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The Renormalization Group of the Model of A^4-Coupling in the Abstract Approach of Quantum Field Theory

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Abstract. For the model of A^4-interaction the postulates of the renormalization group are stated within the abstract approach of quantum field theory. In the massive case these postulates follow if an on-shell formulation of the model is assumed to exist. For the massless model the postulates of the renormalization group imply that the propagator has a pole at momentum zero. Consequently there is no dynamic mass generation and the propagator is normalizable on the mass shell. It is shown that the S-matrix elements scale with canonical dimensions. A general method of rescaling parameter values is developed which takes into account the possibility of propagator zeros and stationary points of the effective coupling.

1. Introduction

Originally the renormalization group was introduced by Petermann and Stueckelberg within the framework of the perturbative treatment of Lagrangian field theory [1]. The work of Gell-Mann and Low, as well as Bogoliubov and Shirkov, has made evident that the concept of the renormalization group goes far beyond perturbation theory [2, 3]. Consequences for the infrared and ultraviolet behavior of Green's functions were derived which correspond to summing up contributions from all orders of perturbation theory. The results obtained modify or sometimes even contradict the approximative statements of perturbation theory. In recent years renormalization group properties have been extensively studied further [4–19]. In particular, new powerful methods were developed by Callan and Symanzik [6, 7]. Among the many important conclusions of the renormalization group approach the most remarkable one is perhaps the phenomenon of asymptotic freedom which has had an enormous impact on our understanding of elementary particle interactions [17–19].

The purpose of this and a forthcoming paper is to set up the renormalization group within the abstract approach of quantum field theory, independent of

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perturbation theory and without reference to an ultraviolet cut-off. In addition to the usual postulates of abstract field theory[1] some general information is taken from Lagrangian field theory concerning the existence and uniqueness of solutions for the model considered. Perturbation theory is only used where applicable, namely in connection with the asymptotic behavior of Green's functions in the small coupling limit.

In this first paper the model of a neutral scalar field $A(x)$ with quartic interaction will be studied. The renormalization group will be defined as the group of all equivalence transformations

$$A(x) \to z^{1/2} A(x), \qquad z > 0, \tag{1.1}$$

which multiply a field operator by a finite positive number. Obviously, such transformations have no other effect than changing the normalization of the field. The concept of the renormalization group thus merely reflects the arbitrariness in normalizing quantized field operators.

For the massive model a uniquely distinguished normalization of $A(x)$ is suggested by the asymptotic behavior of the system in scattering processes. According to the Haag-Ruelle scattering theory $A(x)$ approaches asymptotic fields $A_{\text{in}}(x)$, $A_{\text{out}}(x)$ for $x_0 \to \pm\infty$ [23, 24]. The field operator $A(x)$ is then normalized on mass shell by applying the standard normalization of a free quantized field to the incoming field operator. This is equivalent to requiring that the residue of the propagator pole equals its free field value. Under this normalization condition it will be postulated that field operators $A(x, g, m)$ exist which describe the model for given renormalized mass $m > 0$ and suitably defined coupling constant g below an appropriate bound. $A(x, g, m)$ corresponds to the conventional perturbative expansion. The precise connection to perturbation theory is made by requiring that the derivatives of the time-ordered Green's functions with respect to g at $g = 0$ coincide with the corresponding expressions of the respective expansions[2].

For discussing the limit of vanishing mass the on-shell normalization is not appropriate since it may cause an infrared singular behavior of the Green's functions for $m \to 0$. The field operator should therefore be normalized off-shell employing a normalization mass $\kappa^2 < 0$. Moreover, the symmetry point of the vertex function defining the coupling constant should be chosen off the mass shell. By suitable redefinition of field operator and coupling constant it will be verified that such an off-shell formulation of the massive model is always possible. The properties of the renormalization group then follow in the form of statements on the existence and uniqueness of the field operator for given mass m, coupling constant g and normalization mass κ. This implies the differential equation of the renormalization group in their standard form [4, 8].

The derivation of the renormalization group properties thus obtained is similar to the approach chosen by Wilson in his presentation of the Gell-Mann Low

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1 For the formulation of the postulates of abstract quantum field theory see the monographs [20–22]

2 This requirement excludes solutions of the model which are not linked to the perturbative expansion in the weak coupling limit. An example of such a solution has been given by McCoy and Wu for the massless Thirring model [25]. It is of particular interest that this solution shows the phenomenon of mass generation

analysis of quantum electrodynamics [10]. Starting point in Wilson's work is the conventional renormalization scheme of quantum electrodynamics from which an off-shell formulation is constructed which remains valid in the limit of vanishing electron mass. Renormalization group properties then hold for the off-shell formulation which imply the Gell-Mann Low equation of the photon propagator.

For the massless A^4-model it would not be justified a priori to normalize the field operator on the mass shell. For asymptotic fields cannot be defined in the perturbative treatment since the propagator develops a singularity of the form

$$\Delta'_F \sim \frac{\lg^\alpha k^2}{k^2}. \tag{1.2}$$

Therefore, the renormalization group properties of the off-shell formulation are postulated for the massless model – by analogy to the massive case in which these properties can be proved. No assumption on the position of the propagator singularity is made since this would not be independent from the other postulates. The absence of an intrinsic mass is indicated by the scaling relations which express the fact that the normalization mass is the only dimensional parameter of the model.

On the basis of these postulates it will be proved that the propagator of the massless model is singular at momentum zero. Therefore, massless modes must be present. Spontaneous mass generation is thus ruled out for solutions of the massless A^4-model which satisfy the postulates of the renormalization group[3]. This is a special case of a general phenomenon found by Gross and Neveu for infrared stable massless models with one coupling constant [26]. Moreover, Symanzik's result follows that this singularity of the propagator must be a pole with a cut also starting at momentum zero[3]. Hence the exact propagator does not suffer from the infrared problems present in perturbation theory. Accordingly the field operator may be normalized on-shell and an S-matrix can be defined. It will be shown that as a consequence the S-matrix elements satisfy Callan-Symanzik equations with no anomaly of the dimension. This confirms a result by Gross and Wess [27, 28] that S-matrix elements of massless models – if they exist – always have canonical dimensions.

In a general treatment of the renormalization group two complications arise which are not present in perturbation theory. The first one concerns the uniqueness of field operators for given parameter values. For massless models with one coupling constant it was shown in [29] that extrema of the effective coupling are compatible with the requirements of the renormalization group. In this case the quantities of the system become multivalued functions of the coupling constant. The extremal values of the effective coupling are zeros of β at which β^{-1} is still integrable. An example of a model showing such behavior was given in [30]. In the present paper it will be discussed for the A^4-model that a similar phenomenon may already occur in the on-shell formulation of a massive theory. The reason is that the functional defining the renormalized coupling constant may have extrema in terms of another parametrization. In such a situation distinct field

3 These statements would not apply to solutions of the model which are not related to conventional perturbation theory in the weak coupling limit (see footnote 2 and [25])

operators exist with equal parameter values g and κ. In view of this the uniqueness postulates of the renormalization group should only be formulated locally for field operators of sufficiently weak coupling. By applying equivalence transformations (1.1) the field operator of the massless model can then be constructed as a (multivalued) function of g and κ for the full range between the weak coupling limit and the first ultraviolet fix point.

Another complication arises if the propagator becomes negative in part of the Euclidean region due to subtractions of the Lehmann representation. For negative values the propagator cannot be normalized in the conventional manner. Appropriate modifications of the normalization procedure will therefore be proposed.

In Sect. 2 the renormalization group properties of the massive A^4-model are derived from the assumption that solutions of the model exist which can be normalized on the mass shell. For the massless model the postulates of the renormalization group are stated in Sect. 3 and used for proving some general consequences. Applications to the propagator and the effective coupling are made in Sect. 4. It is shown that the propagator and the effective coupling are made in Sect. 4. It is shown that the propagator of the massless model has a pole at momentum zero so that mass generation cannot take place. A general method of rescaling parameter values in the presence of propagator zeros and stationary points of the effective coupling is given in Sect. 5. In Sect. 6 some comments on the ultraviolet limit are made. In conclusion it is shown that the S-matrix elements of the massless models scale with canonical dimensions (Sect. 7).

2. Massive Model of A^4-Coupling

In this section we consider the model of a hermitean scalar field A with A^4-coupling describing the interaction of massive neutral particles of spin zero. The discussion of the massless case will be postponed until the following section. The model will be considered for values of a renormalized coupling constant g_0 and mass m from a domain

$$0 \leqq g_0 < \eta_0 \tag{2.1}$$

$$0 \leqq m_- < m < m_+ \tag{2.2}$$

with suitable bounds η_0 and m_+. η_0 and/or m_+ may be $+\infty$. It is assumed that for given parameter values g_0 and m from the domain (2.1–2) a field operator

$$A = A(x, g_0, m^2) \tag{2.3}$$

exists which describes the model of A^4-coupling and satisfies the conditions

$$-i(k^2 - m^2)\Delta'_F(k^2, g_0, m^2) = 1 \quad \text{at} \quad k^2 = m^2 \tag{2.4}$$

$$\Gamma(k_1^0, \ldots, k_4^0, g_0, m^2) = -ig_0 \tag{2.5}$$

for the propagator Δ'_F and the four-point vertex function Γ. $k_1^0, \ldots, k_4^0$ denote a choice of constant vectors on the mass shell $k_j^{0\,2} = m^2$. The conventional symmetry point with

$$k_j^{0\,2} = m^2\,, \quad (k_i^0 + k_j^0)^2 = \tfrac{4}{3}m^2\,, \quad i \neq j\,, \tag{2.6}$$

may for instance be taken provided the vertex function can be continued analytically to this point. In the limit of small coupling the operator $A(x, g_0, m^2)$ should yield the perturbative expansion of the Lagrangian formalism which is invariant under the reflection $A \to -A$. A precise formulation of this connection to perturbation theory will be given below in terms of time-ordered Green's functions[4].

The usual postulates of quantum field theory, such as Lorentz invariance, locality, spectrum conditions, etc. are supposed to hold for the field operator [20–22]. The Wightman functions should satisfy the scaling relations of dimensional analysis

$$W(x_1, \dots, x_n, g_0, a^2 m^2) = a^n W(ax_1, \dots, ax_n, g_0, m^2), \tag{2.7}$$

which express the fact that the dimensionless quantity $m^{-n} W$ only depends on dimensionless variables.

In this section it will be assumed that in momentum space time-ordered functions τ can be constructed from Wightman functions W without subtractions. The Lehmann representation of the propagator then does not require subtractions. This assumption is not essential and is only made in order to simplify the discussion of the present section.

We now state more precisely how the field operators $A(x, g, m)$ are related to standard perturbation theory. In the Lagrangian formulation of the massive A^4-model the expansions of the time-ordered Green's functions with respect to powers of g_0 are uniquely determined by the normalization conditions (2.4–6). The operators $A(x, g, m)$ are assumed to exist with the property that the conditions (2.4–6) hold and that the derivatives of their time-ordered vacuum expectation values with respect to g_0 coincide at $g_0 = 0$ with the corresponding perturbative expressions of the Lagrangian formulation.

The scaling relation (2.7) implies that the model can be extended from (2.2) to any positive value $m > 0$ of the mass so that the domain of admissible parameter values becomes

$$0 \leqq g_0 < \eta_0, \quad m > 0. \tag{2.8}$$

The restriction on the coupling constant may be essential for the existence of a solution $A(x, g_0, m^2)$ since the admissible values of g_0 must belong to the range of values which the vertex function can assume at the chosen reference point. Examples for such a situation will be provided by the following discussion.

For the model considered there is no unique choice for the definition of the renormalized coupling constant. In principle, a coupling constant G could be defined by any finite functional $\Phi(A)$ of the field operator provided it is continuously differentiable as a function of g_0 and behaves like

$$\Phi(A(x, g_0, m^2)) = g_0 + o(g_0^2) \tag{2.9}$$

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4 We do not consider solutions with spontaneous breaking of the reflection symmetry $A \to -A$. The existence of such solutions has not yet been established rigorously in perturbation theory. I am grateful to Dr. Becchi for a discussion of this point

for small values of g_0. The new coupling constant G is then defined by requiring

$$\Phi(A) = G. \tag{2.10}$$

$A'(x, G, m^2)$ denotes the field operator determined by the normalization conditions (2.4) and (2.10). G is a function of the original coupling parameter g_0

$$G = G(g_0) = g_0 + o(g_0^2), \tag{2.11}$$

which can be inverted

$$g_0 = g_0(G) \tag{2.12}$$

in an interval $0 \leqq g_0 < C$ since

$$\frac{\partial G}{\partial g_0} = 1 \quad \text{at} \quad g_0 = 0. \tag{2.13}$$

In this interval

$$A'(x, G(g_0), m^2) \equiv A(x, g_0, m^2).$$

At some value of g_0 the function G may have a maximum. Then g_0 is not uniquely determined for given G so that the model admits distinct field operators with identical normalization conditions (2.4) and (2.10). On the other hand (2.4) with the new condition (2.19) may permit a larger range

$$0 \leqq G < \eta' \tag{2.14}$$

for a unique parametrization of the field operator. If $g(G)$ has extrema in this region the model admits distinct field operators with identical normalization conditions of the conventional type (2.4–6). Let

$$G_1 < G_2 < \ldots < G_a \tag{2.15}$$

be the positions of the extrema of $g_0(G)$ in (2.14). In terms of g_0 the field operator may be parametrized by different branches

$$\begin{aligned} A(x, g_0, 0, m^2) &= A'(x, G, m^2), \qquad 0 \leqq G \leqq G_1 \\ &\ \cdots \\ A(x, g_0, j, m^2) &= A'(x, G, m^2), \qquad G_{j-1} \leqq G \leqq G_j \\ &\ \cdots \\ A(x, g_0, a+1, m^2) &= A'(x, G, m^2), \qquad G_a \leqq G \leqq \eta'. \end{aligned} \tag{2.16}$$

In this notation (2.3) represents the branch $j=0$. The bound η_0 of (2.1) must be chosen less or equal to G_1, the position of the first maximum.

The normalization condition (2.4) states that the propagator $-i\Delta'_F$ has a pole at $k^2 = m^2$ with residue one. This normalization on mass shell is the natural choice from the view point of scattering theory since then the field asymptotically approaches the conventional incoming and outgoing fields. For the discussion of the limit $m \to 0$, however, the normalization on mass shell is problematic. In perturbation theory the propagator and the vertex function develop logarithmic singularities near the mass shell for $m \to 0$. For this reason the possibility of normalizing the field operator off-shell will now be studied, at first for the massive case $m > 0$.

A transformation of the form

$$A'(x) \to A''(x) = z^{1/2} A'(x), \qquad z > 0 \tag{2.17}$$

will be called an equivalence transformation. It is trivial in the sense that it only changes the normalization of the field by a finite positive factor. The renormalization group is defined as the group of all equivalence transformations.

We now apply an equivalence transformation to the field operator (2.3) or (2.16)

$$\begin{aligned} &A(x) = z^{1/2} A(x, g_0, m^2) \\ \text{or} \qquad &\qquad\qquad\qquad\qquad z > 0\,. \\ &A(x) = z^{1/2} A(x, g_0, j, m^2) \end{aligned} \tag{2.18}$$

Let $\mathscr{F}$ denote the family of all operators which can be obtained this way. In the work that follows some properties of the family $\mathscr{F}$ will be derived. First we try to define the coupling constant by an expression which is invariant under the renormalization group. We observe that the quantity

$$V(k_1 \ldots k_4) = i \prod_{j=1}^{4} \sqrt{-i(k_j^2 - m^2) \Delta'_F(k_j)}\, \Gamma(k_1 \ldots k_4) \tag{2.19}$$

is an invariant of the renormalization group. According to Wu the connected time-ordered functions with more than two coordinates can be expanded in the form [31]

$$\tau_{\text{conn}}(k_1 \ldots k_n) = \prod_{j=1}^{n} \sqrt{i\Delta'_F(k_j^2}\, F(V, k_1, \ldots, k_n)\,. \tag{2.20}$$

Here F is given by Wu's renormalized version of the skeleton expansion as a functional of (2.19) alone. The function (2.19) may therefore be considered as a measure for the strength of interaction. Now we define the effective coupling Q as a function of k^2 by

$$Q(k^2) = V(k_1 \ldots k_4) \tag{2.21}$$

with the momenta

$$k_j = k_j(k^2) \tag{2.22}$$

taken at the symmetry point

$$\begin{aligned} &k_j^2(k^2) = k^2\,, \qquad (k_i(k^2) + k_j(k^2))^2 = \tfrac{4}{3} k^2 \\ &\qquad i \neq j\,. \end{aligned} \tag{2.23}$$

By definition Q is an invariant of the renormalization group. Due to (2.4) the relation (2.5) for the coupling constant may be written in the form

$$Q(k^2, g_0, m^2) = g_0 \quad \text{at} \quad k^2 = m^2\,. \tag{2.24}$$

Within the family $\mathscr{F}$ we construct an operator

$$A(x) = z^{1/2} A(x, g_0, m^2)\,, \tag{2.25}$$

which is normalized off the mass shell by the condition

$$-i(k^2-m^2)\Delta'_F(k^2)=1 \quad \text{at} \quad k^2=\kappa^2<m^2\,. \tag{2.26}$$

Since

$$\Delta'_F(k^2)=z\Delta'_F(k^2,g_0,m^2) \tag{2.27}$$

the factor z is given by

$$z^{-1}=-i(\kappa^2-m^2)\Delta'_F(\kappa^2,g_0,m^2)\,. \tag{2.28}$$

If the Lehmann representation of the propagator is unsubtracted, as it is assumed in this section, the value of z is uniquely determined and positive. Since the effective coupling is an invariant of the renormalization group we have

$$Q(k^2)=Q(k^2,g_0,m^2)\,. \tag{2.29}$$

Hence the defining Eqs. (2.5) or (2.24) of the coupling constant becomes

$$Q(k^2)=g_0 \quad \text{at} \quad k^2=m^2\,. \tag{2.30}$$

Since the vertex function becomes singular on the mass shell for $m\to 0$ we redefine the coupling constant by the value

$$g=g\left(g_0,\frac{\kappa^2}{m^2}\right)=Q(\kappa^2)=Q(\kappa^2,g_0,m^2) \tag{2.31}$$

of Q at the normalization mass $\kappa<m$. It is

$$g=g_0+o(g^2)\,. \tag{2.32}$$

Since

$$g\left(0,\frac{\kappa^2}{m^2}\right)=0\,, \qquad \frac{\partial g}{\partial g_0}\left(0,\frac{\kappa^2}{m^2}\right)=1\,. \tag{2.33}$$

g is a positive function for small positive g_0. Therefore it can be inverted,

$$g_0=g_0\left(g,\frac{\kappa^2}{m^2}\right) \quad \text{for} \quad 0\leqq g<\eta \tag{2.34}$$

and the field operator $A(x)$ expressed in terms of the new coupling constant g,

$$A(x)=A(x,g,\kappa^2,m^2)\,, \qquad 0\leqq g<\eta\,, \qquad \kappa^2<m^2\,. \tag{2.35}$$

It is obvious from the derivation that this field operator is uniquely determined by the conditions (2.26) and (2.30) within a local neighborhood of $\mathscr{F}$. In perturbation theory these conditions uniquely determine the time-ordered functions for any

$$\kappa^2\leqq m^2\,, \qquad g>0\,, \tag{2.36}$$

with no restriction to a local neighborhood. But this uniqueness property in this global sense may well be violated independent of perturbation theory.

The results obtained will be summarized by the following statements which hold for the family $\mathscr{F}$ of field operators.

A. Invariance under the Renormalization Group

If the field operator A belongs to the family $\mathscr{F}$ any operator A' obtained by an equivalence transformation

$$A'(x) = z^{1/2} A(x), \qquad z > 0, \tag{2.37}$$

also belongs to $\mathscr{F}$.

B. Existence and Uniqueness of Normalized Field Operators

In the family $\mathscr{F}$ exists a normalized field operator

$$A = A(x, g, \kappa^2, m^2) \tag{2.38}$$

for all parameter values

$$0 \leqq g < \eta, \qquad \kappa^2 \leqq m^2, \qquad m^2 > 0, \tag{2.39}$$

which satisfies the following three conditions B.1–3 for its propagator Δ'_F and four-point vertex function Γ.

B.1. Normalization of the Field Operator

$$-i(k^2 - m^2)\Delta'_F(k^2, g, \kappa^2, m^2) = 1 \quad \text{at} \quad k^2 = \kappa^2 \leqq m^2. \tag{2.40}$$

B.2. Value of the Renormalized Coupling Constant

$$Q(k^2, g, \kappa^2, m^2) = g \quad \text{at} \quad k^2 = \kappa^2 \leqq m^2. \tag{2.41}$$

B.3. Position of Mass Singularities

$$\Delta'_F(k^2, g, \kappa^2, m^2)^{-1} = 0 \quad \text{at} \quad k^2 = m^2. \tag{2.42}$$

In a sufficiently small neighborhood of $A(x, g', \kappa'^2, m^2)$ the conditions (2.40–42) uniquely determine the field operator $A(x, g, \kappa^2, m^2)$[5].

C. Small Coupling Limit

For $g \to +0$ all derivatives of the Green's functions with respect to g exist as distributions in the coordinates and are consistent with the perturbative expansions of the Lagrangian formalism obtained under the conditions (2.40–42).

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5 A neighborhood of $A(x, g', \kappa'^2, m^2)$ could for instance be characterized by

$$A(x) = z^{1/2} A(x, g'', \kappa''^2, m^2)$$

with

$$|g'' - g'| \leqq a, \qquad |\kappa''^2 - \kappa'^2| \leqq b, \qquad |z - 1| \leqq c$$

and a, b, c chosen sufficiently small

D. Scale Invariance

The Wightman functions of the normalized fields satisfy scale invariance in the sense of dimensional analysis

$$W(x_1, \ldots, x_n, g, a^2\kappa^2, a^2m^2) = a^n W(ax_1, \ldots, ax_n, g, \kappa^2, m^2) \tag{2.43}$$

(similarly for time ordered functions).

Field operators related by an equivalence transformation (2.37) are called equivalent. Two pairs of parameter values g_1, κ_1^2 and g_2, κ_2^2 are equivalent

$$g_1, \kappa_1^2 \sim g_2, \kappa_2^2,$$

if the corresponding field operators are equivalent. By these equivalence relations the family $\mathscr{F}$ and the set of all possible parameter pairs are divided into equivalence classes.

From the stated postulates it is straightforward to derive the form of the transformation between two fields belonging to equivalent pairs of parameter values. To this end we introduce the dimensionless function

$$R(k^2, g, \kappa^2, m^2) = -i(k^2 - m^2)\Delta'_F(k^2, g, \kappa^2, m^2). \tag{2.44}$$

As a consequence of statement D on scaling the function R and the effective coupling Q depend only on g and the dimensionless ratios k^2/κ^2 and m^2/κ^2

$$\begin{aligned} R &= R\left(\frac{k^2}{\kappa^2}, g, \frac{m^2}{\kappa^2}\right) \\ Q &= Q\left(\frac{k^2}{\kappa^2}, g, \frac{m^2}{\kappa^2}\right). \end{aligned} \tag{2.45}$$

The normalization conditions (2.40–41) are

$$R\left(u, g, \frac{m^2}{\kappa^2}\right) = 1 \quad \text{at} \quad u = 1, \tag{2.46}$$

$$Q\left(u, g, \frac{m^2}{\kappa^2}\right) = g \quad \text{at} \quad u = 1. \tag{2.47}$$

We now apply an equivalence transformation to the field operator with parameters g_1, κ_1^2

$$A(x) = z^{1/2} A(x, g_1, \kappa_1^2, m^2) \tag{2.48}$$

and try to satisfy the conditions (2.40–42) for $A(x)$ at other values g_2, κ_2^2. Requiring the condition (2.40) to hold at $k^2 = \kappa_2^2$ the value of z follows,

$$\begin{aligned} z^{-1} &= -i(\kappa_1^2 - m^2)\Delta'_F(\kappa_2^2, g_1, \kappa_1^2, m^2) \\ &= R\left(\frac{\kappa_2^2}{\kappa_1^2}, g_1, \frac{m^2}{\kappa_1^2}\right). \end{aligned} \tag{2.49}$$

If the Lehmann representation of the propagator is unsubtracted z exists and is positive for any $\kappa_2^2 = m^2$. Condition (2.41) yields the value of g_2,

$$g_2 = Q\left(\frac{\kappa_2^2}{\kappa_1^2}, g_1, \frac{m^2}{\kappa_1^2}\right). \tag{2.50}$$

The Green's functions of the new operator (2.48) fulfill the normalization conditions (2.40–42) for the parameter values g_2 and κ_2^2. By taking κ_2^2 sufficiently close to κ_1^2 the new operator will be in a neighborhood small enough that the uniqueness postulate applies. Hence

$$A(x) = A(x, g_2, \kappa_2^2, m^2).$$

We thus have the transformation law

$$A(x, g_2, \kappa_2^2, m^2) = z^{1/2} A(x, g_1, \kappa_1^2, m^2) \quad \text{if} \quad g_2, \kappa_2^2 \sim g_1, \kappa_1^2 \tag{2.51}$$

with z and g_2 given by (2.49) and (2.50). For the Fourier transform of the time-ordered functions we find

$$\tau(k_1 \ldots k_n g_2, \kappa_2^2, m^2) = z^{n/2} \tau(k_1 \ldots k_n g_1, \kappa_1^2, m^2). \tag{2.52}$$

Differentiating with respect to κ_1^2 and setting $\kappa_2^2 = \kappa_1^2 = \kappa^2$ the differential equation

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \beta \frac{\partial}{\partial g} + \tfrac{1}{2} n \gamma\right) \tau = 0 \tag{2.53}$$

follows. The coefficients β and γ are given by

$$\beta\left(g, \frac{m^2}{\kappa^2}\right) = \left.\frac{\partial Q\left(u, g, \frac{m^2}{\kappa^2}\right)}{\partial u}\right|_{u=1} \tag{2.54}$$

$$\gamma\left(g, \frac{m^2}{\kappa^2}\right) = \left.\frac{\partial R\left(u, g, \frac{m^2}{\kappa^2}\right)}{\partial u}\right|_{u=1}. \tag{2.55}$$

The invariance of the effective coupling under the renormalization group,

$$Q(k^2, g_2, \kappa_2^2, m^2) = Q(k^2, g_1, \kappa_1^2, m^2) \quad \text{if} \quad g_2, \kappa_2^2 \sim g_1, \kappa_1^2 \tag{2.56}$$

implies the differential equation

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \beta \frac{\partial}{\partial g}\right) Q = 0. \tag{2.57}$$

3. Postulates for the Massless Model

For the massless model of A^4-coupling the propagator and vertex function should not be normalized a priori on mass shell since perturbation theory indicates the possibility of infrared singularities. The off-shell formulation, however, is known to be infrared convergent in perturbation theory. We therefore base the formulation of the massless model on the properties A–D of the off-shell formulation which were derived in the previous section for the massive case. These statements will be formulated below as postulates for the massless model. But no assumption on the position of the mass singularity (statement B.3) will be made. For it will be shown later that the vanishing of the renormalized mass can be derived from the other postulates.

In the present section the possibility will be included that subtractions may be necessary in the defining equations of the time-ordered functions. The Lehmann representation of the propagator, for instance, may involve subtractions. However, a unique prescription for constructing time-ordered functions from Wightman functions should be selected and used in the normalization conditions below. Apart from such obvious requirements the precise definition of the time-ordered functions is not relevant.

We now state the postulates A–D for a family $\mathscr{F}$ of field operators A pertaining to the massless model of A^4-coupling[6].

A. Invariance under the Renormalization Group

If the field operator A belongs to the family $\mathscr{F}$ any operator A' obtained by an equivalence transformation

$$A'(x) = z^{1/2} A(x) \tag{3.1}$$

also belongs to $\mathscr{F}$. Moreover, the time-ordered functions should be constructed such that they transform under the renormalization group as the Wightman functions, namely,

$$\tau'(x_1 \ldots x_n) = z^{n/2} \tau(x_1 \ldots x_n).$$

B. Existence and Uniqueness of Normalized Field Operators

In the family $\mathscr{F}$ exists a normalized field operator

$$A = A(x, g, \kappa^2) \tag{3.2}$$

for the parameter values

$$0 \leqq g < \eta, \qquad \kappa_-^2 \leqq \kappa^2 \leqq \kappa_+^2 < 0,$$

which satisfies the following three conditions B.1–2 for its propagator Δ_F' and four-point vertex function Γ.

B.1. Normalization of the Field Operator

$$-ik^2 \Delta_F'(k^2, g, \kappa^2) = 1 \quad \text{at} \quad k^2 = \kappa^2. \tag{3.3}$$

B.2. Value of the Renormalized Coupling Constant

$$Q(k^2, g, \kappa^2) = g \quad \text{at} \quad k^2 = \kappa^2. \tag{3.4}$$

In a sufficiently small neighborhood of $A(x, g', \kappa^2)$ the conditions (3.3–4) should uniquely determine the field operator $A(x, g, \kappa^2)$.

C. Small Coupling Limit

For $g \to +0$ all derivatives of the Green's functions with respect to g should exist as distributions in the coordinates and be consistent with the perturbative expansions of the Lagrangian formalism obtained by imposing the conditions (3.3–4).

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6 Recently Eckmann and Epstein have shown the existence of time-ordered products assuming an extended form of the Osterwalder-Schrader condition [35]

D. Scale Invariance

The Wightman functions of the normalized field operators satisfy scale invariance in the sense of dimensional analysis

$$W(x_1, \ldots, x_n, g, a^2, \kappa^2) = a^n W(ax_1, \ldots, ax_n, g, \kappa^2) \tag{3.5}$$

(similarly for time-ordered functions).

The effective coupling Q may be defined by (2.21–23) with $m=0$. (3.4) is then equivalent to the condition

$$\Gamma(k_1 \ldots k_4, g, \kappa^2) = -ig \tag{3.6}$$

at the conventional symmetry point off-shell

$$k_j^2 = \kappa^2, \quad (k_i + k_j)^2 = \tfrac{4}{3}\kappa^2, \quad i \neq j.$$

Another convenient choice of the symmetry point is given by the collinear vectors

$$k_j = k_j(k^2), \quad j = 1, 2, 3, 4$$

with

$$k_1 = k_2 = k_3 = k, \quad k_4 = -3k.$$

The coupling constant is then given by

$$\Gamma(k, k, k, -3k) = -ig \quad \text{at} \quad k^2 = \kappa^2, \tag{3.7}$$

or equivalently by (3.4) with the effective coupling

$$Q(k^2) = -i(k^2)^2 \Delta'_F(k^2)^2 \Gamma(k, k, k, -3k). \tag{3.8}$$

The advantage of this definition is that Ruelle's general results on the analytic properties of time-ordered functions in momentum space can be applied immediately [32]. For the discussion of the analytic properties of the effective coupling thus defined see [33]. In particular, Q is regular analytic at any point $k^2 < 0$ where $D'_F(k^2) \neq 0$.

The scaling relation (3.5) implies that the model can be extended from (3.3) to any value $\kappa^2 < 0$ of the normalization mass κ so that the domain of admissible parameter values becomes

$$0 \leqq g < \eta, \quad \kappa^2 < 0.$$

The form of the global renormalization group transformations may be derived from the postulates as in the previous section. The effective coupling Q and the function

$$R(k^2, g, \kappa^2) = -ik^2 \Delta'_F(k^2, g, \kappa^2) \tag{3.9}$$

depend only on g and the dimensionless ratio

$$Q = Q(u, g), \quad R = R(u, g), \quad u = \frac{k^2}{\kappa^2}.$$

The conditions (3.3–4) take the form

$$R(u,g)=1 \quad \text{at} \quad u=1, \tag{3.10}$$

$$Q(u,g)=g \quad \text{at} \quad u=1. \tag{3.11}$$

For equivalent pairs of parameter values

$$g,\kappa^2 \sim g',\kappa'^2$$

one finds the transformation law

$$A(x,g',\kappa'^2)=z^{1/2}A(x,g,\kappa^2), \tag{3.12}$$

$$\tau(k_1\ldots k_n,g',\kappa'^2)=z^{n/2}\tau(k_1\ldots k_n,g,\kappa^2), \tag{3.13}$$

$$z^{-1}=-i\kappa^2\Delta'_F(\kappa'^2,g,\kappa^2)=R\left(\frac{\kappa'^2}{\kappa^2},g\right), \tag{3.14}$$

$$g'=Q\left(\frac{\kappa'^2}{\kappa^2},g\right). \tag{3.15}$$

κ'^2 should be chosen close enough to κ^2 so that the uniqueness assumption applies and $z>1$.

From (3.13) the differential equation (2.53) of the renormalization group follows with the Callan-Symanzik functions

$$\beta(g)=\left.\frac{\partial Q(u,g)}{\partial u}\right|_{u=1} \tag{3.16}$$

$$\gamma(g)=\left.\frac{\partial R(u,g)}{\partial u}\right|_{u=1} \tag{3.17}$$

as coefficients. No differentiability assumptions need be made in connection with (2.53). As a consequence of the scaling law (3.5) the Green's functions may be differentiated with respect to κ^2 any number of times. In deriving (2.53) the existence of the g-derivative follows for all values at which $\beta(g)\neq 0$. By comparison with perturbation theory postulate C implies the Taylor formulae

$$\beta(g)=g^2b(g) \tag{3.18}$$

$$\gamma(g)=g^2c(g) \tag{3.19}$$

with finite limits

$$\begin{aligned} b&=\lim_{g\to 0} b(g)>0 \\ c&=\lim_{g\to 0} c(g). \end{aligned} \tag{3.20}$$

On the basis of the postulates it is expected that the effective coupling $Q(u,g)$ is a monotonously increasing function of u for $0<g<\eta$, $0<u<v$ with suitable bound v. In the remainder of this section a detailed proof of this statement will be given.

We first use the uniqueness postulate in order to show that g', as given by (3.15), is a monotonic function of κ'^2 in a neighborhood of κ^2 provided $0<g<\eta$. To this end we note that by the uniqueness postulate the solution κ'^2 of (3.15) is unique so that g' can be inverted with respect to κ'^2. Since Q is continuous in κ'^2 it follows that g' is monotonic in κ'^2 for the domain considered. Equivalently, $Q(u, g)$ is monotonic in u near $u=1$ for $0<g<\eta$.

It will next be shown that the transformation (3.12–15) which holds locally can be extended to any values g, g' from the interval

$$0<g, g'<\eta \tag{3.21}$$

with arbitrary $\kappa^2<0$. To this end a sequence of parameter values

$$\begin{aligned}
&g_0, \kappa_0^2 \sim g_1, \kappa_1^2 \sim \ldots \sim g_n, \kappa_n^2\,,\\
&A(x, g_{j+1}, \kappa_{j+1}^2)=z_j^{1/2} A(x, g_j, \kappa_j^2)\,,\\
&g_0=g,\ \kappa_0^2=\kappa^2\,; \qquad g_n=g',\ \kappa_n^2=\kappa'^2\,,
\end{aligned} \tag{3.22}$$

will be used with the property that the field operator $A(x, g_{j+1}, \kappa_{j+1}^2)$ lies in the neighborhood of $A(x, g_j, \kappa_j^2)$. Then

$$g, \kappa^2 \sim g', \kappa'^2$$

and the equivalence transformation (3.12–15) exists. It must yet be shown that a sequence (4.2) can always be constructed. Suppose there are values g' with $g<g'<\eta$ for which a sequence (4.2) cannot be given. Then there exists a value h such that the construction is possible for all $g'<h$, but not for $g'>h$. Let $g'<h$ and $g''>h$ be sufficiently close to h. The solutions κ'^2 and κ''^2 of

$$g'=Q\left(\frac{\kappa'^2}{\lambda^2}, h\right), \qquad g''=Q\left(\frac{\kappa''^2}{\lambda^2}, h\right)$$

are continuous functions of g' or g'' respectively near $g'=g''=h$. Moreover,

$$R\left(\frac{\kappa'^2}{\lambda^2}, h\right)^{-1}, \qquad R\left(\frac{\kappa''^2}{\lambda^2}, h\right)^{-1}$$

are continuous near $\kappa'^2=\kappa''^2=\lambda^2$. Choosing g' and g'' close enough to h the field operator $A(x, g', \kappa'^2)$ will be in a neighborhood of $A(x, g'', \kappa''^2)$ which is in contradiction to the hypothesis on h. Similarly (3.22) can be derived for $0<g'<g$. This completes the proof that the equivalence transformation (3.12–13) can be extended to any pair of values g, g' from the interval (3.21).

We may now enlarge the domain where Q is monotonic in u and R positive. For the sequence (3.22) we have

$$g'=Q\left(\frac{\kappa'^2}{\kappa_{n-1}^2}, g_{n-1}\right)=Q\left(\frac{\kappa'^2}{\kappa^2}, g\right), \tag{3.23}$$

$$z=R\left(\frac{\kappa'^2}{\kappa^2}, g\right)^{-1}=z_{n-1} \ldots z_0>0\,. \tag{3.24}$$

Since g' is monotonic in κ'^2 it follows that Q is monotonous in κ'^2/κ^2. For sufficiently small values of g it is $\beta(g)>0$ and therefore Q increasing because of (3.16). By (3.23) Q is increasing as a function of κ'^2/κ^2 for any g from the interval $0<g<\eta$. If κ'^2/κ^2 is made arbitrarily small g' will stay within (3.21) since Q is an increasing function of κ'^2. Moreover, the value $\kappa'^2/\kappa^2=1$ is always permissible since then $0<g=g'<\eta$. Therefore,

$$Q=Q(u,g), \quad 0<g<\eta, \tag{3.25}$$

is a monotonicly increasing function of u in the domain

$$0<u<v, \quad v>1. \tag{3.26}$$

v depends on g, it is the smallest solution of the equation

$$\eta=Q(v,g). \tag{3.27}$$

The inversion of (3.25)

$$u=u(Q,g), \quad 0<g<\eta, \tag{3.28}$$

is a monotonicly increasing function of Q in $0<Q<\eta$. From (3.24) and the domain (3.26) of the variable $u=\kappa'^2/\kappa^2$ it follows that

$$R(u,g)>0 \quad \text{for} \quad 0<u<v,\ 0<g<\eta. \tag{3.29}$$

As a consequence, the effective coupling Q is regular analytic at any point u with $0<u<v$ if $0<g<\eta$ since D'_F does not vanish in this region.

4. Effective Coupling and Propagator

In this section some basic properties of the effective coupling and the propagator will be discussed for the massless model. The postulates stated in the previous section imply the differential equations [see (2.53) and (2.57)]

$$u\frac{\partial Q}{\partial u}=\beta\frac{\partial Q}{\partial g}, \tag{4.1}$$

$$u\frac{\partial R}{\partial u}=\beta\frac{\partial R}{\partial g}+\gamma R \tag{4.2}$$

with the notation

$$Q(u,g)=Q(k^2,g,\kappa^2), \tag{4.3}$$

$$R(u,g)=-ik^2D_F(k^2,g,\kappa^2). \tag{4.4}$$

The coefficients β and γ are the Callan-Symanzik functions defined by (3.16–17) with the weak coupling behavior (3.18–20).

Let

$$g',\kappa'^2 \sim g,\kappa^2$$

be equivalent sets of parameters from the region $0<g$, $g'<\eta$ where the validity of the postulates has been assumed. Then the identities

$$Q(k^2,g',\kappa'^2)=Q(k^2,g,\kappa^2), \tag{4.5}$$

$$R(k^2,g',\kappa'^2)=zR(k^2,g,\kappa^2) \tag{4.6}$$

hold with g and z given by (3.14–15). Differentiating (4.5–6) with respect to k^2 and setting $k^2 = \kappa'^2$, $u = \kappa'^2/\kappa^2$ one finds the relations

$$\beta(Q) = u \frac{\partial Q}{\partial u}, \tag{4.7}$$

$$\gamma(Q) = \frac{u}{R} \frac{\partial R}{\partial u}, \tag{4.8}$$

with

$$Q = Q(u, g), \qquad R = R(u, g).$$

Since Q is regular analytic at any point $u > 0$ below v it follows from (4.7) that β can have only a finite number of zeros in the interval $0 < Q < \eta$, namely

$$\beta(g) = 0 \quad \text{at} \quad g = g^{(1)}, \ldots, g^{(a)} \quad \text{in} \quad 0 < g < \eta. \tag{4.9}$$

At these zeros the effective coupling is stationary [see (4.7)] but does not have an extremum. By (3.28) we may introduce Q and g as independent variables in (4.7–8) and obtain the differential equations

$$\left.\frac{\partial u}{\partial Q}\right|_g = \frac{u}{\beta(Q)}, \tag{4.10}$$

$$\left.\frac{\partial R}{\partial Q}\right|_g = \frac{\gamma(Q)}{\beta(Q)} R \tag{4.11}$$

for the inversion (3.28) and

$$R = R(u(Q, g), g).$$

$|_g$ indicates that the derivative is formed at a constant value of g. With the initial conditions (3.10–11) the differential equations (4.10–11) have the unique solutions

$$u = e^{\int_g^Q \frac{dx}{\beta(x)}}, \tag{4.12}$$

$$R = e^{\int_g^Q \frac{\gamma(x)}{\beta(x)} dx} \tag{4.13}$$

$$0 < g, \qquad Q < \eta, \qquad 0 < u < v.$$

At the zeros (4.9) of β the integrals of the exponentials are convergent since the functions $\lg u$ and $\lg R$ have existing limits for $Q \to g^{(\alpha)}$. The integrability of β^{-1} at such zeros was also shown in [33] as a consequence of the analytic properties of Q.

Alternatively, (4.13) may be derived from the differential equation (4.2). Introducing g and Q as independent variables the differential equation takes the form

$$\left.\frac{\partial R}{\partial g}\right|_Q = -\frac{\gamma(g)}{\beta(g)} R,$$

which with (3.10–11) again leads to (4.13).

The relations (4.12–13) have important consequences for the exact infrared behavior of the model. (4.12) implies

$$\lim_{u \to 0} Q(u, g) = 0 \quad \text{or} \quad \lim_{k^2 \to 0} Q(k^2, g, \kappa^2) = 0\,, \tag{4.14}$$

because of (3.18) and (3.20). With this the infrared limit of R becomes

$$\lim_{u \to 0} R(u, g) = e^{\int_g^0 \frac{\gamma(x)}{\beta(x)} dx}\,. \tag{4.15}$$

Thus the infrared behavior of the propagator is determined by

$$-i \lim_{k^2 \to 0} k^2 D'_F(k^2, g, \kappa^2) = e^{\int_g^0 \frac{\gamma(x)}{\beta(x)} dx}\,. \tag{4.16}$$

The integral on the right-hand side converges since the integrand γ/β approaches the finite limit

$$\lim_{x \to +0} \frac{\gamma(x)}{\beta(x)} = \frac{c}{b}$$

[see (3.18–20)]. Hence the infrared limit (4.16) is finite and the propagator has a pole at $k=0$. This shows the existence of massless one particle states. According to the collision theory developed by Buchholz [34] for massless bosons asymptotic states of these particles can be constructed and the corresponding S-matrix elements are well defined.

Formula (4.16) for the residue

$$r = -i \lim_{k^2 \to 0} k^2 D'_F(k^2, g, \kappa^2) \tag{4.17}$$

was derived for values of the coupling constant from the original domain of definition with $0 < g < \eta$. In general, r satisfies the differential equation

$$\beta \frac{\partial r}{\partial g} + \gamma r = 0\,, \tag{4.18}$$

which can be solved by the methods of the following section. If the effective coupling has extrema in the Euclidean region r becomes a multivalued function of g. For any value of g which can be introduced by an equivalence transformation the residue (4.17) must of course converge since it is a finite multiple of (4.16).

5. Rescaling of Parameter Values

So far field operators were only considered for the original range of coupling parameter values as specified by the postulates. Field operators will now be set up for any value of the coupling parameter which the effective coupling may assume in the Euclidean region. We start out from the normalized field operator $A(x, g_0, \kappa_0^2)$ for the original domain of definition with $0 < g_0 < \eta$ and $\kappa_0^2 < 0$. g_0, κ_0 denote reference values which will be kept fixed in the work that follows. Our aim is to replace the coupling parameter g_0 by any value

$$g = Q(u, g_0)\,, \quad u = \kappa^2/\kappa_0^2\,, \quad \kappa^2 < 0\,, \tag{5.1}$$

of the effective coupling. The problem with introducing g as coupling parameter for the field operators and Green's functions is twofold:

i) The function Q need not be uniquely invertible, so that field operator and Green's functions become multivalued functions of g,

ii) it may not be possible to normalize the propagator to $+1$ at the normalization point.

Moreover, Q may be singular at zeros of D'_F. In order to deal with these difficulties we divide the region of positive u-values up into intervals such that Q is monotonic and D'_F non-vanishing in each interval. To this end let

$$u_j = u_j(g_0), \quad j = 1, 2, \dots \tag{5.2}$$

denote the positions of all extrema of Q and zeros of D'_F, ordered by magnitude,

$$u_j < u_{j+1}, \quad j = 1, 2, \dots . \tag{5.3}$$

Because of (3.26) and (3.29) the first extremum of Q or propagator zero is above one,

$$1 < u_1(g_0), \quad 0 < g_0 < \eta .$$

We further set $u_0 = 0$. In each interval

$$\mathcal{U}_j(g_0) : u_j(g_0) < u < u_{j+1}(g_0), \quad j = 0, 1, 2, \dots \tag{5.4}$$

the function $Q(u, g_0)$ is monotonic. Let $\mathcal{G}_j$ be the range of values which Q assumes in the interval $\mathcal{U}_j$. $\mathcal{G}_j$ is an open interval, possibly extending to $+\infty$ and/or $-\infty$. In $\mathcal{U}_j$ the function (5.1) can be inverted uniquely by a monotonic function

$$u = u^{(j)}(g, g_0) \quad g \in \mathcal{G}_j . \tag{5.5}$$

In each $\mathcal{U}_j$ the propagator does not vanish, hence carries the same sign in the entire interval.

After these preparations we may now define field operators for any $\kappa^2 < 0$ and coupling parameter (5.1) by the equivalence transformations

$$A^{(j)}(x, g, \kappa^2) = z^{1/2} A(x, g_0, \kappa_0^2), \quad 0 < g_0 < \eta \tag{5.6}$$

$$= \left| R\left(\frac{\kappa^2}{k_0^2}, g_0\right) \right| \tag{5.7}$$

if the ratio κ^2/κ_0^2 falls into the interval

$$\kappa^2/\kappa_0^2 \in \mathcal{U}_j(g_0) . \tag{5.8}$$

The parameter values g_0, κ_0^2 and g, κ^2, j are called equivalent,

$$g_0, \kappa_0^2 \sim g, \kappa^2, j . \tag{5.9}$$

The supercript $^{(j)}$ labels different branches with respect to the coupling parameter. $^{(0)}$ denotes the original branch

$$A^{(0)}(x, g, \kappa^2) = A(x, g, \kappa^2) .$$

In (5.7) the absolute value of R is used so that $z>0$. The propagator is then normalized to ± 1 at the normalization point depending on the propagator sign in $\mathcal{U}_j$. The field operator (5.6) satisfies the normalization conditions

$$\left.\begin{aligned} -ik^2(D_F^{(j)})^{-1}(k^2,g,\kappa^2) &= \pm 1 \\ Q^{(j)}(k^2,g,\kappa^2) &= g \end{aligned}\right\} \quad k^2=\kappa^2 . \qquad \begin{aligned}(5.10)\\(5.11)\end{aligned}$$

Under the equivalence transformation (5.6–7) the effective coupling and propagator transform like

$$Q^{(j)}\left(\frac{k^2}{\kappa^2},g\right)=Q\left(\frac{k^2}{\kappa_0^2},g_0\right), \tag{5.12}$$

$$D_F^{(j)}(k^2,g,\kappa^2)=zD_F(k^2,g_0,\kappa_0^2). \tag{5.13}$$

These relations imply that the position of an extremum of Q or a propagator zero transform like

$$\begin{aligned} &u_l^{(j)}(g)=\frac{\kappa_0^2}{\kappa^2}u_l(g_0), \qquad l=1,2,\ldots, \\ &g=Q\left(\frac{\kappa^2}{\kappa_0^2},g_0\right), \qquad u_j(g_0)<\frac{\kappa^2}{\kappa_0^2}<u_{j+1}(g_0). \end{aligned} \tag{5.14}$$

We further set

$$u_0^{(j)}(g)\equiv 0$$

and define intervals $\mathcal{U}_l^{(j)}(g)$ by

$$\begin{aligned} &\mathcal{U}_l^{(j)}(g): u_l^{(j)}(g)<u<u_{l+1}^{(j)}(g) \\ &l=0,1,2,\ldots. \end{aligned}$$

Within each interval $\mathcal{U}_l^{(j)}$ the effective coupling $Q^{(j)}$ is monotonic and the propagator $D_F^{(j)}$ non-vanishing. The range of values which $Q^{(j)}$ assumes in $\mathcal{U}_l^{(j)}$ is denoted by $\mathcal{G}_l$.

It further follows from (5.12) that the values

$$g_l=Q^{(j)}(u_l^{(j)}(g),g)=Q(u_l(g_0),g_0) \tag{5.15}$$

are invariants of the renormalization group. g_l represents an extremal value of the effective coupling and/or the coupling parameter corresponding to a propagator zero provided Q remains finite there. If Q is singular at a propagator zero we have

$$\lim_{u\to u_l(g)} Q^{(j)}(u,g)=\lim_{u_0\to u_l(g_0)} Q(u_0,g_0)=\pm\infty ,$$

where both limits are taken either from above or below. It follows that the limits of the intervals $\mathcal{G}^{(j)}$ are invariants of the renormalization group and thus independent of the chosen normalization point.

The point $u=1$ belongs to the interval $\mathcal{U}_l^{(l)}$. For

$$u_l^{(l)}(g)<1<u_{l+1}^{(l)}(g) \tag{5.16}$$

follows from (5.14).

In the remainder of this section we generalize the differential equations of the Green's functions to arbitrary branches. To this end we consider equivalent parameter values of the same branch,

$$g_2, \kappa_2^2, j \sim g_0, \kappa_0^2 \sim g_1, \kappa_1^2, j. \tag{5.17}$$

Then the equivalence transformation

$$\begin{aligned} \mathrm{A}^{(j)}(\mathrm{x}, \mathrm{g}_2, \kappa_2^2) &= z^{1/2} \mathrm{A}^{(j)}(\mathrm{x}, \mathrm{g}_1, \kappa_1^2) \\ \tau^{(j)}(k_1 \ldots k_n, g_2, \kappa_2^2) &= z^{n/2} \tau^{(j)}(k_1 \ldots k_n, g_1, \kappa_1^2) \end{aligned} \tag{5.18}$$

holds with

$$z = \left| R^{(j)} \left(\frac{\kappa_2^2}{\kappa_1^2}, g_1 \right) \right|^{-1}, \tag{5.19}$$

$$g_2 = Q^{(j)} \left(\frac{\kappa_2^2}{\kappa_1^2}, g_1 \right). \tag{5.20}$$

From this the differential equations of the j-th branch follows,

$$\left(\kappa^2 \frac{\partial}{\partial \kappa^2} + \beta^{(j)} \frac{\partial}{\partial g} + \frac{1}{2} n \gamma^{(j)} \right) \tau^{(j)} = 0 \tag{5.21}$$

with the Callan-Symanzik functions

$$\beta^{(j)}(g) = \left. \frac{\partial Q^{(j)}(u, g)}{\partial u} \right|_{n=1}, \tag{5.22}$$

$$\gamma^{(j)}(g) = \left. \frac{\partial |R^{(j)}(u, g)|}{\partial u} \right|_{n=1}. \tag{5.23}$$

From (5.12) and (5.22) one can derive the extension

$$u \frac{\partial Q(u, g_0)}{\partial u} = \beta^{(j)}(Q(u, g_0)) \tag{5.24}$$

of (4.7) to values u from any interval

$$\mathscr{U}_j : u_j(g_0) < u < u_{j+1}(g_0).$$

Writing (5.12) for the i-th branch in the form

$$Q^{(i)}(u, g) = Q\left(\frac{\kappa^2}{\kappa_0^2} u, g_0 \right), \quad u_i < \frac{\kappa^2}{\kappa_0^2} < u_{i+1} \tag{5.25}$$

and inserting it into (5.24) we find

$$\begin{aligned} & u \frac{\partial Q^{(i)}(u, g)}{\partial u} = \beta^{(j)}(Q^{(i)}(u, g)) \\ & u_j^{(i)}(g) < u < u_{j+1}^{(i)}(g), \end{aligned} \tag{5.26}$$

which generalizes (4.7) to any branch i and values u taken from an arbitrary interval $\mathscr{U}_j$.

From (5.13) and (5.23) we get

$$\gamma^{(j)}(Q(u,g_0)) = \frac{u}{R(u,g_0)} \frac{\partial R(u,g_0)}{\partial u}, \tag{5.27}$$

if

$$u_j(g_0) < u < u_{j+1}(g_0).$$

Inserting (4.25) and

$$R^{(i)}(u,g) = \frac{R\left(\frac{\kappa^2}{\kappa_0^2} u, g_0\right)}{\left|R\left(\frac{\kappa^2}{\kappa_0^2}, g_0\right)\right|}$$

into (5.27) the generalization

$$\begin{aligned} &\gamma^{(j)}(Q^{(i)}(u,g)) = \tfrac{1}{2} \frac{u}{R^{(i)}(u,g)} \frac{\partial R^{(i)}(u,g)}{\partial u} \\ &u_j^{(i)}(g) < u < u_{j+1}^{(i)}(g) \end{aligned} \tag{5.28}$$

of (4.8) follows. The inversions

$$u = u^{(i,j)}(Q^{(i)}, g)$$

of

$$Q^{(i)} = Q^{(i)}(u,g), \qquad u_j(g) < u < u_{j+1}(g)$$

and

$$R^{(i)} = R^{(i)}(u^{ij}(Q^{(i)}, g), g)$$

satisfy the differential equations

$$\left.\frac{\partial u}{\partial Q^{(i)}}\right|_g = \frac{u}{\beta^{(j)}(Q^{(j)})}, \tag{5.29}$$

$$\left.\frac{\partial R^{(i)}}{\partial Q^{(i)}}\right|_g = \frac{\gamma^{(j)}(Q^{(i)})}{\beta^{(j)}(Q^{(i)})} R^{(i)}, \tag{5.30}$$

which generalize (4.10–11). These equations can be integrated piecewise for any interval $\mathcal{G}_j$ of the variable $Q^{(j)}$. For the interval $\mathcal{G}_i$ the solution of (5.29) is

$$u = e^{\int_g^{Q^{(i)}(u,g)} \frac{dx}{\beta^{(i)}(x)}}, \qquad u_i^{(i)}(g) < u < u_{i+1}^{(i)}(g),$$

where the normalization condition (5.11) and the inequality (5.16) were used. For the interval $\mathcal{G}_j$

$$u = u_j e^{\int_{g_j}^{Q^{(i)}(u,g)} \frac{dx}{\beta^{(j)}(x)}},$$

if $j>i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$

$$u=u_{j+1}e^{\int\limits_{g_{j+1}}^{Q^{(i)}(u,g)}\frac{dx}{\beta^{(j)}(x)}},$$

if $j<i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$.

Iterating these relations we get

$$\lg u=\int\limits_{g}^{g_{i+1}}\frac{dx}{\beta^{(i)}}+\int\limits_{g_{i+1}}^{g_{i+2}}\frac{dx}{\beta^{(i+1)}}+\ldots+\int\limits_{g_j}^{Q^{(i)}(u,g)}\frac{dx}{\beta^{(j)}} \tag{5.31}$$

if $j>i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$

$$\lg u=\int\limits_{g}^{g_i}\frac{dx}{\beta^{(i)}}+\int\limits_{g_i}^{g_{i-1}}\frac{dx}{\beta^{(i-1)}}+\ldots+\int\limits_{g_{j+1}}^{Q^{(i)}(u,g)}\frac{dx}{\beta^{(j)}}, \tag{5.32}$$

if $j<i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$.

Similarly (5.30) can be integrated to

$$\lg R^{(i)}(u,g)=\int\limits_{g}^{g_{i+1}}\frac{\gamma^{(i)}}{\beta^{(i)}}dx+\int\limits_{g_{i+1}}^{g_{i+2}}\frac{\gamma^{(i+1)}}{\beta^{(i+1)}}dx+\ldots+\int\limits_{g_j}^{Q^{(i)}(u,g)}\frac{\gamma^{(j)}}{\beta^{(j)}}dx, \tag{5.33}$$

if $j>i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$,

$$\lg R^{(i)}(u,g)=\int\limits_{g}^{g_i}\frac{\gamma^{(i)}}{\beta^{(i)}}dx+\int\limits_{g_i}^{g_{i-1}}\frac{\gamma^{(i-1)}}{\beta^{(i-1)}}dx+\ldots+\int\limits_{g_j+1}^{Q^{(i)}(u,g)}\frac{\gamma^{(j)}}{\beta^{(j)}}dx, \tag{5.34}$$

if $j<i$ and $u_j^{(i)}(g)<u<u_{j+1}^{(i)}(g)$.

6. Remarks on the Ultraviolet Limit

For simplicity we assume that the total number of extrema of Q and zeros of D'_F in the Euclidean region is finite. Let Q be monotonic and D'_F be nonvanishing above $u_0>0$. Then the differential equations (5.29–30) are solved by

$$u=u_0e^{\int\limits_{g_0}^{Q}\frac{dx}{\beta'}}, \tag{6.1}$$

$$R=R_0e^{\int\limits_{g_0}^{Q}\frac{\gamma'}{\beta'}dx}, \tag{6.2}$$

$$g_0=Q(u_0,g),\qquad R_0=R(u_0,g),$$

where $'$ denotes the branch of β and γ which is appropriate above the largest position of an extremum of Q. Since Q is monotonic above u_0 it can have only one accumulation point g_∞ for $u\to\infty$

$$g_\infty=\lim_{u\to\infty}Q(u,g)=\lim_{u\to\infty}Q^{(j)}(u,g). \tag{6.3}$$

g_∞ is infinite or a finite number, independent of g because

$$Q\left(\frac{k^2}{\kappa^2},g\right)=Q\left(\frac{k^2}{\kappa'^2},g'\right)\quad\text{if}\quad g',\kappa'^2\sim g,\kappa^2. \tag{6.4}$$

A negative or vanishing value of g_∞ cannot be excluded unless Q is monotonic for all $u>0$. In the limit $u\to\infty$ (6.1) and (6.3) imply

$$\lim_{Q\to g_\infty} \int_{g_0}^{Q} \frac{dx}{\beta'} = \infty . \tag{6.5}$$

Hence a finite limit g_∞ must be a zero of β'

$$\beta'(g_\infty)=0 \tag{6.6}$$

with non-integrable β'^{-1}. According to (6.2) the propagator is asymptotically determined in the ultraviolet limit by the behavior of β' and γ' near $g=g_\infty$. More detailed information on how β' and γ' behave near $g=g_\infty$ is needed in order to establish a well-defined scale invariant limit theory for $g\to g_\infty$. Such a limit theory cannot exist for $g_\infty=\pm\infty$ since the vertex function would diverge in the Euclidean region by (3.6) or (3.7). It should be stressed that even in case of a finite limit g_∞ for (6.3) a scale invariant limit theory need not exist. On the other hand it cannot be excluded that the system approaches a free field in this limit.

In Sect. 5 it was shown how field quantities can be extended by renormalization group transformations from values g_0, κ_0^2 of the original domain $0<g_0<\eta$, $\kappa_0^2<0$ to functions of the coupling constant g given by any value

$$g=Q\left(\frac{\kappa^2}{\kappa_0^2}, g_0\right), \qquad \kappa^2<0,$$

of the effective coupling. A further extension beyond the limit value $g=g_\infty$ of (6.3) is possible by analytic continuation in g. It is also conceivable that solutions of the model exist which are not related to the perturbative expansion, neither by renormalization group transformations nor by analytic continuation (see [25]).

7. Scattering Amplitudes

We conclude this section by deriving the renormalization group equation for S-matrix elements involving the massless particles which are associated with the propagator pole. The scattering amplitudes are given by

$$\begin{aligned} &A(p_1\dots p_n)\\ &\quad=(-i)^n r^{-n/2} \lim_{k_j\to p_j} \prod_{j=1}^{n} k_j^2 \tau^{\mathrm{conn}}(k_1\dots k_n),\\ &p_j^2=0. \end{aligned} \tag{7.1}$$

τ^{conn} denotes the connected part of the time-ordered function in momentum space before separating the factor $\delta\left(\sum_{j=1} k_j\right)$. r is the residue (4.17) of the propagator pole which satisfies the differential equation (4.18). Combining the renormalization group equation (2.53) with the differential equation

$$\left(2\kappa^2\frac{\partial}{\partial\kappa^2}+\sum_{j=1}^{n} k_j\frac{\partial}{\partial k_j}+3n\right)\tau^{\mathrm{conn}}=0 \tag{7.2}$$

of dimensional scaling one finds the broken scaling equation

$$\sum_{j=1}^{n}\left(4-d+k_j\frac{\partial}{\partial k_j}\right)\tau^{\text{conn}}=2\beta\frac{\partial\tau^{\text{conn}}}{\partial g} \tag{7.3}$$

with the anomalous dimension

$$d(g)=1+\gamma(g). \tag{7.4}$$

Multiplying by $r^{-n/2}\prod_{j=1}^{n}k_j^2$ and commuting this factor with the differential operator

$$\sum_{j=1}^{n}\left(1+p_j\frac{\partial}{\partial p_j}\right)A=2\beta\frac{\partial A}{\partial g} \tag{7.5}$$

follows as the renormalization group equation of the scattering amplitudes. It should be noted that this equation involves the canonical rather than the anomalous dimension. This result is in agreement with a paper by Gross and Wess who obtained the scaling equation (7.5) by investigating some general properties of the energy-momentum tensor [27, 28].

Since the residue of the propagator pole is finite a field operator

$$A_n(x,g,\kappa^2)=r^{-1/2}A(x,g,\kappa^2) \tag{7.6}$$

may be introduced which is normalized on mass shell by the condition

$$-ik^2\Delta'_{Fn}(k^2,g,\kappa^2)=1 \quad \text{at} \quad k^2=0 \tag{7.7}$$

for its propagator. Using the differential equation (4.18) of the residue r the renormalization group equation

$$\left(\kappa^2\frac{\partial}{\partial\kappa^2}+\beta\frac{\partial}{\partial g}\right)\tau_n=0 \tag{7.8}$$

follows again with no anomalous part of the dimension. Obviously the anomaly of the dimension is only due to normalizing off mass shell and should therefore not appear in the broken scaling equation of the scattering amplitudes.

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