the algebra A_1. It will be assumed below that the operator P acts on the vector with weight $M=L$. This fact will be reflected by denoting the operator as P^L.

Let the spectral composition of the irreducible representations of the superalgebra A_1 be found in the representation J. In virtue of cyclicity of the representation J, any vector $|J;LM=L\rangle$ highest relative to the subalgebra A_1 may be presented as

$$|J;LM=L\rangle=\sum_{a,k} C_{a,k}L_-^a R_-^k|J\lambda\rangle \tag{2.9}$$

where k, in virtue of (2.4c), may assume only the values 0, 1, and $L=J-a-\frac{1}{2}k$. Since L is known [14] to be non-negative integer or semi-integer, then J may also assume just the same values. After the action of the operator P^L on both parts of the equality (2.9) and considering the property (2.8b) of the operator P^L, we get

$$\begin{aligned}&|J;LM=L\lambda+k\rangle=N(L)P^LR_-^k|J\lambda\rangle=N(L)R_-^k|J\lambda\rangle\,,\\ &k=0,1\,;\qquad L=J-\tfrac{1}{2}k\,,\end{aligned} \tag{2.10}$$

where $N(L)$ is a certain (still indefinite) factor depending on the normalization conditions. It can readily be seen that the vectors (2.10) at $J \geqq 1/2$ differ from zero and exhibit a definite parity $\lambda + k$.

Thus, we have obtained that the finite-dimensional representation with the highest weight J of the superalgebra $osp(1.2)$, when being narrowed to the subalgebra A_1, is expanded into the direct sum of two IR $L=J$ and $L=J-1/2$. The two representations exhibit multiplicity 1. The representation with $J=0$ is identical; in this case also $L=0$. It follows from this result that any finite-dimensional representation with highest weight J is an irreducible representation (IR). The above conclusions coincide with the results of [8].

The actions of the lowering operators $F_-(M,L)$ of the algebra A_1

$$\begin{gathered}\mathscr{F}_-(M,L) = \left[\frac{(L+M)!}{(2L)!(L-M)!}\right]^{1/2} L_-^{L-M} \\ M = -L, -L+1, \ldots, L\end{gathered} \tag{2.11}$$

on the vectors (2.10) give the complete basis in the space $V(J)$:

$$\{|J; LM\lambda+k\rangle = F_-(M,L)|J\lambda\rangle\} \tag{2.12a}$$

where

$$\begin{gathered}F_-(M,L) = N(L)\mathscr{F}_-(M,L)R_-^k\,, \\ k=0,1\,; \quad L=J-\tfrac{1}{2}k\,; \quad M=-L, -L+1, \ldots, L\,.\end{gathered} \tag{2.12b}$$

It can readily be calculated that the dimension of the IR J is

$$\dim J = 4J+1\,. \tag{2.13}$$

Any IR of the superalgebra $osp(1.2)$ is known to be equivalent to the grade star representation [10, 11], i.e. there exists such Hermitian form $\langle\ |\ \rangle$ in the space $V(J)$ that, relative to it,

$$\begin{aligned}L_\pm^\ddagger &= L_\mp\,, & L_0^\ddagger &= L_0\,, \\ R_+^\ddagger &= (-1)^\varepsilon R_-\,, & R_-^\ddagger &= (-1)^{\varepsilon+1} R_+\,.\end{aligned} \tag{2.14}$$

The operation $\ddagger$ means the grade adjoint operation

$$\langle A^\ddagger f|\varphi\rangle = (-1)^{\alpha(A)\cdot\lambda}\langle f|A\varphi\rangle \tag{2.15}$$

for any homogeneous operator A of parity $\alpha(A)$ and for any homogeneous vectors $f, \varphi \in V(J)$ (λ is the parity of vector f). There exist two classes of the grade star IR's differing in the index $\varepsilon = 0, 1$. The two classes differ essentially in only the parity $\lambda = 0, 1$ ascribed to the highest vector $|J\lambda\rangle$. It can readily be shown that λ and ε are interrelated as

$$\lambda = \varepsilon + 1\,. \tag{2.16}$$

If IR J is grade star representation, then the basis (2.12a, b) is orthonormalized at

$$N(L) = \left[\frac{4^{2(J-L)}\Gamma(2L+1)}{\Gamma(2J+1)}\right]^{1/2}. \tag{2.17}$$

The explicit form of the basis vectors (2.12a, b), (2.17) makes it quite possible to obtain the actions of the operators $L_\pm$, L_0, $R_\pm$ on these vectors:

$$\begin{aligned} &L_0|J;LM\lambda+k\rangle = M|J;LM\lambda+k\rangle\,, \qquad k=2(J-L) \\ &L_\pm|J;LM\lambda+k\rangle = \sqrt{(L\mp M)(L\pm M+1)}|J;LM\pm1\lambda+k\rangle\,, \end{aligned} \tag{2.18a}$$

$$\left.\begin{aligned} R_\pm|J;JM\lambda\rangle &= \mp\tfrac{1}{2}\sqrt{J\mp M}|J;J-\tfrac{1}{2}M\pm\tfrac{1}{2}\lambda+1\rangle\,, \\ R_\pm|J;J-\tfrac{1}{2}M\lambda+1\rangle &= -\tfrac{1}{2}\sqrt{J\pm M+\tfrac{1}{2}}|J;JM\pm\tfrac{1}{2}\lambda\rangle \end{aligned}\right\} \tag{2.18b}$$

The expressions (2.18a, b) coincide with the results obtained in [8].

Use will be made below of the matrices of the operators $A_1=\dfrac{i}{2}(L_++L_-)$, $A_2=\frac{1}{2}(L_+-L_-)$, $A_3=-iL_0$ and $R_\pm$ for representation $J=1/2$. If the highest vector is considered to be odd ($\lambda=1$), these matrices prove to be of the form

$$A_1=\frac{i}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad A_2=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad A_3=\frac{i}{2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix} \tag{2.19a}$$

$$R_+=\frac{1}{2}\begin{pmatrix}0&-1&0\\0&0&0\\-1&0&0\end{pmatrix},\quad R_-=\frac{1}{2}\begin{pmatrix}0&0&1\\-1&0&0\\0&0&0\end{pmatrix}. \tag{2.19b}$$

3. The Projection Operator of the Superalgebra *osp*(1.2)

Let V be the space of the finite-dimensional representation of the superalgebra $osp(1.2)$. It will be set that

$$V_+=\{f\in V|R_+f=0\}\,. \tag{3.1}$$

Let the projection operator $V\to V_+$ be denoted $\mathfrak{P}$. Obviously, this operator has to satisfy the following requirements

$$[L_0,\mathfrak{P}]=0\,,\qquad R_+\mathfrak{P}=0\,, \tag{3.2a}$$

$$\mathfrak{P}^2=\mathfrak{P}\,,\qquad \mathfrak{P}f=f \quad \text{if} \quad f\in V_+\,. \tag{3.2b}$$

We shall find the explicit expression of $\mathfrak{P}$ which will be sought to be of the form

$$\mathfrak{P}=\sum_r C_r(L_0)R_-^rR_+^r\,, \tag{3.3}$$

where $C_r(L_0)$ are the unknown factors depending on the operator L_0. In such a form, the operator $\mathfrak{P}$ satisfies the first of the equalities (3.2a). The factors $C_r(L_0)$ will be found from the remainder conditions (3.2a, b). By substituting the operator (3.3) in the second equality of (3.2a), transposing the operator R_+ with the operators R_-^r, combining R_+ with R_+^r, and collecting the similar terms, we get

$$\begin{aligned} R_+\mathfrak{P}=&\sum_n R_-^{2n}R_+^{2n+1}[C_{2n}(L_0)-\tfrac{1}{4}(2L_0+n+1)C_{2n+1}(L_0)] \\ &+\sum_n R_-^{2n+1}R_+^{2n+2}[\tfrac{1}{4}(n+1)C_{2n+2}(L_0)-C_{2n+1}(L_0)]=0\,. \end{aligned} \tag{3.4}$$

Since the elements $R_-^r \cdot R_+^{r+1}$ at various r are linearly independent, we obtain that the equality (3.4) is valid if

$$\begin{cases} \frac{1}{4}(n+1)C_{2n+2}(L_0) - C_{2n+1}(L_0) = 0, \\ \frac{1}{4}(2L_0 + n + 1)C_{2n+1}(L_0) - C_{2n}(L_0) = 0, \qquad n = 0, 1, 2, \dots . \end{cases} \tag{3.5}$$

The solution for this set of the operator equations at the boundary condition $C_0(L_0) = 1$, which follows from the second equality of (3.2b), is

$$C_{2n}(L_0) = \frac{4^{2n}\Gamma(2L_0+1)}{n!\,\Gamma(2L_0+n+1)}, \tag{3.6a}$$

$$C_{2n+1}(L_0) = \frac{4^{2n+1}\Gamma(2L_0+1)}{n!\,\Gamma(2L_0+n+2)}, \tag{3.6b}$$

where $\Gamma(\ )$ is the operator gamma-function.

Thus, we have obtained that the projection operator $\mathfrak{P}$ of the superalgebra $osp(1.2)$ may be presented in the form (3.3), where the factors $C_r(L_0)$ are given by the formulae (3.6a, b). It can easily be verified that the obtained operator $\mathfrak{P}$ will also satisfy the following equality:

$$\mathfrak{P}L_- = \mathfrak{P}R_- = 0. \tag{3.7}$$

The Clebsch-Gordan coefficients will be conveniently calculated using another form of the operator $\mathfrak{P}$:

$$\mathfrak{P} = RP = PR, \tag{3.8a}$$

where

$$R = 1 + \frac{4\Gamma(2L_0+1)}{\Gamma(2L_0+2)} R_- R_+ . \tag{3.8b}$$

P is the projection operator of the algebra A_1 given by the formulae (2.7a, b). The form (3.8a, b) can be obtained by substituting in (3.3), (3.6a, b) $R_+^{2n+k} = (\frac{1}{4}L_+)^n R_+^k$, $R_-^{2n+k} = (-\frac{1}{4}L_-)^n R_-^k$ $(k = 0, 1)$ and by transposing the operators L_+^n and L_-^n to the right or left.

If the operator $\mathfrak{P}$ acts in the subspace

$$V_{M=J} = \{f \in V | L_0 f = Jf\} \tag{3.9}$$

it will be denoted as $\mathfrak{P}^J$.

4. Clebsch-Gordan Coefficients

Let $V(J_1) = V_0(J_1) \oplus V_1(J_1)$ and, corrispondingly, $V(J_2) = V_0(J_2) \oplus V_1(J_2)$ be the spaces of two IR's J_1 and, correspondingly, J_2 with the bases

$$|J_1; L_1 M_1 \lambda_1 + k_1\rangle = F_-(M_1 L_1)|J_1 \lambda_1\rangle, \tag{4.1a}$$

$$L_1 = J_1, J_1 - \tfrac{1}{2}; \quad M = -L_1, -L_1 + 1, \dots, L_1, \quad k_1 = 2(J_1 - L_1) \tag{4.1b}$$

and, correspondingly,

$$|J_2\,;L_2M_2\lambda_2+k_2\rangle=F_-(M_2L_2)|J_2\lambda_2\rangle \tag{4.2a}$$

$$L_2=J_2,J_2-\tfrac{1}{2}\,;\qquad M_2=-L_2,-L_2+1,\dots,L_2\,;\qquad k_2=2(J_2-L_2)\,. \tag{4.2b}$$

The space of representation of the tensor product $J_1\otimes J_2$ will be denoted as $V(J_1\otimes J_2)$. The vectors

$$|J_1\,;L_1M_1\lambda_1+k_1\rangle|J_2\,;L_2M_2\lambda_2+k_2\rangle$$

at all the admissible values of (4.1b), (4.2b) form the basis in the space $V(J_1\otimes J_2)$.

The operators $L_\pm$, L_0, and $R_\pm$ of representation $J_1\otimes J_2$ are composed of the operators

$$\begin{aligned}L_\pm&=L_\pm(1)+L_\pm(2)\,,\\ L_0&=L_0(1)+L_0(2)\,,\\ R_\pm&=R_\pm(1)+R_\pm(2)\,.\end{aligned} \tag{4.3}$$

It will be noted that

$$\begin{aligned}&[L_m(1),L_n(2)]_-=0\,,\qquad (m,n=+,0,-)\,,\\ &[R_p(1),R_q(2)]_+=0\,,\qquad (p,q=+,-)\,.\end{aligned} \tag{4.4}$$

At a fixed value of i, the operators $L_\pm(i)$, $L_0(i)$, $R_\pm(i)$ acts on the vectors of the subspace $V(J_i)$. It should be borne in mind that

$$A(2)|\lambda_1+k_1\rangle|\lambda_2+k_2\rangle=(-1)^{\alpha\cdot(\lambda_1+k_1)}|\lambda_1+k_1\rangle\cdot(A(2)|\lambda_2+k_2\rangle)$$

where, α is the parity of the operator $A(2)$.

Consider the vector

$$\mathfrak{P}^J F_-(M_1L_1)|J_1\lambda_1\rangle F_-(M_2L_2)|J_2\lambda_2\rangle \tag{4.5}$$

where $J=M_1+M_2$. After substituting here the expressions $L_-(1)=L_--L_-(2)$, $R_-(1)=R_--R_-(2)$ in the lowering operator $F_-(M_1L_1)$ and considering the equalities (3.7), we get

$$\begin{aligned}&\mathfrak{P}^J|J_1\,;L_1M_1\lambda_1+k_1\rangle|J_2\,;L_2M_2\lambda_2+k_2\rangle\\ &\quad\simeq\mathfrak{P}^J|J_1\lambda_1\rangle|J_2\,;L_2'J-J_1\lambda_2+k_2'\rangle\end{aligned} \tag{4.6a}$$

which immediately gives the admissible values of J

$$J_1-J_2\leqq J\leqq J_1+J_2\,. \tag{4.6b}$$

Similarly, it can be obtained that

$$\begin{aligned}&\mathfrak{P}^J|J_1\,;L_1M_1\lambda_1+k_1\rangle|J_2\,;L_2M_2\lambda_2+k_2\rangle\\ &\quad\simeq\mathfrak{P}^J|J_1\,;L_1'J-J_2\lambda_1+k_1'\rangle|J_2\lambda_2\rangle\end{aligned} \tag{4.7a}$$

and

$$J_2-J_1\leqq J\leqq J_1+J_2\,. \tag{4.7b}$$

By combining (4.6b) and (4.7b), we get

$$|J_1-J_2|\leqq J\leqq J_1+J_2. \tag{4.8}$$

After calculating the total number of the vectors of the form

$$F_-(M,L)\mathfrak{P}^J|J_1\lambda_1\rangle|J_2;L_2J-J_1\lambda_2+k_2\rangle \tag{4.9}$$

at all the admissible values of J, L, M we obtain

$$\sum(4J+1)=(4J_1+1)(4J_2+1). \tag{4.10}$$

The summation here has been made over all integer and semi-integer values of J satisfying the inequalities (4.8). It can be seen that the number of the vectors (4.9) is exactly the dimension of the representation $J_1\otimes J_2$, i.e. the vectors (4.9) form the basis in the space $V(J_1\otimes J_2)$.

Thus, we have obtained that the space of representation $V(J_1\otimes J_2)$ can be expanded into the direct sum of the subspaces

$$V(J_1\otimes J_2)=\bigoplus_J V(J). \tag{4.11}$$

In each $V(J)$ the IR of weight J is operative and the weight J takes on all the integer and semi-integer values satisfying the inequalities (4.8). This conclusion coincides with the results obtained in [8].

The coefficients interrelating two bases

$$\begin{aligned}&|J;LM\mu\rangle\\&\quad=\sum(J_1L_1M_1\lambda_1+k_1,J_2L_2M_2\lambda_2+k_2|JLM\mu)\\&\quad\quad\cdot|J_2;L_1M_1\lambda_1+k_1\rangle|J_2;L_2M_2\lambda_2+k_2\rangle\end{aligned} \tag{4.12}$$

are the Clebsch-Gordan coefficients (CGC) of the superalgebra $osp(1.2)$. Here, μ denotes the parity of the vector $|J;LM\mu\rangle$ which is determined by the parity of the right part: $\mu=\lambda_1+\lambda_2+k_1+k_2$. [It will be shown below that the right part of (4.12) display a definite parity.]

The expressions of CGC are determined by a particular bilinear Hermitian form set in the space $V(J_1\otimes J_2)$. We shall examine two cases.

(A) The IR's J_1 and J_2 are the grade star representations of the same class $\varepsilon=0,1$. In the space $V(J_1\otimes J_2)$ the form

$$\langle f_1\otimes f_2|\varphi_1\otimes\varphi_2\rangle=(-1)^{\alpha(f_2)\alpha(\varphi_1)}\langle f_1|\varphi_1\rangle\langle f_2|\varphi_2\rangle \tag{4.13}$$

is set, where $f_1,\varphi_1\in V(J_1)$, $f_2,\varphi_2\in V(J_2)$ are the homogeneous elements (odd or even); $\alpha(\ )$ is function of parity on the spaces $V(J_1)$ and $V(J_2)$; $\langle f|\varphi\rangle$ is the Hermitian form on these spaces. It can readily be verified that the representation $J_1\otimes J_2$ with respect to the form (4.13) is the grade star representation of the class ε.

The expression (4.9) can be used to find the formula for calculating CGC:

$$\begin{aligned}&(J_1L_1M_1\lambda_1+k_1,J_2L_2M_2\lambda_2+k_2|JLM\mu)\\&\quad=N(L_2,\lambda_i,k_i)\langle J_1;L_1M_1\lambda_1+k_1|\langle J_2;L_2M_2\lambda_2+k_2|\\&\quad\quad\cdot F_-(M,L)\mathfrak{P}^J|J_1\lambda_1\rangle|J_2;L_2J-J_1\lambda_2+k_2\rangle,\end{aligned} \tag{4.14a}$$

where

$$N(L_2\lambda_i k_i)=(-1)^{(\lambda_2+k_2)(\lambda_1+k_1)}[|\langle J_1\lambda_1|\langle J_2;L_2J-J_1\lambda_2+k_2| \cdot \mathfrak{P}^J|J_1\lambda_1\rangle|J_2;L_2J-J_1\lambda_2+k_2\rangle|]^{-1/2}. \tag{4.14b}$$

In the given case, it should be set that $\lambda_1=\lambda_2=\lambda$ in (4.14a, b). The calculations made using these formulas have shown that CGC of the superalgebra $osp(1.2)$ can be factorized into two factors:

$$(J_1L_1M_1\lambda_1+k_1, J_2L_2M_2\lambda_2+k_2|JLM\mu) = \left(\begin{array}{cc||c} J_1 & J_2 & J \\ L_1\lambda+k_1 & L_2\lambda+k_2 & L\mu \end{array}\right)\cdot(L_1M_1L_2M_2|LM). \tag{4.15}$$

The first factor ($\|$) is independent of the projections M_1 and M_2 and will be called the scalar factor. The second factor $(L_1M_1L_2M_2\|LM)$ is the known CGC of the group SU(2). The explicit expressions of the scalar factors are presented in Table A.

Table A. Scalar factors $\left(\begin{array}{cc||c} J_2 & J_2 & J \\ L_1\lambda+k_2 & L_2\lambda+k_2 & L\mu \end{array}\right)$

	(J_1+J_2+J) integer		$(J_1=J_2+J)$ semi-integer	
$L_1\lambda+k_1\ \ L_2\lambda+k_2$ \ $L\mu$	$L=J\quad \mu=0$	$L=J-\frac{1}{2}\quad \mu=1$	$L=J\quad \mu=1$	$L=J-\frac{1}{2}\quad \mu=0$
$J_1\lambda,\ J_2\lambda$	$\left[\frac{J_1+J_2+J+1}{2J+1}\right]^{1/2}$	0	0	$(-1)^{\lambda+1}\left[\frac{J_1+J_2-J+\frac{1}{2}}{2J}\right]^{1/2}$
$J_2\lambda,\ J_2-\frac{1}{2}\lambda+1$	0	$(-1)^{\lambda}\left[\frac{-J_1+J_2+J}{2J}\right]^{1/2}$	$\left[\frac{J_1-J_2+J+\frac{1}{2}}{2J+1}\right]^{1/2}$	0
$J_1-\frac{1}{2}\lambda+1,\ J_2\lambda$	0	$\left[\frac{J_1-J_2+J}{2J}\right]^{1/2}$	$(-1)^{\lambda+1}\left[\frac{-J_1+J_2+J+\frac{1}{2}}{2J+1}\right]^{1/2}$	0
$J_1-\frac{1}{2}\lambda+1,\ J_2-\frac{1}{2}\lambda+1$	$(-1)^{\lambda+1}\left[\frac{J_1+J_2-J}{2J+1}\right]$	0	0	$\left[\frac{J_1+J_2+J+\frac{1}{2}}{2J}\right]^{1/2}$

It can readily be verified that the obtained scalar factors for the form (4.13) are orthogonal and normalized, but only to ± 1:

$$\sum_{L_1L_2}(-1)^{(\lambda+k_1)(\lambda+k_2)}\left(\begin{array}{cc||c} J_1 & J_2 & J \\ L_1\lambda+k_1 & L_2\lambda+k_2 & Lk_1+k_2 \end{array}\right) \cdot\left(\begin{array}{cc||c} J_1 & J_2 & J' \\ L_1\lambda+k_1 & L_2\lambda+k_2 & Lk_1+k_2 \end{array}\right)=\pm 1\delta_{JJ'} \tag{4.16}$$

The expressions of the scalar factors (Table A) coincide with the results obtained in [11] by the other method.

Consider now the case

(B) In the space $V(J_1\otimes J_2)$, the form

$$\langle f_1\otimes f_2|\varphi_1\otimes\varphi_2\rangle=\langle f_1|f_2\rangle\langle\varphi_1|\varphi_2\rangle \tag{4.17}$$

is determined, where $f_1, \varphi_1 \in V(J_1)$ and $f_2, \varphi_2 \in V(J_2)$. If $V(J_1)$ and $V(J_2)$ are the Hilbert spaces, then $V(J_1 \otimes J_2)$ is also the Hilbert space. In this case, CGC are calculated the formula (4.14a) where $N(L_2\lambda_i k_i)$ should be replaced by the expression

$$\begin{aligned} N(L_2\lambda_i k_i) = [\langle J_1\lambda_1| \langle J_2; L_2J - J_1\lambda_2 + k_2| \\ \cdot (F_-(ML)\mathfrak{P}^J)^+ F_-(ML)\mathfrak{P}^J |J_1\lambda_1\rangle |J_2; L_2J - J_1\lambda_2 + k_2\rangle]^{-1/2}. \end{aligned} \tag{4.18}$$

Here $(+)$ denotes the Hermitian conjugation. The calculated scalar factors

$$\left(\begin{matrix} J_1 & J_2 \\ L_1\lambda_1 + k_1 & L_2\lambda_2 + k_2 \end{matrix} \middle\| \begin{matrix} J \\ L\mu \end{matrix}\right)$$

are presented in Table B.

Table B. Scalar factors $\left(\begin{matrix} J_2 & J_2 \\ L_1\lambda_1 + k_1 & L_2\lambda_2 + k_2 \end{matrix} \middle\| \begin{matrix} J \\ L\,\mu \end{matrix}\right)$

	(J_1+J_2+J) integer		(J_1+J_2+J) semi-integer	
$L_2\lambda_1+k_1, L_2\lambda_2+k_2$	$L=J, \mu=\lambda_1+\lambda_2$	$L=J-\frac{1}{2}, \mu=\lambda_1+\lambda_2+1$	$I=J\ \mu=\lambda_1+\lambda_2+1$	$L=J-\frac{1}{2}\ \mu=\lambda_1+\lambda_2$
$J_1\lambda_1, J_2\lambda_2$	$\left[\frac{J_1+J_2+J+1}{2J_1+2J_2+1}\right]^{1/2}$	0		$(-1)^{\lambda_1+1}\left[\frac{J_1+J_2-J+\frac{1}{2}}{2J_1+2J_2+1}\right]^{1/2}$
$J_1\lambda_1, J_2-\frac{1}{2}\lambda_2+1$	0	$(-1)^{\lambda_1}\left[\frac{-J_1+J_2+J}{2J}\right]^{1/2}$	$\left[\frac{J_1-J_2+J+\frac{1}{2}}{2J+1}\right]^{1/2}$	0
$J_1-\frac{1}{2}\lambda_1+1, J_2\lambda_2$	0	$\left[\frac{J_1-J_2+J}{2J}\right]^{1/2}$	$(-1)^{\lambda_1+1}\left[\frac{-J_1+J_2+J+\frac{1}{2}}{2J+1}\right]^{1/2}$	0
$J_1-\frac{1}{2}\lambda_1+1,$ $J_2-\frac{1}{2}\lambda_2+1$	$(-1)^{\lambda_1+1}\left[\frac{J_1+J_2-J}{2J_1+2J_2+1}\right]$	0	0	$\left[\frac{J_1+J_2+J+\frac{1}{2}}{2J_1+2J_2+1}\right]^{1/2}$

The normalization condition for the scalar factor is of the form

$$\sum_{L_1, L_2} \left(\begin{matrix} J_1 & J_2 \\ L_1\lambda_1 + k_1 & L_2\lambda_2 + k_2 \end{matrix} \middle\| \begin{matrix} J \\ L\mu \end{matrix}\right) \left(\begin{matrix} J_1 & J_2 \\ L_1\lambda_1 + k_1 & L_2\lambda_2 + k_2 \end{matrix} \middle\| \begin{matrix} J \\ L\mu \end{matrix}\right) = 1. \tag{4.19}$$

It can be seen from Table B that the scalar factor (hence CGC) are not orthogonal in the case of the form (4.17).

5. The Groups *OSP*(1.2) and *UOSP*(1.2)

Let $\mathfrak{G}$ be the complex Grassmann algebra with N generators $\xi_1, \xi_2, \xi_3, \ldots$ $(\xi_i\xi_j + \xi_j\xi_i = 0)$. N may be both finite and infinite. The element

$$\eta = \sum_{m \geq 0} \sum_{i_1 < i_2 < \ldots < i_m} \eta_{i_1 i_2 \ldots i_m} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m} \tag{5.1}$$

is called even if the addends with even m in (5.1) differ from zero and odd if the addends with odd m differ from zero. The set of even elements will be denoted $\mathfrak{G}_0$, and the set of odd elements $\mathfrak{G}_1$ $(\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1)$. Both odd and even elements are called homogeneous. The parity function $\alpha(\eta)$ is defined on homogeneous elements by the formula:

$$\alpha(\eta)=\begin{cases}0, & \text{if} \quad \eta\in\mathfrak{G}_0\\ 1, & \text{if} \quad \eta\in\mathfrak{G}_1\,.\end{cases} \tag{5.2}$$

Let the Grassmann envelope of Lie superalgebra $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$ be denoted $\mathfrak{A}(\mathfrak{G})=\mathfrak{A}_0(\mathfrak{G}_0)\oplus\mathfrak{A}_1(\mathfrak{G}_1)$. By definition, $\mathfrak{A}(\mathfrak{G})$ consists of formal linear combinations $\sum\eta_i e_i$, where $\{e_i\}$ is a basis of $\mathfrak{A}$, $\eta_i\in\mathfrak{G}$; the elements e_i and η_i at each fixed i are of the same parity. The commutator of the arbitrary elements $X=\sum_i \eta_i e_i$, $Y=\sum_j \eta'_j e_j$ is defined by formula

$$[X,Y]=\sum_{i,j}\eta_i\eta'_j[e_i,e_j] \tag{5.3}$$

Here $[e_i,e_j]$ means the commutator in the Lie superalgebra $\mathfrak{A}$. It can readily be verified that $\mathfrak{A}(\mathfrak{G})$ is the usual Lie algebra.

Return now to the complex Lie superalgebra $osp(1.2)$. Let $osp(1.2)$ be realized by the third-order matrices with basis (2.19a, b). Obviously, the elements of the algebra $osp(1.2;\mathfrak{G})=osp_0(1.2;\mathfrak{G}_0)\oplus osp_1(1.2;\mathfrak{G}_1)$ will be the matrices of the form

$$X=\begin{pmatrix} a_{11} & -\eta_1 & \eta_2\\ -\eta_2 & a_{22} & a_{23}\\ -\eta_1 & a_{32} & a_{33}\end{pmatrix} \tag{5.4}$$

at $a_{11}=0$, $a_{22}=-a_{33}$. The elements a_{ij} are taken from $\mathfrak{G}_0$ and η_1, η_2 from $\mathfrak{G}_1$.

It will be required further that a semilinear mapping $\eta\to\eta^\square$ be set in the Grassmann algebra $\mathfrak{G}$, with the properties

$$\begin{aligned} &\mathfrak{G}_0^\square=\mathfrak{G}_0\,, \qquad \mathfrak{G}_1^\square=\mathfrak{G}_1\,,\\ &(\nu\eta)^\square=\nu^\square\eta^\square\,, \qquad (c\cdot\eta)^\square=\bar{c}\cdot\eta^\square\,,\\ &(\eta^\square)^\square=(-1)^{\alpha(\eta)}\eta \end{aligned} \tag{5.5}$$

for homogeneous ν, η and complex c. The operation $(\square)$ of such kind was treated in [9] and exists explicitly at an even number of the generators N of the algebra $\mathfrak{G}$ or at infinite N.

Let the grade adjoint operation $(\ddagger)$ be introduced in the Lie superalgebra $osp(1.2)$ [10, 11]. The operation $(\ddagger)$ satisfies the following conditions

$$x^\ddagger\in osp_i(1.2) \quad \text{if} \quad x\in osp_i(1.2)\,, \qquad i=\{0,1\}\,, \tag{5.6a}$$

$$(c_1x+c_2y)^\ddagger=\bar{c}_1x^\ddagger+\bar{c}_2y^\ddagger\,, \tag{5.6b}$$

$$[x,y]^\ddagger=(-1)^{\alpha(x)\alpha(y)}[y^\ddagger,x^\ddagger]\,, \tag{5.6c}$$

$$(x^\ddagger)^\ddagger=(-1)^{\alpha(x)}x \tag{5.6d}$$

for all homogeneous elements x, y and complex c_1, c_2. It is known that there exist two such operations:

$$L_{\pm}^{\ddagger}=L_{\mp}, \quad L_0^{\ddagger}=L_0, \quad R_{-}^{\ddagger}=(-1)^{\varepsilon+1}R_{+}, \quad R_{+}^{\ddagger}=(-1)^{\varepsilon}R_{-} \tag{5.7}$$

which differ in indices $\varepsilon=0,1$.

The operation $(\ddagger)$, in combination with the operation $(\square)$ in the Grassmann algebra $\mathfrak{G}$, induces the adjoint operation $(+)$ in the Lie algebra $osp(1.2;\mathfrak{G})$ through the formula

$$(\eta_1 x+\eta_2 y)^{+}=\eta_1^{\square}x^{\ddagger}+\eta_2^{\square}y^{\ddagger}. \tag{5.8}$$

It can readily be verified that

$$\begin{aligned}&(X^{+})^{+}=X,\\ &[X,Y]^{+}=[Y^{+},X^{+}], \qquad X,Y\in osp(1.2;\mathfrak{G}).\end{aligned} \tag{5.9}$$

If X is the third-order matrix of the form (5.4), then

$$X^{\square}=\begin{pmatrix} a_{11}^{\square} & \eta_2^{\square} & \eta_1^{\square}\\ -\eta_1^{\square} & a_{23}^{\square} & a_{32}^{\square}\\ \eta_2^{\square} & a_{23}^{\square} & a_{33}^{\square}\end{pmatrix} \tag{5.10}$$

for $\varepsilon=0$.

Let the "real" subalgebra

$$uosp(1.2;\mathfrak{G})\equiv\{H\in osp(1.2;\mathfrak{G})|H^{+}=-H\} \tag{5.11}$$

be singled out in the Lie algebra $osp(1.2;\mathfrak{G})$.

The arbitrary element H of this subalgebra is of the form

$$H=a_1A_1+a_2A_2+a_3A_3+\eta^{\square}R_{+}+\eta R_{-}, \tag{5.12a}$$

where

$$\begin{aligned}&A_1=\frac{i}{2}(L_{+}+L_{-}), \qquad A_2=\tfrac{1}{2}(L_{+}-L_{-}),\\ &A_3=-iL_0, \quad A_i^{\ddagger}=-A_i, \quad R_{+}^{\ddagger}=R_{-}, \quad R_{-}^{\ddagger}=-R_{+},\\ &a\in\mathfrak{G}_0, \quad a_i^{\square}=a_i, \quad \eta\in\mathfrak{G}_1.\end{aligned} \tag{5.12b}$$

The subalgebra $uosp(1.2;\mathfrak{G})$ is the analogue of the compact real form of the Lie algebra $osp(1.2;\mathfrak{G})$. It may be shown that the very $uosp(1.2;\mathfrak{G})$ is the Grassmann envelope of not a single Lie superalgebra.

The set of all nonsingular matrices of the form (5.4) (where the trace of matrice X may differ from zero) will be denoted $PL(1.2)$. This set forms the group with conventional operation of matrix multiplication.

The group $OSP(1.2)$ is determined to be the exponential mapping of the Grassmann envelope $osp(1.2;\mathfrak{G})$ of the superalgebra $osp(1.2)$ realized in the form of matrices (5.4) to the group $PL(1.2)$:

$$OSP(1.2)=\{\exp X|X\in osp(1.2;\mathfrak{G})\}. \tag{5.13}$$

The analogue of the compact form $UOSP(1.2)$ of the group $OSP(1.2)$ is given by the condition

$$UOSP(1.2)=\{\exp H|H\in uosp(1.2;\mathfrak{G})\}. \tag{5.14}$$

Considered further will be the group $UOSP(1.2)$. It follows from the general theory [3] of the Lie groups with Grassmann structure that the arbitrary element $g\in UOSP(1.2)$ may be presented in the form of the product of the elements from the one-parameter subgroups

$$g=u\xi \tag{5.15a}$$

$$u=\exp(a_1A_1)\exp(a_2A_2)\exp(a_3A_3), \quad (a_i^\square=a_i\in\mathfrak{G}), \tag{5.15b}$$

$$\xi=\exp(\eta^\square R_+ +\eta R_-), \quad (\eta\in\mathfrak{G}_1). \tag{5.15c}$$

Later on, we shall use the expansion

$$u=\exp(\varphi A_3)\exp(\theta A_1)\exp(\psi A_3), \tag{5.16}$$

where

$$\begin{aligned}
\theta&=\theta_0+\tilde{\theta}=\theta^\square, & 0&\leqq\theta_0\leqq\pi,\\
\varphi&=\varphi_0+\tilde{\varphi}=\varphi^\square, & 0&\leqq\varphi_0<2\pi,\\
\psi&=\psi_0+\tilde{\psi}=\psi^\square, & -2\pi&\leqq\psi_0<2\pi.
\end{aligned} \tag{5.17}$$

The parameters θ_0, φ_0, and ψ_0 are the conventional Euler angles for the group SU(2); $\tilde{\theta}$, $\tilde{\varphi}$, and $\tilde{\psi}$ are the nilpotent even variables of the Grassmann algebra $\mathfrak{G}$. It may be shown that, if the parameters a_1 and a_2 in (5.15) are nilpotent, then the expansion (5.16) is not always valid. However, the set of the elements, for which the expansion (5.16) is not valid, are of a real dimension smaller than the dimension of the entire group. Thus, disregarding this set, the arbitrary element $g\in UOSP(1.2)$ may be presented as

$$g=\exp(\varphi A_3)\exp(\theta A_2)\exp(\psi A_3)\exp(\eta^\square R_+ +\eta R_-). \tag{5.18}$$

Substituting here the explicit expressions (2.19a, b) of the matrices A_1, A_3, $R_\pm$, we get

$$g=\begin{pmatrix} 1+\frac{1}{4}\eta^\square\eta & -\frac{1}{2}\eta^\square & \frac{1}{2}\eta \\ g_{21} & (1-\frac{1}{8}\eta^\square\eta)u_{22} & (1-\frac{1}{8}\eta^\square\eta)u_{23} \\ g_{31} & (1-\frac{1}{8}\eta^\square\eta)u_{32} & (1-\frac{1}{8}\eta^\square\eta)u_{33} \end{pmatrix} \tag{5.19a}$$

where

$$\begin{aligned} g_{21}&=-\tfrac{1}{2}\eta u_{22}-\tfrac{1}{2}\eta^\square u_{23},\\ g_{31}&=-\tfrac{1}{2}\eta u_{32}-\tfrac{1}{2}\eta^\square u_{33} \end{aligned} \tag{5.19b}$$

and the matrix $\|u_{pq}\|$ is of the form [13]:

$$u=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta/2)\exp\left(\frac{i(\varphi+\psi)}{2}\right) & i\sin(\theta/2)\exp\left(\frac{i(\varphi-\psi)}{2}\right) \\ 0 & i\sin(\theta/2)\exp\left(\frac{i(\psi-\varphi)}{2}\right) & \cos(\theta/2)\exp\left(-\frac{i(\varphi+\psi)}{2}\right) \end{pmatrix}, \tag{5.19c}$$

The formulae relating the Euler angles to the matrix elements

$$\begin{aligned} \cos\theta &= 2u_{22}u_{33}-1\,, \\ \sin\theta\exp(i\varphi) &= -2iu_{22}u_{23}\,, \\ \sin\theta\exp(i\psi) &= -2iu_{22}u_{32} \end{aligned} \tag{5.20}$$

will be useful for further reasoning.

6. Representation of the Group *UOSP*(1.2)

Let $V=V_0\oplus V_1$ be the linear graded space representation of a certain Lie superalgebra $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$; $V(\mathfrak{G})=V_0(\mathfrak{G}_0)\oplus V_1(\mathfrak{G}_1)$ be the Grassmann envelope of subspace V. [$V_i(\mathfrak{G}_i)$ be the linear envelope of V_i over $\mathfrak{G}_i$.] The representation of the Grassmann envelope $\mathfrak{A}(\mathfrak{G})=\mathfrak{A}_0(\mathfrak{G}_0)\oplus\mathfrak{A}_1(\mathfrak{G}_1)$ and the corresponding group with Grassmann structure is the homomorphism of these objects to the set of linear operators given in $V(\mathfrak{G})$. Here we see the complete analogy with the conventional Lie group. It is clear that any finite-dimensional representation of the superalgebra $\mathfrak{A}$ may be restored up to the representation of the corresponding Lie group with Grassmann structure. If the representation of the Lie superalgebra is irreducible, it will also be irreducible for the group.

Let $T(g)$ be IR of the group $UOSP(1.2)$ in the space, which is the Grassmann envelope of space $V(J)$,

$$V(J;\mathfrak{G})=V_0(J;\mathfrak{G}_0)\oplus V_1(J;\mathfrak{G}_1)\,. \tag{6.1}$$

Let the operator matrix $T(g)$ be found in the basis $|J;LM\lambda+k\rangle$ ($L=J, J-\frac{1}{2}$; $M=-L,\dots,L$; $k=2(J-L)$)

$$T(g)|J;LM\lambda+k\rangle=\sum_{L'M'}T^{J\lambda}_{L'M',LM}(g)|J;L'M'\lambda+k'\rangle\,. \tag{6.2}$$

Since

$$T(g)=T(u)T(\xi) \tag{6.3}$$

[see the expansion (5.15)], we have

$$\begin{aligned} T^{J\lambda}_{L'M',LM}(g)=\sum_{L''M''}&\langle J;L'M'\lambda+k'|T(u)|J;L''M''\lambda+k''\rangle \\ &\cdot\langle J;L''M''\lambda+k''|T(\xi)|J;LM\lambda+k\rangle\,. \end{aligned} \tag{6.4}$$

Considering that the matrix elements of the operator $T(u)$ are diagonal in L, we get

$$T^{J\lambda}_{L'M',LM}(g)=\sum_{M''} D^{L'}_{M'M'}(u)\langle J;L'M''\lambda+k'|T(\xi)|J;LM\lambda+k\rangle, \tag{6.5}$$

where $\|D^{L'}_{M'M''}(u)\|$ is the conventional matrix of finite rotations of IR L' of the group SU(2). This matrix is a function of the Grassmann variables φ, θ, ψ. The second matrix element in (6.5) can readily be calculated. As a result, we get:

$$T^{J\lambda}_{JM',JM}(\Omega,\eta^{\square},\eta)=(1-\tfrac{1}{4}J\cdot\eta^{\square}\eta)D^{J}_{M',M}(\Omega), \tag{6.6a}$$

$$T^{J\lambda}_{J-1/2M',J-1/2M}(\Omega,\eta^{\square},\eta)=(1+\tfrac{1}{4}(J+\tfrac{1}{2})\eta^{\square}\eta)D^{J-1/2}_{M',M}(\Omega), \tag{6.6b}$$

$$\begin{aligned}T^{J\lambda}_{J-1/2M',JM}(\Omega,\eta^{\square},\eta)=&-\tfrac{1}{2}\sqrt{J-M}\,\eta^{\square}D^{J-1/2}_{M'M+1/2}(\Omega)\\&+\tfrac{1}{2}\sqrt{J+M}\,\eta D^{J-1/2}_{M'M-1/2}(\Omega),\end{aligned} \tag{6.6c}$$

$$\begin{aligned}T^{J\lambda}_{JM',J-1/2M}(\Omega,\eta^{\square},\eta)=&-\tfrac{1}{2}\sqrt{J+M+\tfrac{1}{2}}\,\eta^{\square}D^{J}_{M'M+1/2}(\Omega)\\&-\tfrac{1}{2}\sqrt{J-M+\tfrac{1}{2}}\,\eta D^{J}_{M'M-1/2}(\Omega),\end{aligned} \tag{6.6d}$$

$(\Omega=\varphi,\theta,\varphi)$.

Now, we shall find the character $\chi_J(g)$ of IR J of the group $UOSP(1.2)$. It will be reminded that the character $\chi(g)$ of the finite-dimensional linear representation $T(g)$ of the group G is the supertrace of the representation matrix: $\chi(g)=\operatorname{str}T(g)$ [3]. Similarly to the case of the conventional Lie groups, the representation character $\chi(g)$ is completely determined by the eigenvalue of the Cartan subgroup [$\exp(tA_3)$, $t\in\mathfrak{G}_0$ in our case]. In the case of IR J of the group $UOSP(1.2)$, the character $\chi_J(g)$ is of the form

$$\chi_J(t)=(-1)^{\lambda}\sum_{M=-J}^{J}e^{-iMt}+(-1)^{\lambda+1}\sum_{M=J-1/2}^{J-1/2}e^{-iMt} \tag{6.7a}$$

the first and second sums to the right in (6.7a) are the characters $\tilde{\chi}_L(g)$ of IR's $L=J$, $J-1/2$ of the SU(2) group. We find eventually:

$$\chi_J(t)=(-1)^{\lambda}\tilde{\chi}_J(t)+(-1)^{\lambda+1}\tilde{\chi}_{J-1/2}(t)=(-1)^{\lambda}\frac{\cos(J+\frac{1}{4})t}{\cos\frac{1}{4}t}, \tag{6.7b}$$

$(t\in\mathfrak{G}_0)$.

Let $\mathfrak{B}$ be the linear space of the Grassmann analytical functions $f(g)$ [4] given on the group $UOSP(1.2)$. The representation of the group $UOSP(1.2)$ is related to the space $\mathfrak{B}$ through the operators of the right and left shifts

$$T(g)f(g')=f(g^{-1}g'), \tag{6.8a}$$

$$T(g)f(g')=f(g'g), \tag{6.8b}$$

$$f(g')\in\mathfrak{B};\qquad g,g'\in UOSP(1.2).$$

It may be shown that a two sides invariant measure $d\mu(g)$ exists on the group $UOSP(1.2)$ [3], i.e.

$$\int f(g)d\mu(g)=\int f(g'g)d\mu(g)=\int f(gg')d\mu(g)=\int f(g^{-1})d\mu(g). \tag{6.9}$$

If $\mathfrak{B}$ consists of the functions for which the integral $\int f^2(g)d\mu(g)$ exists, then the representation (6.8a) is called the left regular, and (6.8b) the right regular.

Let the density $\varrho(g)$ of the measure $d\mu(g)$ be calculated:

$$d\mu(g)=\varrho(\varphi,\theta,\psi,\eta^{\square},\eta)d\varphi d\theta d\psi d\eta^{\square}d\eta\,. \tag{6.10}$$

It is known [3] that the density $\varrho(g)$ of a Lie group with Grassmann structure is closely associated with the invariant forms of the first-order on such group. Just as in the case of the conventional Lie groups, the number of such forms is the same as the number of the parameters in the group. All the independent first-order forms ω_k are related to the common invariant

$$\xi g^{-1}dg\xi^{-1}=u^{-1}du+d\xi\cdot\xi^{-1} \tag{6.11}$$

where u and ξ are the elements (5.15b, c). The relationship (6.11) is of the form [3]

$$\begin{aligned}\xi g^{-1}dg\xi^{-1}&=\sum_{i=1}^{3}\omega_i A_i+\tilde{\omega}_1R_++\tilde{\omega}_2R_-\,,\\ \omega_i&=a_{i1}d\varphi+a_{i2}d\theta+a_{i3}d\psi\,,\qquad(i=1,2,3)\,,\\ \tilde{\omega}_j&=c_{j1}d\eta^{\square}+c_{j2}d\eta\,,\qquad(j=1,2)\,.\end{aligned} \tag{6.12}$$

The density $\varrho(\varphi,\theta,\psi,\eta^{\square},\eta)$ is expressed through the determinants of the matrices

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix},\qquad C=\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix} \tag{6.13}$$

by the formula

$$\varrho(\varphi,\theta,\psi,\eta^{\square},\eta)=\frac{\det A}{\det C}\,. \tag{6.14}$$

The calculations give the expression

$$\varrho(\varphi,\theta,\psi,\eta^{\square},\eta)=\frac{1}{4\pi}(1-\tfrac{1}{4}\eta^{\square}\eta)\sin\theta\,. \tag{6.15}$$

The factor $\dfrac{1}{4\pi}$ has been selected on the basis of the normalizing condition

$$\int d\mu(g)=1\,. \tag{6.16}$$

It can easily be verified that the matrix elements $T^{J\lambda}_{L'M',LM}(g)$ (6.6a–d) form the orthonormalized set of the functions

$$\begin{aligned}\int T^{J\lambda}_{L'M',LM}(g)\bar{T}^{J'\lambda}_{\tilde{L}'\tilde{M}',\tilde{L}\tilde{M}}(g)d\mu(g)&\\ =\delta_{J,J'}\delta_{L',\tilde{L}'}\delta_{L,\tilde{L}}\delta_{M'\tilde{M}'}\delta_{M\tilde{M}}\,.&\end{aligned} \tag{6.17}$$

Here the horizontal bar over $T^{J'\lambda}_{\tilde{L}'\tilde{M}',\tilde{L}\tilde{M}}(g)$ means the operation $(\square)$ in the space of the Grassmann analytical functions. It will be noted that the right part of (6.17) differs from the case of the group SU(2) by the absence of the factor $1/(2J+1)$ where $(2J+1)$ is the dimension of IR J of the group SU(2). It may be demonstrated that $T^{J\lambda}_{L'M',LM}(g)$ form the complete set of basis functions in the space $\mathfrak{B}$.

7. The Infinitesimal Operators of Regular Representations

Now, we shall find the infinitesimal operators of the left and right regular representations.

Let $\omega(t)$ be the one-parameter subgroup $UOSP(1.2)$. The operators of the left and right regular representations corresponding to the elements of this subgroup will transform the functions $f(g)$ into $T^l(\omega(t))\cdot f(g)=f(\omega^{-1}(t)g)$ and $T^r(\omega(t))\cdot f(g)=f(g\omega(t))$. Therefore, the infinitesimal operators of the representations $T^l(g)$ and $T^r(g)$ corresponding to the one-parameter subgroup $\omega(t)$ will transform the function $f(g)$ into the values $\frac{\vec{d}}{dt}f(\omega^{-1}(t)g)$ and $f(g\omega(t))\frac{\overleftarrow{d}}{dt}$ at $t=0$. Here, $\frac{\vec{d}}{dt}$ and, correspondingly, $\frac{\overleftarrow{d}}{dt}$ denote that the left and, correspondingly, right derivatives are taken. It will be reminded that, when the function $f(g)$ is differentiated with respect to even or odd variables, the right and left derivatives are respectively identical or generally speaking different [4]. It is clear that the infinitesimal operators of the representations $T^l(g)$ and $T^r(g)$ are determined, in any case, in the space of infinitely differentiable functions on the group $UOSP(1.2)$.

If the new parameters of the elements $\omega^{-1}(t)g$ and $g\omega(t)$ are designated $\varphi(t)$, $\theta(t)$, $\varphi(t)$, $\eta^{\square}(t)$, $\eta(t)$ then the infinitesimal operators A^l_ω and A^r_ω corresponding to the subgroup $\omega(t)$ are of the form

$$A^l_\omega=\varphi'(0)\frac{\vec{\partial}}{\partial\varphi}+\theta'(0)\frac{\vec{\partial}}{\partial\theta}+\psi'(0)\frac{\vec{\partial}}{\partial\psi}+\eta'^{\square}(0)\frac{\vec{\partial}}{\partial\eta^{\square}}+\eta'(0)\frac{\partial}{\partial\eta} \tag{7.1}$$

for the left, and

$$A^r_\omega=\frac{\overleftarrow{\partial}}{\partial\varphi}\varphi'(0)+\frac{\overleftarrow{\partial}}{\partial\theta}\theta'(0)+\frac{\overleftarrow{\partial}}{\partial\psi}\psi'(0)+\frac{\overleftarrow{\partial}}{\partial\eta^{\square}}\eta'^{\square}(0)+\frac{\overleftarrow{\partial}}{\partial\eta}\eta'(0) \tag{7.2}$$

for the right regular representations. The operator A^l_ω acts on the function from the left $A^l_\omega\cdot f=(A^l_\omega f)$, and A^r_ω from the right $A^r_\omega\cdot f=(fA^r_\omega)$. The values $\varphi'(0)$, $\theta'(0)$, $\psi'(0)$, $\eta'^{\square}(0)$, $\eta'(0)$ in (7.1) are the left-hand, and in (7.2) the right-hand, derivatives in t at $t=0$. Thus, it can be seen that the calculations of A^l_ω and, correspondingly, A^r_ω reduce to calculations of the left and, correspondingly, right derivatives of $\varphi'(t)$, $\theta'(t)$, $\psi'(t)$, $\eta'^{\square}(t)$, $\eta(t)$ at $t=0$ for the element $\omega^{-1}(t)g$ and, correspondingly, $g\omega(t)$.

The derivatives of the Euler angles $\varphi'(t)$, $\theta'(t)$, $\psi'(t)$ should be calculated using the formulas (5.20), where φ, θ, ψ depend on t, while the matrices u are determined from the expansion $\omega^{-1}(t)g=u\xi$ for the left, and $g\omega(t)=u\xi$ for the right regular representations.

The explicit expressions of the infinitesimal operators A^l_1, A^l_2, A^l_3 corresponding to the one-parameter subgroups $\exp(tA_i)$ $(i=1,2,3)$ for the left regular representation are the same as the corresponding operators for the group SU(2). The only difference is that, in the case of the group $UOSP(1.2)$ the Euler angles are the Grassmann variables. The explicit expressions of the operators L^l_+, L^l_-, L^l_0 (see the equalities 5.12b) are

$$L^l_+ = -e^{i\varphi}\left(i\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} - \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right),$$

$$L^l_- = -e^{-i\varphi}\left(i\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} + \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right), \tag{7.3}$$

$$L^l_0 = -i\frac{\partial}{\partial\varphi}.$$

The subgroup $\omega(v^\square, v) = \exp(v^\square R_+ + vR_-)$ is in correspondence at once with two infinitesimal operators R^l_+ and R^l_- :

$$R^l_+ f(g) = -\frac{\vec{\partial}}{\partial v^\square} f(\omega^{-1}(v^\square, v)g),$$

$$R^l_- f(g) = \frac{\vec{\partial}}{\partial v} f(\omega^{-1}(v^\square, v)g). \tag{7.4}$$

The calculations give the following expressions of R^l_+ and R^l_- :

$$\begin{aligned} R^l_+ = {} & \tfrac{1}{4}(\eta^\square u_{33} + \eta u_{32})L^l_+ + \tfrac{1}{4}(\eta^\square u_{23} + \eta u_{22})L^l_0 \\ & + (1 - \tfrac{1}{8}\eta^\square\eta)u_{22}\frac{\partial}{\partial\eta^\square} - (1 - \tfrac{1}{8}\eta^\square\eta)u_{23}\frac{\vec{\partial}}{\partial\eta}, \end{aligned} \tag{7.5}$$

$$\begin{aligned} R^l_- = {} & \tfrac{1}{4}(\eta^\square u_{23} + \eta u_{22})L^l_- - \tfrac{1}{4}(\eta^\square u_{33} + \eta u_{32})L^l_0 \\ & - (1 - \tfrac{1}{8}\eta^\square\eta)u_{33}\frac{\vec{\partial}}{\partial\eta} + (1 - \tfrac{1}{8}\eta^\square\eta)u_{32}\frac{\vec{\partial}}{\partial\eta^\square}, \end{aligned} \tag{7.6}$$

where the matrix $\|u_{pq}\|$ is of the form (5.19c). It can readily be verified that the operators (7.3), (7.5), (7.6) satisfy the commutational relations (2.4a, b, c).

The following expressions have been obtained for the infinitesimal operators of the right regular representation:

$$L^r_+ = H^r_+ - \frac{\overleftarrow{\partial}}{\partial\eta^\square}\eta,$$

$$L^2_- = H^r_- - \frac{\overleftarrow{\partial}}{\partial\eta}\eta^\square,$$

$$L^r_0 = H^r_0 + \tfrac{1}{2}\frac{\overleftarrow{\partial}}{\partial\eta}\eta - \tfrac{1}{2}\frac{\partial}{\partial\eta^\square}\eta^\square, \tag{7.7}$$

$$\begin{aligned} R^r_+ &= L^r_+\frac{\eta^\square}{4} - L^r_0\frac{\eta}{4} + \frac{\overleftarrow{\partial}}{\partial\eta^\square}(1 - \tfrac{1}{8}\eta^\square\eta) \\ &= H^r_+\frac{\eta^\square}{4} - H^r_0\frac{\eta}{4} + \frac{\overleftarrow{\partial}}{\partial\eta^\square}(1 + \tfrac{1}{4}\eta^\square\eta), \end{aligned}$$

$$\begin{aligned} R^r_- &= -L^r_-\frac{\eta}{4} - L^r_0\frac{\eta^\square}{4} + \frac{\overleftarrow{\partial}}{\partial\eta}(1 - \tfrac{1}{8}\eta^\square\eta) \\ &= -H^r_-\frac{\eta}{4} - H^r_0\frac{\eta^\square}{4} + \frac{\partial}{\partial\eta}(1 + \tfrac{1}{4}\eta^\square\eta). \end{aligned}$$

Here H^r_+, H^r_-, H^r_0 are the infinitesimal operators of the right regular representation of the group SU(2) depending on the Grassmann variables φ, θ, ψ and are set by the formulae (7.3) after the replacements $\varphi \to -\psi$, $\psi \to -\varphi$, $L_i \to H_i$ $(i=+,-,0)$. It can readily be verified the commutational relations (2.4a, b, c) are also valid for the operators (7.7) considering that all the operators act on the function $f(g)$ from the right. In its turn, this means that $[L^r_\pm, L^r_0] = \pm L^r_0$, $[L^r_-, L^r_+] = 2L^r_0$ etc.

The Laplace-Kazimir operator Δ of the superalgebra $osp(1.2)$ is of the form [8]

$$\Delta = \tfrac{1}{2}L_+L_- + \tfrac{1}{2}L_-L_+ + R_+R_- - R_-R_+ + L_0^2 \tag{7.8}$$

and commutates with all the elements of the superalgebra $osp(1.2)$. The formula (7.3)–(7.7) will be used to obtain the expression of Δ through the Grassmann variables φ, θ, ψ, $\eta^\square$, η. In this case, $\Delta^r = \Delta^l \equiv \Delta$ i.e. just what should be expected. The explicit form of the operator Δ is

$$\begin{aligned}\Delta &= (1+\tfrac{1}{8}\eta^\square\eta)\Delta_{\mathrm{SU}(2)} + \tfrac{1}{2}\frac{\overleftarrow{\partial}}{\partial\eta^\square}\eta^\square + \tfrac{1}{2}\frac{\overleftarrow{\partial}}{\partial\eta}\eta + 2\frac{\overleftarrow{\partial}}{\partial\eta}\frac{\overleftarrow{\partial}}{\partial\eta^\square}(1-\tfrac{1}{4}\eta^\square\eta)\\ &= (1+\tfrac{1}{8}\eta^\square\eta)\Delta_{\mathrm{SU}(2)} + \tfrac{1}{2}\eta^\square\frac{\overrightarrow{\partial}}{\partial\eta^\square} + \tfrac{1}{2}\eta\frac{\overrightarrow{\partial}}{\partial\eta} + 2(1-\tfrac{1}{4}\eta^\square\eta)\frac{\overrightarrow{\partial}}{\partial\eta}\frac{\overrightarrow{\partial}}{\partial\eta^\square},\end{aligned} \tag{7.9a}$$

where

$$\Delta_{\mathrm{SU}(2)} = -\left[\frac{\partial^2}{\partial\theta^2} + \mathrm{ctg}\,\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} - 2\cos\theta\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right] \tag{7.9b}$$

is the Laplace-Kazimir operator of the group SU(2) when the Euler angles are the Grassmann variables.

Similarly to the case of the group SU(2), the elements of the (LM)-column of the operator matrix $T(g)$, i.e. the functions

$$T^{J\lambda}_{L'M',LM}(g) \qquad (L'=J, J-\tfrac{1}{2};\ M=-L', -L'+1, \dots, L)$$

form the basis $|J; L'M'\lambda+k'\rangle = T^{J\lambda}_{L'-M',LM}(g)$ $(k'=2(J-L'))$ of the left regular IR of weight J of the group $UOSP(1.2)$. Similarly, the elements of (LM)-line form the basis of the right regular IR of weight J: $|J; LM\lambda+k\rangle = T^{J\lambda}_{L'M',LM}(g)$, $(k=2(J-L))$.

The functions $T^{J\lambda}_{L'M',LM}(g)$ are the eigenfunctions of the Laplace-Kazimir operator Δ:

$$\Delta T^{J\lambda}_{L'M',LM}(g) = J(J+\tfrac{1}{2})T^{J\lambda}_{L'M',LM}(g)\,. \tag{7.10}$$

Besides that,

$$\begin{aligned}R^l_+ T^{J\lambda}_{J-J,LM}(g) &= 0\,,\\ R^r_+ T^{J\lambda}_{L'M',JJ}(g) &= 0\,.\end{aligned} \tag{7.11}$$

The explicit function $T^{J\lambda}_{J-J,JJ}(g)$ may also be found by solving the Eqs. (7.10) and (7.11). The rest matrix elements $T^{J\lambda}_{L'M',LM}(g)$ will be obtained by acting on $T^{J\lambda}_{J-J,JJ}(g)$ by the powers of the lowering operators R^l_-, R^r_-.

If the Eqs. (2.18) are used, where the operators $R_\pm$ are substituted for $R^l_\pm$ and $|J; L'M'\lambda+k'\rangle = T^{J\lambda}_{L'-M',LM}(g)$ is of the form (6.6a–d), we shall obtain the recurrent relations for the functions $D^J_{M'M}$ of the group SU(2). With this purpose, the factors at $\eta^\square$, η, $\eta^\square\eta$ and the term which does not comprise these variables, should be reparately equated to zero. One of such relations is

$$\sqrt{J+M}\, D^J_{M',M} = \sqrt{J+M'}\, u_{33} D^{J-1/2}_{M'-1/2\, M-1/2} + \sqrt{J-M'}\, u_{23} D^{J-1/2}_{M'+1/2\, M-1/2}. \quad (7.12)$$

The relations of such kind are known to be the formulas of coupling of two D-functions of the group SU(2) with moments $J-\frac{1}{2}$ and $\frac{1}{2}$.

Acknowledgement. One of the authors (T.V.N.) is thankful to Professor Yu. F. Smirnov for informative discussions.

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Communicated by Ya. G. Sinai

Received June 1, 1980