Commun. Math. Phys. 76, 1–26 (1980)

Communications in Mathematical Physics
© by Springer-Verlag 1980

Nonlinear Dynamics of the Infinite Classical Heisenberg Model: Existence Proof and Classical Limit of the Corresponding Quantum Time Evolution

Pierre A. Vuillermot*†
School of Mathematics, Georgia Institute of Technology, Atlanta, 30332 USA

Abstract. For any initial spin configuration we prove the existence, unicity and regularity properties of the solution of Hamilton's equations for the infinite classical Heisenberg model with stable interactions, along with the essential selfadjointness of the associated Liouville operator. We also prove new properties of SU(2)-coherent states which, together with the concept of Trotter approximations for one-parameter (semi-) groups, are used to show that this dynamics is nothing but the classical limit of the time evolution generated by the infinite quantum (suitably normalized) Heisenberg Hamiltonian. The classical limit emerges when the spin magnitude S goes to infinity while Plank's constant $\hbar$ goes to zero, their product $S\hbar$ remaining fixed. The main results are stated in the form of intertwining relations between the classical and the quantum unitary group.

Introduction

Rigorous results about the time evolution of systems with infinitely many degrees of freedom are rather scarce, although the last few years have revealed new contributions in the field by various authors, among whom Lanford, Lebowitz and Lieb [1] who considered a lattice system of anharmonic oscillators of arbitrary dimension, and more recently Dobrushin and Fritz [2] who considered one- and two dimensional continuous systems with singular interactions. We consider here the classical Heisenberg model with stable interactions. In section 1 we prove the existence, unicity and regularity properties of the solution of Hamilton's equations for any initial spin configuration, and essential selfadjointness of the corresponding Liouville operator. We then derive in section 2 new properties of SU(2)-coherent states which, along with the concept of Trotter approximations for one-parameter contraction (semi-) groups, are used to prove con-

* Work supported in part by the Swiss National Science Foundation under Grant 820-436-76 and in part by the U.S. Department of Energy under contract EG-77-C-03-1538.

† Also at Physical Dynamics, Inc., La Jolla, CA 92037 USA

0010-3610/80/0076/0001/$05.20

vergence of the time evolution generated by the infinite quantum suitably normalized Heisenberg Hamiltonian towards that of the classical model described in section 1. This classical limit emerges when the spin magnitude S goes to infinity while Plank's constant $\hbar$ goes to zero, their product $S\hbar$ remaining fixed. Parts of these results have been announced in [4].

I. Hamiltonian Dynamics of the Infinite Classical Heisenberg Model with Stable Interactions

We identify the ν-dimensional lattice with $\mathbb{Z}^\nu$ and, with each site $r \in \mathbb{Z}^\nu$, we associate a unit-vector $\mathbf{s}_r \in \mathcal{S}_r^2 \subset \mathbb{R}_r^3$ where $\mathcal{S}_r^2$ and $\mathbb{R}_r^3$ stand for the two-dimensional unit-sphere and the three-dimensional Euclidean space respectively. Write Λ for any finite region of $\mathbb{Z}^\nu$, $|\Lambda|$ for its cardinality, and consider the classical Heisenberg model defined from the Hamilton function

$$-\mathrm{h}_\Lambda = \frac{1}{2} \sum_{r,r' \in \Lambda} j(r-r')\mathbf{s}_r \cdot \mathbf{s}_{r'} \tag{I.1}$$

with real couplings $j(r)$ satisfying the stability condition

$$M_\nu(\mathbf{j}) = \sum_{r \in \mathbb{Z}^\nu} |j(r)| < +\infty \tag{I.2}$$

and $j(-r) = j(r)$ as well as $j(0) = 0$. Consider the topological product $\mathcal{S}^{2|\Lambda|} = \prod_{r \in \Lambda} \mathcal{S}_r^2$ equipped with the symplectic probability measure $d\mu_\Lambda \equiv (4\pi)^{-|\Lambda|} \prod_{r \in \Lambda} \times d(\cos\theta_r) d\phi_r$ where each $\phi_r \in [0, 2\pi)$ and each $\theta_r \in [0, \pi]$. This corresponds to the phase space structure on $\mathcal{S}^{2|\Lambda|}$ with $\{q_r\}_{r \in \Lambda} = \{\cos\theta_r\}_{r \in \Lambda}$ as generalized coordinates, and $\{p_r\}_{r\in\Lambda} = \{\phi_r\}_{r\in\Lambda}$ as generalized momenta. Consider now the complex Hilbert space $\mathcal{H}_\Lambda \equiv \mathcal{L}^2_{\mu_\Lambda}(\mathcal{S}^{2|\Lambda|}; \mathbb{C})$ and define the Liouville operator

$$\mathrm{L}_\Lambda = i \sum_{r\in\Lambda} \left\{ \frac{\partial \mathrm{h}_\Lambda}{\partial \phi_r} \frac{\partial}{\partial(\cos\theta_r)} - \frac{\partial \mathrm{h}_\Lambda}{\partial(\cos\theta_r)} \frac{\partial}{\partial\phi_r} \right\} \tag{I.3}$$

on the domain $\mathcal{D}(\mathrm{L}_\Lambda) = \mathcal{C}^{(1)}(\mathcal{S}^{2|\Lambda|}; \mathbb{C}) \equiv \mathcal{C}_\Lambda^{(1)}$ (once continuously differentiable complex-valued functions on $\mathcal{S}^{2|\Lambda|}$) where it is symmetric. Define then the Poisson bracket of two $\mathcal{C}_\Lambda^{(1)}$-functions by

$$[f, g]_\Lambda = \sum_{r\in\Lambda} \left\{ \frac{\partial f}{\partial\phi_r} \frac{\partial g}{\partial(\cos\theta_r)} - \frac{\partial f}{\partial(\cos\theta_r)} \frac{\partial g}{\partial\phi_r} \right\} \tag{I.4}$$

We have in particular

$$[\mathrm{s}_r^x, \mathrm{s}_{r'}^y]_\Lambda = \delta_{r,r'} \mathrm{s}_r^z \tag{I.5}$$

and cyclically in x, y and z, for each $r, r' \in \Lambda$. These relations are formally identical to the usual SU(2)-commutation relations of the spin operators in quantum mechanics, and will play an important role in section 2. Expressions (I.3) and (I.4) can be rewritten in terms of the vectors $\mathbf{s}_r$'s. Namely,

$$\mathrm{L}_\Lambda = i \sum_{r\in\Lambda} \mathbf{s}_r \cdot \frac{\partial \mathrm{h}_\Lambda}{\partial \mathbf{s}_r} \times \frac{\partial}{\partial \mathbf{s}_r} \tag{I.6}$$

and

$$[f,g]_A = \sum_{r\in A} \mathbf{s}_r \cdot \frac{\partial f}{\partial \mathbf{s}_r} \times \frac{\partial g}{\partial \mathbf{s}_r} \tag{I.7}$$

as an elementary calculation shows. Thus

$$\mathrm{L}_A f = i[\mathrm{h}_A, f]_A \tag{I.8}$$

for each $f\in\mathscr{D}(\mathrm{L}_A)$ and in particular for $f = \mathrm{s}^x_\alpha, \mathrm{s}^y_\alpha, \mathrm{s}^z_\alpha$ for some α we get the flow equations in vector form

$$\mathrm{L}_A(\mathbf{s}_\alpha) = i\mathbf{s}_\alpha \times \sum_{r\in A} j(\alpha - r)\mathbf{s}_r = i\dot{\mathbf{s}}_{\alpha,A} \tag{I.8'}$$

on $\mathscr{S}^{2|A|}$, where the dot stands for the derivative with respect to time. Equations (I.8′) describe a Larmor precession of $\mathbf{s}_\alpha$ in the effective "magnetic field" $\sum_{r\in A} j(\alpha - r)\mathbf{s}_r$ due to the Heisenberg interaction (I.1). Finally, we shall denote by $\mathfrak{A}_A$ the local algebra of observables of all the continuous complex-valued functions on $\mathscr{S}^{2|A|}$, namely $\mathfrak{A}_A = \mathscr{C}(\mathscr{S}^{2|A|};\mathbb{C})$.

The thermodynamic limit will be performed along an increasing sequence of boxes $\{A_n\}_{n=1}^\infty$ converging to $\mathbb{Z}^\nu$ in the sense of inclusion: whenever $n \leqq n'$ we have $A_n \subseteq A_{n'}$ and for any finite A there exists an integer $N(A)$ such that $N(A) \leqq n$ implies $A \subseteq A_n$. We shall write simply $\mathfrak{A}_n$ for $\mathfrak{A}_{A_n}$, $\mathscr{H}_n$ for $\mathscr{H}_{A_n}$ and so on. We identify the configuration space of the infinite system with $\mathscr{S}^{2|\mathbb{Z}^\nu|} \equiv \prod_{r\in\mathbb{Z}^\nu} \mathscr{S}^2_r$, compact in the product topology. The equations of motion defining the Hamiltonian flow are simply given by the infinite-volume version of (I.8′), namely

$$\dot{\mathbf{s}}_\alpha = \mathbf{s}_\alpha \times \sum_{r\in\mathbb{Z}^\nu} j(\alpha - r)\mathbf{s}_r \tag{I.9}$$

for each $\alpha\in\mathbb{Z}^\nu$, which makes sense by virtue of the stability condition (I.2). The algebra of observables $\mathfrak{A}$ is defined as $\mathfrak{A} \equiv \mathscr{C}(\mathscr{S}^{2|\mathbb{Z}^\nu|};\mathbb{C})$. Observe that $\mathfrak{A}$ is nothing but the quasi-local algebra generated by the $\mathfrak{A}_n$'s by Stone–Weierstrass theorem [5]. Finally we identify the Hilbert space $\mathscr{H}$ of the infinite system with $\mathscr{H} \equiv \mathscr{L}^2_{\mu_{\mathbb{Z}^\nu}}(\mathscr{S}^{2|\mathbb{Z}^\nu|};\mathbb{C})$, where $\mu_{\mathbb{Z}^\nu}$ denotes the infinite product measure $d\mu_{\mathbb{Z}^\nu} \equiv \prod_{r\in\mathbb{Z}^\nu} (4\pi)^{-1} \times d(\cos\theta_r)d\phi_r$. Observe that $\mathscr{H}_n$ can be interpreted as the GNS-space associated with the C^*-algebra $\mathfrak{A}_n$ and the state μ_n, while $\mathscr{H}$ can be interpreted as the GNS-space associated with the C^*-algebra $\mathfrak{A}$ and the state $\mu_{\mathbb{Z}^\nu}$ (see [6] for a definition of these concepts). Now observe that for any $A' \subset A$ and $f\in\mathscr{C}^{(1)}_{A'}$, relation (I.6) gives

$$\begin{aligned}\mathrm{L}_A f &= i \sum_{\alpha\in A'} \mathbf{s}_\alpha \cdot \frac{\partial \mathrm{h}_A}{\partial \mathbf{s}_\alpha} \times \frac{\partial f}{\partial \mathbf{s}_\alpha} \\ &= i \sum_{\alpha\in A'} \mathbf{s}_\alpha \cdot \left\{ \sum_{r\in A} j(\alpha - r)\mathbf{s}_r \right\} \times \frac{\partial f}{\partial \mathbf{s}_\alpha}\end{aligned} \tag{I.10}$$

Choosing then $A' = A_n$ for some n and $f\in\mathscr{C}^{(1)}_n$, we define the Liouville operator $\mathrm{L}_{\mathbb{Z}^\nu}$ of the infinite system on the dense domain $\mathscr{D}(\mathrm{L}_{\mathbb{Z}^\nu}) = \bigcup_{n=1}^\infty \mathscr{C}^{(1)}_n$ in $\mathscr{H}$ by its action on local $\mathscr{C}^{(1)}$-functions, namely

$$\mathrm{L}_{\mathbb{Z}^\nu} f = i \sum_{\alpha\in\Lambda_n} \mathbf{s}_\alpha \cdot \left\{ \sum_{r\in\mathbb{Z}^\nu} j(\alpha - r)\mathbf{s}_r \right\} \times \frac{\partial f}{\partial \mathbf{s}_\alpha} \tag{I.11}$$

This expression again makes sense by virtue of (I.2), and defines a symmetric operator on $\mathscr{H}$. Now write $\mathbf{s}_{\Lambda_n} = \{\mathbf{s}_\alpha\}_{\alpha\in\Lambda_n} \in \mathbb{R}^{3|\Lambda_n|}$ for a spin configuration in Λ_n and $\mathbf{s}_{\mathbb{Z}^\nu}$ for a spin configuration of the infinite system. Our first result is concerned with the existence, the unicity and the regularity properties of the solution of (I.9), which are proved in the following

Theorem 1.1. *For each $n\in\mathbb{N}$ and each initial spin configuration $\mathbf{s}_{\mathbb{Z}^\nu}(0) = (\mathbf{s}_{\Lambda_n}(0); \mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n}(0))$ there exists a unique global solution $\mathbf{s}_{\Lambda_n}(t; \mathbf{s}_{\Lambda_n}(0))$ of the system of equations given by* (I.8′) *for $\alpha\in\Lambda_n$ and by $\dot{\mathbf{s}}_\alpha = 0$ for $\alpha\in\mathbb{Z}^\nu/\Lambda_n$. The map $\mathbf{s}_{\Lambda_n}(\cdot;\cdot)$ from $\mathbb{R}\times\mathscr{S}^{2|\Lambda_n|}$ into $\mathscr{S}^{2|\Lambda_n|}$ is moreover jointly $\mathscr{C}^\infty$ in $(t;\mathbf{s}_{\Lambda_n}(0))$ and the limit*

$$\mathbf{s}_{\alpha,\mathbb{Z}^\nu}(t; \mathbf{s}_{\mathbb{Z}^\nu}(0)) = \lim_{n\to\infty} \mathbf{s}_{\alpha,\Lambda_n}(t; \mathbf{s}_{\Lambda_n}(0)) \tag{I.12}$$

exists for all $t\in\mathbb{R}$, uniformly on bounded t-intervals, namely

$$\lim_{n\to\infty} \sup_{|t|\leqq t_0 < +\infty} \| \mathbf{s}_{\alpha,\Lambda_n}(t; \mathbf{s}_{\Lambda_n}(0)) - \mathbf{s}_{\alpha,\mathbb{Z}^\nu}(t; \mathbf{s}_{\mathbb{Z}^\nu}(0)) \| = 0 \tag{I.13}$$

for each $\alpha\in\mathbb{Z}^\nu$ and each $t_0\in\mathbb{R}^+$. The expression (I.12) *represents moreover the unique global solution of* (I.9).

Proof. Starting from (I.8′) we get (with $\mathrm{L}_{\Lambda_n} \equiv \mathrm{L}_n$)

$$\begin{aligned} &\| i\mathrm{L}_n(\mathbf{s}_\alpha) - i\mathrm{L}_n(\boldsymbol{\tau}_\alpha) \| \\ &\quad = \Big\| \mathbf{s}_\alpha \times \sum_{r\in\Lambda_n} j(\alpha - r)\mathbf{s}_r - \boldsymbol{\tau}_\alpha \times \sum_{r\in\Lambda_n} j(\alpha - r)\boldsymbol{\tau}_r \Big\| \\ &\quad = \Big\| \mathbf{s}_\alpha \times \sum_{r\in\Lambda_n} j(\alpha - r)(\mathbf{s}_r - \boldsymbol{\tau}_r) + (\mathbf{s}_\alpha - \boldsymbol{\tau}_\alpha) \times \sum_{r\in\Lambda_n} j(\alpha - r)\boldsymbol{\tau}_r \Big\| \\ &\quad \leqq \sum_{r\in\Lambda_n} |j(\alpha - r)| \, \| \mathbf{s}_r - \boldsymbol{\tau}_r \| + \| \mathbf{s}_\alpha - \boldsymbol{\tau}_\alpha \| \sum_{r\in\Lambda_n} |j(\alpha - r)| \\ &\quad \leqq 2M_\nu(\mathbf{j}) \| \mathbf{s}_{\Lambda_n} - \boldsymbol{\tau}_{\Lambda_n} \| \end{aligned} \tag{I.14}$$

with $\| \mathbf{s}_{\Lambda_n} \| = \sum_{r\in\Lambda_n} \| \mathbf{s}_r \|$, using (I.2) and elementary inequalities. L_n is thereby a Lipschitz vectorfield on $\mathscr{S}^{2|\Lambda_n|}$. Local existence, unicity and the regularity properties of the finite-volume solution then follow from standard arguments (see for instance [7]), in particular from a contraction mapping argument applied to the map

$$(M\mathbf{f}_\alpha)_{\Lambda_n}(t; \mathbf{s}_{\Lambda_n}(0)) = \mathbf{s}_\alpha(0) + \int_0^t du (i\mathrm{L}_n \mathbf{f}_\alpha)(u) \tag{I.15}$$

defined on a suitable complete metric space of $\mathscr{S}^{2|\Lambda_n|}$-valued functions around $(0; \mathbf{s}_{\Lambda_n}(0))$, whose fixed point

$$(M\mathbf{s}_\alpha)_{\Lambda_n} = \mathbf{s}_{\alpha,\Lambda_n} \tag{I.16}$$

is precisely solution of (I.8′). Extension to a global unique solution $\mathbf{s}_\alpha(t; \mathbf{s}_{\Lambda_n}(0))$

for each α and each $t \in \mathbb{R}$ is then immediate in this case, since $\mathcal{S}^{2|\Lambda_n|}$ is compact, and the regularity properties follow from the fact that L_n is a $\mathscr{C}^\infty$-vectorfield on $\mathcal{S}^{2|\Lambda_n|}$. Fixing now once and for all the initial condition $\mathbf{s}_{\mathbb{Z}^\nu}(0) = (\mathbf{s}_{\Lambda_n}(0); \mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n}(0))$ and observing that (I.16) reads

$$\mathbf{s}_{\alpha,\Lambda_n}(t;\mathbf{s}_{\Lambda_n}(0)) = \mathbf{s}_\alpha(0) + \int_0^t du\, \mathbf{s}_\alpha(u) \times \sum_{r\in\Lambda_n} j(\alpha - r)\mathbf{s}_r(u) \tag{I.17}$$

we get

$$\mathbf{s}_{\alpha,\Lambda_n}(t;\mathbf{s}_{\Lambda_n}(0)) - \mathbf{s}_{\alpha,\Lambda_m}(t;\mathbf{s}_{\Lambda_m}(0)) = \int_0^t du\, \mathbf{s}(u) \times \sum_{r\in\Lambda_n/\Lambda_m} j(\alpha - r)\mathbf{s}_r(u) \tag{I.18}$$

for $m < n$, namely $\Lambda_m \subset \Lambda_n$, and consequently

$$\| \mathbf{s}_\alpha(t;\mathbf{s}_{\Lambda_n}(0)) - \mathbf{s}_\alpha(t;\mathbf{s}_{\Lambda_m}(0)) \| \leqq t \sum_{r\in\Lambda_n/\Lambda_m} |j(\alpha - r)| \tag{I.19}$$

The sequence $(\mathbf{s}_\alpha(t;\mathbf{s}_{\Lambda_m}(0))\}_{m=1}^\infty$ is then Cauchy for each $t > 0$ by virtue of (I.2), which proves existence of (I.12) for each α and $t \in \mathbb{R}$ (the extension to $t < 0$ is trivial). Now for each $\varepsilon > 0$ and $|t - t'| \leqq \varepsilon M_\nu^{-1}(j)$ we have again from (I.17) the estimate

$$\| \mathbf{s}_{\alpha,\Lambda_n}(t;\mathbf{s}_{\Lambda_n}(0)) - \mathbf{s}_{\alpha,\Lambda_n}(t';\mathbf{s}_{\Lambda_n}(0)) \| \leqq M_\nu(\mathbf{j}) |t - t'| \leqq \varepsilon \tag{I.20}$$

The family $\{\mathbf{s}_{\alpha,\Lambda_n}(t;\mathbf{s}_{\Lambda_n}(0))\}_{n=1}^\infty$ is thus equicontinuous, and the uniform convergence (I.13) then follows from Ascoli's first theorem [8]. To show that (I.12) is solution of (I.9), we first observe that the sequence $\{ \sum_{r\in\Lambda_n} j(\alpha - r)\mathbf{s}_r(u)\}_{n=1}^\infty$ is also equicontinuous (same argument as that leading to (I.20)), which proves its uniform convergence to $\sum_{r\in\mathbb{Z}^\nu} j(\alpha - r)\mathbf{s}_r(u)$; this function is thus continuous on $\mathcal{S}^{2|\mathbb{Z}^\nu|}$, or equivalently an element of the quasi-local algebra $\mathfrak{A}$. On the other hand we have

$$\| \mathbf{s}_\alpha(u) \times \sum_{r\in\Lambda_n} j(\alpha - r)\mathbf{s}_r \| \leqq M_\nu(j) \tag{I.21}$$

uniformly in n. Taking then the limit when $n \to \infty$ on both sides of (I.17), a $\mathscr{L}^1([0;t];du)$-dominated convergence argument shows that (I.12) is a solution of

$$\mathbf{s}_{\alpha,\mathbb{Z}^\nu}(t;\mathbf{s}_{\mathbb{Z}^\nu}(0)) = \mathbf{s}_\alpha(0) + \int_0^t du\, \mathbf{s}_\alpha(u) \times \sum_{r\in\mathbb{Z}^\nu} j(\alpha - r)\mathbf{s}_r(u) \tag{I.22}$$

or equivalently of (I.9), and unicity follows again from a contraction mapping argument. This proves the theorem.

Having got a complete description of the flow generated by (I.1) on $\mathcal{S}^{2|\mathbb{Z}^\nu|}$, we now face the problem of determining a well defined time evolution for all the quasi-local observables. This will be done in several steps. Observe first that one may consider $\mathscr{H}_n \subseteq \mathscr{H}_{n'}$ for $n \leqq n'$ and $\mathscr{H}_n \subseteq \mathscr{H}$ for each n. Introduce then the sequence $\{P_n\}_{n=1}^\infty$ or orthogonal projections from $\mathscr{H}$ onto $\mathscr{H}_n$, namely

$$(P_n f)(\mathbf{s}_{\Lambda_n}) = \int_{\mathcal{S}^{2|\mathbb{Z}^\nu/\Lambda_n|}} d\mu_{\mathbb{Z}^\nu/\Lambda_n}(\mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n}) f(\mathbf{s}_{\Lambda_n};\mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n}) \tag{I.23}$$

We have the following

Proposition 1.2. *The sequence* $\{\mathscr{H}_n, \mathrm{P}_n\}_{n=1}^{\infty}$ *is a Trotter approximation for* $\mathscr{H}$, *and the sequence* $\{\mathrm{L}_n\}_{n=1}^{\infty}$ *of the finite-volume Liouville operators converges strongly (in the sense of Trotter) to* $\mathrm{L}_{\mathbb{Z}^\nu}$.

Proof. The first statement above means that

$$|||\mathrm{P}_n||| \leqq \mathrm{M} \tag{I.24}$$

uniformly in n and

$$\lim_{n\to\infty} \| \mathrm{P}_n f \|_{\mathscr{H}_n} = \| f \|_{\mathscr{H}} \tag{I.25}$$

for each $f \in \mathscr{H}$, where $|||\cdot|||$ in (I.24) is the usual operator norm. The second one is equivalent to

$$\lim_{n\to\infty} \| \mathrm{P}_n \mathrm{L}_{\mathbb{Z}^\nu} f - \mathrm{L}_n \mathrm{P}_n f \|_{\mathscr{H}_n} = 0 \tag{I.26}$$

for each $f \in \mathscr{D}(\mathrm{L}_{\mathbb{Z}^\nu})$ (see for instance [9]–[11] for further details). The proof of (I.24) with $\mathrm{M} = 1$ is obvious from the definition (I.23) since $\mu_{\mathbb{Z}^\nu/\Lambda_n}$ is a probability measure. To prove (I.25) observe that $\mathscr{H}_n \subseteq \mathscr{H}_{n'}$ implies $\mathrm{P}_n \leqq \mathrm{P}_{n'}$ for $n \leqq n'$ in the sense of quadratic forms since $\mathrm{P}_{n'}\mathrm{P}_n = \mathrm{P}_n = \mathrm{P}_n\mathrm{P}_{n'}$, which gives $(\mathrm{P}_{n'} - \mathrm{P}_n)^2 = \mathrm{P}_{n'} - \mathrm{P}_n \geqq 0$. There exists therefore a bounded symmetric operator P on $\mathscr{H}$ such that

$$\lim_{n\to\infty} \| \mathrm{P}_n f - \mathrm{P} f \|_{\mathscr{H}} = 0 \tag{I.27}$$

for each $f \in \mathscr{H}$. Now for $f \in \bigcup_{n=1}^{\infty} \mathscr{H}_n$ we have $\mathrm{P}_n f = f$ from some n since $\mu_{\mathbb{Z}^\nu/\Lambda_n}$ is a probability measure on $\mathscr{S}^{2|\mathbb{Z}^\nu/\Lambda_n|}$, and so $\mathrm{P} = \mathbb{1}$ (identity operator) on $\bigcup_{n=1}^{\infty} \mathscr{H}_n$. Consequently we have $\mathrm{P} = \mathbb{1}$ everywhere on $\mathscr{H}$ since $\bigcup_{n=1}^{\infty} \mathscr{H}_n$ is dense in $\mathscr{H}$. Relation (I.27) then reads

$$\lim_{n\to\infty} \| \mathrm{P}_n f - f \|_{\mathscr{H}} = 0 \tag{I.28}$$

for each $f \in \mathscr{H}$, which implies (I.25). To prove (I.26) we take $f \in \mathscr{C}_n^{(1)}$ for some n, choose $m > n$ and rewrite (I.11) as

$$(\mathrm{L}_{\mathbb{Z}^\nu} f)(\mathbf{s}_{\mathbb{Z}^\nu}) = i \sum_{\alpha\in\Lambda_n} \mathbf{s}_\alpha \cdot \{ \sum_{r\in\Lambda_m} j(\alpha - r)\mathbf{s}_r + \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} j(\alpha - r)\mathbf{s}_r \} \times \frac{\partial f}{\partial \mathbf{s}_\alpha} \tag{I.29}$$

$$= \mathrm{L}_m f + i \sum_{\alpha\in\Lambda_n} \mathbf{s}_\alpha \cdot \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} j(\alpha - r)\mathbf{s}_r \times \frac{\partial f}{\partial \mathbf{s}_\alpha} \tag{I.30}$$

Using then (I.23) and observing that $\mathrm{L}_m \mathrm{P}_m f = \mathrm{L}_m f$ we get

$$(\mathrm{P}_m \mathrm{L}_{\mathbb{Z}^\nu} f - \mathrm{L}_m \mathrm{P}_m f)(\mathbf{s}_{\Lambda_m}) = i \sum_{\alpha\in\Lambda_n} \mathbf{s}_\alpha \cdot \{ \int_{\mathscr{S}^{2|\mathbb{Z}^\nu|}} d\mu_{\mathbb{Z}^\nu}(\mathbf{s}_{\mathbb{Z}^\nu}) \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} j(\alpha - r)\mathbf{s}_r \} \times \frac{\partial f}{\partial \mathbf{s}_\alpha} \tag{I.31}$$

For the absolute value of (I.31) we then get the estimate

$$|(P_m L_{\mathbb{Z}^\nu} f - L_m P_m f)(s_{\Lambda_m})| \leqq \sum_{\alpha\in\Lambda_n} \left\| \frac{\partial f}{\partial \mathbf{s}_\alpha} \right\| \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} |j(\alpha - r)|$$
$$\leqq \sum_{\alpha\in\Lambda_n} K_\alpha \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} |j(\alpha - r)| \tag{I.32}$$

where $\|\cdot\|$ denotes the usual euclidean norm in $\mathbb{R}^3$ as in theorem (I.1). The K_α's are bounds on $\left\| \frac{\partial f}{\partial \mathbf{s}_\alpha} \right\|$ uniform in $\mathbf{s}_{\Lambda_n}$ since each one of the components of $\frac{\partial f}{\partial \mathbf{s}_\alpha}$ is continuous by assumption on f. Consequently we have for the $\mathscr{L}^2$-norm $\|\cdot\|_{\mathscr{H}_m}$ of (I.31) the estimate

$$\| P_m L_{\mathbb{Z}^\nu} f - L_m P_m f \|^2_{\mathscr{H}_m} = \int_{\mathscr{S}^{2|\Lambda_m|}} d\mu_{\Lambda_m}(\mathbf{s}_{\Lambda_m}) |(P_m L_{\mathbb{Z}^\nu} f - L_m P_m f)(\mathbf{s}_{\Lambda_m})|^2$$
$$\leqq \{ \sum_{\alpha\in\Lambda_n} K_\alpha \sum_{r\in\mathbb{Z}^\nu/\Lambda_m} |j(\alpha - r)| \}^2 \tag{I.33}$$

which implies (I.26) because of (I.2). This proves the proposition.

We next consider the problem of the selfadjoint extensions of L_n (and $L_{\mathbb{Z}^\nu}$). Observe first that (I.1) reads in polar coordinates

$$-h_{\Lambda_n} = \tfrac{1}{2} \sum_{r,r'\in\Lambda_n} j(r - r')\{\sin\theta_r \sin\theta_{r'} \cos(\phi_r - \phi_{r'}) + \cos\theta_r \cos\theta_{r'}\} \tag{I.34}$$

and is consequently invariant under the reflection $\phi_r \to -\phi_r$. The conjugation J_n on $\mathscr{H}_n$ defined by

$$(J_n f)(\cos\theta_1, \ldots, \cos\theta_{|\Lambda_n|}; \phi_1, \ldots, \phi_{|\Lambda_n|})$$
$$= f(\cos\theta_1, \ldots, \cos\theta_{|\Lambda_n|}; -\phi_1, \ldots -\phi_{|\Lambda_n|}) \tag{I.35}$$

and leaving $\mathscr{D}(L_n)$ invariant then satisfies the commutation relation $[J_n, L_n]_- = 0$ on $\mathscr{D}(L_n)$. L_n has therefore equal deficiency indices by a Von Neumann's theorem [12] and thereby at least one self-adjoint extension [13]. This is obviously a version of the usual "time-reversal" argument [12] since we have defined the generalized momenta by $p_r \equiv \phi_r$ for each r. The fact that L_n (and $L_{\mathbb{Z}^\nu}$) actually has a unique selfadjoint extension (essential selfadjointness) is described in the following

Theorem 1.3. *L_n is essentially selfadjoint on $\mathscr{D}(L_n)$ for each n, and $L_{\mathbb{Z}^\nu}$ is essentially selfadjoint on $\mathscr{D}(L_{\mathbb{Z}^\nu})$. Furthermore for any initial spin configuration $\mathbf{s}_{\mathbb{Z}^\nu}(0) = (\mathbf{s}_{\Lambda_n}(0); \mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n}(0))$ on $\mathscr{S}^{2|\mathbb{Z}^\nu|}$ and each $t\in\mathbb{R}$ we have the strongly continuous unitary groups*

$$(U_n(t)f)(\mathbf{s}_{\Lambda_n}(0)) = f(\mathbf{s}_{\Lambda_n}(t; \mathbf{s}_{\Lambda_n}(0)) = (\exp[iL_n^* t] f)(\mathbf{s}_{\Lambda_n}(0)) \tag{I.36}$$

on $\mathscr{H}_n$ and similarly

$$(U(t)f)(\mathbf{s}_{\mathbb{Z}^\nu}(0)) = f(\mathbf{s}_{\mathbb{Z}^\nu}(t; \mathbf{s}_{\mathbb{Z}^\nu}(0))) = (\exp[iL^*_{\mathbb{Z}^\nu} t] f)(\mathbf{s}_{\mathbb{Z}^\nu}(0)) \tag{I.37}$$

on $\mathscr{H}$, where $L_n^* = L_n^{**}$ and $L^*_{\mathbb{Z}^\nu} = L^{**}_{\mathbb{Z}^\nu}$ denote respectively the unique selfadjoint extensions of L_n and $L_{\mathbb{Z}^\nu}$. Moreover, the sequence of unitaries $\{U_n(t)\}_{n=1}^\infty$ converges strongly (in the sense of Trotter) to $U(t)$ uniformly on every bounded t-interval, *namely*

$$\lim_{n\to\infty} \sup_{|t|\leqq t_0<+\infty} \| P_n U(t) f - U_n(t) P_n f \|_{\mathscr{H}_n} = 0 \tag{I.38}$$

for each $f \in \mathscr{H}$ *and each* $t_0 \in \mathbb{R}^+$.

Proof. The flow $s_{\Lambda_n}(t; s_{\Lambda_n}(0))$ described in theorem (I.1) generates a strongly continuous unitary group on $\mathscr{H}_n$ through the first relation in (I.36) and Liouville's theorem. Essential selfadjointness of L_n and the fact that $U_n(t)$ is actually generated by L_n^* then follow from an adaptation of known arguments [12]: First, $f \in \mathscr{D}(L_n)$ implies $U_n(t) f \in \mathscr{D}(L_n)$ by the regularity property of theorem I.1, as well as

$$\frac{dU_n(t)}{dt}(t=0) f(s_{\Lambda_n}) = (L_n f)(s_{\Lambda_n}) \tag{I.39}$$

by definition of L_n, pointwise on $\mathscr{S}^{2|\Lambda_n|}$; now $f \in \mathscr{D}(L_n)$ has a uniformly bounded derivative on $\mathscr{S}^{2|\Lambda_n|}$ and $s_{\alpha,\Lambda_n}(t; s_{\Lambda_n}(0))$ has the same property in any compact t-interval around the origin (see for instance the argument leading to (I.20)). We then have

$$\lim_{t\to 0} \left\| \frac{dU_n(t)}{dt} f - L_n f \right\|_{\mathscr{H}_n} = 0 \tag{I.40}$$

by a dominated convergence argument. All this implies $L_n^* = L_n^{**}$ and then the second equality in (I.36), as a consequence of a theorem by Nelson [14]. Essential selfadjointness of $L_{\mathbb{Z}^\nu}$ and (I.37) can be proved similarly. Finally (I.38) follows from (I.26) and Trotter's stability theorem [9]–[11]. This completes the proof.

Observe that the flow $s_{\Lambda_n}(t; s_{\Lambda_n}(0))$ generates also a strongly continuous one-parameter group of automorphisms $\alpha_n(t)$ of the C^*-algebra $\mathfrak{A}_n$ through the definition

$$(\alpha_n(t) f)(s_{\Lambda_n}(0)) = f(s_{\Lambda_n}(t; s_{\Lambda_n}(0))) \tag{I.41}$$

Similarly for the infinite system with the definition

$$(\alpha_{\mathbb{Z}^\nu}(t) f)(s_{\mathbb{Z}^\nu}(0)) = f(s_{\mathbb{Z}^\nu}(t; s_{\mathbb{Z}^\nu}(0))) \tag{I.42}$$

Thus, what theorem (I.3) describes are simply the unitary implementations of these groups of automorphisms in the corresponding GNS-spaces $\mathscr{H}_n$ and $\mathscr{H}$ corresponding to the states μ_{Λ_n} and $\mu_{\mathbb{Z}^\nu}$, respectively. We shall follow exactly the same pattern for the time evolution of the quantum model we describe in the next section.

II. Dynamics of the Quantum Heisenberg Model with Stable Interactions: Rephrasing Old Results

Consider now the quantum Heisenberg Hamiltonian

$$-H_n(S) = \frac{1}{2} \sum_{r,r' \in \Lambda_n} j(r-r') S_r \cdot S_{r'} \tag{II.1}$$

acting on $\tilde{\mathscr{H}}_n(\mathrm{S}) = \bigotimes_{r\in\Lambda_n} \mathbb{C}^{(2\mathrm{S}+1)}$, with couplings satisfying the same conditions as in section I, in particular the stability condition (I.2), and where $\mathrm{S} = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. The local algebra of observables $\mathfrak{A}_n(\mathrm{S})$ is identified with the space of the linear (bounded) operators on $\tilde{\mathscr{H}}_n(\mathrm{S})$ equipped with the uniform norm, and is generated by the Pauli matrices obeying the SU(2)-commutation relations

$$[\mathrm{S}_r^x, \mathrm{S}_{r'}^y]_- = i\delta_{r,r'}\mathrm{S}_r^z \tag{II.2}$$

and their cyclic permutations. The finite-volume time evolution is defined by

$$\alpha_n(t;\mathrm{S})\mathrm{A} = \exp[it\mathrm{H}_n(\mathrm{S})]\mathrm{A}\exp[-it\,\mathrm{H}_n(\mathrm{S}))] \tag{II.3}$$

for each local A, and with $\mathrm{A}\in\mathfrak{A}_m(\mathrm{S})$ for $m < n$, the limit

$$\alpha_{\mathbb{Z}^\nu}(t;\mathrm{S})\mathrm{A} = \lim_{n\to\infty} \alpha_n(t;\mathrm{S})\mathrm{A} \tag{II.3'}$$

exists for all $t\in\mathbb{R}$ as a well defined element of the quasilocal algebra $\mathfrak{A}(\mathrm{S})$ generated by the $\mathfrak{A}_n(\mathrm{S})$'s ([15], [16]); it defines eventually a strongly continuous group of automorphisms on $\mathfrak{A}(\mathrm{S})$. Obviously $\alpha_n(t;\mathrm{S})$ is the exact quantum mechanical counterpart of $\alpha_n(t)$ in (I.41), and $\alpha_{\mathbb{Z}^\nu}(t;\mathrm{S})$ that of $\alpha_{\mathbb{Z}^\nu}(t)$ in (I.42). Consider now the central state

$$\rho_{n,\mathrm{S}}(\cdot) = (2\mathrm{S}+1)^{-|\Lambda_n|}\,\mathrm{Tr}_n(\cdot) \tag{II.4}$$

on $\mathfrak{A}_n(\mathrm{S})$, where Tr_n stands for trace on $\tilde{\mathscr{H}}_n(\mathrm{S})$, and equip the vector space of all the linear (bounded) operators on $\tilde{\mathscr{H}}_n(\mathrm{S})$ with the Hilbert–Schmidt sesquilinear form

$$(\mathrm{A}, \mathrm{B})_{n,\mathrm{S}} = \rho_{n,\mathrm{S}}(\mathrm{A}\mathrm{B}^*) \tag{II.5}$$

We shall write $\mathscr{H}_n(\mathrm{S})$ for the corresponding Hilbert space, which is nothing but the GNS-space of $\mathfrak{A}_n(S)$ associated to the state (II.4). Elementary considerations show indeed that the one-parameter family $\alpha_n(t;\mathrm{S})$ is unitarily implemented in $\mathscr{H}_n(\mathrm{S})$ by

$$\mathrm{U}_n(t;\mathrm{S}) = \exp[it\,\mathrm{ad}\,\mathrm{H}_n(\mathrm{S})] \tag{II.6}$$

with $\mathrm{ad}\,\mathrm{H}_n(\mathrm{S})(\mathrm{A}) = [\mathrm{H}_n(\mathrm{S}), \mathrm{A}]_-$, which is obviously selfadjoint with respect to (II.5); moreover, we have

$$\mathrm{U}_n(t;\mathrm{S})\Pi_n(\mathrm{A})\mathrm{U}_n^{-1}(t;\mathrm{S}) = \Pi_n(\alpha_n(t;\mathrm{S})\mathrm{A}) \tag{II.7}$$

and

$$\mathrm{U}_n(t;\mathrm{S})\omega_n = \omega_n \tag{II.8}$$

with the morphisms $\Pi_n(\mathrm{A})\mathrm{B} = \mathrm{AB}$ and the cyclic vector $\omega_n = \mathbb{1}_n$ (identity operator). One has a similar construction for the infinite system. Consider indeed $\mathrm{A}\in\mathscr{H}_n(\mathrm{S})$ and $n' > n$; we then have from (II.4) the consistency relation

$$\rho_{n',\mathrm{S}}(\mathrm{A}\otimes\mathbb{1}_{\Lambda_{n'}/\Lambda_n}) = \rho_{n,\mathrm{S}}(\mathrm{A}) \tag{II.9}$$

and consequently the inductive limit-state $\rho_\mathrm{S} \equiv \lim_{n\to\infty} \rho_{n,\mathrm{S}}$ exists on $\mathfrak{A}(\mathrm{S})$ (see for instance [17]). We shall identify the Hilbert space $\mathscr{H}(\mathrm{S})$ of the infinite system with the GNS-space of $\mathfrak{A}(\mathrm{S})$ associated with the state ρ_{S}. The state ρ_{S} being clearly time-translation invariant since the $\rho_{n,\mathrm{S}}$'s are, the one-parameter family $\alpha_{\mathbb{Z}^\nu}(t;\mathrm{S})$ can also be unitarily implemented in $\mathscr{H}(\mathrm{S})$ through

$$\mathrm{U}_{\mathbb{Z}^\nu}(t;\mathrm{S})\Pi(\mathrm{A})\mathrm{U}_{\mathbb{Z}^\nu}^{-1}(t;\mathrm{S}) = \Pi(\alpha_{\mathbb{Z}^\nu}(t;\mathrm{S})\mathrm{A}) \tag{II.10}$$

and

$$\mathrm{U}_{\mathbb{Z}^\nu}(t;\mathrm{S})\omega = \omega \tag{II.11}$$

with $\Pi(\mathrm{A})\cdot\mathrm{B} = \mathrm{AB}$ and $\omega = \mathbb{1}$ (identity on $\mathscr{H}(\mathrm{S})$) by Segal's theorem (see for instance [6]). We shall denote by $\mathrm{ad}_{\mathbb{Z}^\nu}\mathrm{H}(\mathrm{S})$ its self-adjoint generator. Now observe that we have again $\mathscr{H}_n(\mathrm{S}) \subseteq \mathscr{H}_{n'}(\mathrm{S})$ for $n \leq n'$ and also, up to some natural identifications, $\mathscr{H}_n(\mathrm{S}) \subseteq \mathscr{H}(\mathrm{S})$ for each n. Considering then the family of orthogonal projections $\mathrm{P}_n(\mathrm{S})$ from $\mathscr{H}(\mathrm{S})$ onto $\mathscr{H}_n(\mathrm{S})$, one can play a similar game as in proposition (I.2) to prove that $\{\mathscr{H}_n(\mathrm{S}), \mathrm{P}_n(\mathrm{S})\}_{n=1}^{\infty}$ is a Trotter approximation sequence for $\mathscr{H}(\mathrm{S})$. Omitting the details we end up with the following.

Theorem II. 1. *The sequence of unitaries* $\{\mathrm{U}_n(t;\mathrm{S}) = \exp[it\,\mathrm{ad}\,\mathrm{H}_n(\mathrm{S})]\}_{n=1}^{\infty}$ *with selfadjoint generator* $\mathrm{ad}\,\mathrm{H}_n(\mathrm{S})$ *converges strongly, in the sense of Trotter, to the unitary group*

$$\mathrm{U}_{\mathbb{Z}^\nu}(t;\mathrm{S}) = \exp[it\,\mathrm{ad}_{\mathbb{Z}^\nu}\mathrm{H}(\mathrm{S})] \tag{II.12}$$

with selfadjoint generator $\mathrm{ad}\,\mathrm{H}_{\mathbb{Z}^\nu}(\mathrm{S})$, *defined by* (II.10) *and* (II.11). *Moreover, the convergence is locally uniform on* $\mathbb{R}$, *namely*

$$\lim_{n\to\infty}\ \sup_{|t|\leq t_0} \|\mathrm{P}_n(\mathrm{S})\mathrm{U}_{\mathbb{Z}^\nu}(t;\mathrm{S})\mathrm{A} - \mathrm{U}_n(t;\mathrm{S})\mathrm{P}_n(\mathrm{S})\mathrm{A}\|_{\mathscr{H}_n(\mathrm{S})} = 0 \tag{II.13}$$

for each $\mathrm{A}\in\mathscr{H}(\mathrm{S})$ *and each* $t_0\in\mathbb{R}^+$.

Comparison of theorems (I.3) and (II.1) now shows that the Hilbert space $\mathscr{H}_n(\mathrm{S})$, the generator $\mathrm{ad}\,\mathrm{H}_n(\mathrm{S})$ and the group $\mathrm{U}_n(t;\mathrm{S})$ are the natural counterparts of their classical analogues $\mathscr{H}_n$, the generator L_n^* and the unitary group $\mathrm{U}_n(t;\mathrm{S})$. A similar remark obviously holds for the infinite system. The precise connection between them will be examined now. This will be done in several steps with the help of the SU (2)-coherent state formalism, and with new Trotter approximations of a far less intuitive nature than those considered so far.

III. SU(2)-Coherent States and the Trotter Approximation of Classical Observables by the Quantum Ones

For each $r\in\mathbb{Z}^\nu$ define the SU(2)-coherent state

$$|\Omega_r\rangle = \exp\left[\frac{\theta_r}{2}\{\mathrm{S}_r^- e^{i\phi_r} - \mathrm{S}_r^+ e^{-i\phi_r}\}\right]|\mathrm{S}_r\rangle \tag{III.1}$$

where $|\mathrm{S}_r\rangle$ is the spin-up state $\mathrm{S}_r^z|\mathrm{S}_r\rangle = \mathrm{S}|\mathrm{S}_r\rangle$ in $\mathbb{C}_r^{(2\mathrm{S}+1)}$ and $\mathrm{S}_r^\pm = \mathrm{S}_r^x \pm i\mathrm{S}_r^y$. Write $|\Omega_{\Lambda_n}\rangle = \bigotimes_{r\in\Lambda_n} |\Omega_r\rangle$ and define the map $T_{n,\mathrm{S}}$ from $\mathscr{H}_n$ onto $\mathscr{H}_n(\mathrm{S})$ for each

n by

$$T_{n,\mathrm{S}}f=(2\mathrm{S}+1)^{|\Lambda_n|}\int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n})f(\mathbf{s}_{\Lambda_n})|\Omega_{\Lambda_n}\rangle\langle\Omega_{\Lambda_n}| \tag{III.2}$$

Having

$$\rho_{n,\mathrm{S}}(|\Omega_{\Lambda_n}\rangle\langle\Omega'_{\Lambda_n}|)=(2\mathrm{S}+1)^{-|\Lambda_n|}\langle\Omega'_{\Lambda_n}|\Omega_{\Lambda_n}\rangle \tag{III.3}$$

and

$$|\langle\Omega'_{\Lambda_n}|\Omega_{\Lambda_n}\rangle|^2=\prod_{\alpha\in\Lambda_n}\left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2\mathrm{S}} \tag{III.4}$$

from [18], writing $\| \ \|_{n,\mathrm{S}}$ for the Hilbert–Schmidt norm defined above we get

$$\|T_{n,\mathrm{S}}f\|^2_{n,\mathrm{S}}=(2\mathrm{S}+1)^{-|\Lambda_n|}\mathrm{Tr}_{\Lambda_n}[(T_{n,\mathrm{S}}f)(T_{n,\mathrm{S}}f)^*] \tag{III.5}$$

$$=(2\mathrm{S}+1)^{|\Lambda_n|}\int_{\mathscr{S}^{2|\Lambda_n|}\times\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n})d\mu_{\Lambda_n}(\mathbf{s}'_{\Lambda_n})f(\mathbf{s}_{\Lambda_n})\overline{f(\mathbf{s}'_{\Lambda_n})}\prod_{\alpha\in\Lambda_n}\left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2\mathrm{S}} \tag{III.6}$$

Applying then Schwarz inequality to $f\otimes\bar{f}$ and $(2\mathrm{S}+1)^{|\Lambda_n|}\cdot\prod_{\alpha\in\Lambda_n}\left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2\mathrm{S}}$ on the product space $\mathscr{S}^{2|\Lambda_n|}\times\mathscr{S}^{2|\Lambda_n|}$ we get

$$\|T_{n,\mathrm{S}}f\|_{n,\mathrm{S}}\leqq\|f\|_{\mathscr{H}_n} \tag{III.7}$$

since

$$(2\mathrm{S}+1)^{|\Lambda_n|}\int_{\mathscr{S}^{2|\Lambda_n|}\times\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n})d\mu_{\Lambda_n}(\mathbf{s}'_{\Lambda_n})\prod_{\alpha\in\Lambda_n}\left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2\mathrm{S}}=1 \tag{III.8}$$

as an elementary calculation shows. Consequently we have

$$|||T_{n,\mathrm{S}}|||\leqq 1 \tag{III.9}$$

uniformly in n and S for the corresponding operator norm. Observe now that by virtue of [18] we have the particular representation

$$\mathbb{1}_{\Lambda_{n'}/\Lambda_n}=(2\mathrm{S}+1)^{|\Lambda_{n'}|-|\Lambda_n|}\int_{\mathscr{S}^{2(|\Lambda_{n'}|-|\Lambda_n|)}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_{n'}/\Lambda_n})|\Omega_{\Lambda_{n'}/\Lambda_n}\rangle\langle\Omega_{\Lambda_{n'}/\Lambda_n}| \tag{III.10}$$

for the identity operator on $\mathscr{H}_{\Lambda_{n'}/\Lambda_n}(\mathrm{S})$, whenever $n'>n$. We get therefore the consistency relation

$$T_{n',\mathrm{S}}(f\otimes\mathbb{1}_{\Lambda_{n'}/\Lambda_n})=T_{n,\mathrm{S}}f\otimes 1_{\Lambda_{n'}/\Lambda_n} \tag{III.11}$$

for each $f\in\mathscr{H}_n$, where $1_{\Lambda_{n'}/\Lambda_n}$ stands for the identity function in $\mathscr{H}_{\Lambda_{n'}/\Lambda_n}$. One can then define unambiguously T_S on $\bigcup_{n=1}^{\infty}\mathscr{H}_n$ by $T_\mathrm{S}=T_{n,\mathrm{S}}$ on each $\mathscr{H}_n$, and we have

$$\|T_\mathrm{S}f\|_{\mathscr{H}(\mathrm{S})}=\|T_{n,\mathrm{S}}f\|_{\mathscr{H}_n(\mathrm{S})}\leqq\|f\|_{\mathscr{H}_n(\mathrm{S})}=\|f\|_{\mathscr{H}} \tag{III.12}$$

for each $f \in \bigcup_{n=1}^{\infty} \mathcal{H}_n$ by (III.7), implying

$$|||T_S||| \leqq 1 \tag{III.13}$$

uniformly in S. An extension by continuity then defines T_S as a bounded operator from $\mathcal{H}$ into $\mathcal{H}(S)$ since $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is dense in $\mathcal{H}$. The preceding construction now allows us to prove the following

Theorem III.2. *For each n the sequence* $\{\mathcal{H}_n(S), T_{n,S}\}_{S=1}^{\infty}$ *is a Trotter approximation for* $\mathcal{H}_n$, *and similarly the sequence* $\{\mathcal{H}(S), T_S\}_{S=1}^{\infty}$ *is a Trotter approximation for* $\mathcal{H}$.

Proof. The first statement means that

$$|||T_{n,S}||| \leqq M_n \tag{III.14}$$

uniformly in S and that

$$\lim_{S\to\infty} \| T_{n,S} f \|_{\mathcal{H}_n(S)} = \| f \|_{\mathcal{H}_n} \tag{III.15}$$

for each $f \in \mathcal{H}_n$. The relations defining the second statement are similar. Relation (III.14) has been proved already with $M_n = 1$ for each n (relation III.9). To prove (III.15) we choose $f \in \mathfrak{A}_n$ and define

$$F_S(\mathbf{s}_{\Lambda_n}) = (2S+1)^{|\Lambda_n|} \int_{\mathcal{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}'_{\Lambda_n}) \overline{f(\mathbf{s}'_{\Lambda_n})} \prod_{\alpha\in\Lambda_n} \left\{ \frac{\mathbf{s}_\alpha \cdot \mathbf{s}'_\alpha + 1}{2} \right\}^{2S} \tag{III.16}$$

We have

$$\| T_{n,S} f \|^2_{\mathcal{H}_n(S)} = \int_{\mathcal{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n}) f(\mathbf{s}_{\Lambda_n}) F_S(\mathbf{s}_{\Lambda_n}) \tag{III.17}$$

from (III.6) and Fubini's theorem. With $\mathbf{s} \in \mathcal{S}^2$ fixed consider now the family of one-site measures

$$d\nu_S(\mathbf{s};\mathbf{s}') = (2S+1) \left\{ \frac{\mathbf{s}\cdot\mathbf{s}' + 1}{2} \right\}^{2S} d\mu(\mathbf{s}') \tag{III.18}$$

and denote by $\delta_\mathbf{s}$ the Dirac measure at $\mathbf{s}$. One has convergence of the ν_S's toward $\delta_\mathbf{s}$ in the vague topology of measures when $S \to \infty$, namely

$$\lim_{S\to\infty} \int_{\mathcal{S}^2} d\nu_S(\mathbf{s};\mathbf{s}') g(\mathbf{s}') = g(\mathbf{s}) \tag{III.19}$$

for each one-site continuous g. To prove this statement, we choose an orthogonal system of coordinates whose z-axis coincides with $\mathbf{s}$. Expression (III.19) is then equivalent to

$$\lim_{S\to\infty} \frac{2S+1}{4\pi} \int_0^{2\pi} d\phi' \int_{-1}^{+1} ds'^z \left\{ \frac{s'^z + 1}{2} \right\}^{2S} g(\phi'; s'^z) = g(\mathbf{s}) \tag{III.20}$$

with $\mathbf{s} = (0;0;1)$. With the change of variable $t^z = S(s'^z - 1)$, the left-hand side of (III.20) can be rewritten as

$$\frac{2S+1}{4\pi S}\int_0^{2\pi} d\phi' \int_{-2S}^{0} dt^z\left\{\frac{t^z}{2S}+1\right\}^{2S} g\left(\phi';\frac{t^z}{S}+1\right)$$
$$=\frac{2S+1}{4\pi S}\int_0^{2\pi} d\phi' \int_{-\infty}^{0} dt^z\left\{\frac{t^z}{2S}+1\right\}^{2S} g\left(\phi';\frac{t^z}{S}+1\right)\chi_{[-2S;0]}(t^z) \tag{III.21}$$

where $\chi_{[-2S;0]}$ stands for the characteristic function of $[-2S;0]$. Writing now

$$f_S(\phi';t^z)=\frac{2S+1}{4\pi S}\left\{\frac{t^z}{2S}+1\right\}^{2S} g\left(\phi';\frac{t^z}{S}+1\right)\chi_{[-2S;0]}(t^z) \tag{III.22}$$

for the integrand above, we have

$$\lim_{S\to\infty} f_S(\phi';t^z)=(2\pi)^{-1}\exp[t^z]g(\phi';1) \tag{III.23}$$

for any fixed ϕ', pointwise everywhere in t^z on $(-\infty;0]$ by continuity of g. Furthermore we get from (III.22) the estimate (uniform in S)

$$|f_S(\phi';t^z)|\leqq(2\pi)^{-1}\|g\|_\infty \exp[t^z] \tag{III.24}$$

where $\|g\|_\infty$ denotes the uniform norm of g on $\mathscr{S}^2$. Since $\exp[t^z]\in\mathscr{L}^1((-\infty;0];dt)$, a $\mathscr{L}^1((-\infty;0];dt)$-dominated convergence argument then allows one to conclude that

$$\lim_{S\to\infty}\int_{-\infty}^{0} dt^z f_S(\phi';t^z)=(2\pi)^{-1}g(\phi';1)\int_{-\infty}^{0} dt^z\exp[t^z]$$
$$=(2\pi)^{-1}g(\phi';1) \tag{III.25}$$

for each ϕ'. Consequently we have

$$\lim_{S\to\infty}\int_0^{2\pi} d\phi'\int_{-\infty}^{0} dt^z f_S(\phi';t^z)=(2\pi)^{-1}\int_0^{2\pi} d\phi' g(\phi';1) \tag{III.26}$$

by a similar argument on $[0,2\pi)$. Now we have

$$g(\phi';s'^z)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} C_{lm}P_l^m(s'^z)\exp[im\phi'] \tag{III.27}$$

(uniformly on $\mathscr{S}^2$), where the P_l^m's are the associated Legendre polynomials. Consequently $g(\phi';1)$ does not depend on ϕ' since $P_l^m(1)=\delta_{m,0}$ which, according to (III.26), proves that

$$\lim_{S\to\infty}\int_0^{2\pi} d\phi'\int_{-\infty}^{0} dt^z f_S(\phi';t^z)=g(\phi;1) \tag{III.28}$$

for all $\phi\in[0,2\pi)$ or equivalently (III.20) since $\mathbf{s}=(0;0;1)=(\sin\theta\cos\phi;\sin\theta\sin\phi;\cos\theta)$ implies $\theta=0$ while ϕ remains arbitrary in $[0,2\pi)$. The sequence of product measures

$$\prod_{\alpha\in\Lambda_n} d\nu_S(\mathbf{s}_\alpha;\mathbf{s}'_\alpha)=(2S+1)^{|\Lambda_n|}\prod_{\alpha\in\Lambda_n}\left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2S} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n}) \tag{III.29}$$

then converges in the vague topology to $\delta_{\mathbf{s}_{\Lambda_n}}$, which means that

$$\lim_{S\to\infty} F_S(\mathbf{s}_{\Lambda_n}) = \overline{f(\mathbf{s}_{\Lambda_n})} \tag{III.30}$$

pointwise everywhere on $\mathscr{S}^{2|\Lambda_n|}$ according to (III.16). Consequently we have

$$\lim_{S\to\infty} f(\mathbf{s}_{\Lambda_n})F_S(\mathbf{s}_{\Lambda_n}) = |f(\mathbf{s}_{\Lambda_n})|^2 \tag{III.31}$$

Writing now $\|f\|_\infty$ for the uniform norm of f on $\mathscr{S}^{2|\Lambda_n|}$ we get the estimate

$$\begin{aligned} |f(\mathbf{s}_{\Lambda_n})F_S(\mathbf{s}_{\Lambda_n})| &\leqq \|f\|_\infty |F_S(\mathbf{s}_{\Lambda_n})| \\ &\leqq \|f\|_\infty^2 \end{aligned} \tag{III.32}$$

uniformly in S according to (III.16) and the identity

$$(2S+1)^{|\Lambda_n|} \int d\mu_{\Lambda_n}(\mathbf{s}'_{\Lambda_n}) \prod_{\alpha\in\Lambda_n} \left\{\frac{\mathbf{s}_\alpha\cdot\mathbf{s}'_\alpha+1}{2}\right\}^{2S} \equiv 1 \tag{III.33}$$

Since $\mathscr{S}^{2|\Lambda_n|}$ is compact, a dominated convergence argument again then allows us to conclude that

$$\lim_{S\to\infty} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n}) f(\mathbf{s}_{\Lambda_n})F_S(\mathbf{s}_{\Lambda_n}) = \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n})|f(\mathbf{s}_{\Lambda_n})|^2 \tag{III.34}$$

which is precisely (III.15) for $f\in\mathfrak{A}_n$ (see III.17). The general case with $f\in\mathscr{H}_n$ follows from a density argument and the uniformity of (III.9) in S. One can prove that

$$\lim_{S\to\infty} \|T_S f\|_{\mathscr{H}(S)} = \|f\|_{\mathscr{H}} \tag{III.35}$$

by similar arguments. This completes the proof.

In what follows, we shall denote by $|\Omega(S)\rangle$ the coherent state associated to a spin S. We now prove the following

Proposition III. 3. *For each* $S > \frac{1}{2}$ *we have the one-site relations*

$$\begin{aligned} &\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S)|S^z|\Omega(S)\rangle \\ &= 2S\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2} - S|\langle \Omega(S)|\Omega'(S)\rangle|^2 \\ &= S\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2} \\ &\quad -\frac{i}{2}\langle \Omega(S)|\Omega'(S)\rangle \frac{\partial}{\partial\phi'}\langle \Omega'(S)|\Omega(S)\rangle \end{aligned} \tag{III.36}$$

and the complex conjugate expressions

$$\begin{aligned} &\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^z|\Omega'(S)\rangle \\ &= 2S\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2} - S|\langle \Omega(S)|\Omega'(S)\rangle|^2 \end{aligned}$$

$$= S\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2}$$
$$+\frac{i}{2}\langle \Omega'(S)|\Omega(S)\rangle \frac{\partial}{\partial\phi'}\langle \Omega(S)|\Omega'(S)\rangle \tag{III.37}$$

Relations of the same kind hold for

$$\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^{(x,y)}|\Omega'(S)\rangle.$$

Proof. We prove (III.36). We have

$$|\Omega(S)\rangle = \sum_{M=-S}^{+S} C^S_M(\theta;\phi)|M\rangle \tag{III.38}$$

with

$$C^S_M(\theta;\phi) = \binom{2S}{M+S}^{1/2}\left(\cos\frac{\theta}{2}\right)^{S+M}\left(\sin\frac{\theta}{2}\right)^{S-M}\exp[i(S-M)\phi] \tag{III.39}$$

and $S^z|M\rangle = M|M\rangle$ from [18]. Consequently

$$\langle \Omega'(S)|\Omega(S)\rangle = \sum_{M=0}^{2S} \overline{C^S_{M-S}(\theta';\phi')}C^S_{M-S}(\theta;\phi) \tag{III.40}$$

and

$$\langle \Omega'(S)|S^z|\Omega(S)\rangle = \sum_{M=0}^{2S} M\overline{C^S_{M-S}(\theta';\phi')}C_{M-S}(\theta;\phi) - S\langle \Omega'(S)|\Omega(S)\rangle \tag{III.41}$$

Now we have

$$\sum_{M=0}^{2S} M\overline{C^S_{M-S}(\theta';\phi')}C^S_{M-S}(\theta;\phi)$$
$$= \frac{d}{dx}\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\exp[x] + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp[i(\phi-\phi')]\right)^{2S} \quad (x=0) \tag{III.42}$$

from (III.39) and the binomial expansion; thus

$$\langle \Omega'(S)|S^z|\Omega(S)\rangle$$
$$= 2S\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp[i(\phi-\phi')]\right)^{2(S-(1/2))}\cos\frac{\theta}{2}\cos\frac{\theta'}{2}$$
$$- S\langle \Omega'(S)|\Omega(S)\rangle$$
$$= 2S\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle\cos\frac{\theta}{2}\cos\frac{\theta'}{2} - S\langle \Omega'(S)|\Omega(S)\rangle \tag{III.43}$$

which proves the first relation in (III.36). The fact that (III.43) is equal to

$$S\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle\cos\frac{\theta}{2}\cos\frac{\theta'}{2} - \frac{i}{2}\frac{\partial}{\partial\phi'}\langle \Omega'(S)|\Omega(S)\rangle \tag{III.44}$$

follows from an elementary computation; expression (III.44) is indeed equal to

$$S\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}-\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp[i(\phi-\phi')]\right)$$
$$=S\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle\cdot$$
$$\cdot\left(2\cos\frac{\theta}{2}\cos\frac{\theta'}{2}-\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\exp[i(\phi-\phi')]\right)\right)$$

which is (III.43). This proves the proposition.

Combination of the preceding results now leads to the following

Proposition III. 4. *For each one-site continuous function f and $\gamma = x, y, z$ we have*

$$\lim_{S\to\infty}\frac{2S+1}{S}\int_{\mathscr{S}^2} d\mu(\mathbf{s}')f(\mathbf{s}')\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^\gamma|\Omega'(S)\rangle = s^\gamma f(\mathbf{s}) \tag{III.45}$$

and for each one-site $\mathscr{C}^{(1)}$-function g we have

$$\begin{aligned}\lim_{S\to\infty}(2S+1)\int_{\mathscr{S}^2} d\mu(\mathbf{s}')g(\mathbf{s}')\{&\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S)|S^\gamma|\Omega(S)\rangle\\ &-\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^\gamma|\Omega'(S)\rangle\}\\ =&-i[s^\gamma, g](\mathbf{s})\end{aligned} \tag{III.46}$$

Proof. We prove the proposition for $\gamma = z$, the rest is similar. Write

$$F(\mathbf{s};\mathbf{s}')=\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{i(\phi-\phi')}\right)\cos\frac{\theta}{2}\cos\frac{\theta'}{2} \tag{III.47}$$

Using the first relation in (III.37) we then have

$$\begin{aligned}&\frac{2S+1}{S}\int d\mu(\mathbf{s}')f(\mathbf{s}')\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^z|\Omega'(S)\rangle\\ &=(2S+1)\int d\mu(\mathbf{s}')f(\mathbf{s}')\{2\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\\ &\quad-|\langle \Omega(S)|\Omega'(S)\rangle|^2\}\\ &=(2S+1)\int d\mu(\mathbf{s}')2f(\mathbf{s}')F(\mathbf{s};\mathbf{s}')|\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle|^2\\ &\quad-(2S+1)\int d\mu(\mathbf{s}')f(\mathbf{s}')|\langle \Omega(S)|\Omega'(S)\rangle|^2\end{aligned}$$

Since both f and F are continuous we then have from theorem (III.2)

$$\begin{aligned}&\lim_{S\to\infty}\frac{2S+1}{S}\int d\mu(\mathbf{s}')f(\mathbf{s}')\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^z|\Omega'(S)\rangle\\ &=\lim_{S\to\infty}(2S+2)\int d\mu(\mathbf{s}')2f(\mathbf{s}')F(\mathbf{s};\mathbf{s}')|\langle \Omega(S)|\Omega'(S)\rangle|^2-f(\mathbf{s})\\ &=(2F(\mathbf{s};\mathbf{s})-1)f(\mathbf{s})=s^z f(\mathbf{s})\end{aligned} \tag{III.48}$$

since

$$\begin{aligned}&\int d\mu(\mathbf{s}')2f(\mathbf{s}')F(\mathbf{s};\mathbf{s}')|\langle \Omega(S)|\Omega'(S)\rangle|^2\\ &=\int d\mu(\mathbf{s}')2f(\mathbf{s}')F(\mathbf{s};\mathbf{s}')\left\{\frac{\mathbf{s}\cdot\mathbf{s}'+1}{2}\right\}^{2S}=O(S^{-1})\end{aligned} \tag{III.49}$$

for S large. This proves (III.45). Now from (III.36) and (III.37) we get

$$S\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2}$$

$$-S\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2}$$

$$= -\frac{i}{2}\{\langle \Omega(S)|\Omega'(S)\rangle \frac{\partial}{\partial\phi'}\langle \Omega'(S)|\Omega(S)\rangle + \langle \Omega'(S)|\Omega(S)\rangle \frac{\partial}{\partial\phi'}\langle \Omega(S)|\Omega'(S)\rangle\} \tag{III.50}$$

Consequently we have by proposition III.3 the relations

$$(2S+1)\int d\mu(\mathbf{s}')g(\mathbf{s}')\{\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S)|S^z|\Omega(S)\rangle$$
$$-\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^z|\Omega'(S)\rangle\}$$
$$=(2S+1)\int d\mu(\mathbf{s}')g(\mathbf{s}')\{2S\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S-\tfrac{1}{2})|\Omega(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2}$$
$$\left.-2S\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S-\tfrac{1}{2})|\Omega'(S-\tfrac{1}{2})\rangle \cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right\}$$
$$=-i(2S+1)\int d\mu(\mathbf{s}')g(\mathbf{s}')\{\langle \Omega(S)|\Omega'(S)\rangle\frac{\partial}{\partial\phi'}\langle \Omega'(S)|\Omega(S)\rangle$$
$$+\langle \Omega'(S)|\Omega(S)\rangle\frac{\partial}{\partial\phi'}\langle \Omega(S)|\Omega'(S)\rangle\} \tag{III.51}$$

Now integration by parts in the last expression shows that (III.51) can still be written as

$$2i(2S+1)\int d\mu(\mathbf{s}')\frac{\partial g}{\partial\phi'}(\mathbf{s}')|\langle \Omega(S)|\Omega'(S)\rangle|^2$$
$$+i(2S+1)\int d\mu(\mathbf{s}')g(\mathbf{s}')\{\langle \Omega'(S)|\Omega(S)\rangle\frac{\partial}{\partial\phi'}\langle \Omega(S)|\Omega'(S)\rangle$$
$$+\langle \Omega(S)|\Omega'(S)\rangle\frac{\partial}{\partial\phi'}\langle \Omega'(S)|\Omega(S)\rangle\} \tag{III.52}$$

since the boundary terms vanish. Combination of (III.51) and (III.52) consequently shows that

$$\lim_{S\to\infty}(2S+1)\int d\mu(\mathbf{s}')g(\mathbf{s}')\{\langle \Omega(S)|\Omega'(S)\rangle\langle \Omega'(S)|S^z|\Omega(S)\rangle$$
$$-\langle \Omega'(S)|\Omega(S)\rangle\langle \Omega(S)|S^z|\Omega'(S)\rangle\}$$
$$=\lim_{S\to\infty} i(2S+1)\int d\mu(\mathbf{s}')\frac{\partial g}{\partial\phi'}(\mathbf{s}')\left\{\frac{\mathbf{s}\cdot\mathbf{s}'+1}{2}\right\}^S$$
$$=i\frac{\partial g}{\partial\phi'}(\mathbf{s}) \tag{III.53}$$

according to theorem (III.2) (relation (III.19)). This proves (III.46) by our definition (1.4) of the Poisson Bracket. This achieves the proof.

Now consider $|\Omega_{\mathbb{Z}^\nu}(S)\rangle = \bigotimes_{r\in\mathbb{Z}^\nu} |\Omega_r(S)\rangle$ which is well defined on $\tilde{\mathscr{H}}(S) = \bigotimes_{r\in\mathbb{Z}^\nu} \mathbb{C}_r^{(2S+1)}$ since $\prod_{r\in\mathbb{Z}^\nu} \langle \Omega_r(S)|\Omega_r(S)\rangle = 1$. We have the following

Theorem III.3. *Consider the normalised quantum Heisenberg model*

$$-H_n(S) = (2S)^{-1} \sum_{r,r'\in\Lambda_n} j(r-r')\mathbf{S}_r\cdot\mathbf{S}'_r \tag{III.54}$$

where the couplings obey the various conditions given above. Then the sequence $\{\mathrm{ad}\, H_n(S)\}_{S=1/2}^{\infty}$ *converges to* L_n *in the sense of Trotter pointwise everywhere on* $\mathscr{S}^{2|\Lambda_n|}$; *in other words for each* $f\in\mathscr{D}(L_n)$ *and each spin configuration* $\mathbf{s}_{\mathbb{Z}^\nu} = (\mathbf{s}_{\Lambda_n}; \mathbf{s}_{\mathbb{Z}^\nu/\Lambda_n})$ *we have*

$$\lim_{S\to\infty} |\langle \Omega_{\Lambda_n}(S)|\mathrm{ad}\, H_n(S)T_n(S)f - T_n(S)L_n f|\Omega_{\Lambda_n}(S)\rangle| = 0 \tag{III.55}$$

Similarly, for the infinite system we have

$$\lim_{S\to\infty} |\langle \Omega_{\mathbb{Z}^\nu}(S)|\mathrm{ad}_{\mathbb{Z}^\nu} H(S)T(S)f - T(S)L_{\mathbb{Z}^\nu} f|\Omega_{\mathbb{Z}^\nu}(S)\rangle| = 0 \tag{III.56}$$

for all $f\in\mathscr{D}(L_{\mathbb{Z}^\nu})$.

Proof. We prove (III.55) (relation (III.56) can be obtained by taking the infinite volume limit of (III.55)). Since

$$\langle \Omega_{\Lambda_n}(S)|T_n(S)L_n f|\Omega_{\Lambda_n}(S)\rangle$$
$$= (2S+1)^{|\Lambda_n|} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n})(L_n f)(\mathbf{s}_{\Lambda_n}) \prod_{r\in\Lambda_n} \left\{\frac{\mathbf{s}_r\cdot\mathbf{s}'_r + 1}{2}\right\}^{2S} \tag{III.57}$$

we get

$$\lim_{S\to\infty} \langle \Omega_{\Lambda_n}(S)|T_n(S)L_n f|\Omega_{\Lambda_n}(S)\rangle = (L_n f)(\mathbf{s}_{\Lambda_n}) \tag{III.58}$$

from theorem (III.2) (relations (III.16) and (III.30)). We now prove that

$$\lim_{S\to\infty} \langle \Omega_{\Lambda_n}(S)|\mathrm{ad}\, H_n(S)T_n(S)f|\Omega_{\Lambda_n}(S)\rangle = (L_n f)(\mathbf{s}_{\Lambda_n}) \tag{III.59}$$

for all $f\in\mathscr{D}(L_n)$. We have

$$\langle \Omega_{\Lambda_n}(S)|\mathrm{ad}\, H_n(S)T_n(S)f|\Omega_{\Lambda_n}(S)\rangle$$
$$= (2S+1)^{|\Lambda_n|}\sum_{\gamma} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}'_{\Lambda_n})f(\mathbf{s}'_{\Lambda_n})\{\langle \Omega'_{\Lambda_n}(S)|\Omega_{\Lambda_n}(S)\rangle\langle \Omega_{\Lambda_n}(S)|H_n^{(\gamma)}(S)|\Omega'_{\Lambda_n}(S)\rangle$$
$$-\langle \Omega_{\Lambda_n}(S)|\Omega'_{\Lambda_n}(S)\rangle\langle \Omega'_{\Lambda_n}(S)|H_n^{(\gamma)}|\Omega_{\Lambda_n}(S)\rangle\} \tag{III.60}$$

from the definitions, with

$$H_n(S) = \sum_{\gamma=x,y,z} H_n^{(\gamma)}(S) \tag{III.61}$$

and

$$H_n^{(\gamma)}(S) = -(2S)^{-1} \sum_{r,r' \in \Lambda_n} j(r-r') S_r^{(\gamma)} S_{r'}^{(\gamma)}. \tag{III.61'}$$

Now consider the z-part of (III.60) in which we substitute (III.61'); we get

$$\begin{aligned}
&(2S)^{-1}(2S+1)^{|\Lambda_n|} \sum_{r,r' \in \Lambda_n} j(r-r') \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s'_{\Lambda_n}) f(s'_{\Lambda_n}) \prod_{j \in \Lambda_n/\{r,r'\}} \left\{ \frac{s_j \cdot s'_j + 1}{2} \right\}^{2S} \cdot \\
&\cdot \{ \langle \Omega'_r(S) | \Omega_r(S) \rangle \langle \Omega_r(S) | S_r^z | \Omega'_r(S) \rangle \langle \Omega'_{r'}(S) | \Omega_{r'}(S) \rangle \langle \Omega_{r'}(S) | S_{r'}^z | \Omega'_{r'}(S) \rangle \\
&- \langle \Omega_r(S) | \Omega'_r(S) \rangle \langle \Omega'_r(S) | S_r^z | \Omega_r(S) \rangle \langle \Omega_{r'}(S) | \Omega'_{r'}(S) \rangle \langle \Omega'_{r'}(S) | S_{r'}^z | \Omega_{r'}(S) \rangle \}
\end{aligned} \tag{III.62}$$

upon using the properties of the coherent states stated above. Now choose f of the form

$$f = \bigotimes_{r \in \Lambda_n} f_r \tag{III.63}$$

where each f_r denotes a one-site $\mathscr{C}^{(1)}$-function and apply the identity $A_r B_{r'} - C_r D_{r'} = A_r(B_{r'} - D_{r'}) + D_{r'}(A_r - C_r)$ to the expression in the bracket of (III.62); we then get

$$\begin{aligned}
&\langle \Omega_{\Lambda_n}(S) | \mathrm{ad}\, H_n(S) T_n(S) f | \Omega_{\Lambda_n}(S) \rangle \\
&= (2S+1)^{|\Lambda_n|-2} \sum_{r,r' \in \Lambda_n} j(r-r') \int_{\mathscr{S}^{2(|\Lambda_n|-2)}} d\mu_{\Lambda_n/\{r,r'\}}(s_{\Lambda_n/\{r,r'\}}) \Big(\bigotimes_{\alpha \in \Lambda_n \{r,r'\}} f_\alpha \Big) (s_{\Lambda_n/\{r,r'\}}) \\
&\cdot \prod_{j \in \Lambda_n/\{r,r'\}} \left\{ \frac{s_j \cdot s'_j + 1}{2} \right\}^{2S} \cdot \\
&\cdot \left\{ \frac{2S+1}{S} \int_{\mathscr{S}^2} d\mu_r(s'_r) f_r(s'_r) \langle \Omega'_r(S) | \Omega_r(S) \rangle \langle \Omega_r(S) | S_r^z | \Omega'_r(S) \rangle \right. \\
&\cdot \int_{\mathscr{S}^2} d\mu_{r'}(s'_{r'}) f_{r'}(s'_{r'}) [\langle \Omega'_{r'}(S) | \Omega_{r'}(S) \rangle \langle \Omega_{r'}(S) | S_{r'}^z | \Omega'_{r'}(S) \rangle \\
&- \langle \Omega_{r'}(S) | \Omega'_{r'}(S) \rangle \langle \Omega'_{r'}(S) | S_{r'}^z | \Omega_{r'}(S) \rangle] \\
&+ \frac{2S+1}{S} \int_{\mathscr{S}^2} d\mu_{r'}(s'_{r'}) f_{r'}(s'_{r'}) \langle \Omega'_{r'}(S) | \Omega_{r'}(S) \rangle \langle \Omega_{r'}(S) | S_{r'}^z | \Omega'_{r'}(S) \rangle \\
&\cdot \int_{\mathscr{S}^2} d\mu_r(s'_r) f(s'_r) [\langle \Omega'_r(S) | \Omega_r(S) \rangle \langle \Omega_r(S) | S_r^z | \Omega'_r(S) \rangle \\
&\left. - \langle \Omega_r(S) | \Omega'_r(S) \rangle \langle \Omega'_r(S) | S_r^z | \Omega_r(S) \rangle] \right\}
\end{aligned} \tag{III.64}$$

Applying then proposition (III.4) and theorem (III.2) (combination of (III.16) and (III.34)) to (III.64) we get

$$\lim_{S \to \infty} \langle \Omega_{\Lambda_n}(S) | \mathrm{ad}\, H_n(S) T_n(S) f | \Omega_{\Lambda_n}(S) \rangle = i[h_{\Lambda_n}^{(z)}, f](s_{\Lambda_n}) \tag{III.65}$$

with $h_n^{(z)} = -\frac{1}{2} \sum_{r,r' \in \Lambda_n} j(r-r') s_r^z s_{r'}^z$, which is the z-part of (III.59) according to (I.8) for functions of the form (III.63). The extension of each $f \in \mathscr{D}(L_n)$ follows from similar arguments. The proof is similar for $\gamma = x, y$ according to proposition

(III.4). This proves (III.59) which, combined with (III.58), gives (III.55). Similarly, an infinite volume argument gives (III.56). This completes the proof.

Along the same lines, we now prove the following

Proposition III.5. *Under the same conditions as in the preceding theorem, we have*

$$\lim_{S\to\infty} (\mathrm{ad}\, H_n(S) T_n(S) f, T_n(S) g)_{\mathscr{H}_n(S)} = (L_n f, g)_{\mathscr{H}_n} \tag{III.66}$$

for each $f \in \mathscr{D}(L_n)$ *and each* $g \in \mathscr{H}_n$. *Similarly for the infinite system we get*

$$\lim_{S\to\infty} (\mathrm{ad}_{\mathbb{Z}^\nu} H(S) T(S) f, T(S) g)_{\mathscr{H}(S)} = (L_{\mathbb{Z}^\nu} f, g)_{\mathscr{H}} \tag{III.67}$$

for each $f \in \mathscr{D}(L_{\mathbb{Z}^\nu})$ *and for each* $g \in \mathscr{H}$.

Proof. We prove (III.66); by density, it is sufficient to prove it for $g \in \mathfrak{A}_n$. Since

$$\begin{aligned} &\mathrm{ad}\, H_n(S) T_n(S) f \\ &= (2S+1)^{|\Lambda_n|} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s'_{\Lambda_n}) f(s'_{\Lambda_n}) \{ H_n(S) |\Omega'_{\Lambda_n}(S)\rangle\langle\Omega'_{\Lambda_n}(S)| - |\Omega'_{\Lambda_n}(S)\rangle \\ &\quad \cdot \langle\Omega'_{\Lambda_n}(S)| H_n(S) \} \end{aligned} \tag{III.68}$$

and

$$T_n(S) g = (2S+1)^{|\Lambda_n|} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s_{\Lambda_n}) g(s_{\Lambda_n}) |\Omega_{\Lambda_n}(S)\rangle\langle\Omega_{\Lambda_n}(S)| \tag{III.69}$$

by definition, we get

$$\begin{aligned} &(\mathrm{ad}\, H_n(S) T_n(S) f, T_n(S) g)_{\mathscr{H}_n(S)} = (2S+1)^{-|\Lambda_n|} \mathrm{Tr}_n (\mathrm{ad}\, H_n(S) T_n(S) f)(T_n(S) g)^* \\ &= \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s_{\Lambda_n}) G_S(s_{\Lambda_n}) \overline{g(s_{\Lambda_n})} \end{aligned} \tag{III.70}$$

with

$$\begin{aligned} &G_S(s_{\Lambda_n}) \\ &= (2S+1)^{|\Lambda_n|} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s'_{\Lambda_n}) f(s'_{\Lambda_n}) \{ \langle\Omega'_{\Lambda_n}(S)|\Omega_{\Lambda_n}(S)\rangle\langle\Omega_{\Lambda_n}(S)| H_n(S) |\Omega'_{\Lambda_n}(S)\rangle \\ &\quad - \langle\Omega_{\Lambda_n}(S)|\Omega'_{\Lambda_n}(S)\rangle\langle\Omega'_{\Lambda_n}(S)| H_n(S) |\Omega_{\Lambda_n}(S)\rangle \} \end{aligned} \tag{III.71}$$

upon using Fubini's theorem. Consequently we have

$$G_S(s_{\Lambda_n}) = \langle\Omega_{\Lambda_n}(S)| \mathrm{ad}\, H_n(S) T_n(S) f |\Omega_{\Lambda_n}(S)\rangle \tag{III.72}$$

from (III.68) and (III.71), so that

$$\lim_{S\to\infty} G_S(s_{\Lambda_n}) = (L_n f)(s_{\Lambda_n}) \tag{III.73}$$

pointwise everywhere on $\mathscr{S}^{2|\Lambda_n|}$ by theorem (III.3) (relation III.59). We then have $\| G_S \|_\infty = O(1)$ for large S so that there exists a constant K independent of S satisfying the estimate

$$| G_S(s_{\Lambda_n}) \overline{g(s_{\Lambda_n})} | \leqq K \| g \|_\infty \tag{III.74}$$

for each $g \in \mathfrak{A}_n$. Moreover we get

$$\lim_{S \to \infty} G_S(s_{\Lambda_n})\overline{g(s_{\Lambda_n})} = (L_n f)(s_{\Lambda_n})\overline{g(s_{\Lambda_n})} \tag{III.75}$$

from (III.73) so that the relation

$$\lim_{S \to \infty} \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s_{\Lambda_n}) G_S(s_{\Lambda_n})\overline{g(s_{\Lambda_n})} = \int_{\mathscr{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(s_{\Lambda_n})(L_n g)(s_{\Lambda_n})\overline{g(s_{\Lambda_n})} \tag{III.76}$$

follows from a dominated convergence argument. Relation (III.76) is precisely (III.66) according to (III.70). An infinite volume argument gives (III.67). This completes the proof.

Similarly, one can prove the following result, whose proof will be omitted.

Proposition III. 6. *Under the same conditions as in the preceding statements we have*

$$\lim_{S \to \infty} \| \mathrm{ad}\, H_n(S) T_n(S) f \|_{\mathscr{H}_n(S)} = \| L_n f \|_{\mathscr{H}_n} \tag{III.77}$$

for each $f \in \mathscr{D}(L_n)$ *and similarly*

$$\lim_{S \to \infty} \| \mathrm{ad}_{\mathbb{Z}^\nu} H(S) T(S) f \|_{\mathscr{H}(S)} = \| L_{\mathbb{Z}^\nu} f \|_{\mathscr{H}} \tag{III.78}$$

for each $f \in \mathscr{D}(L_{\mathbb{Z}^\nu})$.

Now we have the following

Theorem III.4. *Under the same conditions as in the preceding statements, we have*

$$\lim_{S \to \infty} \| \mathrm{ad}\, H_n(S) T_n(S) f - T_n(S) L_n f \|_{\mathscr{H}_n(S)} = 0 \tag{III.79}$$

for all $f \in \mathscr{D}(L_n)$, *and similarly for the infinite system*

$$\lim_{S \to \infty} \| \mathrm{ad}_{\mathbb{Z}^\nu} H(S) T(S) f - T(S) L_{\mathbb{Z}^\nu} f \|_{\mathscr{H}(S)} = 0 \tag{III.80}$$

for all $f \in \mathscr{D}(L_{\mathbb{Z}^\nu})$.

Proof. Consider (III.79). We have

$$\begin{aligned} &\| \mathrm{ad}\, H_n(S) T_n(S) f - T_n(S) L_n f \|^2_{\mathscr{H}_n(S)} \\ &\quad = \| \mathrm{ad}\, H_n(S) T_n(S) f \|^2_{\mathscr{H}_n(S)} - (\mathrm{ad}\, H_n(S) T_n(S) f, T_n(S) L_n f)_{\mathscr{H}_n(S)} \\ &\qquad - (T_n(S) L_n f, \mathrm{ad}\, H_n(S) T_n(S) f)_{\mathscr{H}_n(S)} + \| T_n(S) L_n f \|^2_{\mathscr{H}_n(S)} \end{aligned} \tag{III.81}$$

Now when $S \to \infty$, the first term converges to $\| L_n f \|^2_{\mathscr{H}_n}$ according to proposition (III.6), the second and the third one to $- \| L_n f \|^2_{\mathscr{H}_n}$ according to proposition (III.5) (relation III.66) in which we choose $g = L_n f$, and the fourth one to $\| L_n f \|^2_{\mathscr{H}_n}$ again by theorem (III.2) (relation (III.15). Relation (III.80) can be proved similarly. This completes the proof.

We now extend the infinitesimal results of this section to global ones.

IV. Intertwining Relations Between the Quantum and the Classical Unitary Group

We still write $U_n(t;S)$ and $U(t;S)$ for the unitary groups generated by ad $H_n(S)$ and $\mathrm{ad}_{\mathbb{Z}^\nu} H(S)$ respectively, where $H_n(S)$ now refers to the normalised model (III.54). We first establish the following.

Proposition IV.1. *For each $f \in \mathscr{H}_n$ define $g^\lambda = (\lambda - L_n^*)^{-1} f$ with $\lambda > 0$. Then we have*

$$\int_0^\infty dte^{-\lambda t}\{U_n(t;S)T_n(S)f - T_n(S)U_n(t)f\}$$

$$= \int_0^\infty dte^{-\lambda t}U_n(t;S)\{\mathrm{ad}\, H_n(S)T_n(S)g^\lambda - T_n(S)L_n^* g^\lambda\} \tag{IV.1}$$

for each $t > 0$ and each S. A similar relation holds for the infinite system, namely

$$\int_0^\infty dt\, e^{-\lambda t}\{U(t;S)T(S)f - T(S)U(t)f\}$$

$$= \int_0^\infty dte^{-\lambda t}U(t;S)\{\mathrm{ad}_{\mathbb{Z}^\nu} H(S)T(S)g^\lambda - T(S)L^* g^\lambda\} \tag{IV.2}$$

for all $f \in \mathscr{H}$, with $g^{(\lambda)} = (\lambda - L^*)^{-1} f$ in this case.

Proof. We prove (IV.1). We have

$$g^\lambda = (\lambda - L_n^*)^{-1} f = \int_0^\infty dte^{-\lambda t}U_n(t)f \tag{IV.3}$$

which is the Laplace transform formula for the resolvant ([19]). Define the quantized observables

$$F_S = T_n(S)f \tag{IV.4}$$

$$G_S^{(\lambda)} = T_n(S)g^\lambda \tag{IV.5}$$

together with

$$F_S^\lambda = (\lambda - \mathrm{ad}\, H_n(S))G_S^\lambda \tag{IV.6}$$

Then we have the identity

$$\int_0^\infty dte^{-\lambda t}\{U_n(t;S)T_n(S)f - T_n(S)U_n(t)f\}$$

$$= \int_0^\infty dte^{-\lambda t}U_n(t;S)(F_S - F_S^\lambda) + \int_0^\infty dte^{-\lambda t}\{U_n(t;S)F_S^\lambda - T_n(S)U_n(t)f\} \tag{IV.7}$$

Now we have

$$\int_0^\infty dte^{-\lambda t}\{U_n(t;S)F_S^\lambda - T_n(S)U_n(t)f\}$$

$$= (\lambda - \mathrm{ad}\, H_n(S))^{-1}F_S^\lambda - T_n(S)g^\lambda = 0 \tag{IV.8}$$

from (IV.5) and (IV.6), and

$$\begin{aligned} F_S - F_S^\lambda &= T_n(S)f - (\lambda - \text{ad } H_n(S))T_n(S)g^\lambda \\ &= T_n(S)(\lambda - L_n^*)g^\lambda - (\lambda - \text{ad } H_n(S))T_n(S)g^\lambda \\ &= \text{ad } H_n(S)T_n(S)g^\lambda - T_n(S)L_n^* g^\lambda \end{aligned} \tag{IV.9}$$

Substitution of (IV.8) and (IV.9) into (IV.7) gives (IV.1).

The proof of (IV.2) is similar. Proposition (IV.1) is thus complete. Combination of proposition (IV.1) and theorem (III.4) now leads to the following

Proposition IV.2. *We have*

$$\lim_{S\to\infty} \left\| \int_0^\infty dt e^{-\lambda t}\{U_n(t;S)T_n(S)f - T_n(S)U_n(t)f\} \right\|_{\mathscr{H}_n(S)} = 0 \tag{IV.10}$$

for each $f \in \mathscr{H}_n$ *and*

$$\lim_{S\to\infty} \left\| \int_0^\infty dt e^{-\lambda t}\{U(t;S)T(S)f - T(S)U(t)f\} \right\|_{\mathscr{H}(S)} = 0 \tag{IV.11}$$

for the infinite system, with $f \in \mathscr{H}$.

Proof. From proposition (IV.1) we get the estimates

$$\begin{aligned} &\left\| \int_0^\infty dt e^{-\lambda t}\{U_n(t;S)T_n(S)f - T_n(S)U_n(t)f\} \right\|_{\mathscr{H}_n(S)} \\ &\leqq \int_0^\infty dt e^{-\lambda t} \| U_n(t;S)\{\text{ad } H_n(S)T_n(S)g^\lambda - T_n(S)L_n^* g^\lambda\} \|_{\mathscr{H}_n(S)} \\ &\leqq \lambda^{-1} \| \text{ad } H_n(S)T_n(S)g^\lambda - T_n(S)L_n^* g^\lambda \|_{\mathscr{H}_n(S)} \end{aligned} \tag{IV.12}$$

which implies (IV.10) by theorem (III.4) (relation (III.84)). This completes the proof.

Combination of the preceding results now allows one to prove the following

Theorem IV. 1. *Consider the normalised quantum Heisenberg model* (III.64). *Then we have*

$$\lim_{S\to\infty} \sup_{|t|\leqq t_0} \| U_n(t;S)T_n(S)f - T_n(S)U_n(t)f \|_{\mathscr{H}_n(S)} = 0 \tag{IV.13}$$

for each $f \in \mathscr{H}_n$ *and each* $t_0 \in \mathbb{R}^+$. *Similarly for the infinite system we have*

$$\lim_{S\to\infty} \sup_{|t|\leqq t_0} \| U(t;S)T(S)f - T(S)U(t)f \|_{\mathscr{H}(S)} = 0 \tag{IV.14}$$

for each $f \in \mathscr{H}$.

Proof. Consider (IV.13), the proof of (IV.14) is similar. We have

$$\begin{aligned} &\| U_n(t;S)T_n(S)f - T_n(S)U_n(t)f \|_{\mathscr{H}_n(S)} \\ &\leqq \| T_n(S)f \|_{\mathscr{H}_n(S)} + \| T_n(S)U_n(t)f \|_{\mathscr{H}_n(S)} \leqq 2 \| f \|_{\mathscr{H}_n} \end{aligned} \tag{IV.15}$$

upon using (III.12) and the unitarity property. Thus with

$$h_S(t) = U_n(t;S)T_n(S)f - T_n(S)U_n(t)f \tag{IV.16}$$

we see that $\| h_S(t) \|_{\mathscr{H}_n(S)}$ is uniformly bounded in S and t, so that

$$0 \leqq \liminf_{S\to\infty} \sup_{|t|\leqq t_0} \| h_S(t) \|_{\mathscr{H}_n(S)} \leqq \limsup_{S\to\infty} \sup_{|t|\leqq t_0} \| h_S(t) \|_{\mathscr{H}_n(S)} < +\infty \tag{IV.17}$$

Moreover we have for $t, t' \in \mathbb{R}^+$

$$h_S(t) - h_S(t') = \{U_n(t;S) - U_n(t';S)\}T_n(S)f - T_n(S)\{U_n(t) - U_n(t')\}f$$
$$= \int_{t'}^{t} d\xi U_n(\xi;S) \operatorname{ad} H_n(S)T_n(S)f - T_n(S)\int_{t'}^{t} d\xi U_n(\xi)L_n^* f \tag{IV.18}$$

and consequently the estimate

$$\| h_S(t) - h_S(t') \|_{\mathscr{H}_n(S)} \leqq |t - t'| \{ \| \operatorname{ad} H_n(S)T_n(S)f \|_{\mathscr{H}_n(S)} + \| L_n^* f \|_{\mathscr{H}_n} \} \tag{IV.19}$$

where we have used (III.12) and the unitarity property once again. Now by theorem (III.4) we have $\| \operatorname{ad} H_n(S)T_n(S)f \|_{\mathscr{H}_n(S)} = O(1)$ for large S, so that there exists a constant K independent of S with

$$\| h_S(t) - h_S(t') \|_{\mathscr{H}_n(S)} \leqq K |t - t'| \tag{IV.20}$$

Thus for each $\varepsilon > 0$ we have $\| h_S(t) - h_S(t') \|_{\mathscr{H}_n(S)} \leqq \varepsilon$ whenever $|t - t'| \leqq \delta = \frac{\varepsilon}{k}$. The sequence $\{h_S\}_{S=1}^{\infty}$ of $\mathscr{H}_n(S)$-valued functions is thereby equicontinuous on $\mathbb{R}$ (the extension of our considerations to negative times is trivial since we are dealing with unitary groups instead of general contraction semi-groups). Now for any linear bounded functional κ_S on $\mathscr{H}_n(S)$ with norm uniformly bounded in S, we have

$$|\kappa_S(h_S(t))| \leqq C \| h_S(t) \|_{\mathscr{H}_n(S)} \tag{IV.21}$$

for some C independent of S, and

$$|\kappa_S(h_S(t)) - \kappa_S(h_S(t'))| \leqq C \| h_S(t) - h_S(t') \|_{\mathscr{H}_n(S)} \tag{IV.22}$$

for each $t, t' \in \mathbb{R}$, so that the sequence of real-valued functions $\{\kappa_S(h_S(t))\}_{S=1/2}^{\infty}$ is in turn, uniformly bounded in S and t and equicontinuous. Moreover we have by linearity

$$\left| \int_0^{\infty} dt e^{-\lambda t} \kappa_S(h_S(t)) \right| \leqq C \left\| \int_0^{\infty} dt e^{-\lambda t} h_S(t) \right\|_{\mathscr{H}_n(S)} \tag{IV.23}$$

so that

$$\lim_{S\to\infty} \int_0^{\infty} dt e^{-\lambda t} \kappa_S(h_S(t)) = 0 \tag{IV.24}$$

by (IV.10) (or (IV.11). Consequently we have

$$\lim_{S\to\infty} \sup_{|t|\leqq t_0} \kappa_S(h_S(t)) = 0 \tag{IV.25}$$

by the uniqueness of the Laplace transform and Ascoli's first theorem ([8]–[20]). Now one can choose a sequence of such linear functionals κ_S such that

$$\lim_{S\to\infty} \sup_{|t|\leq t_0} \kappa_S(\mathscr{h}_S(t)) = \lim_{S\to\infty} \sup \sup_{|t|\leq t_0} \| \mathscr{h}_S(t) \|_{\mathscr{H}_n(S)} \tag{IV.26}$$

Combination of (IV.26), (IV.25) and (IV.17) then leads to

$$\lim_{S\to\infty} \sup_{|t|\leq t_0} \| \mathscr{h}_S(t) \|_{\mathscr{H}_n(S)} = 0 \tag{IV.27}$$

which is precisely (IV.13) according to the definition (IV.16). This completes the proof.

The following statements can still be obtained by polarization (theorem IV.2) and by another Laplace transform argument (Theorem IV.3). The proofs are omitted.

Theorem IV. 2. *Under the same conditions as in the preceding theorem, we have*

$$\lim_{S\to\infty} (U_n(t;S)T_n(S)f, T_n(S)g)_{\mathscr{H}_n(S)} = (U_n(t)f, g)_{\mathscr{H}_n} \tag{IV.28}$$

for each $f, g \in \mathscr{H}_n$, and similarly

$$\lim_{S\to\infty} (U(t;S)T(S)f, T(S)g)_{\mathscr{H}(S)} = (U(t)f, g)_{\mathscr{H}} \tag{IV.29}$$

for each $f, g \in \mathscr{H}$.

Theorem IV.3. *Under the same conditions as in the preceding theorems, one has*

$$\lim_{S\to\infty} \sup_{|t|\leq t_0} |\langle \Omega_{\Lambda_n}(S)| U_n(t;S)T_n(S)f - T_n(S)U_n(t)f | \Omega_{\Lambda_n}(S)\rangle| = 0 \tag{IV.30}$$

for all $t_0 \in \mathbb{R}^+$ and all $f \in \mathscr{H}_n$. Similarly for the infinite system, one has

$$\lim_{S\to\infty} \sup_{|t|\leq t_0} |\langle \Omega_{\mathbb{Z}^\nu}(S)| U(t;S)T(S)f - T(S)U(t)f | \Omega_{\mathbb{Z}^\nu}(S)\rangle| = 0 \tag{IV.31}$$

for all $f \in \mathscr{H}$.

Remarks and Open Problems. The preceding theorems exhibit the connection between the quantum unitary group $U(t;S)$ and the classical one $U(t)$; the latter one is the Trotter limit of the former one when $S \to \infty$. Although we chose a unit system in which $\hbar = 1$ throughout, one can readily convince oneself that the correct limiting procedure to deal with otherwise is $S \to \infty$, $\hbar \to 0$ with $S\hbar$ fixed. Observe now that the quantization operator $T_n(S)$ in (III.2) and its infinite volume version (III.12) are, group-theoretically speaking, intertwining operators between the quantum and the classical unitary group (see for instance (IV.13) and (IV.14)). More precisely, our theorems show that the two unitary representations $U(t;S)$ and $U(t)$ of the additive real line become equivalent in the limit $S \to \infty$.

An interesting open problem is the extension of our results, for instance in the spirit of [21], to D-dimensional classical and quantum spin systems with $D \geq 4$. In that case, the flow equations (I.8′) (or (I.9)) should be replaced by

$$\dot{\mathbf{s}}_\alpha = [\mathbf{s}_\alpha, \mathbf{H}_\alpha] \tag{IV.32}$$

with $D \times D$ antisymmetric matrices $\mathbf{s}$ and $\mathbf{H}$, where $[\,,]$ denotes the usual matrix commutator, in other words the Lie bracket corresponding to SO(D). For $D = 3$ indeed, (IV.32) would reduce to (I.8′) with the effective magnetic field $\mathbf{H}_\alpha =$

$\sum_r j(\alpha - r)\mathbf{s}_r$ since $\mathbb{R}^3$, equipped with the vector product, becomes a Lie algebra isomorphic to that of SO(3) (the Lie algebra of all 3×3-antisymmetric matrices).

Observe furthermore that our theorem (IV.2) can be used to compare classical and quantum time–time correlation functions in statistical mechanics; if we choose for instance

$$g = Z_n^{-1} \exp[-\mathrm{h}_n] f,\ Z_n = \int_{\mathcal{S}^{2|\Lambda_n|}} d\mu_{\Lambda_n}(\mathbf{s}_{\Lambda_n}) \exp[-\mathrm{h}_n(\mathbf{s}_{\Lambda_n})] \tag{IV.33}$$

in (IV.28), we get

$$(\mathrm{U}_n(t)f, g)_{\mathscr{H}_n} = \langle f(t)\, f(0) \rangle_{\Lambda_n} \tag{IV.34}$$

where $\langle\ \rangle_{\Lambda_n}$ stands for thermal average.

Let us finally mention that we have not been able so far to prove sharper statements than those in theorem (IV.1) about the classical limit, for instance intertwining relations such as (IV.13) and (IV.14) where the normalised Hilbert–Schmidt norm would be replaced by the *uniform* operator norm. Along the same lines, it would be interesting to see whether the sequence $\{\mathfrak{A}(S), T(S)\}_{S=1}^{\infty}$ is a Trotter approximation for $\mathfrak{A}$, where both $\mathfrak{A}(S)$ and $\mathfrak{A}$ are considered as C^*-algebras for their respective uniform norm.

Acknowledgements. The author would like to thank Professor E. H. Lieb for discussions, Professor B. Simon for communicating to him his results in [21] prior publication, the U.S. Department of Energy and the Swiss National Science Foundation for financial support under contracts EG-77-C-03-1538 and 820-436-76. He also thanks Mrs. Sharon Wilson for the very expert typing of the manuscript.

References

1. Lanford, O.E., Lebowitz, J. L., Lieb, E. H.: J. Stat. Phys. **16**, 453–461 (1977)
2. Dobrushin, R. L., Fritz, J.: Commun. Math. Phys. **55**, 275–292 (1977)
3. Dobrushin, R. L., Fritz, J.: Commun. Math. Phys. **57**, 67–81 (1977)
4. Vuillermot, P. A.: Lett. Nuovo Cimento **24**, 333–338 (1979)
5. Lang, S.: Analysis II, London, Amsterdam, Paris: Addison Wesley (1968)
6. Ruelle, D.: Statistical mechanics, rigorous results. New York: Benjamin, 1969
7. Arnold, V.: Ordinary differential equations. Cambridge, Massachusetts: MIT Press, 1973
8. Schwartz, L.: Cours d'analyse. Paris: Hermann, 1967
9. Trotter, A. F.: Pac. J. Math. **8**, 887–919 (1958)
10. Kurtz, T. G.: Funct. Analy. **3**, 354–375 (1969)
11. Hepp, K., Lieb, E. H.: Ann. Phys. **76**, 360–404 (1973)
12. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. II New York: Academic Press, 1977
13. Riesz, F., Nagy, B.: Leçons d'analyse fonctionelle. Paris: Gauthiers-Villars 1968
14. Nelson, E.: Math. **70**, 572–615 (1959)
15. Robinson, D. W.: Commun. Math. Phys. **7**, 337–348 (1968)
16. Streater, R. F.: Commun. Math. Phys. **6**, 233–247 (1967)
17. Sakai, S.: C*-algebras and W*-algebras, Berlin, Heidelberg, New York: Springer 1971
18. Lieb, E. H.: Commun. Math. Phys. **31**, 327–340 (1973)
19. Yosida, K.: Functional analysis, Berlin, Heidelberg, New York: Springer 1965
20. Schwartz, L.: Theorie des distributions. Paris: Herman, 1973
21. Simon, B.: Commun. Math. Phys. **71**, 247–276 (1980)

Communicated by E. Lieb

Received January 21, 1980