Commun. Math. Phys. 76, 149–163 (1980)

Communications in
Mathematical
Physics

Wall and Boundary Free Energies

III. Correlation Decay and Vector Spin Systems

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Abstract. The asymptotic free energy of planar walls and boundaries is analyzed for scalar and vector spin systems. Under the hypothesis of correlation decay, various alternative definitions are found to be equivalent in the thermodynamic limit and independent of the "associated" walls. Furthermore, a torus, or box having periodic boundary conditions, is shown to have no boundary or surface free energy. For vector spin systems with n-component spins, existence of the thermodynamic limit is shown for $n=2$ and "positive" interactions.

9. Introduction

The problem of the existence, uniqueness and properties of the free energy associated with a wall or boundary has been considered in two previous papers [1,2], to be referred[1] to below as I and II. In this paper, we consider (i) the boundary free energy in the one phase region, in particular, under the assumption of a uniform correlation decay law; (ii) the boundary free energy for a system of *vector* spins.

It has already been noted in I [see hueristic counterexample: in Sect. 2.7] that the free energy per unit area, $f_\times(K, W, \tilde{W}, \Omega)$, of a planar wall cut in the domain Ω is generally dependent on the associated wall potentials, $\tilde{W}$, imposed on the original boundaries of Ω, whenever the thermodynamic state is on a first order phase boundary so that the system may exhibit two-phase behavior. However, in a one-phase region, which for ferromagnetic spin systems is generally characterized by $T>T_c$ or $h \neq 0$, we expect the free energy associated with a wall to be independent of the associated walls. It is generally expected that, in the one-phase region, the correlations between two sets of spins, s_A and s_B, will vanish as the separation,

1 This paper is written as a direct continuation of Part II [Caginalp, G., Fisher, M.E.: Commun. Math. Phys. **65**, 247–280 (1979)]. Accordingly the numbering of sections and equations runs straight on from Part II

0010-3616/80/0076/0149/$03.00

$d(A, B)$, is taken to infinity. Some rigorous results exist along these lines for nearest and next-nearest neighbor potentials (see Sect. 10.1 and [3] for details). The main assumption we will make in Sect. 10.1 is the correlation decay assumption, $\mathbf{M}_\delta(\delta > 0)$, under which spin correlations decay as $[d(A,B)]^{-d-\delta}$ in a uniform manner. With this assumption the restriction to ferromagnetic spin interactions is unnecessary. The basic result attained in Sect. 10 is the existence and uniqueness (i.e. independence from the associated wall potentials, $\tilde{W}$) of the limiting boundary free energy, $f_\times^\infty(K, W)$.

In particular, under appropriate conditions [see Theorem 10.3.2], one has equality between the two standard limiting boundary free energies, $f_\times^0(K,W)$ and $f_\times^*(K,W)$, for free and superferromagnetic associated walls respectively. Partly as a consequence of this result, one has the equivalence of various alternative definitions such as the periodic boundary free energy. An interesting result that follows as a consequence is that a torus has no boundary free energy, i.e. the expansion of the free energy of a torus in the form

$$\bar{F}(\Omega) \equiv V(\Omega) f(\Omega) = V(\Omega) f_\infty(K) + \sum_\alpha A_\alpha(\Omega) f_\times(K, W_{(\alpha)}) + o(\sum A_\alpha) \tag{9.1}$$

exhibits no area term.

The topic of vector spin systems is discussed in Sect. 11. We consider two-component spins initially and define the analog of ferromagnetic interactions, known as positive interactions. Briefly, this is the requirement that

$$K_{ij}^x \geqq |K_{ij}^y|, \tag{9.2}$$

where $K_{ij} = (K_{ij}^x, K_{ij}^y)$ is the interaction between spins i and j. For positive interactions, we utilize inequalities due to Ginibre [4] to obtain a subadditivity result, upon specifying conditions on walls and associated walls. The wall conditions "subfree" and "superferromagnetic" are now redefined and the existence theorems for these two sets of boundary conditions are proved. Other results previously obtained for scalar spins are similarly generalized to the two-component spin case.

10. Decay of Spin Correlations

The results of the first part of this paper are based on an assumption regarding the asymptotic decay of spin correlations which is discussed below.

10.1. Correlation Decay Assumption and Some Consequences

In Sect. 2.7 we discussed the role of the associated boundary conditions, and gave an intuitive counterexample for the complete independence of the wall free energy, $f_\times$, of the associated wall potentials. Quite generally, one expects that the boundary free energy will be (i) independent of the associated walls when the thermodynamic state of the lattice system is within a one-phase region but (ii) may be dependent on the associated wall conditions when the thermodynamic state is on a first order phase boundary so that the system may exhibit two-phase behavior. In the one-phase region, which for ferromagnetic spin system is generally characterized by $T>T_c$ or $h\neq 0$ (where h is the magnetic field), we expect that the correlations between two sets of spins, s_A and s_B, will vanish as the separation, $d(A,B)$, is taken to infinity, i.e.

$$\langle s_A s_B\rangle - \langle s_A\rangle\langle s_B\rangle \to 0 \quad \text{as} \quad d(A,B)\to\infty\,. \tag{10.1.1}$$

There exist rigorous results for Ising ferromagnets (with nearest and next-nearest neighbor potentials) which state that for (a) $h\neq 0$, or (b) T sufficiently high as to not be a limit point of zeros of the grand canonical partition function, the asymptotic decay (10.1.1) holds (for an infinite lattice), and is exponential in the separation distance.

A result, by Lebowitz and Penrose [5], establishing a connection between analyticity of free energy and exponential decay has been used by various authors [6] to show asymptotic decay for spin systems with short-range interactions and lattice gas systems with hard cores. Although these results concern single spin decay, it has been shown by Lebowitz [8], under appropriate conditions, that if the pair correlation decay is bounded by $\bar{u}(r_{ij})$ where r_{ij} is the distance between spins i and j, then

$$|\langle s_A s_B\rangle - \langle s_A\rangle\langle s_B\rangle| \leqq |A||B|\bar{u}[d(A,B)]\,. \tag{10.1.2}$$

We state now the main condition we will use regarding the decay of correlations. Given a set of interactions $K=\{K_A\}$ and wall and associated wall potentials $W=\{W_A\}$ and $\tilde{W}=\{\tilde{W}_A\}$ we require:

$\mathbf{M}_\delta$ *Correlation Decay.* For any sequence of domains, $\{\Omega_i\}$, if s_A and s_B are sets of spins separated by $d(A,B)$, then for each Ω_i

$$|\langle s_A s_B\rangle - \langle s_A\rangle\langle s_B\rangle| \leqq C\|A\|\,\|B\|\,[d(A,B)]^{-d-\delta}\,, \tag{10.1.3}$$

where the expectation $\langle\cdot\rangle$ is taken with respect to $\mathscr{H}(\zeta,\eta)=\mathscr{H}_0(\Omega_i)+\zeta\mathscr{H}(\Omega_i)+\eta\mathscr{H}'$ where $\mathscr{H}'$ is either part of the Hamiltonian $\mathscr{H}_0(K)$ for a region outside of Ω_i, or a part of the periodic Hamiltonian $\mathscr{H}_0(K^\Pi,\Omega_i)$ not including $\mathscr{H}_0(K,\Omega_i)$ itself.

By the rigorous results quoted above [7], some examples of systems satisfying the correlation decay condition $\mathbf{M}_\delta$ are (i) Ising ferromagnets with nearest neighbor interactions and $T>T_c$ and (ii) systems with ferromagnetic pair interactions which decay as $r^{-3d-\delta}$ and a nonzero magnetic field imposed.

10.2. Boundary Free Energy for Nonferromagnetic Interactions of Finite Range

It is possible, at this point, to drop the assumption regarding the ferromagnetic nature of the bulk interactions and wall and associated wall potentials and prove the following.

Theorem 10.2.1. *Existence of limiting boundary free energy. For a sequence of boxes* $\{\Lambda_{\mathbf{L},N_1,N_2}\}$ *in a system satisfying the uniform correlation bound* **A**, *the correlation decay condition* $\mathbf{M}_\delta$ *with* $\delta>0$, *and having bulk and wall potentials, K and W, of finite* *range, R^∞ and $R^\times$, respectively, and of finite degree, p, subject to free associated boundary conditions, $\tilde{W} \equiv 0$, the limiting boundary free energy*

$$\lim_{\mathbf{L}, N_1, N_2 \to \infty} f_\times(K, W, \tilde{W} \equiv 0, \Lambda_{\mathbf{L}, N_1, N_2}) = f^0_\times(K, W), \tag{10.2.1}$$

exists and is independent of the order in which the limits are taken.

Before proving Theorem 10.2.1, we state and prove a result on nearly subadditive and monotonic functions. This is the analog of Lemma 4.3.3 for subadditive and monotonic functions and is preliminary to proving Theorem 10.2.1.

Lemma 10.2.1. *Nearly subadditive and monotonic functions. Let $f(\mathbf{x}, \mathbf{y})$ be a function on $\mathbb{Z}_+^{d'} \times \mathbb{Z}_+^{d''}$, where $\mathbb{Z}_+$ is the set of positive integers such that the following hold:*

$$\text{(i)} \quad |f(\mathbf{x}, \mathbf{y})| \leqq c_1, \tag{10.2.2}$$

$$\text{(ii)} \quad |x_j f(x_j) + x'_j f(x'_j) - (x_j + x'_j) f(x_j + x'_j)| \leqq c_2, \tag{10.2.3}$$

$$\text{(iii)} \quad |f(y_j + y'_j) - f(y_j)| \leqq \sum_{k = y_j + 1}^{y_j + y'_j} \varepsilon_k^{(j)}, \tag{10.2.4}$$

where c_1 and c_2 are constants and $\{\varepsilon_k^{(j)}\}$ have the property that

$$\sum_{k=1}^{\infty} \varepsilon_k^{(j)} < \infty \tag{10.2.5}$$

for each j. [Note that suppression of other indices in $f(x_j)$, etc. indicates they are held fixed in inequalities (10.2.3) and (10.2.4).] Then the limit of $f(\mathbf{x}, \mathbf{y})$ exists as $\mathbf{x}, \mathbf{y} \to \infty$ *(independently of how the limits are taken).*

Proof. Let $\mathbf{x} \equiv (x_1, x_2, \ldots, x_j, \ldots, x_{d'})$, $\mathbf{x}' \equiv (x_1, x_2, \ldots, x'_j, \ldots, x_{d'})$ and similarly for $\mathbf{y}$ and $\mathbf{y}'$. From (10.2.3) it follows that

$$\begin{aligned} &x_1 x_2 \ldots (x_j + x'_j) \ldots x_{d'} f(x_j + x'_j) \\ &\quad \leqq x_1 x_2 \ldots x_j \ldots x_{d'} f(x_j) + x_1 x_2 \ldots x'_j \ldots x_{d'} f(x'_j) + x_1 \ldots x_{j-1} x_{j+1} \ldots x_{d'} c_2 . \end{aligned} \tag{10.2.6}$$

We define the function

$$k(\mathbf{x}, \mathbf{y}) = \mathrm{f}(\mathbf{x}, \mathbf{y}) + c_2 \|x\|^{-1} \sum_{j=1}^{d'} \prod_{i \neq j} x_i + \sum_{j=1}^{d''} \sum_{k = y_j + 1}^{\infty} \varepsilon_k^{(j)} \tag{10.2.7}$$

and observe that the existence of $\lim k(\mathbf{x}, \mathbf{y})$ as $\mathbf{x}, \mathbf{y} \to \infty$ implies the conclusion since the last two terms on the right hand side of (10.2.7) vanish. Defining the function

$$K(\mathbf{x}, \mathbf{y}) = \|x\| f(\mathbf{x}, \mathbf{y}), \tag{10.2.8}$$

it follows from (10.2.6) that

$$K(x_j + x'_j, \mathbf{y}) \leqq K(x_j, \mathbf{y}) + K(x'_j, \mathbf{y}), \tag{10.2.9}$$

where the other components are suppressed as they remain fixed throughout.

Similarly, one has, by (10.2.4) that

$$f(\mathbf{x}, y_j + y_j') \leqq \mathrm{f}(\mathbf{x}, y_j) + \sum_{j=1}^{d''} \sum_{k=y_j+1}^{y_j+y_j'} \varepsilon_k^{(j)}, \qquad (10.2.10)$$

from which it follows that

$$k(\mathbf{x}, y_j + y_j') \leqq k(\mathbf{x}, y_j) \quad (y_j \geqq 0, \quad \text{all} \quad j). \qquad (10.2.11)$$

The function $k(\mathbf{x}, \mathbf{y})$ is also bounded as a result of (10.2.2); hence the hypotheses of Theoem 4.3.3 are satisfied and the limit of $k(\mathbf{x},\mathbf{y})$ as $\mathbf{x},\mathbf{y} \to \infty$ exists and consequently so does the limit of $f(\mathbf{x},\mathbf{y})$. □

Proof of Theorem. Under the hypotheses stated, boundedness follows from Proposition 3.2.2. We consider box domains, $\Lambda_{\mathbf{L},N}$, of crossectional area $|\mathbf{L}| = L_1 L_2 \ldots L_{d-1}$ and length N (see Sects. 2.2 and 4.1 for precise definitions). Recapitulating notation we write an intermediate (reduced) Hamiltonian depending on the parameter ζ as

$$\bar{\mathscr{H}}^{\zeta}(\Lambda) = \bar{\mathscr{H}}(\Lambda) + \zeta \mathscr{W}(\Lambda), \qquad (10.2.11)$$

where the wall potential $W(\Lambda)$ for free boundary conditions is

$$\mathscr{W}(\Lambda) = \sum_{A \subsetneq \Lambda} W_A s_A. \qquad (10.2.12)$$

The boundary free energy in linearized form is

$$2|\mathbf{L}| f_{\times}(K, W, \tilde{W} = 0, \Lambda) = \int_0^1 d\zeta \langle \mathscr{W} \rangle_{\Lambda}^{\zeta}, \qquad (10.2.13)$$

where $\langle \cdot \rangle_{\Lambda}^{\zeta}$ denotes an expectation computed with $\bar{\mathscr{H}}^{\zeta}(\Lambda)$.

If we let $\mathscr{H}' = \sum_{B \subsetneq \Lambda} K_B' s_B$ denote an additional (reduced) Hamiltonian, then one has the following relationship between expectations:

$$\begin{aligned} \langle s_A \rangle_{\bar{\mathscr{H}} + \bar{\mathscr{H}}'} - \langle s_A \rangle_{\bar{\mathscr{H}}} &= \int d\eta \frac{d}{d\eta} \langle s_A \rangle^{\eta} \\ &= \int d\eta \sum_{B \subsetneq \Lambda'} K_B' [\langle s_A s_B \rangle^{\eta} - \langle s_A \rangle^{\eta} \langle s_B \rangle^{\eta}], \end{aligned} \qquad (10.2.14)$$

where $\langle s_A \rangle_{\bar{\mathscr{H}} + \bar{\mathscr{H}}'}$ and $\langle s_A \rangle^{\eta}$ are expectations taken with the Hamiltonians $\bar{\mathscr{H}} + \bar{\mathscr{H}}'$ and $\bar{\mathscr{H}}^{\eta} = \bar{\mathscr{H}} + \eta \bar{\mathscr{H}}'$ respectively. The difference between two boundary free energies with the same wall potentials $\mathscr{W}$ but with Hamiltonians differing by $\bar{\mathscr{H}}'$ may be written:

$$\begin{aligned} &2|\mathbf{L}| f_{\times}(\bar{\mathscr{H}} + \bar{\mathscr{H}}', W) - 2|\mathbf{L}| f_{\times}(\bar{\mathscr{H}}, W) \\ &= \int_0^1 d\zeta \int_0^1 d\eta \sum_{A \subsetneq \Lambda} \sum_{B \subsetneq \Lambda'} W_A K_B' [\langle s_A s_B \rangle^{\zeta,\eta} - \langle s_A \rangle^{\zeta,\eta} \langle s_B \rangle^{\zeta,\eta}]. \end{aligned} \qquad (10.2.15)$$

Using this formalism we first establish $f_{\times}$ as a nearly monotonic function. We consider a box domain $\Lambda_{\mathbf{L},N_1+N_2}$ $(N_1 + N_2 = N)$ as illustrated previously in Fig. 7c of I. Augmenting the box domain on the N_1 side of one layer which we denote by Λ' with Hamiltonian $\mathscr{H}'$ we may write:

$$2|\mathbf{L}|f_\times(\mathbf{L}, N_1+1, N_2) - 2|\mathbf{L}|f_\times(\mathbf{L}, N_1, N_2)$$
$$= \int_0^1 d\zeta \int_0^1 d\eta \sum_{A \subseteq \Lambda} W_A \sum_B K_B [\langle s_A s_B\rangle^{\zeta,\eta} - \langle s_A\rangle^{\zeta,\eta}\langle s_B\rangle^{\zeta,\eta}], \qquad (10.2.16)$$

where the second sum ranges over $B \subset \Lambda' \cup \Lambda \cdot \Lambda'$. By the decay assumption $\mathbf{M}_\delta$, one has

$$|\langle s_A s_B\rangle^{\zeta,\eta} - \langle s_A\rangle^{\zeta,\eta}\langle s_B\rangle^{\zeta,\eta}| \leqq u[d(A,B)] \quad (\text{all } \zeta, \eta), \qquad (10.2.17)$$

where $u(x) \leqq Cx^{-\delta}$. Denoting the double sum by

$$S_1(\mathbf{L}, N_1, N_2) = \sum_{A,B} u[d(A,B)] \qquad (10.2.18)$$

we bound the sums by integrals. Since the interactions and wall potentials are by assumption of finite range, R^∞ and $R^\times$, respectively, one may take these into account via a single constant C, below, and the situation does not differ substantially from that of nearest neighbor interactions. Letting $x \in \mathbb{R}^{d'}$ lie on the plane of the wall, $\mathscr{P}$, while $\mathbf{y} \in \mathbb{R}^{d'}$ lies on the plane $\mathscr{P}_N$, separating Λ and Λ', we may bound $S_1(\mathbf{L}, N_1, N_2)$ as follows:

$$\sum_{N_1=1}^{\infty} |S_1(\mathbf{L}, N_1, N_2)| \leqq \frac{C}{2|\mathbf{L}|} \int_1^\infty dz \int_1^\infty dx_1 \dots \int_1^\infty dx_{d'} \int_1^\infty dy_1 \dots \int_1^\infty dy_{d'}$$
$$\cdot [(x_1-y_1)^2 + \dots + (x_{d'}-y_{d'})^2 + z^2]. \qquad (10.2.19)$$

By a change of variables $v_j \equiv x_j + y_j$ and $w_j = x_j - y_j$, followed by a change to d-dimensional spherical coordinates so that

$$r^2 = (x_1-y_1)^2 + \dots + (x_{d'}-y_{d'})^2 + z^2, \qquad (10.2.20)$$

one may bound the multidimensional integral in (10.2.19) by $C' \int_1^\infty r^{d-1} dr (r^2)^{-\delta/2} < \infty$ if $\delta > d$. Hence, we have shown

$$|f_\times(\mathbf{L}, N_1+N', N_2) - f_\times(\mathbf{L}, N_1, N_2)| \leqq \sum_{k=N_1+1}^{N_1+N'} \varepsilon_k^{(j)},$$

where $\sum_k \varepsilon_k^{(j)} < \infty$, thus establishing (10.2.4) for the boundary free energy.

We now proceed to show $f_\times$ is nearly subadditive. Consider the situation where two boxes, Λ and Λ', are of dimensions $N_1, L_1, \dots, L_{d'}$ and $N_2, L'_1, L_2 \dots L_{d'}$, respectively, are placed on top of one another, as shown in Fig. 7b of I. The contribution to the wall potential arising from sets $B \in \Lambda \cdot \Lambda'$ is bounded by $C_1 \cdot L_2 L_3 \dots L_{d'}$. Using the same line of reasoning as in the near monotonicity we obtain

$$|L_1 L_2 \dots L_{d'} f_\times(L_1) + L'_1 L_2 \dots L_{d'} f_\times(L'_1) - (L_1+L'_1) L_2 \dots L_{d'} f_x(L_1+L'_1|$$
$$C_1 L_2 \dots L_{d'} + C_2 S_2(L_1, L'_1, L_2, \dots, L_{d'}, N_1, N_2), \qquad (10.2.21)$$

where S_2 is defined like S_1 earlier by

$$S_2 = \sum_{A,B} u[d(A,B)] \tag{10.2.22}$$

with the sum on A running over the sets A such that $A \underset{\sim}{\subset} \Lambda$ and $A \underset{\sim}{\subset} \Lambda'$ such that $W_A \neq 0$, and the sum on B running over $B \underset{\sim}{\subset} \Lambda \cdot \Lambda' \cup \Lambda'$ such that $K_B \neq 0$.

Bounding the sum by an integral, one has

$$S_2 \leqq \int_{\substack{\infty \\ |x_1| \geqq 1}}^{\infty} dx_1 \int_1^{L_2} dx_2 \ldots \int_1^{L_{d'}} dx_{d'} \int_1^{L_2} dy_2 \ldots \int_1^{L_{d'}} dy_{d'} \int_{\substack{-\infty \\ |y_{d'}| \geqq 1}}^{\infty} dy_d u(|\mathbf{x}-\mathbf{y}|) \tag{10.2.23}$$

subject to the further restriction that $|x_j - y_j| \geqq 1$ for all j, and where

$$u(|\mathbf{x}-\mathbf{y}|) \equiv [x_1^2 + y_d^2 + (x_2 - y_2)^2 + \ldots + (x_{d'} - y_{d'})^2]^{-\delta/2}. \tag{10.2.24}$$

Using a change of coordinates to $v_j = x_j + y_j$, $w_j = x_j - y_j$ followed by a change to spherical coordinates, one has

$$|S_2(L_1, L_1', L_2, \ldots, L_{d'}, N_1, N_2)| \leqq C_2' L_2 \ldots L_{d'}. \tag{10.2.25}$$

Hence, the bounds (10.2.21) and (10.2.25) establish the near subadditivity result (10.2.3) for the boundary free energy. The hypotheses of Lemma 10.2.1 are thus satisfied and the conclusion follows. □

Remark 10.2.1. It follows as a corollary of the proof of Theorem 10.2.1 that if the spins are saturating, then the limiting boundary free energies subject to associated boundary conditions which are simple superferromagnetic, $\tilde{W}^*$, and free, $\tilde{W} \equiv 0$, are equal, i.e.

$$f_\times^*(K,W) = f_\times^0(K,W). \tag{10.2.26}$$

Remark 10.2.2. Under the hypotheses of Lemma 7.2.1 and Theorem 10.2.1 the limiting free energy for a multiple wall is equivalent to that of a free wall, i.e. using the notation of Sect. 7,

$$\lim_{\Omega \to \infty} \bar{f}_\times(K, W, \tilde{W}) = \sum_j \lambda_j^\infty f_\times^0(K, W_j). \tag{10.2.27}$$

Under the hypotheses of Lemma 8.2.4 and Theorem 10.2.1

$$f_\times^\Pi(K) = f_\times^0(K, W^0), \tag{10.2.28}$$

where W^0 is the wall potential for a free wall.

This completes our study of nonferromagnetic interactions of finite range, which will be generalized to interactions of infinite range in the following section.

10.3. Boundary Free Energy for Long Range Nonferromagnetic Interactions

Using the theorems for finitely ranged interactions we now extend the results of the previous section to infinite-range interactions. In analogy to condition $\mathbf{M}_\delta$ we introduce a new condition $\mathbf{N}_\delta$ which states, roughly, that given a set of infinite-range interactions and wall and associated wall potentials, there must exist a sequence of finite-range potentials which satisfy $\mathbf{M}_\delta$ and which converge to the original potential. The convergence is with respect to the metric induced by the norm of a Banach space, $\mathscr{B}^\times$, which will be discussed below.

Definition 10.3.1. The *Banach space of interactions*, $\mathscr{B}^\times$, is the set of translationally invariant interactions, $\{K\}$, which satisfy

$$\|K\|_0 \equiv \sum_{i=1}^{q} \sum_{A \ni i} d(A)|K_A|/|A| < \infty\,, \tag{10.3.1}$$

where the sum over i is a sum of sites in one cell. Addition and multiplication by a scalar are defined in the natural way. The subset of $\mathscr{B}^\times$ which consists of interactions $K=\{K_A\}$ such that $K_A=0$ except for finitely many $A \ni 0$ is denoted $\mathscr{B}_0^\times$.

Note that the Banach space $\mathscr{B}^\times$ is related to the norm introduced by condition **F(i)** [see Sect. 2.3]. Indeed, the nature of the norm for $\mathscr{B}^\times$ is necessitated by the conditions required for boundedness of the boundary free energy, which involves **F(i)**. The usual stability norm (condition **B** of Sect. 1.4) differs from the wall stability norm by an additional distance factor $d_\perp(A)$. In particular, it is not difficult to see that for pair potentials, the interactions must decay as $r^{-d-1-\delta}(\delta>0)$ or faster, instead of the usual $r^{-d-\delta}$ required for bulk stability.

The overall strategy is then to focus on particular finite dimensional and compact subsets of $\mathscr{B}^\times$. By establishing appropriate continuity properties and using the results obtained earlier, which are sufficient to prove that the convergence on $\mathscr{B}_0^\times$ is uniform, we can extend the results by completion on a metric space to all of $\mathscr{B}^\times$.

The precise decay condition to be used is:

$\mathbf{N}_\delta$ *Correlation Decay for Long Range Interactions.* For a set of bulk, wall and associated wall potentials, $K=\{K_A\}$; $W=\{W_A\}$ and $\tilde{W}=\{\tilde{W}_A\}$ there exists an infinite set of sequences $K^{(i)}$, $W^{(i)}$, $\tilde{W}^{(i)}$, $(i=1,2,\ldots)$ contained in $\mathbf{M}_\delta \cap B_0^\times$ such that $K^{(i)}\to K$, $W^{(i)}\to W$ and $\tilde{W}^{(i)}\to\tilde{W}$ in norm as $i\to\infty$.

Under the most general circumstances the condition N_δ is not easy to verify explicitly. However, a large class of systems included in the results of Iagolnitzer and Souillard will satisfy this condition. In particular, in a system with ferromagnetic pair interactions which decay as $r^{-3d-\delta}$ and which is subject to a non-zero magnetic field h, the truncations of the potentials still contain the non-zero magnetic field and are thus contained in $\mathbf{M}_\delta$; thus the system with the full long range potentials satisfies $\mathbf{N}_\delta$. A similar situation is expected to hold in ferromagnetic systems for $T>T_c$ although there are as yet no rigorous results here for long range interactions.

Theorem 10.3.1. *Existence of boundary free energy for long-range interactions.*

Let $K=\{K_A\}$ and $W=\{W_A\}$ be a set of interactions and decoupling wall potentials, respectively, which satisfy boundedness conditions **A** (see Sect. 1.4) and **F** (Sect. 2.3) and defining conditions **D** and **E** (Sects. 2.2 and 2.3) and the additional correlation decay condition $\mathbf{N}_\delta(\delta>d)$, and let $\{\Lambda_{\mathbf{L},N_1,N_2}\}$ be a sequence of boxes subject to free associated wall conditions. Then the limiting boundary free energy exists and

$$\lim_{\Lambda,N_1,N_2\to\infty} f_\times(K,W,\Lambda_{\mathbf{L},N_1,N_2})=f_\times^\infty(K,W). \tag{10.3.2}$$

On a set of interactions, K, and wall potentials, W, which are bounded, the convergence is uniform and $f_\times^\infty(K,W)$ is uniformly continuous (where convergence and continuity are in the sense discussed above).

The proof of Theorem 10.3.1 is a consequence of a series of remarks and lemmas which follow.

Remarks 10.3.1. (a) The subspace $\mathscr{B}_0^\times$ is dense in $\mathscr{B}$.

(b) If $\mathscr{B}_{M_1}^\times$ is the subset of $\mathscr{B}^\times$ consisting of $K\in\mathscr{B}^\times$ such that $K_A=0$ expect for at most M_1 sets A, then $\mathscr{B}_{M_1}^\times$ is a finite dimensional Banach space. The subset $\mathscr{B}_{M_1,M_2}^\times$, which is defined as the set of $K=\{K_A\}\in\mathscr{B}_{M_1}^\times$ such that $\sup_A|K_A|\leqq M_2$, is compact with respect to the metric $\|\cdot\|$.

(c) If $\mathscr{B}^\times(Q)$ is the subset consisting of $K\in\mathscr{B}^\times$ such that $K_A=0$ unless $\|A\|\leqq Q$, then $\mathscr{B}^\times(Q)$ is also a Banach space. The subsets $\mathscr{B}_{M_1}^\times(Q)$ and $\mathscr{B}_{M_1,M_2}^\times(Q)$ are defined as in (b), and have the same properties. The following is a direct consequence of these definitions and remarks.

Proposition 10.3.1. *If $K\in\mathscr{B}_{M_1,M_2}(Q)$ then the expectation $\langle S_A\rangle_\Omega^{\zeta\eta}$, where Ω is a finite region, is continuous in ζ and η and uniformly continuous in K with respect to the metric induced by the Banach space norm $\|\cdot\|$.*

These ideas may be applied to the boundary free energy in the form of (10.2.13) upon utilizing Lebesque's dominated convergence theorem and compactness. The necessary result is stated in

Proposition 10.3.2. *Let (X,ϱ) be a compact metric space and $g(x,y):X\times[0,1]\to\mathbb{R}$ a function which is continuous in x and y separately and such that $|g(x,y)|<C$ for all x and y.*

If $h(x)\equiv\int_0^1 g(x,y)dy$ then $h(x)$ is uniformly continuous in x.

Consider now a sequence of domains, $\{\Lambda_n\}$, which we will take to be box domains. Let $f_n^\times(K)$ denote the boundary free energy for a decoupling free energy for a decoupling wall in the domain Λ_n, defined through (10.2.13) for free associated boundary conditions ($W\equiv0$).

Lemma 10.3.1. *For bulk potentials, K satisfying condition A and $K\in\mathscr{B}_{M_1,M_2}(Q)$ and wall potentials which are decoupling, $f_n^\times(K)$ is a uniformly continuous function of K for each fixed n.*

Proof. The function

$$g(K,\zeta)=\sum_{A\subset\Lambda_1^{(n)}\cdot\Lambda_2^{(n)}} K_A\langle s_A\rangle^\zeta(K), \tag{10.3.3}$$

where $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ are the two subdomains into which Λ_n is divided, is bounded by $M_1^2\|S\|^{M_1}$. Hence by Proposition 10.3.2 and the expression (10.2.13) the conclusion follows.

We state a further result on the sequence of functions $f_n^{\times}(K)$ which is a corollary of the proof of Theorem 10.2.1.

Lemma 10.3.2. *Under the hypotheses of Theorem 10.2.1* $f_n^{\times}(K) \to f_\infty^{\times}(K)$ *uniformly in* K *(i.e. with respect to the norm* $\|\cdot\|_0$*).*

Additional properties of $\{f_n^{\times}\}$ and f_∞ follow.

Lemma 10.3.3. *Under the hypotheses of Theorem* 10.3.1, *the functions* $\{f_n^{\times}(K)\}$ *are uniformly equicontinuous and* $f_\infty(K)$ *is uniformly continuous.*

Proof. The uniform continuity of $f_\infty(K)$ follows from Lemma 10.3.1 and Lemma 10.3.2 since a sequence of uniformly continuous functions in a set in a metric space which converge uniformly to a limit function implies uniform continuity of the limit function. The uniform equicontinuity of $\{f_n^{\times}(K)\}$ follows as a result of the proposition: a uniformly convergent sequence of uniformly continuous function in a set in a metric space is a uniformly equicontinuous family.

We state a final elementary proposition which deals with equicontinuous functions on a dense subspace. □

Proposition 10.3.3. *Let* $\{f_n\}$ *be a sequence of uniformly equicontinuous functions from one metric space,* (X, ϱ), *to another,* (Y, d) *with* Y *complete. If for a dense set* $D \subset X$, $f_n(x) \to f(x)$ *uniformly in* D, *then* $f_n(x)$ *converges uniformly to a limit function which is uniformly continuous.*

Proof of Theorem 10.3.1. The proof consists of applying Proposition 10.3.3 to $\{f_n^{\times}(K)\}$ with the properties proved in Lemma 10.3.3. The dense set, D, consists of $\mathscr{B}_{M_1,M_2}^{\times}(Q)$ while the space Y is $\mathscr{B}_{\infty,M_1,M_2}^{\times}(Q)$. Note that the role of Q is simply to control the number of repeats in a set A where $K_A \neq 0$. □

The theorem may be generalized to include more generally shaped domains and more general wall and associated wall potentials. We state below a general result along these lines.

Theorem 10.3.2. *Long range interactions with general domains and wall potentials. Let* K, W, $\tilde{W}$ *be a set of interactions, wall and associated wall potentials respectively, which satisfy the boundedness conditions* **A** and **F** *and the defining conditions* **D** *and* **E** *and the correlation decay condition,* $\mathbf{N}_\delta(\delta > 0)$. *Let* $\{\Omega_n\}$ *be a sequence of simple domains. Then the limiting boundary free energy exists and is independent of the associated wall potentials,*

$$\lim_{n\to\infty} f_\times(K, W, \tilde{W}, \Omega_n) = f_\times^\infty(K, W). \tag{10.3.4}$$

10.4. Periodic Boundary Free Energy with the Correlation Decay Assumption

The periodic boundary free energy, $f_\times^\pi = f_\times(K, W; \Pi_{\mathbf{L}})$ where $\Pi_{\mathbf{L}}$ is a torus form of size $\mathbf{L} = (L_1, \ldots, L_d)$, has been defined in Sect. 8, where several bounds and partial results have been obtained. A basic question is whether the periodic boundary free energy is asymptotically equal to the usual wall free energy with free associated walls. When this is the case, i.e. when $f_\times^\pi = f_\times^0$, then by Proposition 8.2.5, one has the result that a torus (or box with periodic boundary conditions) has no boundary free energy, so that the free energy of the torus has the form

$$F(K,\Pi_{\mathbf{L}}) \equiv F(K^{\pi},\Lambda_{\mathbf{L}}) = |\Lambda_{\mathbf{L}}| f_{\infty}(K) + o(\Lambda_{\mathbf{L}}). \tag{10.4.1}$$

For finite-range interactions, the equality $f_{\times}^{\pi} = f_{\times}^{0}$ has been shown in Sect. 10.2 under the correlation decay condition $\mathbf{M}_{\delta}$ as a consequence of the equality $f_{\times}^{*} = f_{\times}^{0}$. For infinitely ranged interactions, however, the proof is more complicated. The proof uses uniformity of the periodic interactions in a way which will be discussed below. First, we must define the concept of quasi-periodic interactions. To make this precise we define some notation.

Let $\Lambda(\ell)$, $\ell = (l_1, \ldots, l_d)$, be the domain specified by lattice vectors $\mathbf{R} = \sum_{\alpha=1}^{d} n_{\alpha}\mathbf{a}_{\alpha}$ where $0 \leqq n_{\alpha} \leqq l_{\alpha}$, $l_{\alpha} \in Z$, and $\mathbf{a}_{\alpha}$ are defined in Sect. 1.1. We can divide the lattice, as in [3], into congruent images, $\Lambda_t(l)$, labelled by $\mathbf{t} = (t_1, \ldots, t_d)$, $t_i \in Z$, where $\Lambda_{\mathbf{0}}(\ell)$ is the original box, and define a set of lattice vectors by $\ell_{\mathbf{t}} = \sum_{\alpha=1}^{d} t_{\alpha} l_{\alpha} \mathbf{a}_{\alpha}$. Given a collection of sites $A = \{x_1, x_2, \ldots, x_{N(A)}\}$, $x_i \in \Lambda_{\mathbf{0}}(\mathbf{l})$ (possibly with repeats), a periodic collection, $A^{\pi} \subset L$, is a collection of sites which are drawn from $\{x_1 + \ell_{t_1}, \mathbf{x}_2 + \ell_{t_2}, \ldots, \mathbf{x}_{N(A)} + \ell_{t_{N(A)}}\}$ with the constraint that at least one of the t_i vanishes.

The set of all such collections A^{π} will be denoted $\{A^{\pi}\}$. Given a set of potentials, $K = \{K_A\}$, the periodic interaction potential $K^{\pi} = \{K^{\pi}(A)\}$ for the domain $\Lambda(\ell)$ is then defined by

$$K^{\pi}(A) = \sum_{B \in \{A^{\pi}\}} K_B. \tag{10.4.2}$$

Definition 10.4.1. Quasi-periodic potentials. Given a set of potentials $K = \{K_A\}$ we define the corresponding quasi-periodic potentials K_R^{π} by

$$K_R^{\pi}(A) = \sum_{B \in \{A_R^{\pi}\}} K_B, \tag{10.4.3}$$

where A_R^{π} are drawn from $\{\mathbf{x} + \ell_{t_1}, \mathbf{x}_2 + \ell_{t_2}, \ldots, \mathbf{x}_{N(A)} + \ell_{t_{N(A)}}\}$ as in ordinary periodic potentials with restriction that $|\ell_{t_j}| \leqq R$.

In addition to the norm $\|\cdot\|_0$ [(2.3.9)] given in **F(i)** we define a set of periodic norms $\|\cdot\|_{\Lambda(\ell)}$ by

$$\|K\|_{\Lambda(\ell)} = \sum_{\substack{B \ni 0 \\ B \in \{A^{\pi}\}}}' |K_B|, \tag{10.4.4}$$

where the prime indicates that not all of the ℓ_{t_i} are zero.

By viewing periodicity as repeated images and counting translations, one has the following relationship between the two norms.

Proposition 10.4.1. *Given a potential* $K = \{K(A)\}$, *if* $\|K\|_0 < \infty$, *then*

$$|\Lambda(l)| \, \|K\|_{\Lambda(\ell)} \leqq A[\Lambda(\ell)] \, \|K\|_0. \tag{10.4.5}$$

We can define a quasi-periodic boundary free energy with K^π replaced by K^π_R, i.e., we write

$$f_\times^{\pi,R}[K,\Lambda(\ell)]=\frac{F[K,\Lambda(\ell)]-F[K_R^\pi,\Lambda(\ell)]}{A[\Lambda(\ell)]}. \tag{10.4.6}$$

Using the notation $K_R=\{K_R(A)\}$ for potentials truncated at R, we have:

Lemma 10.4.1. *Under the hypotheses of Proposition 10.4.1,*

$$f_\times^\pi[K,\Lambda(\ell)]-f_\times^{\pi,R}[K,\Lambda(\ell)]|\leqq C\|K-K_R\|_0, \tag{10.4.7}$$

independently of $\Lambda(\ell)$.

Proof. One considers the Hamiltonians on $\Lambda(\ell)$ with potentials K, K^π, and K^π_R respectively. Upon using bounds analogous to those in Sect. 1.4 one has

$$|f_\times^{\pi,R}(K,\Lambda)-f_\times^\pi(K,\Lambda)|\leqq\frac{1}{A(\Lambda)}\langle\mathscr{H}(K_R^\pi)-\mathscr{H}(K^\pi)\rangle\leqq\frac{|\Lambda|}{A(\Lambda)}\|K-K_R\|_{\Lambda(\ell)}. \tag{10.4.8}$$

Using Proposition 10.4.1 we obtain (10.4.7). □

As a consequence we have the following uniformity result:

Lemma 10.4.2. *For any potential $K=\{K(A)\}$ for which $\|K\|_0<\infty$,*

$$f_\times^{\pi,R}(K,\Lambda(\ell))\to f_\times^\pi(K,\Lambda(\ell))\quad\text{as}\quad R\to\infty \tag{10.4.9}$$

independently of $\Lambda(\ell)$.

The implication of the preceding two lemmas is that the quasi-periodic potentials may be treated much like ordinary wall potentials of finite range. The equicontinuity result (10.4.7) and the uniformity result (10.4.10) may be used with the techniques of Sects. 7 and 10.3 to establish the basic result:

Theorem 10.4.1. *Equivalence of periodic free energy with original definition.*

In a symmetric system (i.e., one for which $K_A=K_{\mathscr{R}A}$ for a set of operations carrying one wall into another), satisfying the conditions for boundedness, namely, **A** and **F**, and the condition $\mathbf{N}_\delta(\delta>0)$, if $\Lambda(\ell)\to\infty$ through a simple sequence of domains, then the limiting periodic free energy, $f_\times^\pi$ exists, and

$$\lim_{\Lambda(\ell)\to\infty}f_\times^\pi(K,\Lambda(\ell))=f_\times^0(K,W^0), \tag{10.4.10}$$

where W^0 is the wall potential for a free wall. For a set of interactions which are bounded, the convergence is uniform and $f_\times^0$ is uniformly continuous in K.

11. Boundary Free Energy for Vector Spin Systems

Having concluded our discussion of scalar spin systems we now turn to systems with vector-valued spins and consider, in particular, the extension of some of the previous results to the two-component case.

11.1. Vector Spin Systems

To each site i in the lattice we associate a vector $\mathbf{s}_i = (\mathbf{s}_i^x, \mathbf{s}_i^y) \in \mathbb{R}^2$ with unit length (i.e., $\|s_i\| = 1$) and call it a two-component (classical) spin. The spin is parameterized by the points of the unit sphere in $\mathbb{R}^2$ and we choose, as its distribution, the normalized uniform measure on this sphere. Since the basic restrictions on the range of validity of these results will be those imposed on the vector-spin inequalities, one can often take a more general Haar measure (see e.g., Ginibre [4]).

The (reduced) Hamiltonian for such a system is given by:

$$\bar{\mathscr{H}}(\Omega) = \sum_{A \subset \Omega} \left[\dot{K}_A \prod_{i,j \in A} (s_i^x s_j^x + s_i^y s_j^y) + \ddot{K}_A \prod_{i,j \in A} (s_i^x s_j^x - s_i^y s_j^y) \right] + \sum_{i \in \Omega} h_i s_i^x , \tag{11.1.1}$$

which is a generalization of the usual XY model Hamiltonian.

The basic tools in analyzing the boundary free energies are the correlation inequalities due to Ginibre [4]. For the two-component system of N spins, let the spin s_i be described by $s_i = (\cos\theta_i, \sin\theta_i)\, 0 \leqq \theta \leqq 2\pi$ and write $\Theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $P \equiv (p_1, p_2, \ldots, p_N)$, $p_i \in \mathbb{Z}$ and $\Theta \cdot P = \sum_{i=1}^{N} \theta_i \cdot p_i$. Let $\mathscr{F}_2$ be the set of functions on the s_i which can be expanded in powers of $\cos P \cdot \Theta$ with positive coefficients only. We now recall the following proposition due to Ginibre (and discussed in this form by Kunz et al. [9]) and make the following definition.

Proposition 11.1.1. *If $\bar{\mathscr{H}} \in \mathscr{F}_2$ and $f, g \in \mathscr{F}_2$ then*

$$\langle f \rangle \geqq 0 , \tag{11.1.2}$$

$$\langle fg \rangle - \langle f \rangle \langle g \rangle \geqq 0 . \tag{11.1.3}$$

Definition 11.1.1. Hamiltonians of positive type. A plane rotator Hamiltonian, $\bar{\mathscr{H}}(\Omega)$, of positive type is one which is of the form of (11.1.1) with $\dot{K}_A$, $\ddot{K}_A$, and h_i all nonnegative.

The Hamiltonian of positive type plays a role similar to that of the ferromagnetic Hamiltonian in the scalar spin models, as we have the analog of the Griffiths inequalities, since a Hamiltonian of positive type is in $\mathscr{F}_2$. One can also consider more general Hamiltonians involving higher powers of s_i.

The results of Sect. I can be carried over to the two-component case with the introduction of a new norm

$$\|K_A\| = |\dot{K}_A| + |\ddot{K}_A| . \tag{11.1.4}$$

Likewise, the definitions of free walls, subfree walls and ferromagnetic walls are modified by requiring that the previous conditions on W_A and K_A now apply for $\dot{W}_A$, $\dot{K}_A$ and $\ddot{W}_A$, $\ddot{K}_A$, in a similar manner. The wall Hamiltonians $\mathscr{W}(\Omega)$, are defined in analogy with (3.1.5) and (3.1.6) as the difference of wall Hamiltonians of positive type.

11.2. Properties and Existence of Boundary Free Energy for Vector Spin Systems

The basic property of boundedness for vector spin systems is as stated in Proposition 3.2.2 with the conditions **A**, **F**, **G(i)**, and **H** in the context of this section. Likewise, the properties of *boundary convexity*, and *negativity* and *monotonicity* are identical to the corresponding Propositions 3.2.1 and 3.2.3 with the understanding that the variables with respect to which these properties hold are $\dot{W}_A$, $\ddot{W}_A$, $\dot{K}_A$, and $\ddot{K}_A$.

The basic inequalities needed for proving the existence theorems are the analogs of Propositions 3.3.1 and 3.3.2. Using Proposition 11.1.1 these results remain valid with the concepts of subfree, superferromagnetic, etc., redefined as discussed above and with the *extended subfree* (and *superferromagnetic*) consistently relations now required for both $\dot{W}$ and $\ddot{W}$.

The various existence theorems of I and II may now be proved for vector spin systems. We state one of the main results:

Theorem 11.2.1. *Single box domains for vector spin systems. For a box domain,* $\Lambda_{\mathbf{L}}$, *of sides* $L_1, L_2, \ldots, L_d$ *blocks in length, with a Hamiltonian of positive type (see Definition 11.1.1) and bulk potentials,* K, *satisfying condition* **A** *and ferromagnetic wall potentials, which are subfree as defined above and satisfy conditions* **D**, **E**, **F**, *and* $\mathbf{C}_\tau$ *with* $\tau > 1$, *but do not necessarily respect the minimal symmetry requirements* **D**(*iii*) *and* **E**(*iii*), *the total (reduced) free energy,* $F(\Lambda_L) \equiv F(K, \{W_\alpha\}, \Lambda_L)$, *verifies*

$$\lim_{\mathbf{L}\to\infty} [F(\Lambda_L) - |\Lambda_L| f_\infty(K)]/2A_{\mathbf{L}} = \sum_{\alpha=1}^{d} \lambda_\alpha^\infty f_\times^0(K, W_{(\alpha)}), \tag{11.2.1}$$

where the limit $\mathbf{L} = (L_1, L_2, \ldots, L_d) \to \infty$ *may be taken in any way, while* $f_\infty(K)$ *is the limiting bulk free energy, the total wall or boundary area is*

$$A_{\mathbf{L}} = \sum_{\gamma=1}^{2d} |\mathbf{L}_{(\gamma)}| \quad \text{with} \quad |\mathbf{L}_{(\gamma)}| = \sum_{\beta \neq \gamma}^{d} L_\beta, \tag{11.2.2}$$

and the limiting wall ratios are given by

$$\lambda_\alpha^\infty = \lim_{\mathbf{L}\to\infty} (|\mathbf{L}_{(\alpha)}|/A_{\mathbf{L}}) = \lim_{\mathbf{L}\to\infty} \left[2L_\alpha \sum_{\beta=1}^{d} (1/L_\beta)\right]^{-1}. \tag{11.2.3}$$

If the lattice is symmetric, then (11.5.1) *simplifies to*

$$\bar{F}(\Lambda) \equiv V(\Lambda) f(\Lambda) = V(\Lambda) f_\infty(K) + A(\Lambda) f_\times^0(K, W) + o[A(\Lambda)]. \tag{11.2.4}$$

11.3. Systems with Varying Numbers of Components

It is of interest to enquire as to the relation, if any, between the boundary free energies of a system with n-component spins and the corresponding system with $(n-1)$-component spins. Indeed for suitably restricted sets of interactions we can establish various inequalities. Although we have not proven the existence of the thermodynamic limit for $n > 2$ the inequalities are nevertheless valid for any domain, Ω, and hence for lim sup and lim inf as $\Omega \to \infty$.

For an isotropic system of spins with pair interactions of the form $\sum_{\alpha} K_{ij}^{(\alpha)} s_i^{(\alpha)} s_j^{(\alpha)}$, where α denotes the components of the vectors, and $K_{ij}^{(\alpha)} \geqq 0$, one has an inequality [9],

$$\langle s_i \cdot s_j \rangle_{n-1} \geqq \langle s_i \cdot s_j \rangle_n, \tag{11.3.1}$$

where the subscripts denote number of components. One may use this result in conjunction with the previously established properties of the boundary free energy to show that

$$f_{\times}^{\pm}(n-1; K, W, \tilde{W}, \Omega) \geqq f_{\times}^{\pm}(n; K, W, \tilde{W}, \Omega) \tag{11.3.2}$$

(where n and $n-1$ denote the number of spin components in the system) providing the walls and associated walls are of ferromagnetic character and $K_{ij}^{(\alpha)} \geqq 0$ and $h_i \geqq 0$.

A partial result may be obtained in the other direction for $n=2$ under special circumstances for anisotropic systems. The details may be found in [3].

This concludes our discussion of two-component systems. At present we cannot extend the existence theorems to the n-component case for $n > 2$ since the inequalities corresponding to Proposition 11.1.1 have not yet been established.

Acknowledgments. It is a pleasure to thank Professors Michael E. Fisher, James Glimm, Leonard Gross, and Pierre-Andre Vuillermot for numerous discussions during the course of this work. The support of the National Science Foundation through Grant PHY-78-08066 at The Rockefeller University is gratefully acknowledged. The author also thanks the Baker Laboratory and the Material Sciences Center for its support during the initial part of this work.

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Communicated by E. Lieb

Received January 30, 1980; in revised form March 27, 1980