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"Time Inversion" and Mobility of Many Particle Systems

Jacek Waniewski

Institute of Theoretical Physics, Division of Mathematical Physics, Warszawa, Poland

Abstract. The unitary operations which can be generated on many particle states in non-relativistic quantum mechanics are discussed. These operations depend on an arbitrary external field which is in the experimenter's control, whereas the pairwise potential of interaction between the particles is fixed. The various kinds of systems of N identical particles interacting via the potentials $V_I = \sum_{k,j} r_{kj}^p w(r_{kj})$ are studied. For every system in question, the semigroup spanned by evolution transformations is proved to contain all the unitary operators in the Hilbert space of states. In particular, it is shown that the natural evolution operation can be reversed by a certain prescribed sequence of maneouvres involving only external fields.

1. Introduction

In quantum theories some non-obvious truths concerning the mathematical formalism are usually taken for granted. Thus, pure states are represented by vectors in a Hilbert space $\mathscr{H}$. Observables are self-adjoint operators. However, some questions concerning the "economy" of the mathematical language arise. Is every vector $\psi \in \mathscr{H}$ indeed necessary to describe some physical state which can be effectively created? Moreover, can one prove that to every self-adjoint operator there corresponds some effective measuring prescription? Is'nt it so, to the contrary, that physically essential observables are only the customarily considered quantities like energy, momentum, spin etc.? A similar question can be raised about the dynamical aspect of the theory. Evolution transformations of a quantum system are represented by unitary operators. However, can all the unitary operators be interpreted as the dynamical operations?

In spite of their apparent obviousness these questions are non trivial even in the one particle theory [1, 2]. They become more important in the theory of composite systems, due to interrelations with the Einstein-Podolsky-Rosen paradox and the corresponding distinction between the states of the first and the second kind [3]. The arguments concerning the particle production and the joint

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probability distribution show the necessity of the second kind states (symmetrized or antisymmetrized tensor products of one-particle states) in the theory of indistinguishable particles [4, 5]. However, a question arises again, are all such states physically meaningful? Consistently, what is the set of physically meaningful observables? The answer to this problems happens to be conditioned by dynamics: it requires the knowledge of operations which are dynamically achievable [1, 6, 7].

For the Schrödinger particle some of the above questions were raised by Lamb [1]. He has proved that the evolution transformations allow one to produce an arbitrary wave function out of any given initial state. A stronger hypothesis was already formulated by von Neumann [8]. He expressed the belief that every unitary operator is a dynamical transformation. The question about the achievable operations was studied first by Lubkin for spin systems [7]. The particular spin transformations of practical relevance were considered by Haeberlen and Waugh [9]. The dynamical operations ("global mobility") of the single Schrödinger particle have been then determined by Mielnik by showing that all the unitary operators are achievable [2].

In the case of many particle systems the problem of mobility is more complicated and wasn't yet solved. Here, the dynamical operations might a priori be restricted by the fact that we have only the freedom to maneouvre with external potentials, while the mutual interaction potential V_I is given and is not changed by the experimenter's intervention. In spite of this we shall show that for a wide class of potentials V_I the corresponding many particle system (gas) has a curious property, which to some extend contradicts the intuitions about the loss of information during the time evolution. As it turns out, for the finite particle gasses there exists a prescription of maneouvring external fields which permits one to invert any finite time evolution process, forcing the gas micro-configuration to return to its initial state. This prescription is state independent and therefore can be used without knowing what the initial state was. The existence of such prescription is subsequently shown to posses further dynamical consequences, for it implies the operational achievability of any unitary transformation in the corresponding Hilbert space.

2. Mobility

Below we consider the system of $N<\infty$ identical nonrelativistic spinless particles. The Hilbert space $\mathscr{H}$ is the symmetrized (or antisymmetrized) tensor product of N one-particle spaces. The Hamiltonians will be assumed in the form $H=T+V_I+V_{ex}$, where $T=\sum_k p_k^2/2$ is the kinetic energy, $V_I=\sum_{k,j} v(r_{kj})$, $r_{kj}=|q_k-q_j|$, is the fixed potential of mutual interaction, $V_{ex}=\sum_k V(q_k)$ is the potential energy in an external field[1]. To avoid mathematical difficulties we restrict ourselves to such

1 We use the following abbreviations:

$$p_k=(p_k^\nu),\ q_k=(q_k^\nu),\ \sum_\nu=\sum_{\nu=1}^{3},\ \sum_k=\sum_{k=1}^{N},\ \sum_{k,j}=\sum\sum_{1\leq k<j\leq N}$$

interactions V_I, that $D(T)\cap D(V_I)\supset C_0^\infty(\mathbb{R}^{3N})$ and the operators V_I and $H_0=T+V_I$ are essentially self adjoint on $C_0^\infty(\mathbb{R}^{3N})$. The Hamiltonians $H=H_0+V_{ex}$, which are essentially self-adjoint on $C_0^\infty(\mathbb{R}^{3N})$, will be considered and called *admissible.*

We would like to describe the behaviour of the system in arbitrary time-dependent external fields (global mobility) [2]. Mathematically this corresponds to the semigroup of transformations $\mathscr{D}$ spanned by all the unitary operators $\exp\{-itH\}$, $t\geqq 0$, where H are the admissible Hamiltonians with fixed H_0. We shall take $\mathscr{D}$ to be closed in the strong operator topology. The resulting evolution operations are constructed below by iterated limiting transitions in sequences of "elementary" dynamical transformations. Such constructions cannot be considered practical receipts. However, due to metrizability of the strong operator topology on the operator unit ball in the separable Hilbert space (see e.g. [10]) every element of $\mathscr{D}$, which is achieved by any iterated limiting transition is as well achievable by a single sequence of simple evolution operations. Thus, in the whole rest of the paper we shall take the effective physical sense of $\mathscr{D}$ for granted.

One of the most obvious strong limits of the evolution operations are the "shock transformations" [1, 2, 9]:

$$s\text{-}\lim_{t\to 0}\exp\left\{-it\left(H_0+\frac{\alpha}{t}V_{ex}\right)\right\}=\exp\{-i\alpha V_{ex}\} \tag{1}$$

well defined for any bounded V_{ex}. Since any other V_{ex} can be approximated by bounded functions (see the Trotter-Kato and the spectral theorems [11]), therefore $\exp\{i\alpha V_{ex}\}$ belongs to $\mathscr{D}$ for arbitrary V_{ex} and $\alpha\in\mathbb{R}$.

Another class, the "dilatation transformations" is obtained as follows. The identity:

$$\exp\{A\}\exp\{B\}\exp\{-A\}=\exp\{B+[A,B]+\tfrac{1}{2}[A,[A,B]]+\ldots\} \tag{2}$$

applied for $B=-it\left(T+V_I-\dfrac{\alpha^2}{4}\sum_k q_k^2\right)$ and $A=i\dfrac{\alpha}{2}\sum_k q_k^2$ yields:

$$\exp\left\{-it\left(T+V_I-\frac{\alpha}{2}\sum_k(p_kq_k+q_kp_k)\right)\right\}\in\mathscr{D}\,. \tag{3}$$

Now, putting in (3) $\alpha=-\beta/t$ and taking the limit $t\to 0$ one obtains:

$$\Delta_\beta=\exp\left\{-i\frac{\beta}{2}\sum_k(p_kq_k+q_kp_k)\right\}\in\mathscr{D},\qquad \beta\in\mathbb{R}\,. \tag{4}$$

The operators Δ_β have been named "dilatations" because of the property [11, 12]:

$$\Delta_\beta\begin{pmatrix}q_k\\p_k\end{pmatrix}\Delta_{-\beta}=\begin{pmatrix}e^\beta & q_k\\ e^{-\beta} & p_k\end{pmatrix}\qquad k=1,\ldots,N\,. \tag{5}$$

The dilatations allow one to show dynamical achievability of the "pure kinetic" part of evolution $\exp\{-i\gamma T\}$, $\gamma\geqq 0$. With this aim consider the operators:

$$\begin{aligned}U_\beta^t&=\Delta_\beta\exp\{-itH_0\}\Delta_{-\beta}\\&=\exp\{-it(\Delta_\beta T\Delta_{-\beta}+\Delta_\beta V_I\Delta_{-\beta})\}\\&=\exp\{-it(e^{-2\beta}T+V_I(e^\beta q_1,\ldots,e^\beta q_N))\}\in\mathscr{D}\,.\end{aligned} \tag{6}$$

Putting in (6) $t=\gamma e^{2\beta}$ one has:

$$\operatorname*{s-lim}_{\beta\to-\infty} U^t_\beta=\exp\{-i\gamma T\}\in\mathscr{D}, \quad \gamma\geqq 0, \tag{7}$$

provided that V_I fulfils the following condition:

$$\operatorname*{str.r.lim}_{\beta\to-\infty}(T+e^{2\beta}V_I(e^\beta q_1, \ldots, e^\beta q_N))=T, \tag{8}$$

where the limit is taken in the strong resolvent sense [11].

The semigroup spanned by $\exp\{-i\gamma T\}$, $\gamma\geqq 0$, and the shock transformations $\exp\{i\alpha V_{ex}\}$, $\alpha\in\mathbb{R}$, contains all the operators of the form $\exp\left\{i\alpha\sum_k A(k)\right\}$, $\alpha\in\mathbb{R}$, where A is an arbitrary one-particle observable (self-adjoint operator in the one-particle Hilbert space) [2, 6, 13]. As a consequence, the operators $\exp\{i\gamma T\}$ are included in $\mathscr{D}$ for all $\gamma\in\mathbb{R}$.

This allows one to construct the "pure interaction" part of evolution $\exp\{-i\gamma V_I\}$, $\gamma\geqq 0$, with the use of the Lie-Trotter formula [11]:

$$\begin{aligned}&\operatorname*{s-lim}_{n\to\infty}\left(\exp\left\{i\frac{\gamma}{n}T\right\}\exp\left\{-i\frac{\gamma}{n}(T+V_I)\right\}\right)^n\\ &\quad=\exp\{-i\gamma V_I\}\in\mathscr{D}, \quad \gamma\geqq 0.\end{aligned} \tag{9}$$

In general, the self-adjoint operator A will be called *achievable* if all the transformations $\exp\{-i\alpha A\}$, $\alpha\geqq 0$, belong to $\mathscr{D}$ [7]. The admissible Hamiltonian $H_0=T+V_I$ will be called *decomposable* if T and V_I are achievable. The sufficient condition on decomposability of H_0 can be given as follows:

Lemma 1. *Let $H_0=T+V_I$, $V_I=\sum_{k,j}v(r_{kj})$, be admissible and let there exist such numbers $a>0$, $b\geqq 0$ and $p>-3/2$, that*

$$|v(r)|\leqq br^p \quad \textit{for} \quad r\in(0,a). \tag{10}$$

Then H_0 is decomposable.

Proof. We show that (8) holds. Because of essential self-adjointness of T on $C_0^\infty(\mathbb{R}^{3N})$ it is sufficient to prove [11] that for every $\psi\in C_0^\infty(\mathbb{R}^{3N})$:

$$\lim_{\beta\to-\infty} e^{2\beta}\|V_I(e^\beta q_1, \ldots, e^\beta q_N)\psi\|=0. \tag{11}$$

But:

$$\begin{aligned}\|V_I(e^\beta q_1, \ldots. e^\beta q_N)\psi\| &\leqq \sum_{k,j}\|v(e^\beta r_{kj})\psi\|\\ &\leqq BN(N-1)e^{-3\beta/2}\left(\int_0^{e^\beta A} dr\, r^2|v(r)|^2\right)^{1/2},\end{aligned} \tag{12}$$

for some constants A, B depending on ψ. Using (10) one has for $\beta\leqq\ln(a/A)$:

$$e^{2\beta}\|V_I(e^\beta q_1, \ldots, e^\beta q_N)\psi\|\leqq Ce^{(2+p)\beta}\xrightarrow[\beta\to-\infty]{}0. \quad \text{QED} \tag{13}$$

The study of $\mathcal{D}$ happens to be related to the problem of the effective "time-inversion": Is it possible to generate operationally the transformations inverse to the natural evolution?

3. Operational Invertibility of Evolution: Elastic Interactions

If the evolution $\exp\{-itH\}$ can be operationally inverted and therefore, $-H$ is achievable, we shall say that H is *invertible*. One has:

Lemma 2. *Let for an admissible external field the Hamiltonian $H_0+\tilde{V}_{ex}$ be invertible. Then every admissible Hamiltonian H_0+V_{ex} is also invertible.*

Proof. The lemma follows from achievability of $-(V_{ex}-\tilde{V}_{ex})$ [see shock transformations (1)] and the Lie-Trotter formula:

$$\begin{aligned}&\operatorname*{s-lim}_{n\to\infty}\left(\exp\left\{i\frac{t}{n}(H_0+\tilde{V}_{ex})\right\}\exp\left\{i\frac{t}{n}(V_{ex}-\tilde{V}_{ex})\right\}\right)^n\\ &\quad=\exp\{it(H_0+V_{ex})\}\in\mathcal{D}. \quad \text{QED}\end{aligned} \tag{14}$$

The simplest application of this lemma is found for the systems with the elastic interactions, where the inversion of the free evolution can be shown by the direct computation.

Lemma 3. *If $V_I=\frac{a}{2}\sum_{k,j}(q_k-q_j)^2$, then $H_0=T+V_I$ is invertible.*

Proof. The case $a=0$ follows immediately from [2]. Assume $a\neq 0$. Since V_I fulfils the condition (8) the results of Sect. 2 hold. Our aim is to show that $\mathcal{D}$ contains $\exp\{-i\gamma V_I\}$ for any real γ, not only for $\gamma\geqq 0$ [see (9)]. Note, that $\mathcal{D}$ contains the following transformations:

$$\begin{aligned}F^\lambda &=\exp\left\{-i\frac{3}{2}\pi\left(\frac{\lambda}{2}T+\frac{1}{2\lambda}\sum_k q_k^2\right)\right\}, \quad \lambda\neq 0,\\ Z^\alpha &=\exp\left\{-i\frac{\alpha}{2}\sum_k q_k^2\right\}, \quad \alpha\in\mathbb{R},\\ W_a^\gamma &=\exp\left\{-i\gamma a\sum_{k,j}q_kq_j\right\}=\exp\{-i\gamma V_I\}Z^{\gamma a(N-1)}, \quad \gamma\geqq 0,\end{aligned} \tag{15}$$

which act on q_k, p_k, $k=1,2,\ldots,N$, according to the prescriptions:

$$\begin{aligned}&F^\lambda\begin{pmatrix}q_k\\p_k\end{pmatrix}F^{-\lambda}=\begin{pmatrix}-\lambda p_k\\ \lambda^{-1}q_k\end{pmatrix}\\ &Z^\alpha\begin{pmatrix}q_k\\p_k\end{pmatrix}Z^{-\alpha}=\begin{pmatrix}q_k\\ p_k-\alpha q_k\end{pmatrix}\\ &W_a^\gamma\begin{pmatrix}q_k\\p_k\end{pmatrix}W_a^{-\gamma}=\begin{pmatrix}q_k\\ p_k+\gamma a\sum_{j\neq k}q_j\end{pmatrix}.\end{aligned} \tag{16}$$

Now let:

$$A_a^\gamma = \begin{cases} (F^{-1/\gamma a} W_a^\gamma)^5 F^{-1/\gamma a} & \text{for} \quad N=2 \\ F^{\gamma a/(N-2)} Z^{\gamma a/(N-1)} W_a^{\gamma/(N-2)^2} F^{-\gamma a} & \\ \quad \cdot W_a^{\gamma(N-2)} F^{-\gamma a} W_a^\gamma F^{-\gamma a} & \text{for} \quad N \geqq 3. \end{cases} \tag{17}$$

The operators A_a^γ are elements of $\mathscr{D}$ for any $\gamma > 0$. The direct calculations based on (16) prove that:

$$A_a^\gamma \begin{pmatrix} q_k \\ p_k \end{pmatrix} A_a^{-\gamma} = W_a^{-\gamma} \begin{pmatrix} q_k \\ p_k \end{pmatrix} W_a^\gamma. \tag{18}$$

This implies the operator identity $A_a^\gamma = e^{i\alpha} W_a^{-\gamma}$ [2], and so $\mathscr{D}$ includs $(W_a^\gamma)^{-1} = W_a^{-\gamma}$.

The transformations which invert the free evolution can be composed by means of the Lie-Trotter formula:

$$\underset{n\to\infty}{s\text{-lim}} \left(\exp\left\{ i\frac{\gamma}{n} T \right\} W_a^{-\gamma/n} Z^{-\gamma/n} \right)^n = \exp\{ i\gamma(T + V_I)\} \in \mathscr{D}. \quad \text{QED} \tag{19}$$

The calculations of Lemma 3 can be done due to the particular shape of the interaction potentials. However, the mechanism responsible for the inversion works also for more general interactions. This can be seen by applying abstract methods based on the recurrence property.

4. Recurrence Theorem

The quantum analogue of the classical recurrence theorem of Poincare has been formulated by Bocchieri and Loinger [14]. It concerns trajectories in the state space. However, one can obtain also an operator form of the recurrence theorem. Here the Lubkin's observation is relevant [7]. One has:

Lemma 4. *Let the self-adjoint operator A has purely discrete spectrum. Then the closure in the strong operator topology of the semigroup $\{U^s = \exp\{isA\} : s \geqq 0\}$ contains the whole group $\{U^s : s \in \mathbb{R}\}$.*

Proof. It is sufficient to prove that $\{U^s : s \geqq 0\}$ is a relatively compact set [7]. This can be done with the use of the Ascoli theorem [15]. Since the unitary operators are equicontinuous (as the linear ones) we need only to show that for every $\psi \in \mathscr{H}$ the set $\mathscr{A}_\psi = \{U^s \psi : s \geqq 0\}$ is relatively compact. But $U^s \psi = \sum_m \exp\{is\lambda_m\} c_m \psi_m$, where λ_m are eigenvalues of A and ψ_m – its eigenvectors, and so, $\mathscr{A}_\psi$ is relatively compact indeed [16]. QED

Lemma 4 is applicable for Hamiltonians with purely discrete spectrum. The assumptions guaranteeing discreteness of the spectrum of Hamiltonian can be found in [11]. Basing on these results one has the following recurrence theorem:

Theorem 1. *Let* $H = T + V_I + V_{ex}$, *where* $V_I = \sum_{k,j} (v_1 + v_2)(r_{kj})$. *If*

(a) $v_1 \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $v_2 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $v_2(r) \geqq -ar^2 - b$,

(b) $V_{I2} + V_{ex} \in L^1_{\mathrm{loc}}(\mathbb{R}^{3N})$, $V_{I2} + V_{ex} \geqq 0$ *and* $V_{I2} + V_{ex} \to \infty$, *where* $V_{I2} = \sum_{k,j} v_2(r_{kj})$,

then

(1) H_0 *and* H *are admissible,*

(2) *for any* $t > 0$ *one can choose such a sequence of time moments* $t_k > 0$, $k = 1, 2, \ldots$, *that the evolution transformations* $\exp\{-it_k H\}$ *approoximate the operator* $\exp\{itH\}$. *Thus,* H *is invertible.*

Theorem 1 means that a wide class of systems has the recurrence property, at least if "trapped" in a sufficiently strong attractive external field. Consequently, Lemma 2 allows one to reduce the study of mobility to the description of the group spanned by the evolution operations (motion group) [2, 6, 7].

5. Motion Group: Homogeneous Interaction Potentials

Note, that for decomposable and invertible dynamics the closure $\mathscr{M}$ of the motion group is the same as the closure $\tilde{\mathscr{M}}$ of the group spanned by all the operators $\exp\{i\alpha V_I\}$ and $\exp\left\{i\alpha \sum_k A(k)\right\}$, where $\alpha \in \mathbb{R}$ and A are one-particle observables (see Sect. 2). Below, we consider $\tilde{\mathscr{M}}$ for interaction potentials homogeneous in the particle distance:

Lemma 5. *Let* $V_I = a \sum_{k,j} r^p_{kj}$, *where* $a \neq 0$, $p \neq 0$ *and* $p > -2$. *Then* $\tilde{\mathscr{M}}$ *contains all the unitary operators on* $\mathscr{H}$.

Proof. Because of the Lie-Trotter formulas we can restrict ourselves to calculating the Lie algebra $\mathscr{L}$ spanned by iV_I and $i\sum_k A(k)$ [2, 6, 7, 13]. First we consider the special case of the elastic interactions:

Case $p = 2$. Since $i \sum_{k,j} \sum_{\nu=1}^{3} q^\nu_k q^\nu_j = (i/2a)\left(V_I - (N-1)\sum_k q^2_k\right)$ is in $\mathscr{L}$, then

$$\begin{aligned}
&\left[i \sum_k B(k), \left[i \sum_k A(k), i \sum_{k,j} \sum_\nu q^\nu_k q^\nu_j\right]\right] \\
&\quad = -i \sum_{k,j} \sum_\nu ([B(j), [A(j), q^\nu_j]]\, q^\nu_k \\
&\qquad + [B(k), q^\nu_k]\, [A(j), q^\nu_j]) \in \mathscr{L}\,.
\end{aligned} \tag{20}$$

Taking for A and B symmetric polynomials of q^ν, p^ν, $\nu = 1, 2, 3$, one can obtain arbitrary skew-symmetric polynomial of $6N$ variables q^ν_k, p^ν_k invariant under the permutations of the indices $(1, \ldots, N)$. The skew-adjoint closures of some of these polynomials form a dense subset in the space of the skew-adjoint operators in $\mathscr{H}$ [13]. Therefore, every unitary operator (as generated by a skew-adjoint operator) belongs to $\tilde{\mathscr{M}}$.

Case $p \neq -1$. It is sufficient to show that $\tilde{\mathcal{M}}$ contains $\exp\left\{i\alpha \sum_{k,j} r_{kj}^2\right\}$. First we prove that if $i\sum_{k,j} r_{kj}^p \in \mathcal{L}$, then $i\sum_{k,j} r_{kj}^{p+2} \in \mathcal{L}$. Let's denote $f_k = f(q_k)$, $f_{k,\nu} = \partial f_k / \partial q_k^\nu$ and calculate the following commutators:

$$\left[\frac{i}{2}\sum_k (f_k p_k^\nu + p_k^\nu f_k), i\sum_{k,j} r_{kj}^p\right]$$

$$= ip\sum_{k,j}(f_k - f_j)(q_k^\nu - q_j^\nu) r_{kj}^{p-2} \in \mathcal{L}, \tag{21}$$

$$\left[\frac{i}{2}\sum_k (g_k p_k^\nu + p_k^\nu g_k), i\sum_{k,j}(f_k - f_j)(q_k^\nu - q_j^\nu) r_{kj}^{p-2}\right]$$

$$= i\sum_{k,j}\{(g_k f_{k,\nu} - g_j f_{j,\nu})(q_k^\nu - q_j^\nu) r_{kj}^{p-2} + (f_k - f_j)$$

$$\cdot (g_k - g_j)(r_{kj}^{p-2} + (p-2)(q_k^\nu - q_j^\nu)^2 r_{kj}^{p-4})\} \in \mathcal{L}. \tag{22}$$

Moreover, $\mathcal{L}$ includs $i\sum_{k,j}(f_k - f_j)(g_k - g_j) r_{kj}^{p-2}$, as a linear combination of elements of the type (21) and (22). From this follows that also $i\sum_{k,j} r_{kj}^{p+2}$ belongs to $\mathcal{L}$, for r_{kj}^4 is equal to a linear combination of polynomials of the form $(f_k - f_j)(g_k - g_j)$. By induction one obtains that $i\sum_{k,j} r_{kj}^{p+2n}$, $n = 1, 2, \ldots$, are in $\mathcal{L}$. Let's now take $s > 0$, e.g. $s = p$ if $p > 0$ and $s = p + 2$ if $-2 < p < 0$. One has:

$$\left[\left[\frac{i}{2}\sum_k p_k^2, i\sum_{k,j} r_{kj}^{s+2}\right], i\sum_{k,j} r_{kj}^s\right]$$

$$= is(s+2)\sum_{k,j} r_{kj}^{2s} \in \mathcal{L}, \tag{23}$$

and by induction: $i\sum_{k,j} r_{kj}^{ns} \in \mathcal{L}$ for $n = 1, 2, \ldots$. Thus, $\tilde{\mathcal{M}}$ includs $\exp\left\{i\alpha \sum_{k,j} r_{kj}^{ns}\right\}$ [13]. Since the function r^s, $s > 0$, distinguishes the points of $[0, \infty)$, then, by the Stone-Weierstrass theorem, the algebra generated by r^s is dense in the space of all the continuous functions $F(r)$. Therefore, $\tilde{\mathcal{M}}$ contains all the operators $\exp\left\{i\alpha \sum_{k,j} F(r_{kj})\right\}$. In particular, $\exp\left\{i\alpha \sum_{k,j} r_{kj}^2\right\} \in \tilde{\mathcal{M}}$.

Case $p = -1$ (Coulomb potential). It is now sufficient to show that $i\sum_{k,j} r_{kj}$ belongs to $\mathcal{L}$. The above discussion [see (21), (22)] applied to $p = -1$ yields:

$$i\sum_{k,j}\{r_{kj} - 3(q_k^\nu - q_j^\nu)^2 r_{kj}^{-1}\} \in \mathcal{L}.$$

Then:

$$\left[\frac{i}{2}\sum_k (q_k p_k + p_k q_k), i\sum_{k,j}\{r_{kj}^{-3}(q_k^\nu - q_j^\nu)^2 r_{kj}^{-1}\}\right]$$
$$= i\sum_{k,j}\left\{-5r_{kj}^{+3}\sum_\nu (q_k^\nu - q_j^\nu)^4 r_{kj}^{-3}\right\} \in \mathscr{L}. \tag{24}$$

On the other hand, the particular case of (21) for $f = q^\nu$ implies that $\mathscr{L}$ contains $i\sum_{k,j}(q_k^\nu - q_j^\nu)^2 r_{kj}^{-3}$. Therefore one can repeat the calculations, starting in (21) out of $i\sum_{k,j}(q_k^\nu - q_j^\nu)^2 r_{kj}^{-3}$ instead of $i\sum_{k,j} r_{kj}^{-1}$ and obtain:

$$i\sum_{k,j}\left\{-9r_{kj} + 15\sum_\nu (q_k^\nu - q_j^\nu)^4 r_{kj}^{-3}\right\} \in \mathscr{L}. \tag{25}$$

Hence, $i\sum_{k,j} r_{kj}$ belongs to $\mathscr{L}$ as the linear combination of (24) and (25). QED

6. Mobility Theorem: Modified Homogeneous Potentials

We can collect our results to the following "mobility theorem":

Theorem 2. *Let $V_I = a\sum_{k,j} r_{kj}^p$, where $p \neq 0$, $p > -3/2$ and $a \neq 0$ if $-3/2 < p \leqq 2$ or $a > 0$ if $p > 2$. Then the mobility semigroup $\mathscr{D}$ contains all the unitary operators.*

This theorem can be generalized to the class of modified homogeneous potentials of the form $r^p w(r)$. Namely:

Theorem 3. *Let $H_0 = T + a\sum_{k,j} r_{kj}^p w(r_{kj})$, $a \neq 0$, $p \neq 0$, $p > -3/2$, be an admissible and decomposable Hamiltonian. If there exist constants $b \geqq 0$, $c > 0$, $\varepsilon > 0$ such that at least one of the following conditions holds:*

(a) $|w(r) - 1| \leqq br^\varepsilon$ for $r \in (0, c)$,

(b) $|w(r) - 1| \leqq br^{-\varepsilon}$ for $r \in (c, \infty)$ and $|w(r) - 1| r^{p+1} \in L^2(0, c)$,

then $\mathscr{D}$ contains all the unitary operators.

Proof. We shall show, that

$$\text{str.r.lim}\, e^{-\beta p}\Delta_\beta V_I \Delta_{-\beta} = a\sum_{k,j} r_{kj}^p, \tag{26}$$

where $\beta \to -\infty$ if (a) holds, or $\beta \to +\infty$, if (b) holds. Then, $\exp\left\{-i\alpha a\sum_{k,j} r_{kj}^p\right\}$ belong to $\mathscr{D}$ and the theorem follows from Lemma 5.

One has the following sufficient condition on (26):

$$\lim e^{-(3+2p)} \int_0^{e^\beta A} dr |w(r) - 1|^2 r^{2(p+1)} = 0 \tag{27}$$

for any $A>0$ (compare the proof of Lemma 1). It is easy to check, that (a) or (b) yields (27). QED

7. General Remarks

A particular implication of our theorems is the existence of an operational "time-inversion", which might seem paradoxal. It is usually taken for granted that since the exact knowledge of many particle microstate is impossible, hence one can manipulate the thermodynamical magnitudes only. This point of view has been questioned in discussion about the foundations of statistical physics. One of familiar objections is called the Loschmidt-Zermelo paradox [17]. The known Boltzmann's answer to that paradox was based on two assumptions: 1) the practical impossibility of changing at the moment the velocities of all the particles and 2) a very long recurrence-time in the case of large particle number. However, some limitations of these arguments have been experimentally seen for spin systems. It turns out that spin system, in a way, can be forced "to go back in time" recovering its initial microstate (spin echo) [9]. Waugh expressed the opinion that a similar effect might be provoked also in general particle gasses [9]. What we have shown in our work is precisely the consistency of this conjecture with the dynamical laws of nonrelativistic quantum mechanics. The obtained here dynamical reversibility is not like the Loschmidt's time inversion, i.e. the inversion of particle momenta, which is an anti-symplectic mapping of the phase space and cannot be a dynamical transformation. The corresponding operation in quantum mechanics (the complex conjugation of wave functions) is anti-linear. What shows our study is the existence of a different kind of "put-back" operation. It consists in a certain prescription of maneuvering the external fields. The resulting transformation of states is unitary and is inverse to the natural evolution. As a consequence, the past state of the system can be reconstructed without the knowledge of the initial state. The required information is only the interaction potential and the number of particles involved.

The relation to thermodynamics is worth mentioning. Our results hold for arbitrary finite number N of particles, but nothing is said about the limit $N\to\infty$. Even for finite N the recurrence-time is enormously long. The greater N, the more difficult the operation might become. Moreover, the system considered is open, as Hamiltonians with time-dependent external fields are involved. Therefore, the operational invertibility does not mean the breaking of the thermodynamical laws, though, perhaps, it prevents their too universal interpretation.

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