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A Connection Between ν-Dimensional Yang–Mills Theory and $(\nu - 1)$-Dimensional, Non-Linear σ-Models

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Abstract. We study non-linear σ-models and Yang–Mills theory. Yang–Mills theory on the ν-dimensional lattice $\mathbb{Z}^\nu$ can be obtained as an integral of a product over all values of one coordinate of non-linear σ-models on $\mathbb{Z}^{\nu-1}$ in random external gauge fields. This exhibits two possible mechanisms for confinement of static quarks one of which is that clustering of certain two-point functions of those σ-models implies confinement of static quarks in the corresponding Yang–Mills theory. Clustering is proven for all one-dimensional σ-models, for the $U(n) \times U(n)$ σ-models, $n = 1, 2, 3, \ldots$, in two dimensions, and for the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ σ-models for a large range of couplings $g^2 \gtrsim \mathrm{O}(\nu)$. Arguments pertinent to the construction of the continuum limit are discussed. A representation of the expectation of Wilson loops in terms of expectations of random surfaces bounded by the loops is derived when the gauge group is $\mathrm{SU}(2)$, $U(n)$ or $\mathrm{O}(n)$, $n = 1, 2, 3, \ldots$, and connections to the theory of dual strings are sketched.

1. Connections Between σ-Models and Yang–Mills Theory: Description of the Basic Ideas

In this paper we propose to study ν-dimensional (lattice) Yang–Mills theory, in terms of $(\nu - 1)$-dimensional (lattice) σ-models in random external gauge fields. Our main results are the ones described in the abstract. One can also apply our scheme to the study of $\mathbb{Z}(2)$ lattice gauge theories in three and four dimensions and relate them to a two-dimensional Ising model with random couplings in one direction, but this is not studied in this paper. Furthermore, we study a weak coupling limit of Yang–Mills theory relating this theory to *linear* σ-models in an external gauge field, in one dimension less. It appears to provide a *lower bound* on the confining potential—i.e. an *upper bound* on expectations of Wilson loop observables—with a convergent continuum limit. This bound is rigorous in the

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abelian case. In the non-abelian case, it appears to be related to naive perturbation theory and, therefore, it should describe the short distance behaviour of the theory correctly. We show that confinement of static quarks, *always assumed to transform non-trivially under the center of the gauge group*, in ν-dimensional Yang–Mills theory is a consequence of two possible mechanisms:

(1) Clustering of certain two-point functions of the $(\nu-1)$-dimensional σ-model in external gauge field (see Sects. 2–5). This leads to permanent confinement in all two-dimensional and in three-dimensional $U(n)$ Yang–Mills theories and suggests that, for arbitrary, non-abelian gauge Lie groups, the confining potential in $\nu=3$ dimensions is always linear, for arbitrary coupling. For the critical temperature of two-dimensional, non-linear, non-abelian σ-models is expected to be zero, with exponential clustering at positive temperature.

(2) A cancellation between "random phases", depending on the external gauge fields of the long range order in those two-point functions of the $(\nu-1)$-dimensional σ-models. We have arguments suggesting that only this second mechanism can lead to confinement in four-dimensional, continuum gauge theories. See Sects. 6 and 7. We propose to study aspects of ν-dimensional continuum gauge theories by means of the Gaussian weak coupling limit of the $(\nu-1)$-dimensional σ-models mentioned before. That limit suggests, e.g. the correct kind of *normal ordering* of the Wilson loop observables (traces of holonomy operators associated with closed loops) that might enable one to construct the continuum limit of expectations of products of "normal ordered" Wilson loops. This is discussed in Sect. 7, especially for $\nu=3$.

Throughout this paper we systematically adopt the Euclidean description of quantum field theory. Thereby, Yang–Mills theory and non-linear σ-models are converted into classical statistical mechanics systems. The reconstruction of a quantum field theory from the latter is accomplished by means of a Feynman-Kac formula, resp. Osterwalder–Schrader reconstruction [1]. (In the case of lattice theories, Osterwalder–Schrader reconstruction requires the existence of a positive semi-definite transfer matrix which follows from reflection positivity. This and other foundational topics are discussed at length, e.g. in [2, 3, 4]). In the following, "dimension" means the dimension of the Euclidean space-time (lattice). We only consider compact gauge groups, denoted G.

Various analogies and connections between non-linear σ-models and Yang–Mills theory, have been emphasized in the literature. Apart from the well-known ones between two-dimensional σ-models, in particular the $\mathbb{C}P^{N-1}$ models of refs. [5, 6], and Yang–Mills theory in four dimensions (e.g. conformal invariance at the classical level, field theories with constraints and non-trivial topological properties, instantons, asymptotic freedom, etc.) we mention a rather deep analogy that emerges, at the classical level, from formulating these theories in terms of fields with values in a Grassmannian. The corresponding σ-models are the $G_{N,n}(\mathbb{C})$-models of [7, 8], the Yang–Mills theories are the pure $U(n)$-theories. This analogy is stressed in [7, 8]. It is inspired by the work on self-dual Yang–Mills fields in [9–11]. It is potentially useful for further analysis of classical Yang–Mills theory, e.g. the construction of conserved currents, but does not appear to be promising at the quantum level [7]. Therefore we do not use it in this paper.

Relevant for our purposes are the following very simple connections (*not* analogies) between ν-dimensional Yang–Mills theory and $(\nu - 1)$-dimensional, non-linear σ-models:

1.1. Two-Dimensional Yang–Mills Theory and One-Dimensional σ-Models

Two-dimensional, pure Yang–Mills theory with gauge group G is equivalent to a product over all values of one coordinate, e.g. the imaginary time, of *independent*, one-dimensional, non-linear σ-models with fields taking values in G. (To see this one is advised to consider a two-dimensional lattice Yang–Mills theory and to choose the axial gauge, $A_1 = 0$). These one-dimensional σ-models simply describe Brownian motion on the group G. Therefore they can be solved explicitly, even in the continuum limit. (Their transfer matrix is generated by a Casimir operator). Thus, the calculation of expectations of products of Wilson loop observables in a two-dimensional, pure Yang–Mills theory is reduced to calculating correlation functions for Brownian motion on G which, in turn, can be reduced to calculating Clebsch–Gordan coefficients. See [2, 12].

In this paper we describe a related, albeit more complex, higher dimensional generalization of the two-dimensional strategy, relating Yang–Mills theory to a non-linear σ-model. It exhibits a promising line of attack that might enable one to "solve" the three- and four-dimensional $\mathbb{Z}(2)$-theories and to construct the continuum limit of the three-dimensional, pure $U(n)$ theories in the $n \to \infty$ limit. Sect. 4. These theories ought to be the simplest ones.

1.2. Classical Yang–Mills Theory and Classical σ-Models

Let U be some irreducible, unitary representation of a compact Lie group G. Consider a $(\nu - 1)$-dimensional, non-linear σ-model with fields, $g(x)$, taking values in $U(G)$. The Euclidean action of the model is given by

$$\beta A^{\sigma}_{\nu-1} = \beta \sum_{j=1}^{\nu-1} \int d^{\nu-1}x \, \mathrm{tr}(|g^*(\underline{x})(\partial_j g)(\underline{x})|^2) \tag{1.1}$$

The action $A^{\sigma}_{\nu-1}$ is clearly invariant under the transformation $g(\underline{x}) \to bg(\underline{x})t$, with b, t in $U(G)$, i.e. the symmetry group is $G \times G$. Coupling the field $g(\underline{x})$ to an external gauge field means converting the global action of the symmetry group, $G \times G$, into a local one; i.e. one must specify a $G \times G$ connection, (A, B), with $A_j \in \mathscr{G}$, $B_j \in \mathscr{G}, j = 1, \ldots, \nu - 1$, $\mathscr{G}$ the representation U of the Lie algebra of G, in order to be able to parallel transport $g(\underline{x})$.

The coupling of the field $g(\underline{x})$ to the external gauge field (A, B) is now accomplished by the standard minimal substitution, i.e. one replaces ∂_j by a covariant derivative, D_j, defined by

$$g^* D_j g = g^* \partial_j g + g^* A_j g - B_j . \tag{1.2}$$

The action is replaced by

$$\beta A^{\sigma}_{\nu-1}(A, B) = \beta \sum_{j=1}^{\nu-1} \int d^{\nu-1}x \, \mathrm{tr}(|g^*(\underline{x})(D_j g)(\underline{x})|^2) \tag{1.3}$$

Next, we want to study a weak coupling (low temperature) limit described by: $\beta \equiv \beta(\varepsilon) = f\varepsilon^{-1}, \beta A^{\sigma}_{\nu-1} = O(\varepsilon), \varepsilon \to 0$. On the classical level, this limit is obtained as follows: One chooses

$$g(\underline{x}) = e^{\varepsilon X(\underline{x})}, \tag{1.4}$$

where $X(\underline{x})$ is a C_0^∞ function on $\mathbb{R}^{\nu-1}$ with values in $\mathscr{G}$. Then, to first order in ε,

$$g^* D_j g = A_j - B_j + \varepsilon\{\partial_j X + [A_j, X]\} + O(\varepsilon^2)$$

In order for the action $f\varepsilon^{-1}A^{\sigma}_{\nu-1}$ to be $O(\varepsilon)$ we must require that

$$B_j(\underline{x}) = A_j(\underline{x}) + \varepsilon C_j(\underline{x}) + O(\varepsilon^2), \tag{1.5}$$

where A_j and C_j are C_0^∞ functions with values in $\mathscr{G}$. We then have

$$g^* D_j g = \varepsilon\{\partial_j X + [A_j, X] - C_j\} + O(\varepsilon^2), \tag{1.6}$$

so that $f\varepsilon^{-1}A^{\sigma}_{\nu-1}(A, B) = O(\varepsilon)$,

Next, choose

$$\begin{aligned} A_j(\underline{x}) = A_j(\underline{x}, t),\ C_j(\underline{x}) &= \varepsilon^{-1}(A_j(\underline{x}, t+\varepsilon) - A_j(\underline{x}, t)) \\ &\equiv \partial_\nu^{\text{fin.}} A_j(\underline{x}, t) \\ &= \partial_\nu A_j(x, t) + O(\varepsilon), \end{aligned} \tag{1.7}$$

where t is a parameter in $\mathbb{Z}_\varepsilon = \{\varepsilon n : n \in \mathbb{Z}\}$, and $A_j(\underline{x}, t)$ is a C_0^∞ function on $\mathbb{R}^\nu$ with values in $\mathscr{G}$. We also change our notation: $X(\underline{x}) \equiv A_\nu(\underline{x}, t)$. Then

$$\begin{aligned} (g^* D_j g)(\underline{x}, t) &= \varepsilon\{\partial_j A_\nu + \partial_\nu^{\text{fin.}} A_j + [A_j, A_\nu]\}(\underline{x}, t) + O(\varepsilon^2) \\ &\equiv \varepsilon F_{j\nu}^{\text{fin.}}(\underline{x}, t) + O(\varepsilon^2) \end{aligned} \tag{1.8}$$

The action is then given by

$$\beta(\varepsilon) A^{\sigma}_{\nu-1}(A(t), B(t)) = f\varepsilon \sum_{j=1}^{\nu-1} \int d^{\nu-1}x \operatorname{tr}(|F_{j\nu}^{\text{fin.}}(\underline{x}, t)|^2) + O(\varepsilon^2) \tag{1.9}$$

Next, we assign to the external gauge field $A(t) = (A_j(t))$ on $\mathbb{R}^{\nu-1}$ an action equal to the $(\nu-1)$-dimensional Yang–Mills action,

$$A_{\nu-1}(A(t)) \equiv \varepsilon f A^{YM}_{\nu-1}(A(t)) = \varepsilon f \sum_{1 \leq i < j \leq \nu-1} \int d^{\nu-1}x \operatorname{tr}(|F_{ij}(\underline{x}, t)|^2).$$

The total action for fixed t is then given by

$$A_{\text{tot.}}(\varepsilon, t) = \beta(\varepsilon) A^{\sigma}_{\nu-1}(A(t), A(t+\varepsilon)) + \varepsilon f A^{YM}_{\nu-1}(A(t)), \tag{1.10}$$

and the total action by

$$A_{\text{tot.}}(\varepsilon) = \sum_{t \in \mathbb{Z}_\varepsilon} A_{\text{tot.}}(\varepsilon, t) \tag{1.11}$$

If $A_\nu(\underline{x}, t)$ and $A_j(\underline{x}, t)$, $j = 1, \ldots, \nu-1$, are the restrictions of a C_0^∞ connection $\mathscr{A} = (A, A_\nu)$ over $\mathbb{R}^\nu$ to $\mathbb{R}^{\nu-1} \times \mathbb{Z}_\varepsilon$ we have

$$\begin{aligned} A_\nu^{YM} &= \lim_{\varepsilon \to 0} A_{\text{tot.}}(\varepsilon) \\ &= \lim_{\varepsilon \to 0} \sum_{t \in \mathbb{Z}_\varepsilon} \{\beta(\varepsilon) A^{\sigma}_{\nu-1}(A(t), A(t+\varepsilon)) + \varepsilon f A^{YM}_{\nu-1}(A(t))\} \end{aligned} \tag{1.12}$$

which by (1.9)–(1.11) is the standard, ν-dimensional Yang–Mills action of $\mathscr{A}$.

Let $x = (x_1, \ldots, x_{\nu-1}, x_\nu)$, $x_1 = t \in \mathbb{Z}_\varepsilon$, and let x_ν be the time coordinate. Moreover, return to Minkowski space, i.e. a hyperbolic metric. Then the Euler–Lagrange (field) equations corresponding to the action $A_{\text{tot.}}(\varepsilon)$, $\varepsilon > 0$, are a system of infinitely many coupled p.d.e.'s, labelled by $t \in \mathbb{Z}_\varepsilon$. They describe infinitely many, $(\nu-1)$-dimensional non-linear σ-models coupled through $(\nu-1)$-dimensional, external Yang–Mills fields.

This observation may be useful to construct weak solutions to the Cauchy problem for ν-dimensional, classical Yang–Mills, $\nu = 3, 4$, using a compactness argument to construct an $\varepsilon = 0$ limit, given the solutions for arbitrary $\varepsilon > 0$[1].

Quantum mechanically, equations (1.10) and (1.11) appear to tell us that ν-dimensional Yang–Mills theory is, for $\varepsilon > 0$, a product of infinitely many, non-linear σ-models coupled through external gauge fields which are distributed according to $(\nu-1)$-dimensional Yang–Mills measures. This is substantiated in the remainder of this section; see 1.3 below. Equation (1.9) suggests that, in the limit $\varepsilon \to 0$, the σ-models approach *linear* theories (i.e. $\lim_{\varepsilon \to 0} \varepsilon^{-1}\beta(\varepsilon) A^\sigma_{\nu-1}(A(t), A(t+\varepsilon))$ is quadratic in X), corresponding to Gaussian functional integrals. This is obviously true classically and is the basic, implicit assumption in the standard, perturbative treatment of the theory. In Sect. 7 we prove that it is true quantum mechanically as long as the lattice spacing in the spatial directions ($\perp$ t-direction) is positive and independent of ε.

In low ($\nu \leqq 3$) dimensions and for non-abelian gauge group, G. the limiting theory, as $\varepsilon \to 0$, is approached by a family of products of σ-models which are expected to have positive mass gaps, [13]. This would imply permanent confinement of static quarks by a linear potential in three-dimensional, non-abelian continuum Yang–Mills theory. See Theorem 1.2, Sect. 1.3.

Some aspects of the continuum limit are discussed in Sect. 7 (normal-ordering of Wilson loops, implicit renormalization).

1.3. ν-Dimensional Yang–Mills Theory as a Product of (ν − 1)-Dimensional σ-Models with Random Couplings

In this section we develop the theme of Sect. 1.1 and 1.2 in the context of lattice gauge theories and lattice σ-models. The gauge group, G, is chosen to be a compact group, not necessarily a Lie group. Let χ be some irreducible character of G, and U—or U^χ—the corresponding unitary representation of G. We study models on a simple, cubic lattice $\mathbb{Z}^\nu$, resp. $\mathbb{Z}^{\nu-1}$. In this section, the lattice spacing is unity, but this is unimportant. The "Euclidean" action of pure Yang–Mills theory on $\mathbb{Z}^\nu$ is given by

$$A_\nu^{YM} = -\sum_p \operatorname{Re} \chi(g_{\partial p}), \tag{1.13}$$

where p denotes a plaquette (unit square) of $\mathbb{Z}^\nu$, ∂p is the loop formed by the four

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1 A less speculative application of our scheme says that time-independent instanton solutions of four-dim. Yang–Mills theory are three-dim. Prasad-Summerfield monopoles (We learnt this from M.F. Atiyah).

sides of $p, g_C = \prod_{xy \subset C}^{\circlearrowleft} g_{xy}$ is the ordered product of elements $g_{xy} \in G$, (xy a link in $\mathbb{Z}^\nu$) along a closed loop $C \subset \mathbb{Z}^\nu$.

In order to give (1.13) a rigorous meaning one must restrict the sum, $\sum_p$, to extend only over those plaquettes that belong to some bounded, connected subset Λ of $\mathbb{Z}^\nu$. In an unambiguous context, reference to the region Λ is suppressed in our notation. The a priori distribution of the random group elements, g_{xy}, the *gauge fields* assigned to the links xy, is the Haar measure, dg_{xy}, on G. Given a subset $X \subset \mathbb{Z}^\nu$, we define $g(X) = \{g_{xy} : xy \subset X\}$.

The finite volume (Euclidean vacuum) expectation of the lattice gauge theory described here is given by the measure

$$d\mu_B(g(\Lambda)) \equiv Z_\Lambda^{-1} e^{-\beta A_\nu^{YM}(\Lambda)} D_B g(\Lambda), \tag{1.14}$$

where

$$D_B g(\Lambda) \equiv Dg(\Lambda) = B(g(\partial\Lambda)) \prod_{xy \subset \Lambda} dg_{xy}, \tag{1.15}$$

and $A_\nu^{YM}(\Lambda)$ is given by (1.13), with $\sum_p$ replaced by $\sum_{p \subset \Lambda}$. Moreover, $B(g(\partial\Lambda))$ is an arbitrary, bounded function of $g(\partial\Lambda)$, i.e. of all those gauge fields g_{xy} with $xy \subset \partial\Lambda$. The significance of B is to specify boundary conditions. Especially in $\nu = 2$ dimensions, the physics of the theory may depend crucially on the choice of B; see e.g. [14, 15] and Sect. 2. We warn the reader that, in *contrast* to what one does in classical statistical mechanics, it is sometimes necessary to choose boundary conditions, B, which are *non-positive*; (construction of "θ-vacua"). Then $d\mu_B$ is a "signed" measure. The factor Z_Λ is so chosen that the integral of $d\mu_B(g(\Lambda))$ is unity. In accordance with the announced notation we will write

$$d\mu(g) = Z^{-1} e^{-\beta A_\nu^{YM}} Dg \tag{1.16}$$

if reference to Λ and $B(g(\partial\Lambda))$ is superfluous. The limit in which Λ tends to $\mathbb{Z}^\nu$. in (1.14), is the thermodynamic limit. A thermodynamic limit of $d\mu_B(g(\Lambda))$, (in the sense of w^*-convergence of subsequences), can always be constructed by a standard compactness argument, at least when $B \geqq 0$.

We now proceed to a heuristic description of the main ideas of our approach. The coordinates of a lattice site x are denoted $(x^1, \ldots, x^{\nu-1}, x^\nu) = (i, x^\nu)$, with $i = i(x^1, \ldots, x^{\nu-1}) \in \mathbb{Z}^{\nu-1}$. Let $\Lambda_t = \Lambda \cap \{x : x^\nu = t\}$ and Λ_t^0 be the projection of Λ_t onto $\{x : x^\nu = 0\} \cong \mathbb{Z}^{\nu-1}$. Let $g^h(t)$ denote the collection of all gauge fields in Λ assigned to links xy in Λ_t, i.e. $x^\nu = y^\nu = t$.

These gauge fields are called *horizontal* gauge fields localized at $x^\nu = t$. Let $g_i \equiv g_i^v(t) = g_{(i,t)(i,t+1)}$, with $(i,t)(i,t+1) \subset \Lambda$. The gauge fields $g_i(t)$ are called *vertical* gauge fields localized in the slice $[t, t+1]$. The Yang–Mills action can now be rewritten as

$$A_\nu^{YM}(\Lambda) = -\sum_{t \in \mathbb{Z}} \Big\{ \sum_{p \subset \Lambda_t} \operatorname{Re} \chi(g^h(t)_{\partial p}) + \sum_{ij \subset \Lambda_t^0 \cap \Lambda_{t+1}^0} \operatorname{Re} \chi(g_i(t) g^h_{(i,t+1)(j,t+1)} g_j(t)^{-1} \cdot g^h_{(j,t)(i,t)} \Big\}. \tag{1.17}$$

The first term on the r.s. of (1.17) can be recognized to be a sum of Yang–Mills actions, $A^{YM}_{\nu-1}(g^h(t))$, depending only on horizontal gauge fields in the $(\nu-1)$ dimensional hyperplane at $x^\nu = t$. Next, we interpret the second term on the r.s. of (1.17). We note that the vertical gauge fields in different slices are, a priori, independent from each other. Therefore, reference to t is superfluous, and we abreviate $g_i(t)$ by g_i. Moreover, we set

$$\begin{aligned} t_{ij} &\equiv t_{ij}(t) = (g^h_{(i,t+1)(j,t+1)})^{-1}, \\ b_{ij} &\equiv b_{ij}(t) = g^h_{(j,t)(i,t)}. \end{aligned} \tag{1.18}$$

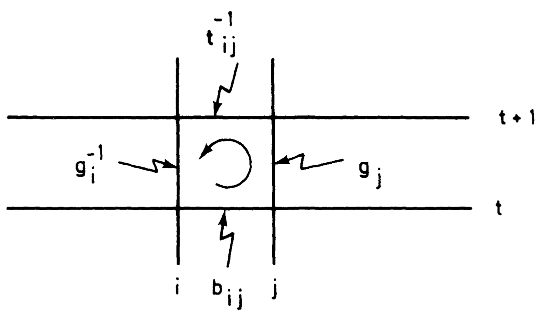

Fig. 1

The second term on the r.s. of (1.17) can now be rewritten as

$$\sum_{t\in\mathbb{Z}} A^\sigma_{\nu-1}(g^h(t), g^h(t+1)),$$

with

$$A^\sigma_{\nu-1}(b,t) = -\sum_{ij\subset\Lambda^0_t\cap\Lambda^0_{t+1}} \operatorname{Re}\chi(g_i^{-1}b_{ij}g_jt_{ij}^{-1}). \tag{1.19}$$

This expression is to be compared with the action of a $(\nu-1)$-dimensional lattice σ-model with fields taking values in G:

$$A^\sigma_{\nu-1} = -\sum_{ij}\operatorname{Re}\chi(g_i^{-1}g_j). \tag{1.20}$$

The global symmetry group of the action $A^\sigma_{\nu-1}$ is the group $G\times G$, acting on the field g as follows:

$$G\times G \ni (b,t): g_i \to b g_i t^{-1}.$$

(Clearly $\chi((bg_it^{-1})^{-1}bg_jt^{-1}) = \chi(g_i^{-1}g_j)$, by the cyclic invariance of χ).

The parallel transport used in definition (1.20) of $A^\sigma_{\nu-1}$ is flat. A non-flat parallel transport is obtained by letting the symmetry group $G\times G$ act locally, i.e. by converting it into a gauge group. Given a curve $\gamma(i,j)\subset\mathbb{Z}^{\nu-1}$ of neighboring links joining a site i to a site j, the parallel transport of $g_i\in G$, localized at i, to the site j along $\gamma(i,j)$ is defined by

$$g_i \to b_{\gamma(i,j)} g_i t^{-1}_{\gamma(i,j)},$$

with

$$b_{\gamma(i,j)}(t_{\gamma(i,j)}) = \prod_{uv \subset \gamma(i,j)}^{\circlearrowleft} b_{uv}(t_{uv}) \tag{1.21}$$

Thus (1.19) is the action of the non-linear σ-model in an external gauge field obtained from (1.20) by minimal substitution; ($\gamma(i,j) = ij$ in (1.21)). The partition function of those σ-models is given by

$$Z^\sigma(b,t) \equiv Z^\sigma_\beta(b,t) = \int e^{-\beta A^\sigma_{\nu-1}(b,t)} \prod_{i \in \Lambda^0_t \cap \Lambda^0_{t+1}} dg_i,$$

$$Z^\sigma \equiv Z^\sigma(\mathbb{1},\mathbb{1}) = \int e^{-\beta A^\sigma_{\nu-1}} \prod_{i \in \Lambda^0_t \cap \Lambda^0_{t+1}} dg_i$$

We set

$$\zeta^\sigma(b,t) = Z^\sigma(b,t)/Z^\sigma. \tag{1.22}$$

We recall the following result of [4, 14].

Theorem 1.1. *For a class of boundary conditions, B, (including periodic and free) specified in* [4]

1) $0 < \zeta^\sigma(b,t) \leqq \zeta^\sigma(\mathbb{1},\mathbb{1}) = 1$;
   (*diamagnetic inequality*)
2) $\zeta^\sigma(b,t)$ *is gauge-invariant, i.e.*
   $\zeta^\sigma(b,t) = \zeta^\sigma(b^h, t^m)$, *with*
   $b^h_{ij} = h_i b_{ij} h_j^{-1}$, $t^m_{ij} = m_i t_{ij} m_j^{-1}$, *where h and m are functions of compact support on* $\mathbb{Z}^{\nu-1}$ *with values in G.* □

We denote by $\langle - \rangle^\sigma_{\nu-1}(b,t)$ the normalized expectation

$$Z^\sigma(b,t)^{-1} \int - e^{-\beta A^\sigma_{\nu-1}(b,t)} \prod_i dg_i \tag{1.23}$$

of the σ-model in the external gauge field (b,t). We let $\langle - \rangle^{YM}_\nu$ denote the ν-dimensional, pure Yang–Mills expectation defined by the measure $d\mu$ introduced in (1.14)–(1.16). Furthermore, we let $d\mu_{\nu-1}(g^h(t))$ be given by (1.14)–(1.16), but with A^{YM}_ν replaced by $A^{YM}_{\nu-1}(\Lambda^0_t) \equiv A^{YM}_{\nu-1}(g^h(t))$. For simplicity, we choose a boundary condition, $B(g(\partial\Lambda))$, which factorizes into functions only depending on $g^h(\partial\Lambda_t)$, resp. $g^\nu_i(t)$, $i \in \partial(\Lambda^0_t \cap \Lambda^0_{t+1})$, $t \in \mathbb{Z}$. It can then be absorbed in the definition of $d\mu_{\nu-1}(g^h(t))$ and of $\langle - \rangle^\sigma_{\nu-1}(g^h(t), g^h(t+1))$ and is suppressed in our notation.

It now follows from (1.17)–(1.20) that

$$\langle - \rangle^{YM}_\nu = \zeta^{-1} \prod_t \{\langle - \rangle^\sigma_{\nu-1}(g^h(t), g^h(t+1)) \cdot \zeta^\sigma(g^h(t), g^h(t+1)) d\mu_{\nu-1}(g^h(t))\}, \tag{1.24}$$

with $\zeta = \int \prod_t \zeta^\sigma(g^h(t), g^h(t+1)) d\mu_{\nu-1}(g^h(t))$.

Under the conditions of Theorem 1.1 (1),

$$0 < \zeta \leqq 1. \tag{1.25}$$

Equation (1.24) is the basic identity exploited in this paper. We apply it to discuss *confinement of static quarks*. For this purpose we define the *Wilson loop observables* which we regard as the basic observables of a Yang–Mills theory: Let U^q be an irreducible representation of G, and χ^q its character. Let C be a closed curve of links in $\mathbb{Z}^\nu$. The Wilson loop observable (the trace of the "holonomy operator" corresponding to C) is defined by

$$W^q(C) \equiv \chi^q(g_C) = \mathrm{tr}\,(U^q(g_C)). \tag{1.26}$$

This defines a random field on the space of closed loops in $\mathbb{Z}^\nu$. We now rewrite it in terms of horizontal and vertical gauge fields.

Let $V(t, C)$ be all those oriented, vertical links in C that belong to the slice $[t, t+1]$, and let $H(t, C) = C \cap \Lambda_t$. Then

$$W^q(C) = \sum_{\underline{m}} \prod_t h^q_{m(t)}[g(H(t, C))]v^q_{m(t)}[g(V(t, C))], \tag{1.27}$$

where $h^q_{m(t)}[g(H(t, C))]$ is a product of matrix elements of $U^q(g^h_{xy})$, $xy \subset H(t, C)$. and $v^q_{m(t)}[g(V(t, C))]$ is a product of matrix elements of $U^q(g^v_{xy})$, $xy \subset V(t, C)$, and $\sum_{\underline{m}}$ is that sum over products of matrix elements—i.e. that contraction scheme—that yields the trace, $\mathrm{tr}(U^q(g_C))$. From (1.24) and (1.27) we derive

$$\begin{aligned}\left\langle \prod_{j=1}^n W^q(C_j) \right\rangle_\nu^{YM} = \zeta^{-1} \sum_{\underline{m}_1 \cdots \underline{m}_n} \int \prod_t \Bigg\{ & \prod_{j=1}^n h^q_{m_j(t)}[g(H(t, C_j))] \\ & \cdot \left\langle \prod_{l=1}^n v^q_{m_l(t)}[g(V(t, C_l))] \right\rangle^\sigma_{\nu-1} (g^h(t), g^h(t+1)) \\ & \cdot \zeta^\sigma(g^h(t), g^h(t+1)) d\mu_{\nu-1}(g^h(t)) \Bigg\}\end{aligned} \tag{1.28}$$

The $n = 1$ expectation provides information about confinement of static quarks, the $n = 2$ expectation about the low-lying excitations of the theory. In a quark confining phase and for a representation U^q that is non-trivial on the center, $\mathscr{Z}$, of the gauge group G one would expect e.g. that

$$\left| \left\langle \prod_{j=1}^n W^q(C_j) \right\rangle_\nu^{YM} \right| \leqq O(\exp[-A(C_1, \ldots, C_n)]), \tag{1.29}$$

where $A(C_1, \ldots, C_n)$ is the total area of the smallest two-dimensional surface bounded by the loops $C_1, \ldots, C_n$. We assert that such an estimate can, in principle, be obtained from (1.28) and a detailed analysis of the cluster properties of the k-point functions of the $(\nu - 1)$-dimensional, non-linear σ-model in an arbitrary external gauge field. For this purpose, we note that

$$|h^q_{m_j(t)}[g(H(t, C_j))]| \leqq 1, \tag{1.30}$$

for all $m_j(t)$ and all t, since $h^q_{m_j(t)}$ is a product of matrix elements of unitary matrices. Moreover, $\sum_{\underline{m}_1, \ldots, \underline{m}_n}$ extends over $d_q^{|C_1| + \cdots + |C_n|}$ terms, where $|C|$ is the number of links contained in C, and d_q is the dimension of the representation U^q.

We now assume that the number of vertical links in C is $\geqq \alpha |C_j|$, for some $\alpha > 0$ and all $j = 1, \ldots, n$. Then, by (1.28), (1.30) and the above arguments, an estimate like (1.29) will follow from decay estimates for

$$\left\langle \prod_{l=1}^{n} v^q_{m_l(t)}[g(V(t,C_l))] \right\rangle^{\sigma}_{\nu-1} (b,t) \tag{1.31}$$

These are ordinary $N(t)$-point functions of the $(\nu-1)$-dimensional, non-linear σ-model in the external gauge field (b,t), with $N(t)=\sum_{j=1}^{n} \operatorname{card}(V(t,C_l))$. Note that $N(t)$ is even, and (1.31) is invariant under $g_j \to e^{i\varphi} g_j$, for $j \in \mathbb{Z}^{\nu-1}$.

Inequality (1.29) will now generally follow from

a) $\langle U^q(g_j)_{kl} \rangle^{\sigma}_{\nu-1}(b,t)=0$,

for all j,k,l and all (b,t); and e.g.

b) the expectation $\langle - \rangle^{\sigma}_{\nu-1}(b,t)$ clusters exponentially, uniformly in b,t. See also Sects. 2, 3, 5.

In order to clarify this discussion we now consider a *special case:* We choose a single, rectangular loop C with vertices at $(0,0)$, $(j,0)$, (j,T), $(0,T)$. Then,

$$\begin{aligned} W^q(C) = \sum_{\underset{\sim}{m},\underset{\sim}{n}} B_{m_0 n_0} \prod_{t=0}^{T-1} U^q(g^v_j(t))_{n_t n_{t+1}} \\ \cdot T_{n_T m_T} \prod_{t=0}^{T-1} U^q(g^v_0(t))^{-1}_{m_t m_{t+1}}, \end{aligned} \tag{1.32}$$

where

$$B_{m_0 n_0} = \left[\prod_{i=0}^{j-1} U^q(g^h_{(i,0)(i+1,0)}) \right]_{m_0 n_0}$$

$$T_{n_T m_T} = \left[\prod_{i=j}^{1} U^q(g^h_{(i,T)(i-1,T)}) \right]_{n_T m_T} \tag{1.33}$$

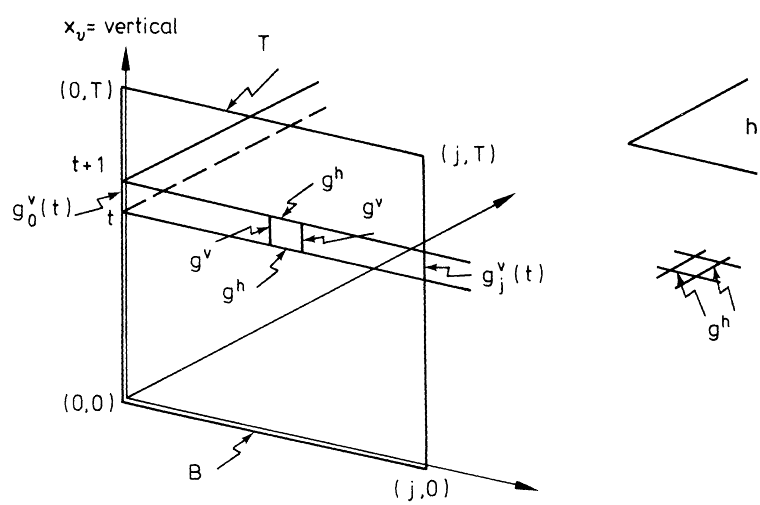

Fig. 2

From our basic identity (1.24) and (1.32), (1.33) it follows that

$$\langle W^q(C)\rangle_\nu^{YM} = \zeta^{-1} \sum_{\underline{m},\underline{n}} \int \prod_t \{d\mu_{\nu-1}(g^h(t))\zeta^\sigma(g^h(t), g^h(t+1))\} B_{mono} T_{n_N m_N} \cdot \prod_{u=0}^{T-1} \langle U^q(g_0)^{-1}_{m_u m_{u+1}} U^q(g_j)_{n_u n_{u+1}} \rangle^\sigma_{\nu-1}(g^h(u), g^h(u+1)). \tag{1.34}$$

We now proceed to estimate the r.s. of (1.34). As shown above, see (1.30), $|B_{mono}| \leqq 1$, $|T_{n_N m_N}| \leqq 1$. (In fact, if the horizontal pieces of C have the direction of a coordinate axis and for a suitable choice of boundary conditions, one can choose an *axial gauge* such that $B = T = \mathbb{1}$). Moreover, the number of terms in $\sum_{\underline{m},\underline{n}} = d_q^{2(T+1)}$.

We now imagine taking the thermodynamic limit, $\Lambda \uparrow \mathbb{Z}^\nu$. Suppose that, in that limit, there is a function $V^q(j)$ diverging to $+\infty$, as $|j| \to \infty$, such that

$$|\langle U^q(g_0)^{-1}_{mn} U^q(g_j)_{kl} \rangle^\sigma_{\nu-1}(b,t)| \leqq e^{-V^q(j)}, \tag{1.35}$$

uniformly in (b, t). Then, by (1.34), (Theorem 1.1.1) and the above estimates,

$$|\langle W^q(C)\rangle_\nu^{YM}| \leqq d_q^{2[T+1]} e^{-TV^q(j)}. \tag{1.36}$$

The Wilson criterion [16] then says that, in this theory, static quarks are confined by a potential $V^{q\bar{q}}(j)$ bounded below by $V^q(j)$. Roughly

$$V^{q\bar{q}}(j) = \lim_{T\to\infty} -\frac{1}{T} \log \langle W^q(C_T)\rangle_\nu^{YM}, \tag{1.37}$$

where $C_T = C$ is the loop depicted in Fig. 2. The correct definition of the potential $V^{q\bar{q}}$ between (infinitely heavy) static quarks may be found in [14]. A slight extension of the above arguments gives

Theorem 1.2. *Let* $V^{q\bar{q}}(j)$ *be defined as in* [14] (*eqs.* (12), (12′), *or as in* (1.37)). *Assume that* (1.35) *holds uniformly in* (b, t) *and choose boundary conditions for which* $\zeta^\sigma(b, t) \geqq 0$, *for all* (b, t). *Then*

$V^{q\bar{q}}(j) \geqq V^q(j)$, *for all* j. □

Inequality (1.35) is a *cluster property* of the U^q-*two-point* function in the $(\nu - 1)$-dimensional, non-linear σ-model in an arbitrary external gauge field. In particular, if $V^q(j) \approx m|j|$, as $|j| \to \infty$, for some $m > 0$, then (1.35) expresses *exponential clustering* of that two-point function. By Theorem 1.2 this implies confinement of static quarks by a linearly rising potential.

We have now completed the proof of our contention that pure Yang–Mills theory in ν-dimensions is equivalent to an integral of a product of $(\nu - 1)$-dimensional, non-linear σ-models in external gauge fields, and we have related clustering in those σ-models to confinement in the Yang–Mills theory.

In the remainder of this paper we are primarily concerned with discussing the cluster properties of $(\nu - 1)$-dimensional, non-linear σ-models in an arbitrary external gauge field. Another mechanism for confinement of static quarks (cancellation of "random phases") is discussed in Sect. 6.

1.4. Summary of Contents of Remaining Sections

In Sect. 2 we discuss general conditions for the clustering of the two-point function of a non-linear σ-model in an external gauge field, i.e. we study the estimate

$$|\langle \bar{U}^q(g_0)_{mn} U^q(g_j)_{kl} \rangle^\sigma_{\nu-1}(b,t)| \leqq e^{-V^q(j)}, \tag{1.38}$$

see (1.35). A necessary condition for $V^q(j) \to \infty$, as $|j| \to \infty$, uniformly in (b,t), is

$$\langle U^q(g_l)_{mn} \rangle^\sigma_{\nu-1}(b,t) = 0, \tag{1.39}$$

for all m, n, all external gauge fields (b,t).
The following result is established in Sect. 2.

Theorem 1.3. *Suppose that the character χ used in the definition* (1.19), (1.20) *of the action $A^\sigma_{\nu-1}$ is the character of a faithful representation of G.*

Then equation (1.39), *for arbitrary* (b,t) *is equivalent to U^q being a representation of G that is non-trivial on the center $\mathscr{Z}$ of the group G.* □

In Yang–Mills theory, the interpretation of this result is that confining representations should be *non-trivial* on $\mathscr{Z}$. This is in accordance with a high temperature (strong coupling) result of [2] and with general wisdom. We note that in zero external gauge field, i.e. for $(b,t) = (\mathbb{1}, \mathbb{1})$,

$$\langle U^q(g_l)_{mn} \rangle^\sigma_{\nu-1}(\mathbb{1}, \mathbb{1}) = 0, \tag{1.40}$$

for every representation U^q of G not containing the trivial one. (This is seen by substituting $g_j g_l^{-1}$ for g_j, for all $j \neq l$, which leaves $dg_j, j \neq l$, invariant).

For non-trivial (b,t), (1.40) is in general *false*. Using (1.39) we then recall standard implications of a high-temperature expansion for clustering, as expressed by (1.38).

We conclude Sect. 2 with some comments on the structure of θ-vacua in general, two-dimensional lattice Higgs theories. We show that the θ-vacua of these theories are labelled by the elements of the center $\mathscr{Z}$ of the gauge group G. (In three dimensions, in the Higgs phase, the characters of $\mathscr{Z}$ generally label topological charge-vortex-super-selection sectors of the theory; see also [18, 14, 19]).

In Sect. 3 we present results specifically concerning the cluster properties of two-dimensional, non-linear σ-models. Our method is based on a slight generalization of the McBryan–Spencer upper bound [20] (for the two-point function of the rotator model) and correlation inequalities of the Ginibre type [21, 22].

Our conclusion is that three-dimensional Yang–Mills theories with gauge group given by an arbitrary Lie group can be expected to have *at least logarithmic* confinement of static quarks. This is proven for $G = U(n), n = 1, 2, 3, \ldots$, recovering a result of [23]; see also [19]. If G is a non-abelian Lie group (e.g. $G = \mathrm{SU}(2)$) we expect *linear* confinement of static quarks, since renormalization group arguments suggest that the two-point function of the two-dimensional, non-linear σ-model in zero external gauge field clusters exponentially, for arbitrary $\beta < \infty$.

One might expect that turning on an external gauge field generally enhances clustering of truncated correlations, so that, by (1.39), (1.38) ought to hold with $V^q(j) = O(|j|)$. Unfortunately, this is in general *false*. For this reason a complete proof of permanent confinement of static quarks by a linear potential in all three-dimensional, pure Yang–Mills theories with a non-abelian, (simple) gauge Lie group will be more subtle than anticipated—if true at all. We also give an argument suggesting that four-dimensional lattice Yang–Mills theories—even non-abelian ones—may generally have a phase transition, as $\beta = g^{-2}$ is varied.

In Sects. 4 and 6 we derive an expansion of the expectation of a product of Wilson loops in terms of expectations of two-dimensional random surfaces bounded by the loops, for ν-dimensional pure Yang–Mills theories with $G = U(n)$ or $O(n), n = 1, 2, 3, \ldots$ or $G = SU(2)$. Our method is based on expanding $U^q - N$-point functions of $(\nu - 1)$-dimensional σ-models in an external gauge field in terms of random walks, [24]. Our expansion relates confinement of static quarks by a linear potential to an exponentially small, statistical weight of random surfaces. We then briefly comment on relations of Yang–Mills theory to dual strings: It can be shown that Yang–Mills theory "converges" to a dual string, as $\beta \to 0$. Hence the low-lying mass spectrum of strongly coupled Yang–Mills theory ($\beta \ll 1$) is expected to resemble the dual string spectrum; (approximate Regge trajectories). We expect that the same is true in the large-n-limit of $U(n)$—or $O(n)$—theories for $\beta = \beta_0 \cdot \beta_n$, $\{\beta_n\}$ suitably chosen and normalized so that $\beta_2 = 1$, $0 < \beta_0$ arbitrary. (We hope to report more details elsewhere).

The end of Sect. 4 concerns an application of the Brascamp–Lieb inequalities [17, 30] to proving lower bounds of β_{critial} for $U(n)$- and $O(n)$-theories. The result is $\beta_{\text{critial}}\ (n) \geqq \beta_c$, for some β_c independent of n (which is somewhat disappointing).

In Sect. 5 we specialize the scheme of Sect. 4 (expansion in random surfaces) to the case of an SU(2) Yang–Mills theory and use it to prove linear confinement of static quarks for all $\beta < \text{const.}/\nu - 2$. In Sect. 6 we distill out of the scheme of Sects. 4 and 5 two basic mechanisms that might lead to permanent confinement of static quarks in Yang–Mills theories; (cluster properties of associated σ-models, resp. cancellation of random phases). The two mechanisms are discussed in some detail, partly rigorously, partly heuristically. In certain respects, Sect. 6 may be the most interesting part of the whole paper. See in particular identity (6.6).

In Sect. 7, we consider $(\nu - 1)$-dimensional, Gaussian (i.e. linear) σ-models in an external gauge field. They are used to describe a hypothetical phase of ν-dimensional Yang–Mills theory which is qualitatively correctly described by perturbation theory. Thus, they ought to provide a correct description of the short distance properties of Yang–Mills theory. The main purposes of that analysis is to gain some insight into how to construct Wilson loop observables in the continuum limit and how to define a scheme for implicit renormalization.

2. Necessary Condition for Clustering of Two-Point Functions in Non-linear σ-Models; θ-Vacua in Two-Dimensional Yang–Mills Theories

2.1. Proof of Theorem 1.3.

In this section we argue that *confining* (or "quark") *representations*, U^q, of the gauge group G are those representations for which

$$\langle U^q(g_l)_{mn} \rangle^{\sigma}_{\nu-1}(b, t) = 0, \tag{2.1}$$

for all l, m, n and all external gauge fields (b, t). Representations violating (2.1) are called *particle* representations. Theorem 1.3 says that, in the strong coupling regime ($\beta \ll 1$), these notions coincide with the ones in [2] where a high temperature expansion for the ν-dimensional Yang–Mills theory is used to distinguish between confining and particle representations; see also [25].

Condition (2.1) is necessary for the clustering of the U^q-two-point function of the ν-dimensional, non-linear σ-model in an arbitrary external gauge field which, in turn, is a sufficient condition for confinement of static quarks transforming according to U^q, (in the sense of Wilson's criterion [16] or its improved version [14]).

We recall that the action of a $(\nu - 1)$-dimensional $G \times G$ non-linear σ-model in an external gauge field is given by

$$A^\sigma_{\nu-1}(b,t) = -\sum_{ij \subset \Lambda} \mathrm{Re}\,\chi(g_i^{-1} b_{ij} g_j t_{ij}^{-1}), \tag{2.2}$$

and the equilibrium expectation, $\langle - \rangle^\sigma_{\nu-1,B}(b,t)$, by the probability measure

$$d\mu^\sigma_{(b,t)}(g) = Z^\sigma_B(b,t)^{-1} B(g_{\partial\Lambda}) e^{-\beta A^\sigma_{\nu-1}(b,t)} \prod_{j\in\Lambda} dg_j, \tag{2.3}$$

where B is a boundary condition only depending on $\{g_j : j \in \partial\Lambda\}$.

First we give a sufficient condition for (2.1). Let $\mathscr{Z}_q$ be a minimal subgroup $\neq \{\mathbb{1}\}$ contained in or equal to the center $\mathscr{Z}$ of the group G with the property that

$$U^q \upharpoonright \mathscr{Z}_q \text{ does } \textit{not} \text{ contain the trivial representation of } \mathscr{Z}_q. \tag{2.4}$$

We assume that the boundary condition B is *invariant* under $\mathscr{Z}_q$, i.e.

$$B(g_{\partial\Lambda}) = B((g\cdot\tau)_{\partial\Lambda}), \tag{2.5}$$

where $(g\cdot\tau)_j = g_j \cdot \tau$, for all j, and τ is some element of $\mathscr{Z}_q$.

Theorem 2.1. *If one assumes (2.4) and (2.5) then*

$$\langle U^q(g_l)_{mn} \rangle^\sigma_{\nu-1,B}(b,t) = 0,$$

for arbitrary (b,t).

Proof. A basic role in the proof is played by the simple identity

$$\int_G dg F(g) = \int_G dg \int_{\mathscr{Z}_q} d\tau F(g\cdot\tau), \tag{2.6}$$

where $F \in L^1(G, dg)$; (a consequence of the right invariance of dg and Fubini's theorem). By (2.3)

$$\begin{aligned}\langle U^q(g_l)_{mn} \rangle^\sigma_{\nu-1,B}(b,t) &= Z^\sigma_B(b,t)^{-1} \int dg_l\, U^q(g_l)_{mn} \int \prod_{j\neq l} dg_j \\ &\quad \cdot \prod_{xy\subset\Lambda} e^{\beta\,\mathrm{Re}\,\chi(g_x^{-1} b_{xy} g_y t_{xy}^{-1})} B(g_{\partial\Lambda}) \\ &= Z^\sigma_B(b,t)^{-1} \int_G dg_l \int_{\mathscr{Z}_q} d\tau\, U^q(g_l\cdot\tau)_{mn} \int \prod_{j\neq l} dg_j \prod_{\substack{xy\subset\Lambda \\ x\neq l\neq y}} e^{\beta\,\mathrm{Re}\,\chi(g_x^{-1} b_{xy} g_y t_{xy}^{-1})}\end{aligned}$$

$$\cdot \prod_{zl} e^{\beta \operatorname{Re} \chi(g_z^{-1} b_{zl} g_l \cdot \tau t_{zl}^{-1})} B(g\partial\Lambda)$$
$$= Z_B^\sigma(b,t)^{-1} \int_G dg_l \int_{\mathscr{Z}_q} d\tau\, U^q(g_l \cdot \tau)_{mn} \int \prod_{j \neq l} dg_j \prod_{xy \subset \Lambda} e^{\beta \operatorname{Re} \chi(\tilde{g}_x^{-1} b_{xy} \tilde{g}_y t_{xy}^{-1})} B(g_{\partial\Lambda}),$$

where $\tilde{g}_x = g_x \tau^{-1}$, for $x \neq l, \tilde{g}_l = g_l$. Here we have used (2.6) and the fact that τ commutes with all b_{xy} and t_{xy}. Since $d\tilde{g}_x = dg_x$, for all x, by right invariance, $B(\tilde{g}_{\partial\Lambda}) = B(g_{\partial\Lambda})$, by the assumed $\mathscr{Z}_q$-invariance of B and $\tilde{g}_l = g_l$, we have

$$\langle U^q(g_l)_{mn} \rangle_{\nu-1,B}^\sigma(b,t) = Z_B^\sigma(b,t)^{-1} \int_G dg_l \int_{\mathscr{Z}_q} d\tau\, U^q(g_l \cdot \tau)_{mn} I(g_l; b, t),$$

with

$$I(g_l; b, t) = I(\tilde{g}_l; b, t) \equiv \int \prod_{j \neq l} d\tilde{g}_j \prod_{xy \subset \Lambda} e^{\beta \operatorname{Re} \chi(\tilde{g}_x^{-1} b_{xy} \tilde{g}_y t_{xy}^{-1})} B(\tilde{g}_{\partial\Lambda}).$$

Next, $U^q(g_l\tau)_{mn} = U^q(g_l)_{mn}\chi^q(\tau)$, by the irreducibility of U^q. Thus

$$\langle U^q(g_l)_{mn} \rangle_{\nu-1,B}^\sigma(b,t) = Z_B^\sigma(b,t)^{-1} \int_G dg_l U^q(g_l)_{mn} I(g_l; b, t) \int_{\mathscr{Z}_q} d\tau\, \chi^q(\tau).$$

By condition (2.4),

$$\int_{\mathscr{Z}_q} d\tau \chi^q(\tau) = 0. \qquad \square$$

We now prove the converse of Theorem 2.1.

Let

$$\mathscr{A}_{\Lambda_0} = \exp\{-\sum_{ij \subset \Lambda_0} \operatorname{Re} \chi(g_i^{-1} b_{ij} g_j t_{ij}^{-1})\}. \tag{2.7}$$

Theorem 2.2. *Suppose that the character* χ *used in the definition of* $A_{\nu-1}^\sigma$ *is faithful and that, for some thermodynamic limit,* $\langle - \rangle_{\nu-1}^\sigma(b,t)$ *of the expectations given by* (2.3) *and arbitrary* (b,t),

$$\langle U^q(g_l)_{mn} \rangle_{\nu-1}^\sigma(b,t) = 0, \tag{2.8}$$

and

$$\langle \mathscr{A}_{\Lambda_0} \rangle_{\nu-1}^\sigma(b,t) \neq 0, \tag{2.9}$$

for arbitrary bounded regions $\Lambda_0 \subset \mathbb{Z}^{\nu-1}$. *Then* U^q *does not contain the trivial representation of the center* $\mathscr{Z}$ *of* G. (*If* U^q *is irreducible this is equivalent to* $U^q \upharpoonright \mathscr{Z} \neq \{\mathbb{1}\}$).

Remark. We note that (2.9) is trivially satisfied if the boundary condition $B(g_{\partial\Lambda})$ is non-negative and $\not\equiv 0$, for all $\Lambda \subset \mathbb{Z}^{\nu-1}$.

Proof. By (2.8)

$$\int \prod_{xy} db_{xy} dt_{xy} \langle U^q(g_l)_{mn} \rangle_{\nu-1}^\sigma(bb', tt')\, F(b,t) = 0, \tag{2.10}$$

for arbitrary, bounded F and arbitrary (b', t'). This equation is basic for our proof.

We choose some bounded region $\Lambda_0 \subset \mathbb{Z}^{\nu-1}$ containing the site l, and $l \notin \partial\Lambda_0$. By taking a conditional expectation with respect to the field configuration inside

Λ_0, i.e. by applying the *DLR* equations [26], we obtain

$$\langle U^q(g_l)_{mn}\rangle^{\sigma}_{v-1}(b,t) = Z_{\Lambda_0}^{-1}\int \prod_{x\in\Lambda_0} dg_x U^q(g_l)_{mn}\cdot \prod_{xy\subset\Lambda_0} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})}\tilde{B}(g_{\partial\Lambda_0};b,t),$$

where $\tilde{B}$ only depends on those b_{xy} and t_{xy} for which xy is outside Λ_0 or on $\partial\Lambda_0$, and Z_{Λ_0} is a normalization factor. Since

$$|Z_{\Lambda_0}| \leqq \exp[\beta v\chi(\mathbb{1})|\Lambda_0|]\,\|\tilde{B}\|_1 < \infty,$$

(2.8) implies

$$\int \prod_{x\in\Lambda_0} dg_x U^q(g_l)_{mn} \prod_{xy\subset\Lambda_0} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})}\tilde{B}(g_{\partial\Lambda_0};b,t) = 0 \tag{2.11}$$

Using the argument leading to (2.10) we see that we may integrate the l.s. of (2.11) over all those gauge fields b_{xy} with the property that $x\in\partial\Lambda_0, y\in\Lambda_1$, with $\Lambda_1 = \Lambda_0\backslash\partial\Lambda_0$, and obtain

$$I(l,\Lambda_1) \equiv \int \prod_{\substack{x\in\partial\Lambda_0\\ y\in\Lambda_1}} db_{xy}\int \prod_{x\in\Lambda_0} dg_x U^q(g_l)_{mn}\cdot \prod_{xy\subset\Lambda_0} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})}\tilde{B}(g_{\partial\Lambda_0};b,t) = 0.$$

By the left and right invariance of db_{xy},

$$\int db_{xy} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})} = \int db_{xy} e^{\beta \operatorname{Re}\chi(b_{xy})} = \text{const.} > 1.$$

Thus

$$0 = I(l,\Lambda_1) = C(b,t)\int \prod_{x\in\Lambda_1} dg_x U^q(g_l)_{mn}\cdot \prod_{xy\subset\Lambda_1} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})},$$

with $|C(b,t)| = |\langle \mathscr{A}_{\Lambda_0}\rangle^{\sigma}_{v-1}(b,t)|$ which is strictly positive, by hypothesis (2.9). Thus

$$E^{\beta}(l,\Lambda_1;b,t) \equiv \int \prod_{x\in\Lambda_1} dg_x U^q(g_l)_{mn}\cdot \prod_{xy\subset\Lambda_1} e^{\beta \operatorname{Re}\chi(g_{\bar{x}}^{-1} b_{xy} g_y t_{\bar{x}\bar{y}}^{-1})} = 0, \tag{2.12}$$

for arbitrary (b,t). The end of the proof is based on

Lemma 2.3. *If* $E^{\beta}(l,\Lambda_1;b,t)=0$, *for arbitrary* (b,t) *and some* $\beta>0$ *then* $E^{\beta'}(l,\Lambda_1;b,t)=0$, *for all* $\beta'\geqq 0$ *and arbitrary* (b,t).

Proof of Lemma 2.3. We claim that, for $\beta>0$ and an arbitrary $\delta>0$, there exists a function $F_\delta\in L^1(G,dg)$ such that

$$\left\| e^{\beta' \operatorname{Re}\chi(g)} - \int db\, e^{\beta \operatorname{Re}\chi(gb^{-1})}F_\delta(b)\right\|_1 < \delta, \tag{2.13}$$

for arbitrary $\beta'\geqq 0$. When $\beta'=0$ this is clear: $\delta=0$, $F_\delta=\text{const.}$ suffice. Thus we may suppose that $\beta'>0$. We consider the Peter–Weyl expansion

$$e^{\beta' \operatorname{Re}\chi(g)} = \sum_{\alpha\in A} c_\alpha(\beta')\chi_\alpha(g); \tag{2.14}$$

here A is a list of the irreducible representations of G. We may choose A to be contained in $\mathbb{Z}$. For arbitrary $\beta'>0$, the coefficients $c_\alpha(\beta')$ are rapidly decreasing in $|\alpha|$. Moreover, by using the power series expansion of the exponential, it is easy to see that $c_\alpha(\beta')\geqq 0$ for all α and $\beta'\geqq 0$, and if $\beta>0$ and $\beta'>0$ then

$$c_\alpha(\beta') > 0 \text{ if and only if } c_\alpha(\beta) > 0.$$

Choosing

$$F_\delta(b) = \sum_{\substack{\alpha \in A \\ |\alpha| \leqq \alpha_\delta < \infty}} (c_\alpha(\beta')/c_\alpha(\beta))\chi_\alpha(b),$$

with $c_\alpha(\beta')/c_\alpha(\beta) = 0$ in case $c_\alpha(\beta') = c_\alpha(\beta) = 0$, we obtain (2.13), provided α_δ is sufficiently large. From (2.12) and (2.13) we conclude that

$$E^{\beta'}(l, \Lambda_1; b, t) = \int E^\beta(l, \Lambda_1; b(b')^{-1}, t) \prod_{xy \subset \Lambda_1} F_\delta(b'_{xy}) db'_{xy} + e(l, \Lambda_1; b, t),$$

where

$$|e(l, \Lambda_1; b, t)| < \nu\delta|\Lambda_1|.$$

Since $E^\beta(l, \Lambda_1; b(b')^{-1}, t) = 0$, for all b, b' and t, the lemma follows by letting $\delta \searrow 0$. □

Since $E^{\beta'}(l, \Lambda_1; b, t) = 0$, for all $\beta' \geqq 0$,

$$F^{\beta'}(l, \Lambda_1; b, t) \equiv \left(\int \prod_{xy \subset \Lambda_1} dg_x \prod_{xy \subset \Lambda_1} e^{\beta' \operatorname{Re} \chi(g_x^{-1} b_{xy} g_y t_{xy}^{-1})} \right)^{-1} \cdot E^{\beta'}(l, \Lambda_1; b, t) = 0 \tag{2.15}$$

for arbitrary (b, t) and $\beta' \geqq 0$
We now choose

$$\Lambda_1 = \{l + n_i e_i + n_j e_j : n_i = 0, 1, n_j = 0, \ldots, N\}, \tag{2.16}$$

where e_i and e_j are two orthogonal unit lattice vectors. We set $l'_n \equiv l + e_i + ne_j$, $l_n \equiv l + ne_j$, and choose

$$\begin{aligned} b_{l'_n l'_{n+1}} &= t_{l'_n l'_{n+1}} = \mathbb{1}, \\ b_{l_n l'_n} &= t_{l_n l'_n} = \mathbb{1}, \\ b_{l_n l_{n+1}} &= t_{l_n l_{n+1}} = h_n, \end{aligned} \tag{2.17}$$

where h_n is an arbitrary element of G, $n = 0, \ldots, N-1$. When β' tends to $+\infty$ the measure

$$(Z_N^{\beta'})^{-1} e^{\beta' \operatorname{Re} \chi(g_{l_N}^{-1} g_{l'_N})} \prod_{n=0}^{N-1} e^{\beta' \{\operatorname{Re} \chi(g_{l_n}^{-1} g_{l'_n}) + \operatorname{Re} \chi(g_{l'_n}^{-1} g_{l'_{n+1}}) + \operatorname{Re} \chi(g_{l_n}^{-1} h_n g_{l_{n+1}} h_n^{-1})\}}$$
$$\cdot \prod_{n=0}^{N} dg_{l_n} dg_{l'_n},$$

where $Z_N^{\beta'}$ is the obvious normalization factor, is a probability measure concentrated on the region $\Omega_N \subset G^{\times 2(N+1)}$ specified by

$$\chi(g_l^{-1} g_{l'_0}) = \chi(g_{l'_0}^{-1} g_{l'_1}) = \chi(g_{l'_1}^{-1} g_{l_1}) = \ldots = \chi(g_{l'_N}^{-1} g_{l_N}) = \chi(\mathbb{1}) \tag{2.18}$$

and

$$\chi(g_{l_n}^{-1} h_n g_{l_{n+1}} h_n^{-1}) = \chi(\mathbb{1}), \qquad \text{for all } n = 0, \ldots, N-1 \tag{2.19}$$

Since $\chi(g) = \chi(\mathbb{1})$ implies $g = \mathbb{1}$, by hypothesis on χ, we conclude that

$$g_l = g_{l'_0} = g_{l'_1} = g_{l_1} = \ldots = g_{l'_N} = g_{l_N},$$

and this and (2.19) yield

$$h_n g_l = g_l h_n, \quad \text{for all } n = 0, \dots, N-1 \tag{2.20}$$

We conclude that, for g_l to belong to Ω_N, for arbitrary $h_0, \dots, h_{N-1}$ and $N < \infty$ it is necessary that

$$hg_l = g_l h, \quad \text{for all } h \in G,$$

i.e. $g_l \in \mathscr{Z}$. Thus, as $\beta' \to +\infty$, $N \to \infty$, and for a suitable choice of $h_0, \dots, h_{N-1}$

$$F^{\beta'}(l, \Lambda_1; b, t) \to \int_{\mathscr{Z}} d\tau\, U^q(\tau)_{mn},$$

with Λ_1 and (b, t) as specified in (2.16), (2.17). Since $F^{\beta'}(l, \Lambda_1; b, t) = 0$, for all β', Λ_1 and (b, t)

$$\int_{\mathscr{Z}} d\tau\, U^q(\tau)_{mn} = 0,$$

or, equivalently,

$U^q \upharpoonright \mathscr{Z}$ does *not* contain the trivial representation of $\mathscr{Z}$. □

Theorem 2.2 shows that if U^q is trivial on the center $\mathscr{Z}$ of G then it is in general *impossible* that

$$\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle (b, t) \to 0, \quad \text{as } |x| \to \infty,$$

for arbitrary (b, t), because, for a suitable choice of (b, t)

$$|\langle U^q(g_0)_{ij} \rangle (b, t)| \quad |\langle U^q(g_x)_{kl} \rangle (b, t)| \quad \geqq \text{const.} > 0$$

for all $x = \xi e$, ξ large enough.

Thus, because of (1.35) and Theorem 1.2, particles transforming under a representation of the gauge group that is trivial on the center cannot be expected to be permanently confined.[2] Motivated by this observation we henceforth constrain our attention to the study of cluster properties of

$$\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle (b, t), \tag{2.21}$$

when $|x| \to \infty$, with U^q a representation of the gauge group that does *not* contain the trivial representation of the center.

The last issue of Sect. 2 is a brief discussion of θ-vacua in two-dimensional non-abelian Higgs theories with Higgs scalars in a representation that is trivial on the center $\mathscr{Z}$ of the gauge group. By this we want to illustrate the importance of complex boundary conditions. We show that such a theory has in general as many physically distinct vacua (θ-vacua) as there are elements in the center $\mathscr{Z}$, and that quarks are in general only confined in the standard $\theta = 0$ vacuum. This is in analogy to what was previously found for abelian theories [14]; see also [19]. For pedagogical reasons we start with a short discussion of the pure $\mathbb{Z}(n)$ models. The action of these models is

$$A(\theta(\Lambda)) = - \sum_{p \subset \Lambda \subset \mathbb{Z}^2} \cos(\theta_{\partial p}), \tag{2.21}$$

---

2 However, the "colour" of such particles is screened by the "colour" of the gauge field. See e.g. [25].

where $\theta_{\partial p} = \theta_{xy} + \theta_{yz} + \theta_{zu} + \theta_{ux}$, if $\partial p = \{xy, yz, zu, ux\}$, and

$$\theta_{xy} = \frac{m}{n} 2\pi, \quad m = 0, \ldots, n-1,$$

for all $xy \subset \Lambda$.

The vacuum expectation is given by a (generally complex-valued) measure $d\mu$, defined by

$$d\mu(\theta(\Lambda)) = Z_\Lambda^{-1} B(\theta(\partial\Lambda)) e^{-\beta A(\theta(\Lambda))} \prod_{xy} d\theta_{xy}, \tag{2.22}$$

with $d\theta$ the normalized counting measure on $\{0, \ldots, n-1\}$, (= Haar measure on $\mathbb{Z}(n)$). As boundary condition, B, we choose

$$B(\theta(\partial\Lambda)) = B_k(\theta(\partial\Lambda)) = \prod_{xy \subset \partial\Lambda} e^{ik\theta_{xy}}, \tag{2.23}$$

$k = 0, \ldots, n-1$.

Since the gauge field is abelian,

$$B_k(\theta(\partial\Lambda)) = \prod_{p \subset \Lambda} e^{ik\theta_{\partial p}}. \tag{2.24}$$

(This is the lattice version of Stokes' theorem).

The vacuum expectation defined by the measure (2.22) with $B = B_k$ is denoted $\langle - \rangle_\Lambda(\beta, k)$. In two dimensions and for $B = B_k$ the "plaquette angles" $\theta_{\partial p}$ with distribution (2.22) are independent random variables. Therefore the existence of the thermodynamic limit

$$\langle - \rangle(\beta, k) = \lim_{\Lambda \uparrow \mathbb{Z}^2} \langle - \rangle_\Lambda(\beta, k)$$

is trivial and so are the facts that $\langle - \rangle(\beta, k)$ is invariant under the symmetries of $\mathbb{Z}^2$ and satisfies reflection positivity, for all k; i.e. $\langle - \rangle(\beta, k)$ is indeed a vacuum expectation.

We now show that, for $k \neq k', \langle - \rangle(\beta, k)$ and $\langle - \rangle(\beta, k')$ are physically different. (The standard vacuum corresponds to $k = 0$).

Let

$$r_{k,l}(\beta) = \frac{\sum_{m=0}^{n-1} e^{i(k-l)(m/n)2\pi} e^{\beta \cos((m/n)2\pi)}}{\sum_{m=0}^{n-1} e^{ik(m/n)2\pi} e^{\beta \cos((m/n)2\pi)}} \tag{2.25}$$

Let $C_1, \ldots, C_N$ be closed loops and $A_1, \ldots, A_N$ the subsets of $\mathbb{Z}^2$ bounded by $C_1, \ldots, C_N$.

Suppose for simplicity, that

$$A_i \cap A_j = \emptyset, \quad \text{for } i \neq j. \tag{2.26}$$

Let

$$W^{q_j}(C_j) = \prod_{xy \subset C_j} e^{-iq_j\theta_{xy}} = \prod_{p \subset A_j} e^{-iq_j\theta_{\partial p}}.$$

Then

$$\langle \prod_{j=1}^{N} W^{q_j}(C_j) \rangle (\beta, k) = \prod_{j=1}^{N} r_{k,q_j}(\beta)^{A_j} \tag{2.27}$$

This is easily generalized to the case where (2.26) is violated.

For $k = 0$, $|r_{0,q_j}(\beta)| < 1$, for all β and all $q_j = 1, \ldots, n-1$.

Thus, in the standard $k = 0$ vacuum, static quarks transforming under a non-trivial, irreducible representation of $\mathbb{Z}(n)$ are permanently confined by a linear potential, and inequality (1.29) holds. However, when $|k - q| < k$, $|r_{k,q}(\beta)| > 1$. Therefore in a $k \neq 0$ vacuum quarks of "n-ality" q, with $|k - q| < k$, repell each other with a linear potential, namely equation (2.25) exhibits "anti-confinement" (liberation) of static quarks of n-ality q, in the most dramatic sense of these words. Put differently, the system in the vacuum state $\langle - \rangle(\beta, k)$, $k \neq 0$, is unstable against coupling to quarks of n-ality q, with $k - q < k$: the state $\langle - \rangle(\beta, k)$ decays into $\langle - \rangle(\beta, k - q)$; the "charges" at infinity are screened.

We now argue that a *two-dimensional Higgs theory* with gauge group $G = \mathrm{SU}(n)$, for example, and Higgs scalars in a representation that is trivial on the center $\mathbb{Z}(n)$ of $\mathrm{SU}(n)$ also has $n = |\mathbb{Z}(n)|$ physically different vacuum expectations, $\langle - \rangle(\beta, k)$, $k = 0, \ldots, n-1$. These expectations are given by the thermodynamic limit of the complex measures

$$d\mu(g(\Lambda)) = Z_\Lambda^{-1} B_k(g(\partial\Lambda)) Z^M(g(\Lambda)) e^{-\beta A_2^{YM}(g(\Lambda))} Dg(\Lambda), \tag{2.28}$$

where

$$B_k(g(\partial\Lambda)) = \prod_{xy \subset \partial\Lambda} U^k(g_{xy}), \tag{2.29}$$

and U^k is a representation of $\mathrm{SU}(n)$ of n-ality k, i.e. $U^k(e^{i\theta}) = e^{ik\theta}$, for $e^{i\theta} \in \mathbb{Z}(n)$. Here, $Z^M(g(\Lambda))$ is a gauge-invariant (non-negative) functional arising by integrating out the Higgs scalars with the property

$$Z(g(\Lambda)) = Z(g \cdot \tau)(\Lambda)), \tag{2.30}$$

for arbitrary $\tau_{xy} \in \mathbb{Z}(n)$, $xy \subset \Lambda$; (2.30) expresses the fact that the Higgs scalars transform trivially under $\mathbb{Z}(n)$, [14, 23].

For $k = 0$, this theory permanently confines static quarks by a linear potential [27, 23].

If the a priori distribution of the Higgs scalars has zero weight at zero field strength the Higgs theory defined in (2.28)–(2.30) converges to the pure $\mathbb{Z}(n)$ lattice gauge theory (2.22), (2.23), as the strength of the coupling of the Higgs scalars to the gauge field tends to ∞, for all $\beta < \infty$ and all Λ. (The proof is standard; convergence is uniform in Λ when $k = 0$). In this limit all boundary conditions $B_k, B'_k, \ldots$ of the same n-ality k are equivalent. That is likely to be true in general, in the thermodynamic limit $\Lambda = \mathbb{Z}^2$, due to screening.

Thus if the coupling of the Higgs scalars to the gauge field is sufficiently strong, the vacuum expectations $\langle - \rangle(\beta, k)$ of the two-dimensional $\mathrm{SU}(n)$ Higgs theory are physically different for different values of k, and we expect the same phenomena (anti-confinement and instability of $\langle - \rangle(\beta, k)$ under coupling to quarks of n-ality q with $|k - q| < k$) as in the pure $\mathbb{Z}_n$ model.

We do not wish to go into details of these arguments, as they are hardly very interesting. (For the modified models of [19] and the abelian models [4, 14] most assertions can be made precise using duality transformations. Notice that there is no need for integrating out the Higgs scalars which we did only to economise on notations. We also recall that, for $2\pi\frac{k}{n}=\pi$, $\langle - \rangle(\beta, k)$ may be doubly degenerate, for suitable coupling constants, and charged super selection sectors may appear [33]).

It is clear how to extend our analysis to arbitrary gauge groups with non-trivial center. In general, one will find as many physically distinct vacua as there are elements in the center, but only the standard vacuum $\langle - \rangle(\beta, 0)$ will permanently confine arbitrary, static quarks transforming non-trivially under the center. The measures with expectation $\langle - \rangle(\beta, k)$, $k \neq 0$, are complex-valued, and they differ from the standard $k=0$ measure only by a boundary condition. The explicit, physical interpretation of those boundary conditions in terms of static (colour) charges at spatial $\pm\infty$ is as in [33]. [In three dimensions, the irreducible characters of the center of G generally label *vortex sectors*, in four dimensions *monopole sectors*; see [18, 19]. The mass gaps on these sectors are given by analogues of surface tensions, as in the case of the soliton sectors of two-dimensional field theories with degenerate vacua].

3. Cluster Properties of Non-linear σ-Models

Let G be a compact Lie group, and let the action of the nonlinear $G \times G$-σ-model in an external $G \times G$-field (b, t) and enclosed in the region $\Lambda \subset \mathbb{Z}^\nu$ be given by

$$A^\sigma_\nu(g; b, t) = - \sum_{ij \subset \Lambda} \operatorname{Re} \chi(g_i^{-1} b_{ij} g_j t_{ij}^{-1}),$$

where χ is a faithful character on G. The expectation in this model at inverse temperature β will be denoted by $\langle \cdot \rangle^\sigma_{G,\nu}(\beta, b, t)$ (suppressing Λ in the notation).

It has been proved in [23] that if χ^q is a character on G which is non-trivial on the center $\mathscr{Z}(G)$, then

$$|\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle^\sigma_{G,\nu}(\beta, b, t)| \leqq \langle U^q(\tau_0) U^q(\tau_x^{-1}) \rangle^\sigma_{\mathscr{Z}(G),\nu}(d\beta, \mathbb{1}, \mathbb{1}), \tag{3.1}$$

where d is the dimension of the irreducible representation U corresponding to χ, and $U^q(\tau)_{ij} = U^q(\tau)\delta_{ij}$ for $\tau \in \mathscr{Z}(G)$, and $1 \leqq i, j \leqq d_q$. Combining this result with (1.24), it follows that *if the $\mathscr{Z}(G) \times \mathscr{Z}(G)$-$\sigma$-model in $\nu - 1$ dimensions clusters for some coupling constant β, then the ν-dimensional Yang–Mills theory with gauge group G and coupling constant β/d confines static quarks.*

Thus, we recover here a result of Mack's [27] ($\nu = 2$), and results in [23] (Theorem 1 and 2).

The following result follows by standard high temperature expansions [2, 31, 32].

Theorem 3.1. *Let χ be the character used in the definition of the action $A^\sigma_{\nu-1}(b, t)$; see* (2.2). *Assume that*

$$1 - e^{-\beta\chi(\mathbb{1})} < \varepsilon < 1$$

for some small $\varepsilon > 0$ *(depending on the Haar measure of the gauge group G and estimated as in* [2]*). Then the two-point function* (2.21) *decays exponentially, as* $|x| \to \infty$.

Remarks

1. By Theorems 1.2 and 2.2, Theorem 3.1 establishes linear confinement of static quarks in an irreducible representation U^q of the gauge group G that is non-trivial on the center $\mathscr{Z}$ of G.
2. Below we apply the Brascamp–Lieb method [17, 30] to prove that for $G = U(n)$ or $O(n)$, χ the character of the fundamental representation of G, the two-point function (2.21) clusters (possibly *not* exponentially) if $\beta < \beta_0$, where β_0 is a positive constant independent of n. In comparison, Theorem 3.1 establishes exponential clustering of the two-point function of the $U(n)$- or $O(n)$-σ-models for $\beta < O(1/n)$.

Recall that in [30] it is proven that the critical inverse temperature, $\beta_c(N)$, of the N-vector models ($= O(N)$ non-linear σ-models on the lattice $\mathbb{Z}^\nu$, $\nu \geqq 3$) obeys

$$\beta_c(N) \geqq N/2\nu. \tag{3.2}$$

The proof follows from a method due to Brascamp and Lieb [17], which boils down to the following estimates:

Let $\varphi \in \mathbb{R}^N$, $S \in S^{N-1}$, and define the real function V on $\mathbb{R}^N$ by

$$e^{V(\varphi)} = \int_{S^{N-1}} e^{\varphi \cdot S} \, d\Omega(S)$$

where $d\Omega$ is the normalized, uniform measure on S^{N-1}. Let $M_V(\varphi)$ denote the norm of the matrix with matrix elements $\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}(\varphi)$. Then, by [17],

$$\sup_{\varphi} M_V(\varphi) \geqq \text{const.}\ \beta_c(N)^{-1} \tag{3.3}$$

It is shown in [30] that

$$\sup_{\varphi} M_V(\varphi) = \frac{1}{N}$$

from which (3.2) follows; (the constant in (3.3) is also estimated in [17]).

The Brascamp–Lieb method can also be applied to general $G \times G$-σ-models (in external gauge fields), in particular to the $O(n) \times O(n)$- or $U(n) \times U(n)$- models. As the reader may easily check, a sufficient condition for $\beta < \beta_c(O(n) \times O(n))$ is the following:

Let V be the function on $\mathbb{R}^{n^2}$ (identified with $\mathscr{M}(n)$) defined by

$$e^{V(\varphi)} = \int_{O(n)} e^{Tr(g^* \varphi)} dg,$$

where dg is the normalized Haar measure on $O(n)$, and let $M_V(\varphi)$ be the norm of the $n^2 \times n^2$-matrix with matrix elements $\frac{\partial^2 V}{\varphi_{ij} \partial \varphi_{kl}}(\varphi)$. Then

$$\sup M_V(\varphi) \geqq \text{const.}\ \beta_c(O(n) \times O(n))^{-1} \tag{3.4}$$

i.e. for $\beta <$ const. $(\sup_{\varphi} M_V(\varphi))^{-1}$, the $O(n) \times O(n)$-σ-model (in an arbitrary external gauge field) has a clustering twopoint function, and hence the corresponding Yang–Mills theory in one dimension more with gauge group $O(n)$ confines static quarks.

Unfortunately, it turns out that the lefthand side of (3.4) does not increase with n, so we can only conclude that

$$\beta_c(0(n) \times 0(n)) \geqq \text{const., for all } n.$$

A similar argument applies for $G = U(n)$. Thus the Brasscamp–Lieb method seems to be insufficient to determine the large-n-asymptotics of $\beta_c(G \times G)$, for $G = O(n)$ or $U(n)$.

It has been remarked in [23] that (3.1) implies that the McBryan–Spencer bound [20] can be applied to $U(n) \times U(n)$-σ-models, or to any $G \times G$-σ-models in two dimensions such that $\mathscr{Z}(G)$ contains a copy of $U(1)$; even in an external $G \times G$-gauge field. For groups whose center does not contain $U(1)$ the situation is more involved. However, we can prove the following

Theorem 3.2. *Suppose that G contains a $U(1)$ subgroup, and that the character χ^q is non-trivial on this subgroup. For free or periodic boundary conditions, the infinite volume two-point functions of the two-dimensional model.*

$$\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle^{\sigma}_{G,2}(\beta, b, \mathbb{1}), \tag{3.5}$$

cluster for all $\beta > 0$ and all b.

Proof. Let us choose representations U^q (of dimension d_q) and U (of dimension d), corresponding to χ^q and χ respectively, such that they map the elements of $U(1) \subset G$ into diagonal matrices; this can obviously be done, and we conclude that there exist integers $k_1^q, \dots, k_{d_q}^q$ and $k_1, \dots, k_d$ such that

$$U^q(h(\theta))_{ij} = \delta_{ij} e^{ik_j^q \theta}, \quad 1 \leqq i, \quad j \leqq d_q,$$

and

$$U(h(\theta))_{ij} = \delta_{ij} e^{ik_j \theta}, \quad 1 \leqq i, j \leqq d,$$

for all $\theta \in [0, 2\pi[$, where $h(\theta)$, $\theta \in [0, 2\pi[$, labels the elements of $U(1) \subset G$.

By using the right-invariance of the Haar measure on G, Fubini's theorem and the cyclicity of χ, we have that

$$\begin{aligned}
&\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} \rangle^{\sigma}_{G,v}(\beta, b, \mathbb{1}) \\
&= Z_v^{\sigma}(\beta, b, 1)^{-1} \int U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} e^{\beta \sum_{ij \subset \Lambda} \operatorname{Re} \chi(g_i^{-1} b_{ij} g_j)} \prod_{i \in \Lambda} dg_i \, . \\
&= Z_v^{\sigma}(\beta, b, 1)^{-1} \int_{U(1)} \prod_{i \in \Lambda} dh_i \int_{G} \prod_{i \in \Lambda} dg_i U^q(g_0 h_0)_{ij} U^q(h_x^{-1} g_x^{-1})_{ji} \cdot e^{\beta \sum_{ij \subset \Lambda} \operatorname{Re} \chi(h_i^{-1} g_i^{-1} b_{ij} g_j h_j)} \\
&= Z_v^{\sigma}(\beta, b, 1)^{-1} \int_{G} \prod_{i \in \Lambda} dg_i \int_{U(1)} \prod_{i \in \Lambda} dh_i U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} U^q(h_0 h_x^{-1})_{jj} \\
&\quad \cdot e^{\beta \sum_{ij \subset \Lambda} \sum_{n=1}^{d} \operatorname{Re} U(h_j h_i^{-1})_{nn} U(g_i^{-1} b_{ij} g_j)_{nn}} \\
&= Z_v^{\sigma}(\beta, b, 1)^{-1} \int_{G} \prod_{i \in \Lambda} dg_i U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} \int_{0}^{2\pi} \prod_{i \in \Lambda} \frac{d\theta_i}{2\pi} e^{ik_j^q(\theta_0 - \theta_x)}
\end{aligned}$$

$$\cdot e^{\beta \sum_{ij \subset \Lambda} \sum_{n=1}^{d} \operatorname{Re} e^{-ik_n(\theta_i - \theta_j)} U(g_i^{-1} b_{ij} g_j)_{nn}} \tag{3.6}$$

where $Z_v^\sigma(\beta, b, 1)$ is a normalisation factor.

At this point we can adopt the method of [23] and apply the correlation inequalities of [22] to conclude that

$$|\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} \rangle_{G,v}^\sigma (\beta, b, \mathbb{1})| \leqq Z_{U(1)}^{-1} 2 \int_0^{2\pi} \prod_{i \in \Lambda} \frac{d\theta_i}{2\pi} \cos(k_j^q(\theta_0 - \theta_x))$$
$$e^{\beta \sum_{ij \subset \Lambda} \sum_{n=1} \cos(k_n(\theta_i - \theta_j))}, \tag{3.7}$$

since $|U(g)_{ij}| \leqq 1$ for all i, j and all $g \in G$.

Now since χ^q is non trivial on $U(1)$, there exists a j_0 such that $k_{j_0}^q \neq 0$, and it is clear that the Mc Bryan–Spencer argument can be applied.

It follows that for any $\beta > 0$ there exists a $C(\beta) > 0$, such that in the infinite volume limit

$$|\langle U(g_0)_{ij_0} U(g_x^{-1})_{j_0 i} \rangle_{G,2}^\sigma (\beta, b, \mathbb{1})| \leqq \text{const.} |x|^{-C(\beta)}. \tag{3.8}$$

To conclude the proof we show that (3.8) implies bounds of the same kind on all the other two-point functions; (in fact some of them are zero).
First, in a finite region Λ we have

$$\begin{aligned}
&\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle_{G,v}^\sigma (\beta, b, 1) \\
&= Z_v^\sigma(\beta, b, 1)^{-1} \int U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} e^{\beta \sum_{ij \subset \Lambda} \operatorname{Re} \chi(g_i^{-1} b_{ij} g_j)} \prod_{j \in \Lambda} dg_j \\
&= Z_v^\sigma(\beta, b, 1)^{-1} \int U^q(g_0 g)_{ij} U^q(g^{-1} g_x^{-1})_{kl} e^{\beta \sum_{ij \subset \Lambda} \operatorname{Re} \chi(g_i^{-1} b_{ij} g_j)} \prod_{j \in \Lambda} dg_j \\
&= \sum_{m,n=1}^{d_q} \langle U^q(g_0)_{im} U^q(g_x^{-1})_{nl} \rangle_{G,v}^\sigma (\beta, b, 1) U^q(g)_{mj} U^q(g^{-1})_{kn},
\end{aligned} \tag{3.9}$$

as a consequence of the right-invariance of the Haar measure and the cyclicity of χ. Next, using the orthonormality of the functions $g \to d_q^{1/2} U^q(g)_{mj}$, $1 \leqq m, j \leqq d_q$, in $L^2(G, dg)$, we get by integrating (3.6) with respect to g that

$$\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{kl} \rangle_{G,v}^\sigma (\beta, b, 1) = \frac{\delta_{jk}}{d_q} \sum_{m=1}^{d_n} \langle U^q(g_0)_{im} U^q(g_x^{-1})_{ml} \rangle_{G,v}^\sigma (\beta, b, \mathbb{1}) \tag{3.10}$$

Thus $\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{jl} \rangle_{G,v}^\sigma (\beta, b, \mathbb{1})$ is independent of j. From this it follows that (3.8) is fulfilled for all j_0's and all i's.

Finally using the left invariance of the Haar measure, and performing the transformation $g_0 \to g g_0$, $g_i \to g_i$ for $i \neq 0$, we get that

$$\langle U^q(g_0)_{ij} U^q(g_x^{-1})_{ji} \rangle_{G,v}^\sigma (\beta, b, 1) = \sum_{m=1}^{d_q} U^q(g)_{im} \langle U^q(g_0)_{mj} U^q(g_x^{-1})_{ji} \rangle_{G,v}^\sigma (\beta, \bar{b}, \mathbb{1}) \tag{3.11}$$

where $\bar{b}_{ij} = b_{ij}$ for $i \neq 0$ and $j \neq 0$, and $\bar{b}_{0j} = g^{-1} b_{0j}$ (or $\bar{b}_{jo} = b_{j0} g$). Now since U^q is irreducible, we can find $g_1, \ldots, g_{d_q} \in G$ such that the vectors $(U^q(g_r)_{i1}, \ldots, U^q(g_r)_{id_q}) \in \mathbb{C}^{d_q}$, $r = 1, \ldots, d_q$, are linearly independent. Since we know that the left hand side of (3.11) fulfills (3.5) when $v=2$, for all b, we can conclude that $\langle U^q(g_0)_{mj} U^q(g_x^{-1})_{ji} \rangle^{\sigma}_{G,2}(\beta, b, 1)$ also obeys a bound of the form (3.5) for all m, j and i. This together with (3.8) ends the proof. □

Theorem 3.2 shows that in the $G \times G$-σ-model in an external gauge field of the form $(b, \mathbb{1})$, where the group G and character χ^q have the required properties, there is no long-range order in two dimensions. The same is wellknown for the N-vector models. In dimensions larger than 2 we have the following

Theorem 3.3. *In $v \geqq 3$ dimensions the $G \times G$-σ-model with $(b, t) = (\mathbb{1}, \mathbb{1})$ has always a phase transition with the property that for β large enough*

$$\langle U(g_0)_{ij} U(g_x^{-1})_{ji} \rangle^{\sigma}_{G,v}(\beta, 1, 1) \geqq \text{const.} > 0$$

uniformly in x, where U has character χ, the same as used in the definition of the action.

Proof. Representing $U(g)$ as a vector in a d^2-dimensional vectorspace (see Sect. 4), the proof is essentially identical to the one given for the N-vector models in [28]. □

The proof of Theorem 3.2 does not work for general external gauge fields (b, t) if G is non-abelian. One way of surmounting this difficulty would be to prove that

$$|\langle \chi^q(g_0 g_x^{-1}) \rangle^{\sigma}_{G,v}(\beta, b, t)| \leqq \langle \chi^q(g_0 g_x^{-1}) \rangle^{\sigma}_{G,v}(\beta, \mathbb{1}, \mathbb{1})$$

for all (b, t). This inequality is true if G is abelian in virtue of the inequaliy (3.1).

In Sect. 7 we argue that this is hardly the case for nonabelian G (e.g. $G = O(3)$), and we show that, in a Gaussian weak coupling limit of the $G \times G$-σ-model, clustering is definitely diminished for certain choices of (b, t). Thus, we have reasons to believe, that apart from abelian, also certain non-abelian lattice Yang–Mills theories in four dimensions may have a phase transition at some $\beta_c < \infty$. Whether quarks are still confined for $\beta > \beta_c$ is then a matter of whether there are strong cancellations of certain "random phases" of the long range order in two-point functions of three dimensional σ-models. See Sect. 6.

4. Expansion of the Expectation of Wilson Loop Observables in Terms of Random Surfaces for $G = O(n)$ or $U(n)$

This section is organized as follows: First we use our basic idea, described in Sect. 1, to write the expectation of a product of Wilson loop observables as the integral of a product of $2k$-point functions of non-linear σ-models. Then we use an expansion of these $2k$-point functions in terms of random horizontal paths joining the $2k$ points pairwise. Such an expansion can be found in [24] for random Gaussian models, and, more generally, for models whose measure is given by an integral of exponentials of (not necessarily real) quadratic forms in the fields, in [29]. We use this for the Haar measures on $O(n)$ and $U(n)$. Now it is clear that when we form the product over all those $2k$-point functions, each class of such paths determines a surface bounded by the loops, since the paths join points on the vertical sides of the loops pairwise. These surfaces get more complicated as the number of loops gets larger, and also as the loops become more general than rectangular ones. But in principle we can write down explicitly the weights of the surfaces for the two groups $O(n)$ and $U(n)$.

Our representation resembles the representation of Green's functions of the dual string in terms of (expectations over) random surfaces. Indeed, when β is very small, the expectations of products of Wilson loops satisfy the Schwinger–Dyson equations for the free dual string Euclidian Green's functions of the same loops, up to terms of order β. This suggests that, in the strong coupling regime ($\beta \ll 1$), the low lying mass spectrum of Yang–Mills theory resembles the mass spectrum (without the tachyon) of a free dual string. (In particular, we expect that it forms approximate Regge trajectories). We believe that the same conclusion ought to hold in the large n limit of $U(n)$- or $O(n)$ theories. (Our ideas are vaguely related to recent proposals of Polyakov [34]).

4.1 The Expansion

Let us first consider $G = O(n)$. We will derive an expansion of the two-point functions for the nonlinear σ-models in an external $G \times G$-gauge field by the method used in [24] and [29].

We will use the identifications of the vector-space $\mathcal{M}(n)$ of $n \times n$-matrices over $\mathbb{R}$ with $\mathbb{R}^{n^2}$ or $\mathbb{R}^n \otimes \mathbb{R}^n$ given by

$$\mathcal{M}(n) \ni g = (g_{\alpha\beta})_{\alpha,\beta=1}^n \to (g_{11}, g_{12}, \ldots, g_{1n}, g_{21}, \ldots, g_{n1} \ldots, g_{nn}) \in \mathbb{R}^{n^2}$$

and

$$\mathbb{R}^n \otimes \mathbb{R}^n \ni X \otimes Y = (X_1, \ldots, X_n) \otimes (Y_1, \ldots, Y_n) \to (X_\alpha Y_\beta)_{\alpha,\beta=1}^n \in \mathcal{M}(n).$$

It is then seen that for $a, b \in \mathcal{M}(n)$ the linear operator on $\mathcal{M}(n)$ given by

$$g \to agb$$

corresponds to $a \otimes b^t$ on $\mathbb{R}^n \otimes \mathbb{R}^n$, where b^t is the transpose of b. In particular $a \otimes b^t$ is orthogonal if a and b are.

Since furthermore,

$$\operatorname{tr}(g^t h) = \sum_{\alpha,\beta} g_{\alpha\beta} h_{\alpha\beta} = \langle g, h \rangle \text{ for } g, h \in \mathcal{M}(n)$$

where $\langle \cdot,\cdot \rangle$ is the natural inner product on $\mathbb{R}^{n^2}$, we get for $g = (g_i)_{i \in \Lambda^0}$, $b = (b_{ij})_{ij \subset \Lambda^0}$, $t = (t_{ij})_{ij \subset \Lambda^0}$, Λ^0 an arbitrary bounded subset of $\mathbb{Z}^{\nu-1}$, that

$$\sum_{ij \subset \Lambda^0} \operatorname{tr}(g_i^t b_{ij} g_j t_{ij}^t) = \sum_{ij \subset \Lambda^0} \langle g_i, b_{ij} \otimes t_{ij} g_j \rangle = (g, \Delta_{b,t} g) + 2(\nu - 1)(g, g) \tag{4.1}$$

where $\Delta_{b,t}$ is the covariant Laplacean on $\mathcal{M}(n)^{|\Lambda^0|}$ defined by

$$-(\Delta_{b,t} g)_i = \sum_{j : ij \subset \Lambda^0} (g_i - b_{ij} \otimes t_{ij} g_j) \equiv \sum_{j : ij \subset \Lambda^0} (g_i - U_{ij} g_j) \tag{4.2}$$

where we have set $U_{ij} = b_{ij} \otimes t_{ij}$, and we suppose $b_{ij}^t = b_{ji}$, $ij \subset \Lambda^0$, $(\cdot,\cdot)$ is the inner product on $\bigoplus_{i \in \Lambda^0} \mathbb{R}^{n^2}$. Furthermore we can suppose suitable boundary conditions imposed, e.g. periodic or free; (see the discussion in Sect. 1).

Next we note that the Haar measure on $O(n)$ has the representation

$$d(g_i) = \prod_{1 \leqq \alpha \leqq \beta \leqq n} \delta((g_i^t g_i)_{\alpha\beta} - \delta_{\alpha\beta}) \prod_{\gamma,\delta=1}^n d(g_i)_{\gamma\delta}$$

$$= \prod_{1 \leqq \alpha \leqq \beta \leqq n} \int_{-\infty}^{\infty} (2\pi)^{-1} e^{-i(\lambda_i)_{\alpha\beta}((g_i^t g_i)_{\alpha\beta} - \delta_{\alpha\beta})} d(\lambda_i)_{\alpha\beta} \prod_{\gamma,\delta=1}^{n} d(g_i)_{\gamma\delta}$$

$$= (2\pi)^{-n(n+1)/2} \int e^{-i\,\mathrm{tr}(\lambda_i^t g_i^t g_i - \lambda_i^t)} \prod_{1 \leqq \alpha \leqq \beta \leqq n} d(\lambda_i)_{\alpha\beta} \prod_{\gamma,\delta=1}^{n} d(g_i)_{\gamma\delta}$$

$$= (2\pi)^{-n(n+1)/2} \int e^{-i\langle g_i, 1 \otimes \lambda_i g_i\rangle + i\,\mathrm{tr}\,\lambda_i} \prod_{1 \leqq \alpha \leqq \beta \leqq n} d(\lambda_i)_{\alpha\beta} \prod_{\gamma,\delta=1}^{n} d(g_i)_{\gamma\delta}$$

$$= (2\pi)^{-n(n+1)/2} \int e^{-i\langle g_i, M_i g_i\rangle + i(1/n)\,\mathrm{tr}\,M_i} dM_i dg_i \tag{4.3}$$

where $dM_i = \prod_{1 \leqq \alpha \leqq \beta \leqq n} d(\lambda_i)_{\alpha\beta}$ is a measure on the set of matrices $M_i = 1 \otimes \lambda_i$ over $\mathbb{R}$ with $\lambda_i = (\lambda_{i,\alpha\beta})_{\alpha,\beta=1}^n$ and $\lambda_{i,\alpha\beta} = \lambda_{i,\beta\alpha}$, $1 \leqq \alpha, \beta \leqq n$, and dg_i is the Lebesgue measure on $\mathbb{R}^{n^2}$.

Using these remarks and the fact that the last term in (4.1) is a constant $2(v-1)n|\Lambda^0|$ if the $g_i's$ are in O(n), we can write

$$\langle (g_0)_{\alpha\beta}(g_x)_{\gamma\delta}\rangle^\sigma_{v-1}(b,t) =$$
$$= (Z^\sigma(b,t))^{-1} \int (g_0)_{\alpha\beta}(g_x)_{\gamma\delta}(2\pi)^{-|\Lambda^0|n(n+1)/2}$$
$$\cdot e^{-2(v-1)n|\Lambda^0|} e^{(1/2)(g,[\beta\Delta_{b,t} - 2iM]g) + i(1/n)\,\mathrm{tr}\,M} dg\,dM$$
$$= (Z^\sigma(b,t))^{-1} \beta^{(1/2)n|\Lambda^0|-1} e^{-2(v-1)n|\Lambda^0|} 2\pi^{-n(n+1)/2|\Lambda^0|} \int (g_0)_{\alpha\beta}(g_x)_{\gamma\delta}$$
$$\cdot e^{(1/2)(g,[\Delta_{b,t} - 2iM]g) + i(\beta/n)\,\mathrm{tr}\,M} dg\,dM$$
$$= (\tilde{Z}^\sigma(b,t))^{-1} \beta^{-1} \int [-\Delta_{b,t} + 2iM]^{-1}_{\alpha\beta,\gamma\delta}(0,x) e^{(1/2)(g,[\Delta_{b,t} - 2iM]g) + i(\beta/n)\,\mathrm{tr}\,M} dg dM \tag{4.4}$$

where we have set $dg = \prod_{i \in \Lambda^0} dg_i$, $M = \bigoplus_{i \in \Lambda^0} M_i$, $dM = \prod_{i \in \Lambda^0} dM_i$ and

$$\tilde{Z}^\sigma(b,t) = (e^{2(v-1)n}(2\pi)^{n(n+1)/2}\beta^{(n/2)})^{|\Lambda^0|} Z^\sigma(b,t)$$
$$= \int e^{(1/2)(g,[\Delta_{b,t} - 2iM]g) + i(\beta/n)\,\mathrm{tr}\,M} dg\,dM.$$

Expanding $(-\Delta_{b,t} + 2iM)^{-1}$ in a Neumann series and using the definition of $\Delta_{b,t}$ (4.2) (see also [24]) we have that

$$(-\Delta_{b,t} + 2iM)^{-1}_{\alpha\beta,\gamma\delta}(0,x) = \sum_{\substack{\omega \\ \omega_i = 0 \\ \omega_f = x}} (2(v-1) + 2iM_0)^{-1} \prod_{i=1}^{|\omega|}$$
$$\cdot U_{\omega(i-1)\omega(i)}(2(v-1) + 2iM_{\omega(i)})^{-1} \tag{4.5}$$

where the sum is over all paths $\omega : \{1, \ldots, |\omega|\} \to \Lambda^0$ for which $\omega(0) \equiv \omega_i = 0$ and $\omega(|\omega|) \equiv \omega_f = x$, and $\omega(i)$ and $\omega(i+1)$ are nearest neighbors for all $i = 0, \ldots, |\omega| - 1$. $|\omega|$ is the length of the path ω. Furthermore let $U_{\omega(-1)\omega(0)} = \mathbb{1}$ for all paths ω.

Using now the representation

$$(2(v-1) + 2iM_j)^{-n} = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t(2(v-1)+2iM_j)} dt$$
$$= (2(v-1))^{-n} \int_0^\infty dt \frac{t^{n-1}}{(n-1)!} e^{-t} e^{-it/(v-1)M_j} \tag{4.6}$$

we get from (4.5)

$$(-\Delta_{b,t}+2iM)^{-1}_{\alpha\beta,\gamma\delta}(0,x) = \sum_{\substack{\omega \\ \omega_i=0 \\ \omega_f=x}} (2(v-1))^{-(|\omega|+1)} \left\{ \prod_{i=0}^{|\omega|} U_{\omega(i-1)\omega(i)} \int_0^\infty dt_i e^{-t_i} e^{-it/(v-1)M_{\omega(i)}} \right\}_{\alpha\beta,\gamma\delta} \tag{4.7}$$

From this we get the desired expansion, namely

$$\langle (g_0)_{\alpha\beta}(g_x)_{\gamma\delta}\rangle^{\sigma}_{v-1}(b,t) = \tilde{Z}^{\sigma}(b,t)^{-1}\beta^{-1}\sum_{\omega}(2(v-1))^{-(|\omega|+1)}\int e^{(1/2)(g,[\Delta_{b,t}-2iM]g)+i(\beta/n)\operatorname{tr}M}\cdot \left\{ \prod_{i=0}^{|\omega|} U_{\omega(i-1)\omega(i)} \int_0^\infty dt_i e^{-t_i} e^{-it_i/(v-1)M_{\omega(i)}} \right\}_{\alpha\beta,\gamma\delta} dg\,dM. \tag{4.8}$$

If this equation is inserted into (1.34) the following expression for the expectation of the rectangular Wilson loop observable depicted in fig. 2, with $\chi^q = \chi =$ the character of the fundamental representation of O(n), results:

$$\langle W(C)\rangle^{YM}_{v} = \cdot\zeta^{-1}\sum_{\underset{\sim}{m},\underset{\sim}{n}}\sum_{\underset{\sim}{\omega}}\int\prod_{t}\{d\mu_{v-1}(g^h(t))\zeta^{\sigma}(g^h(t),g^h(t+1))\}B_{m_0n_0}T_{n_Nm_N} \tag{4.9}$$

$$\cdot\prod_{u=0}^{T-1}\int\Bigg\{(2(v-1))^{-|\omega_u|-1}\tilde{Z}^{\sigma}(g^h(u),g^h(u+1))^{-1}\beta^{-1} \cdot\int e^{(1/2)(g^v(u),[\Delta_{g^h(u),g^h(u+1)}-2iM(u)]g^v(u))}e^{i(\beta/n)\operatorname{tr}M(u)} \cdot\left\{\prod_{i=0}^{|\omega_u|}U_{\omega_u(i-1)\omega_u(i)}\int_0^\infty du_i e^{-u_i}e^{-iu_i/(v-1)M(u)_{\omega_u(i)}}\right\}_{m_{u+1}m_u,n_un_{u+1}} dg^v(u)dM(u)\Bigg\} \tag{4.10}$$

where $\sum_{\omega}$ denotes the sum over all sets $\{\omega_0,\ldots,\omega_{T-1}\}$ of paths, where ω_u is a path in Λ_u, starting at the link $((0,u)(0,u+1))$ and ending at the link $((j,u),(j,u+1))$ for all $u=0,\ldots,T-1$.

Each such set of paths $\omega_0,\ldots,\omega_{T-1}$ determines a surface consisting of vertical plaquettes and bounded by C, and whose intersection with Λ_u is ω_u for each $u=0,\ldots,T-1$, so (4.10) gives an expression for $\langle W(C)\rangle^{YM}_{v}$ as a sum over random surfaces bounded by C. We will comment more on this in Sect. 6.

In the next section we show that the same idea works for $G=\mathrm{SU}(2)$, and gives a rather large range of coupling constants for which the expectation of the Wilson loop observable has area decay.

We close this section with some remarks.

First (4.10) can be generalized to the case of a product of Wilson loop observables. The sum is then over a class of surfaces bounded by those loops.

Second, it is clear that for $G=U(n)$ an analogous expansion can be obtained by identifying $U(n)$ with a submanifold of $\mathbb{C}^{n^2}$ or $\mathbb{C}^n\otimes\mathbb{C}^n$, which is an intersection of quadratic submanifolds, and using a representation of the Haar measure on $U(n)$ analogous to (4.3).

And finally it seems that the integration over the matrix-variables $M(u)$ in (4.10) is a difficult task. The reason that we can perform calculations in the SU(2)-theory is, that the matrices in that case reduce to real numbers.

5. The $SU(2)$ Theory

In this section we choose $G = \mathrm{SU}(2)$ and $\chi^q = \chi$, the character of the fundamental representation of SU(2). Note that χ is real.

We will make use of the homeomorphism $\varphi : S^3 \to \mathrm{SU}(2)$ (S^3 is the 3-sphere of radius 1) defined by

$$\varphi(S^0, S^1, S^2, S^3) = \begin{pmatrix} S^0 + iS^3 & -S^1 + iS^2 \\ S^1 + iS^2 & S^0 - iS^3 \end{pmatrix}, \tag{5.1}$$

to carry out a program analogous to that outlined in Sect. 4. In this case however we have the advantage that φ^{-1} carries the Haar measure on SU(2) to the uniform measure on S^3, which considerably simplifies the calculations. This is due to the fact that the usual five constraints used to specify a SU(2)-matrix from $\mathscr{M}_{\mathbb{C}}(2)$ have been replaced by one single constraint, by a suitable parametrization of SU(2).

We first note that

$$\vec{S}_1 \cdot \vec{S}_2 = \mathrm{tr}(\varphi(\vec{S}_1)^{-1} \varphi(\vec{S}_2)), \quad \forall \vec{S}_1, \vec{S}_2 \in S^3.$$

Furthermore $\chi(g_1^{-1} g_2)$ is invariant under the transformation $g_1 \to h g_1 k^{-1}$ and $g_2 \to h g_2 k^{-1}$, $h, k \in \mathrm{SU}(2)$, so we have

$$\varphi^{-1}(hgk^{-1}) = \mathrm{O}(h, k)\varphi^{-1}(g), \quad \forall g \in \mathrm{SU}(2), \tag{5.2}$$

where $\mathrm{O}(h, k)$ is an orthogonal 4×4-matrix.

Now let Λ^0 be a rectangular region in $\mathbb{Z}^{\nu-1}$, and let $b = (b_{ij})_{ij \subset \Lambda^0}$, $t = (t_{ij})_{ij \subset \Lambda^0}$, with $b_{ij} = b_{ji}^{-1} \in \mathrm{SU}(2)$ and $t_{ij} = t_{ji}^{-1} \in \mathrm{SU}(2)$ for $ij \subset \Lambda^0$. Define the covariant Laplacean $\Delta_{b,t}$ on $(\mathbb{R}^4)^{|\Lambda^0|}$ by

$$-(\Delta_{b,t} X)_i = \sum_{j : ij \subset \Lambda^0} (X_i - \mathrm{O}(b_{ij}, t_{ij}) X_j) \tag{5.3}$$

for $X = (X_i)_{i \in \Lambda^0} \in (\mathbb{R}^4)^{|\Lambda^0|}$.

Then for $g = (g_i)_{i \in \Lambda^0} \in \mathrm{SU}(2)^{|\Lambda^0|}$ and $S = (\vec{S}_i)_{i \in \Lambda^0} = (\varphi^{-1}(g_i))_{i \in \Lambda^0}$ we have by (5.2)

$$\sum_{ij \subset \Lambda^0} \mathrm{tr}(g_i^{-1} b_{ij} g_j t_{ij}^{-1}) = \sum_{ij \subset \Lambda^0} \langle \vec{S}_i, \mathrm{O}(b_{ij}, t_{ij}) \vec{S}_j \rangle = (S, \Delta_{b,t} S) + 2(\nu - 1)|\Lambda^0|.$$

In this section we choose periodic boundary conditions. We note the representation

$$\delta(|\vec{S}_j|^2 - 1)\, d\vec{S}_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda_j(|\vec{S}_j|^2 - 1)}\, d\lambda_j\, d\vec{S}_j$$

of the uniform measure on S^3.

We can now proceed in complete analogy to Sect. 4, and expand the two-point function $\langle S_0^\alpha S_x^\beta \rangle_{\nu-1}^\sigma(b,t)$ in terms of random paths. By using (4.7) and defining

$$\tilde{Z}^\sigma(b,t) = \int e^{(1/2)(S,\Delta_{b,t} S)} \prod_{j\in\Lambda^0} \delta(|\vec{S}_j|^2 - \beta) d\vec{S}_j \tag{5.4}$$

and

$$\tilde{Z}^\sigma(b,t,\omega) = \int e^{(1/2)(S,\Delta_{b,t} S)} \prod_{j\in\Lambda^0} \int du_j \frac{u_j^{n_j(\omega)-1}}{(n_j(\omega)-1)!} e^{-u_j} \delta\left(|\vec{S}_j|^2 - \beta + \frac{u_j}{\nu-1}\right) d\vec{S}_j \tag{5.5}$$

we get

$$\begin{aligned}&\langle S_0^\alpha S_x^\beta \rangle_{\nu-1}^\sigma(b,t)\\ &= \beta^{-1} \sum_{\substack{\omega\\ \omega_i=0\\ \omega_f=x}} (2(\nu-1))^{-|\omega|-1} \frac{\tilde{Z}^\sigma(b,t,\omega)}{\tilde{Z}^\sigma(b,t)} \left(\prod_{i=0}^{|\omega|-1} \mathrm{O}(b_{\omega(i)\omega(i+1)}, t_{\omega(i)\omega(i+1)})\right)_{\alpha\beta}\end{aligned} \tag{5.6}$$

where $dS = \prod_{j\in\Lambda^0} d\vec{S}_j$, $d\lambda = \prod_{j\in\Lambda^0} d\lambda_j$, and $n_j(\omega)$ is the number of times that ω hits the site j. We have used that $\sum_{j\in\Lambda^0} n_j(\omega) = |\omega| + 1$.

The matrix element that enters in (5.6) is bounded in modulus by 1 so from (5.6) we get the estimate

$$|\langle S_0^\alpha S_x^\beta \rangle_{\nu-1}^\sigma(b,t)| \leqq \beta^{-1} \sum_{\substack{\omega\\ \omega_i=0\\ \omega_f=x}} (2(\nu-1))^{-|\omega|-1} \frac{\tilde{Z}^\sigma(b,t,\omega)}{\tilde{Z}^\sigma(b,t)}. \tag{5.7}$$

Let now C be a rectangular loop inside the rectangular region $\Lambda \subset \mathbb{Z}^\nu$ as described in the last part of Sect. 1.3, and let us use the same notations. By (5.1) the term $U^q(g_0)^{-1}_{m_t m_{t+1}} U^q(g_x)_{n_t n_{t+1}}$ appearing in (1.34) is just a sum of four terms of the form $i^r S_0^\alpha S_x^\beta (0 \leqq r, \alpha, \beta \leqq 3)$. Hence by using (5.7) and (1.34), and remembering the remarks following (1.34), we find that

$$\begin{aligned}&|\langle W^q(C)\rangle_\nu^{YM}|\\ &\leqq \zeta^{-1} 4^T \int \Bigg[\prod_t \{d\mu_{\nu-1}(g^h(t)) \zeta^\sigma(g^h(t), g^h(t+1))\}\\ &\cdot \prod_{t=0}^{T-1} \Bigg\{ \beta^{-1} \sum_{\substack{\omega_t\\ (\omega_t)_i\\ (\omega_t)_f}} (2(\nu-1))^{-|\omega_t|-1} \frac{\tilde{Z}^\sigma(g^h(t), g^h(t+1), \omega^t)}{\tilde{Z}^\sigma(g^h(t), g^h(t+1))}\Bigg\}\Bigg]\\ &= \zeta^{-1} 4^T \beta^{-T} \sum_{\omega_0} \dots \sum_{\omega_{T-1}} \Bigg[(2(\nu-1))^{-\sum_{t=0}^{T-1}(|\omega_t|+1)}\\ &\cdot \int \prod_t \{d\mu_{\nu-1}(g^h(t)) \zeta^\sigma(g^h(t), g^h(t+1))\} \prod_{t=0}^{T-1} \frac{\tilde{Z}^\sigma(g^h(t), g^h(t+1), \omega^t)}{\tilde{Z}^\sigma(g^h(t), g^h(t+1))}\Bigg]\\ &= Z^{-1} 4^T \beta^{-T} \sum_{\omega_0} \dots \sum_{\omega_{T-1}} \Bigg[(2(\nu-1))^{-\sum(|\omega_t|+1)}\end{aligned}$$

$$\left. \cdot \int \prod_t \{\beta^{-|\Lambda_t|} \tilde{Z}^\sigma(g^h(t), g^h(t+1), \omega^t) d\mu_{\nu-1}(g^h(t))\} \right] \tag{5.8}$$

where

$$\begin{aligned} Z &= \int \prod_t \{\beta^{-|\Lambda_t|} \tilde{Z}^\sigma(g^h(t), g^h(t+1)) d\mu_{\nu-1}(g^h(t))\} \\ &= \int \prod_t \{d\mu_{\nu-1}(g^h(t)) \int e^{(1/2)\,\beta\,(S_t,\, \Delta_{g^h(t),\, g^h(t+1)} S_t)} \prod_{i \in \Lambda_t} \delta(|(\vec{S}_t)_i|^2 - 1) d(\vec{S}_t)_i\} \end{aligned}$$

It is now easily checked that the multilinear form

$$(f_{t,i}) \to \int \prod_t \{e^{(1/2)(\vec{S}_t,\, \Delta_{g^h(t),\, g^h(t+1)} \vec{S}_t)} \prod_{j \in \Lambda_t} f_{t,j}((\vec{S}_t)_j) d(\vec{S}_t)_j d\mu_{\nu-1}(g^h(t))\}$$

is reflection positive with respect to reflections in pairs of planes through (or between) the sites of $\Lambda \subset \mathbb{Z}^\nu$, so that we can apply chessboard estimates (cf. [4]).

This combined with a thermodynamic estimate shows that, *for* $\beta < \dfrac{1}{2(\nu-1)}$, *the expectation of the Wilson loop observable has area decay.*

Details of these calculations can be found in the appendix to Sect. 5.

6. Basic Mechanisms for Confinement

In this section we distil out of the scheme of Sects. 4 and 5 the basic mechanisms that might lead to permanent confinement of static quarks. The gauge group G is one of the groups SU(2), $U(n)$, O(n), $n = 1, 2, 3, \ldots$, $U(1)$ and O(1) $= \mathbb{Z}(2)$ included. Our discussion is based on eqs. (4.8), resp. (5.4)–(5.6) and (1.34). For simplicity we choose $G = \mathrm{SU}(2)$ (or $U(1)$) and $U^q = U$ to be the fundamental representation of SU(2). Eqs. (4.8), resp. (5.4)–(5.6) then give

$$\langle U^q(g_0)_{mn} U^q(g_x^{-1})_{kl} \rangle^\sigma_{\nu-1}(b, t) = \sum_{\substack{\omega \\ \omega_i = 0 \\ \omega_f = x}} \gamma_\nu^{-|\omega|-1} F(b, t\,|\,\omega) \mathrm{O}(b, t\,|\,\omega)_{mn,kl}, \tag{6.1}$$

with

$$\gamma_\nu = 2(\nu - 1), \quad F(b, t\,|\,\omega) = \frac{\tilde{Z}^\sigma(b, t, \omega)}{\tilde{Z}^\sigma(b, t)}, \tag{6.2}$$

and $\mathrm{O}(b, t\,|\,\omega) = \prod_{s=0}^{|\omega|-1} \mathrm{O}(b_{\omega(s)\omega(s+1)}, t_{\omega(s)\omega(s+1)})$,

where

$$\mathrm{O}(g, h) = U(g)^L U(h^{-1})^R \tag{6.3}$$

$U(g)^L$ is left multiplication by $U(g)$ on $V_U^{\otimes 2}$, $U(h^{-1})^R$ is right multiplication by $U(h^{-1})$ on $V_U^{\otimes 2}$, where $V_U^{\otimes 2}(\supset U(G))$ is the space of all matrices on the vector space V_U that carries the representation U of G. Here g and h are elements in G, and U is the representation of G with character χ which for simplicity we have chosen to be the fundamental representation of G and $G = \mathrm{SU}(2)$ or $U(1)$. Our methods work in general, but when $G = \mathrm{O}(n)$ or $U(n)$, $n \geqq 2$, the factors F and O on the r.s. of (6.1) are tensors which must be contracted. See (4.8).

Clearly, $O(b, t|\omega)$ is a $U(G) \times U(G)$-valued *random phase*.

If we now insert (6.1) into (1.34) we obtain the following representation of $\langle W^q(C)\rangle_v^{YM}$. (For simplicity we choose C to be a rectangular loop in a coordinate plane containing the vertical (x^v-) axis with sides of length $L = |x|$, resp. T). Then

$$\langle W^q(C)\rangle_v^{YM} = \sum_{\underset{\sim}{m},\underset{\sim}{n}} \left(\sum_{S:\partial S = C} \left\langle \prod_{u=0}^{T-1} \left\{ \gamma_v^{-|\omega_u^S|-1} \cdot F(b(u), t(u)|\underset{-u}{\omega^S})O(b(u), t(u)|\underset{-u}{\omega^S})_{m_u m_{u+1}, n_u n_{u+1}} \right\} \cdot B_{mono} T_{n_T m_T} \right\rangle_v^{YM} \right), \tag{6.4}$$

where S belongs to the class of all random surfaces bounded by the loop C("$\partial S = C$") formed out of vertical plaquettes, and, given S, ω_u^S is the path of nearest neighbor vertical links obtained by intersecting S with the slice $\{u \leqq x^v \leqq u+1\}$, (in other words, $S \simeq \{\omega_u^S : 0 \leqq u \leqq T-1\}$), $\underset{-u}{\omega^S}$ is the trace of ω_u^S in the $\{x^v = u\}$ hyperplane, and $\bar{\omega}_u^S$ the one in the $\{x^v = u+1\}$ hyperplane. We now introduce an a priori measure, ρ_v, on the set of all random surfaces bounded by C, by setting

$$\rho_v(S|C) = \prod_{u=0}^{T-1} \gamma_v^{-|\omega_u^S|-1}, \tag{6.5}$$

$(\gamma_v = 2(v-1))$.

Let $C_u(S)$ be the *horizontal loop* in the $\{x^v = u\}$ hyperplane obtained by composing $\bar{\omega}_{u-1}^S$ with $\underset{-u}{\omega^S}$ so as to form a closed loop; $\bar{\omega}_{-1}^S \equiv C \cap \{x^v = 0\}$, $\underset{-T}{\omega^S} \equiv C \cap \{x^v = T\}$. If we now combine (6.2)–(6.5) we readily arrive at the following nice identity.

$$\langle W^q(C)\rangle_v^{YM} = \sum_{S:\partial S = C} \rho_v(S|C) \left\langle \left\{ \prod_{u=0}^{T=1} F(g^h(u), g^h(u+1)^{-1}|\underset{-u}{\omega^S}) \cdot \chi(g_{C_u(S)}) \right\} \chi(g_{C_T(S)}) \right\rangle_v^{YM} \tag{6.6}$$

[Notice that

$$\sum_{i,j} \left(\prod_{xy \subset \bar{\omega}_u^S} U(t_{xy}(u)^{-1}) \right)_{ij} \left(\prod_{x'y' \subset \underset{-}{\omega}_{u+1}^S} U(b_{x'y'}(u+1)) \right)_{ji} = \chi(g_{C_u(S)}),$$

and use (6.3), (6.4).]

Equation (6.6) is a rather powerful and suggestive identity which we recommend to the reader's attention. The previous results of this section (see also the estimates presented in the appendix) show that in the average (with respect to b, t).

$$0 < F(b, t|\omega) \leqq \xi(\beta)^{|x|}, \text{ for } G = SU(2) \text{ or } U(1), \tag{6.7}$$

so that, by (6.1),

$$|\langle U^q(g_0)_{mn} U^q(g_x^{-1})_{kl} \rangle_{v-1}^{\sigma}(b, t)| \leqq O(\exp[|x| \ln \xi(\beta)]), \tag{6.8}$$

for some $\xi(\beta)$ which is *strictly less than* 1, *provided U^q is a quark representation and β is sufficiently small* ($\beta \lesssim O(v^{-1})$). In this case it is enough to bound $|\chi(g_{C_u(S)})|$ by $\chi(\mathbb{1}) = \dim U^q$, for all $u = 0, \ldots, T$, because (6.5)–(6.8) already yield confinement of static quarks by a linear potential ($\geqq -\ln \xi(\beta)|x|$). However, we know from Theorem 3.2 that, for $b = t \equiv \mathbb{1}$, $v \geqq 4$ and β large enough, $F(\mathbb{1}, \mathbb{1}\,|\,\omega)$ cannot satisfy (6.7) with $\xi(\beta) < 1$, since (6.8) is false, namely

$$\langle \operatorname{tr}(U^q(g_0) \cdot U^q(g_x^{-1})) \rangle_{v-1}^{\sigma}(\mathbb{1}, \mathbb{1}) \to \text{const.} > 0, \tag{6.9}$$

as $|x| \to \infty$, no matter whether U^q is a quark- or a particle representation. We have reasons to expect that, for a class of external gauge fields (b, t) of positive measure,

$$\sum_{\substack{\omega \\ \omega_i = 0 \\ \omega_f = x}} \gamma_v^{-|\omega|-1} F(b, t\,|\,\omega) \to M(b, t) > 0, \tag{6.10}$$

as $|x| \to \infty$. (See the discussion in Sect. 7).

If we replace the traces of the random phases by $\chi(\mathbb{1})$ we obtain the upper bound

$$|\langle W(C) \rangle_v^{YM}| \lesssim \chi(\mathbb{1})^{T+1} \left\langle \prod_{u=0}^{T=1} M(b(u), t(u)) \right\rangle_v^{YM},$$

for $T \gg |x| \to \infty$, which does not prove more than perimeter decay, i.e. does not imply confinment. Therefore, for β large and $v \geqq 4$, the only mechanism that might give rise to permanent confinement of static quarks appears to be a *cancellation of the* (*traces over*) *random phases* when taking their expectations.

Such cancellations of random phases, i.e. sharp upper bounds on their expectation value in the Yang–Mills measure, are rather subtle and lie beyond our present methods.

We emphasize however that we can obtain improved upper bounds on $|\langle W(C) \rangle_v^{YM}|$ by taking into account the factor $\prod_{u=0}^{T} \chi(g_{C_u(S)})$ in the expectation on the r.s. of the basic identity (6.6): We first apply a chess board estimate (in the x^v-direction, with reflections in planes between lattice planes) to the r.s. of (6.6) and then refined "thermodynamic" estimates to bound the expression resulting from the chessboard estimate. The general ideas of this method are as in [4, 24, 29] and Sect. 5. The results that emerge are substantially better than the ones of Sect. 5.

We now summarize those results. Detailed statements and (the somewhat lengthy) proofs will appear elsewhere.

By (6.6),

$$|\langle W^q(C) \rangle_v^{YM}| \leqq \sum_{S:\partial S = C} \rho_v(SC) \left| \left\langle \prod_{u=0}^{T-1} F(g^h(u), g^h(u+1)^{-1}\,|\,\omega_u^S) \cdot \prod_{u=0}^{T} \chi(g_{C_u(S)}) \right\rangle_v^{YM} \right|. \tag{6.11}$$

From the chessboard estimate (in the x^v-direction, with reflections at $x^v = \text{const.}$ hyperplanes *between* lattice planes) and slightly subtle upper bounds on $F(g, (g')^{-1}\,|\,\omega)$, viewed as integral kernel of a quadratic form, it follows that

$$\left| \left\langle \prod_{u=0}^{T-1} F(g^h(u), g^h(u+1)^{-1}\,|\,\omega_u^S) \prod_{u=0}^{T} \chi(g_{C_u(S)}) \right\rangle_v^{YM} \right| \leqq \prod_{u=0}^{T-1} \alpha(\omega_u^S) \prod_{u=0}^{T} \mu(C_u(S)), \tag{6.12}$$

where

$$\alpha(\omega) = \prod_{j \in \mathbb{Z}^{v-1}} \frac{[(v-1)\beta]^{n_j(\omega)}}{(n_j(\omega)+1)!}, \tag{6.13}$$

and

$$\mu(C') = \lim_{N \to \infty} \left[\left\langle \prod_{m=0}^{N} \chi(g_{C'(2m)}) \overline{\chi(g_{C'(2m+1)})} \right\rangle_{v,N}^{YM} \right]^{1/2(N+1)} \tag{6.14}$$

where $C' = C'(0)$ is a closed loop in the lattice hyperplane at $x^v = 0$, and $C'(n)$ is the translate of C' in the x^v-direction to the plane at $x^v = n$; $\langle - \rangle_{v,N}^{YM}$ is the Yang–Mills expectation with periodic boundary conditions at $x^v = 0, 2N+2$.

In order to get explicit estimates on $\mu(C')$ one can apply the $\mathbb{Z}(2)$ domination inequality of refs. [23, 19]. For large β one then applies a duality (Fourier) transformation to the resulting expectation in the $\mathbb{Z}(2)$ theory. This reduces the problem to estimating an expectation in a high temperature $\mathbb{Z}(2)$ model which one achieves by a high temperature expansion; see [40]. As a result one finds

$$\mu(C') \leqq e^{-K(\beta)|C'|}, \quad K(\beta) > 0, \tag{6.15}$$

($|C'|$ = length of C'), first for large $\beta < \infty$ and consequently for arbitrary β, by the Griffiths inequality.

More detailed results and proofs of (6.11)–(6.15) will be presented elsewhere. We summarize our estimates in

Theorem 6.1.

$$|\langle W^q(C) \rangle_v^{YM}| \leqq \sum_{S : \partial S = C} \rho_v''(S|C), \tag{6.16}$$

where

$$\rho_v''(S|C) = \rho_v(S|C) \prod_{u=0}^{T-1} \alpha(\underline{\omega}_u^S) \prod_{u=0}^{T} \mu(C_u(S)) \tag{6.17}$$

with α *and* μ *given by* (6.13), (6.14), *resp.*

Remarks. The convergence factor $\prod_{u=0}^{T} \mu(C_u(S))$ is a manifestation of the mechanism of cancellation of random phases.

In the estimates summarized in (6.12)–(6.17) the two mechanisms, the "clustering mechanism" (6.7), (6.8), resp. the cancellation of random phases (6.14), (6.15), conspire.

Our estimates are certainly not optimal, but we expect that the way in which the statistical weight of the product of random phases, $\prod_{u=0}^{T} \chi(g_{C_u(S)})$, is estimated by (6.15) is qualitatively correct for large β, not only for $G = U(1)$, but also for $G = SU(2)$.

It is of interest to test the strength of our methods in various situations, assuming further hypothetical estimates if necessary. One may, e.g. suppose that

(I) $|\langle W(C)\rangle_\nu^{YM}| \leqq \sum_{S:\partial S=C} \rho''_\nu(S|C)$, with

$$\rho''_\nu(S|C) \leqq \rho'_\nu(S|C) \prod_{u=0}^{T} e^{-K|C_u(S)|},$$

where $\rho'_\nu(S|C) \underset{e.g.}{\leqq} \rho_\nu(S|C) \prod_{u=0}^{T-1} \xi^{|\omega_u^S|}$, with $\xi = \xi(\beta) < \sqrt{2(\nu-1)}$ and $K = K(\xi)$ large enough.

(II) One may study a non-relativistic limit (velocity of light $c \gg 1$) of the lattice SU(2) theory. For $c \gg 1$ and β small one finds estimates on $\rho''_\nu(S|C)$ which reveal an intimate connection between the SU(2) theory and a non-relativistic open-string model.

A systematic study of upper bounds on $\rho''_\nu(S|C)$, including (I) and (II), will be initiated elsewhere. (The relevant tool from probability theory is the theory of interacting random walks, resp.—in a formal continuum limit—interacting Brownian paths).

We do not want to end this section without pointing out a drawback of the methods of this section: The difference between the four-dimensional $U(1)$- and the four-dimensional SU(2) lattice gauge theory merely appears as a quantitative one; (e.g. $\alpha_{U(1)}(\omega) > \alpha_{SU(2)}(\omega)$). In contrast, the methods outlined in Sect. 7 *do* point to a qualitative difference between abelian and non-abelian theories.

Although our present estimates for the four-dimensional SU(2) model are far from optimal, one may speculate that, indeed, the "clustering mechanism" (6.7), (6.8) breaks down, in the sense that (6.10) becomes true, at some finite value of β_0, and that for $\beta \gg \beta_0$ the expectation of the Wilson loop ceases to have area decay, even in the SU(2) theory. We emphasize that this would *not* necessarily imply the appearance of coloured physical states in a SU(2) gauge theory with quarks in the spin 1/2 representation, because the colour of sufficiently *light* quarks could be screened completely.

7. Continuous "Time" Formalism and Gaussian σ-Models[3]

This section is somewhat expository. No detailed estimates are presented. A few important technical points are treated in an appendix to Sect. 7.

Throughout this section, G is a compact Lie group, in particular $G = U(1)$ or SU(2). We propose to study the continuous imaginary-time formalism ($x^\nu \in \mathbb{R}$, continuous) for lattice Yang–Mills fields with gauge group G. In the limit of a continuous imaginary time coordinate, ν-dimensional Yang–Mills theory turns out to be related to Gaussian σ-models with fields taking values in the Lie algebra $\mathscr{G}$ of G in an external gauge field $(b, t) \in G \times G$, on a $(\nu-1)$-dimensional lattice. For *quantum theory*, this is a correct and very useful approach, whereas the somewhat complementary approach (continuous space, discrete imaginary time) outlined in

3 Some of the result reported here have been obtained in discussions with E. Seiler.

Sect. 1.2 is problematic. (For $\nu \geqq 3$, it appears to impose unsuitable renormalization conditions, and, moreover, non-perturbative renormalization of $(\nu-1)$-dimensional σ-models in the continuum limit has not yet been carried out for $\nu - 1 \geqq 2$. Notice that the limits, "lattice spacing in time direction" $\searrow 0$, and "lattice spacing in space direction" $\searrow 0$, do not appear to commute for $\nu \geqq 3$; we prefer to take the first limit first).

We start with a lattice $\alpha = \varepsilon\mathbb{Z} \times \delta\mathbb{Z}^{\nu-1}$, $\varepsilon\mathbb{Z} = \{u = \varepsilon n : n \in \mathbb{Z}\}$, and $\delta\mathbb{Z}^{\nu-1} = \{x = \delta y : y \in \mathbb{Z}^{\nu-1}\}$. (We follow the notations of Sects. 1.2 and 1.3).

The Yang–Mills action is now given by

$$A_\nu^{YM} = -\sum_{u\in\varepsilon\mathbb{Z}}\left\{\varepsilon \sum_{p\subset\delta\mathbb{Z}^{\nu-1}} \delta^{\nu-5}\,\mathrm{Re}\,\chi(g^h(u)_{\partial p}) + \varepsilon^{-1}\sum_{ij\subset\delta\mathbb{Z}^{\nu-1}} \delta^{\nu-3}\,\mathrm{Re}\,\chi(g_i(u)b_{ij}(u)g_j(u)^{-1}t_{ij}(u)^{-1})\right\}, \tag{7.1}$$

where $g_i(u) = g_{(i,u)\,(i,u+\varepsilon)}$, $b_{ij}(u) = g_{(i,u)\,(j,u)}$, and $t_{ij}(u) = b_{ij}(u+\varepsilon) = g_{(i,u+\varepsilon)\,(j,u+\varepsilon)}$. See Sect. 1.3, (1.17). (The spatial cutoff, Λ, will be suppressed in our notation). We set

$$A_{\nu-1}^{\sigma}(b,t) = -\sum_{ij\subset\delta\mathbb{Z}^{\nu-1}} \delta^{\nu-3}\,\mathrm{Re}\,\chi(g_i b_{ij} g_j^{-1} t_{ij}^{-1}) \tag{7.2}$$

The Yang–Mills vacuum expectation corresponding to the action (7.1) is given by

$$\langle - \rangle_\nu^{YM}(\beta|\varepsilon,\delta) = \zeta^{-1}\prod_{u\in\varepsilon\mathbb{Z}}\{\langle - \rangle_{\nu-1}^{\sigma}(\beta|\varepsilon,\delta|b(u),t(u)) \cdot \zeta^{\sigma}(\beta|\varepsilon,\delta|b(u),t(u))\,d\mu_{\nu-1}^{\beta,\delta}(b(u))\}, \tag{7.3}$$

where ζ is the partition function of the ν-dimensional Yang–Mills theory, $\zeta^{\sigma}(\beta|\ldots)$ the one of the $(\nu-1)$-dimensional σ-model in an external gauge field (b,t) with action $\varepsilon^{-1}A_{\nu-1}^{\sigma}(b,t)$, normalized such that $\zeta^{\sigma}(\ldots|\mathbb{1},\mathbb{1}) = 1$, $\langle - \rangle_{\nu-1}^{\sigma}(\ldots)$ the vacuum expectation of that model, and $d\mu_{\nu-1}^{\beta,\delta}$ the normalized, $(\nu-1)$-dimensional Yang–Mills measure. We propose to study the leading behaviour of (7.2) and (7.3) for $\varepsilon \ll \delta \leqq 1$. For the study of the limit $\varepsilon \searrow 0$ we set in (7.2)

$$g_i = e^{\varepsilon X_i}, \quad X_i \in \mathcal{G}, \tag{7.4}$$

and

$$dg_i \to dX_i, \tag{7.5}$$

where dg_i is the Haar measure on G, and dX_i is the Lebesgue measure on $\mathcal{G}$, for all $i \in \delta\mathbb{Z}^{\nu-1}$. (See Sect. 1.2). For the action (7.2) we find to first order in ε

$$\begin{aligned}\varepsilon^{-1}A_{\nu-1}^{\sigma}(b,t) = &-\varepsilon^{-1}\sum_{ij}\delta^{\nu-3}\,\mathrm{Re}\,\chi(b_{ij}t_{ij}^{-1})\\ &-\sum_{ij}\delta^{\nu-3}\{\mathrm{Re}\,\chi(X_i b_{ij} t_{ij}^{-1}) - \mathrm{Re}\,\chi(t_{ij}^{-1} b_{ij} X_j)\}\\ &+\varepsilon\sum_{ij}\delta^{\nu-3}\{\mathrm{Re}\,\chi(X_i b_{ij} X_j t_{ij}^{-1}) - 1/2\,\mathrm{Re}\,\chi(X_i^2 b_{ij} t_{ij}^{-1}) - 1/2\,\mathrm{Re}\,\chi(t_{ij}^{-1} b_{ij} X_j^2)\}.\end{aligned} \tag{7.6}$$

The first term on the r.s. of (7.6) is independent of X and can be combined with the

$(\nu-1)$-dimensional Yang–Mills actions for the horizontal gauge fields, $g^h(u) = b(u)$, to yield

$$A^{YM}_{\nu,\,\mathrm{rad.}} \equiv -\sum_{n\in\varepsilon\mathbb{Z}}\Big\{\varepsilon\sum_{p\subset\delta\mathbb{Z}^{\nu-1}}\delta^{\nu-5}\,\mathrm{Re}\,\chi(b(u)_{\partial p})$$

$$+\varepsilon^{-1}\sum_{ij\subset\delta\mathbb{Z}^{\nu-1}}\delta^{\nu-3}\,\mathrm{Re}\,\chi(b_{ij}(u)t_{ij}(u)^{-1})\Big\} \tag{7.7}$$

which is the expression for the ν-dimensional Yang–Mills action in the radiation gauge ($g^v = 1$). (Thus, the remaining terms in (7.6) could be gauged away when $\varepsilon \searrow 0$. We do however not choose the radiation gauge).

Next, suppose that $\{b_{ij}(u)\}_{u\in\varepsilon\mathbb{Z}}$ is the restriction of a smooth gauge field, $b_{ij}(u)$, on $\delta\mathbb{Z}^{\nu-1}\times\mathbb{R}$ to the lattice $\delta\mathbb{Z}^{\nu-1}\times\varepsilon\mathbb{Z}$. Then

$$b_{ij}(u)t_{ij}(u)^{-1} = b_{ij}(u)b_{ij}(u+\varepsilon)^{-1} = \mathbb{1} + \mathrm{O}(\varepsilon),$$

and

$$\lim_{\varepsilon\searrow 0}\varepsilon^{-1}\{b_{ij}(u)t_{ij}(u)^{-1} - \mathbb{1}\} \equiv B_{ij}(u) \tag{7.8}$$

In finite volume, Λ, (fixed on an ε-independent scale), equ. (7.8) holds in the sense of stochastic differential equations for the paths $\{b_{ij}(u)\}$ in the support of the imaginary-time vacuum measure determined by the action $A^{YM}_{\nu,\,\mathrm{rad.}}$, see (7.7), in the limit $\varepsilon = 0$ (which is the path space measure, $d\mu_{\nu,\,\mathrm{rad.}}$, of the ν-dimensional Yang–Mills theory in the continuous-time Hamiltonian formulation, [35]). We define

$$(D_bX)_{ij} = X_i - b_{ij}X_jb_{ij}^{-1}. \tag{7.9}$$

Taking into account (7.8), the second term on the r.s. of (7.6) approaches

$$\varepsilon L_{\nu-1}(b,B) \equiv -\frac{\varepsilon}{2}\sum_{ij}\delta^{\nu-3}\chi((D_bX)_{ij}B_{ij}), \tag{7.10}$$

and the third term on the r.s. of (7.6)

$$\varepsilon A_{\nu-1}(b) \equiv -\frac{\varepsilon}{4}\sum_{ij}\delta^{\nu-3}\chi((D_bX)^2_{ij}), \tag{7.11}$$

as $\varepsilon \searrow 0$, up to $\mathrm{O}(\varepsilon^2)$ terms. In (7.10), (7.11) the summation, $\sum_{ij}$, extends over all ordered nearest neighbors. In finite volume, Λ, the treatment of the $\varepsilon\searrow 0$ limit can in principle be made rigorous. This is a somewhat tedious exercise in manipulating Trotter product formulae and the heat kernel on G. For $G = U(1)$ or $SU(2)$ one can follow [36], where the $\varepsilon\searrow 0$ limit in the radiation gauge is studied.

After having taken $\varepsilon\searrow 0$, one wants to study the limit $\delta\searrow 0$. This problem is at the core of the renormalization theory of Yang–Mills fields. A partial aspect of this problem is the analysis of the $\delta\searrow 0$ limit of the Gaussian σ-models in external gauge field with action $A_{\nu-1}(b) + L_{\nu-1}(b,B)$ and a priori distribution $\prod_i dX_i$, see (7.5), at "inverse temperature" βh, $h > 0$, ($h = h(\delta,\nu)$). In this step the external gauge field is kept fixed. For $G = U(1)$ or $SU(2)$ and $\nu - 1 = 2$, the $\delta\searrow 0$ limit of these models has been constructed in [37], (and the methods of [37] suffice to also analyze the three-dimensional case, of interest in the construction of four-dimensional Yang–Mills fields).

We now recall the main problems arising in the study of the $\delta \searrow 0$ limit and in the analysis of confinement for $\varepsilon = 0$. This requires some more definitions. Let V_χ be the finite dimensional Hilbert space carrying the representation U_χ of G with character χ.

We define

$$\xi = U^\chi(b), \quad b \in G. \tag{7.12}$$

Let $M_\chi \cong V_\chi^{\otimes 2} \supset U(G)$ be the space of all matrices on V_χ. Let Y be an arbitrary M_χ-valued function on $\delta\mathbb{Z}^{v-1}$. We define a (finite difference) covariant gradient by

$$(\nabla_b^{(\delta)} Y)_{ij} \equiv (\nabla_\xi^{(\delta)} Y)_{ij} = \frac{1}{\delta}(Y_i - \xi_{ij} Y_j \xi_{ij}^*), \tag{7.13}$$

with $\xi_{ij} = U^\chi(b_{ij})$. Furthermore, the covariant Laplacean is given by

$$-\Delta_\xi^{(\delta)} = \nabla_\xi^{(\delta)*} \nabla_\xi^{(\delta)}. \tag{7.14}$$

For $\delta = 0$, the superscripts are dropped.

For Y, Z M_χ-valued functions on $\delta\mathbb{Z}^{v-1}$, we define

$$(Y, Z) = \sum_{i \in \delta\mathbb{Z}^{v-1}} \delta^{v-1} \operatorname{tr}(Y_i^* Z_i), \tag{7.15}$$

and similarly for M_χ-valued functions on unordered pairs of nearest neighbors in $\delta\mathbb{Z}^{v-1}$.

Let $\mathscr{G}_\chi = U_\chi(\mathscr{G})$ be the matrices in M_χ which represent the Lie algebra $\mathscr{G}$ of G. Such matrices are henceforth denoted $\Phi, \Psi, \ldots$. In our new notations we get from (7.10), (7.11)

$$\begin{aligned} A_{v-1}(\xi) &= -\tfrac{1}{2}(\Phi, \Delta_\xi^{(\delta)} \Phi), \\ L_{v-1}(\xi, B) &= \delta^{-1} \operatorname{Re}(\nabla_\xi^{(\delta)} \Phi, B), \end{aligned} \tag{7.16}$$

and the uniform measure on $\mathscr{G}_\chi$ is denoted $d\Phi$. The Gaussian vacuum expectation of this model, at "inverse temperature" $\beta h, h > 0$, is denoted $\langle - \rangle_{\beta h}^{(\delta)}(\xi, B)$, and $\zeta^{(\delta)}(\beta h | \xi, B)$ is its partition function, normalized such that $\zeta^{(\delta)}(\beta h | \mathbb{1}, 0) = 1$. When $G = \mathrm{SU}(2)$, χ the isospin 1/2 character, we set $\Phi = i \sum_{\alpha=1}^{3} \varphi^\alpha \sigma_\alpha$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The adjoint representation used in the definition of $\nabla_\xi^{(\delta)}, \Delta_\xi^{(\delta)}$ has isospin 1, and we may now set

$$(\nabla_\xi^{(\delta)} \varphi)_{ij} = \delta^{-1}(\varphi_i - \xi_{ij} \varphi_j).$$

where $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{pmatrix} \in \mathbb{R}^3$ and ξ_{ij} belongs to the isospin 1 representation of SU(2), i.e. to SO(3). Moreover $d\Phi = d^3\varphi$, the Lebesgue measure on $\mathbb{R}^3$. We now propose to study the behaviour of

1) $C_{m,\xi}^{(\delta)} = (-\Delta_\xi^{(\delta)} + m^2)^{-1}$, in particular of its integral kernel, $C_{m,\xi}^{(\delta)}(x, y)$, x, y in $\delta\mathbb{Z}^{v-1}$, for arbitrary $\delta \geqq 0, m \geqq 0$, and arbitrary ξ;

2) $\zeta^{(\delta)}(\beta h|\xi, B)$ as a function of $\delta \geqq 0$, ξ and B, (with ξ, B e.g. of compact support);
3) $\langle (e^{h\Phi_x})_{kl}(e^{-h\Phi_y})_{mn}\rangle^{(\delta)}_{\beta h}(\xi, B)$, xy in $\delta\mathbb{Z}^{\nu-1}$
as a function of $|x-y|$, h, δ, ξ and B. This two-point function is related to the expectation of the Wilson loop, $\langle W^q(C)\rangle^{YM}_\nu$, of the ν-dimensional Yang–Mills theory on $\delta\mathbb{Z}^{\nu-1}\times\mathbb{R}$ (in the limit $\varepsilon=0$) by the formula

$$\langle W^q(C)\rangle^{YM}_\nu = \lim_{\varepsilon\searrow 0}\Bigg\{\zeta_\varepsilon^{-1}\sum_{\underset{\sim}{m},\underset{\sim}{n}}\int d\mu^{(\varepsilon)}_{\nu,\mathrm{rad.}}(b)\prod_{u\in\varepsilon\mathbb{Z}}\zeta^{(\delta)}(\beta\varepsilon|\xi(b), B^{(\varepsilon)})B_{m_0 n_0}T_{n_T m_T}\cdot$$
$$\prod_{u\in\varepsilon\mathbb{Z}\cap[0,T-\varepsilon]}\langle (e^{\varepsilon\Phi_0})_{m_u m_{u+\varepsilon}}(e^{-\varepsilon\Phi_x})_{n_u n_{u+\varepsilon}}\rangle^{(\delta)}_{\beta\varepsilon}(\xi(b), B^{(\varepsilon)})\Bigg\}, \tag{7.17}$$

where $d\mu^{(\varepsilon)}_{\nu,\mathrm{rad.}}(b)$ is the ν-dimensional Yang–Mills measure in the radiation gauge, $\xi(b(u))_{ij} = U^\chi(b(u)_{ij})$, $B^{(\varepsilon)}_{ij}(u) = \varepsilon^{-1}\{\xi_{ij}(u)\xi_{ij}(u+\varepsilon)^{-1} - \mathbb{1}\}$, and $x\in\delta\mathbb{Z}^{\nu-1}$ (independent of u) with $|x| = L$. Formula (7.17) involves the hidden assumption that, for $\delta > 0$, we can first pass to the Gaussian limit of the σ-models and then take the $\varepsilon\searrow 0$ limit. (As remarked already above, we are confident that this can be justified by adapting the techniques of [36]. See also [37, 38]).

Fortunately, problems 1) and 2) have been solved already in [37] for $G = \mathrm{SU}(2)$ and $\nu - 1 = 2$, and the techniques developed there suffice, in principle, to solve them for arbitrary compact gauge groups and $2 \leqq \nu - 1 \leqq 3$. From that reference we infer that

$$\| C^{(\delta)}_{m,\xi}(x, y)\| \leqq \| C^{(\delta)}_{m,\mathbb{1}}(x, y)\|, \text{ for all } \delta, \tag{7.18}$$

(Landau diamagnetism [4, 37]),
and for $\xi_{ij} = \xi^{(\delta)}_{ij} = e^{i\delta\mathscr{B}^{(\delta)}_{ij}}$, where $\mathscr{B}^{(\delta)}_{ij}$ is the restriction of continuous continuum gauge field, $\mathscr{B}$, with values in $\mathscr{G}_\chi$ and of compact support to the lattice $\delta\mathbb{Z}^{\nu-1}$,

$$C^{(\delta)}_{m,\xi}(x, y) \overset{\delta\searrow 0}{\longrightarrow} C_{m,\mathscr{B}}(x, y), \quad \text{in} \quad L^p(\Lambda\times\Lambda), \tag{7.19}$$

for arbitrary bounded, open $\Lambda\subset\mathbb{R}^{\nu-1}$ and $1 \leqq p < p(\nu-1)$, with $p(2) = \infty$, $p(3) = 3$, and for a large class of boundary conditions (e.g. free, periodic, Dirichlet) at $\partial\Lambda$. See [37] for detailed statements and proofs of this and other results. These results suffice to control the limit of $\langle - \rangle^{(\delta)}_{\beta h}(\xi, B^{(\delta)})$, as $\delta\searrow 0$, for ξ as in (7.19) and $B^{(\delta)}$ chosen such that $\delta^{-1}B^{(\delta)} \to B'$, as $\delta\searrow 0$, e.g. in the sup norm.
[As an example, we mention that, for $\delta = 0$,

$$\begin{aligned}\langle \Phi^{kl}_x \Phi^{mn}_y\rangle_{\beta h}(\mathscr{B}, B') &= (\beta h)^{-1} C_{0,\mathscr{B}}(x, y)_{kl,mn}\\ &+ (\beta h)^{-1}(C_{0,\mathscr{B}}\nabla_{\mathscr{B}}B')_{kl}(x)(C_{0,\mathscr{B}}\nabla_{\mathscr{B}}B')_{mn}(y).\end{aligned}$$

For $\nu - 1 = 2$, this identity usually has infrared divergences, unless, e.g. O-Dirichlet data at the boundary of some bounded, open region are introduced in $\Delta_{\mathscr{B}}$ or $\mathscr{B}$ is suitably chosen. For $\nu - 1 \geqq 3$ there are no infrared divergences. The two-point functions in 3) and (7.17) have *no* infrared divergences, even for $\nu - 1 = 2$, but must be ultraviolet-renormalized when $\delta\searrow 0$, for $\nu - 1 \geqq 2$; see below].

Next we study the partition function $\zeta^{(\delta)}(\beta h|\xi^{(\delta)}, B^{(\delta)})$, with $\xi^{(\delta)} = e^{i\delta\mathscr{B}^{\delta}_{ij}}$, $\mathscr{B}^{(\delta)}_{ij}$ the restriction of a smooth $\mathscr{G}_\chi$-valued field $\mathscr{B}$ on $\mathbb{R}^{\nu-1}$ of compact support to $\delta\mathbb{Z}^{\nu-1}$, and $\delta^{-1}B^{(\delta)} \overset{\delta\searrow 0}{\longrightarrow} B'$, in $C^2_0(\mathbb{R}^{\nu-1})$. For $\nu - 1 = 2$ we temporarily introduce

O-Dirichlet data in $\Delta^{(\delta)}_{\xi}$ at the boundary of some δ-independent, bounded open set Λ, in order to eliminate infrared divergences. (Reference to Λ is suppressed in our notation). Let $\Delta^{(\delta)}$ be the usual finite difference Laplacean, and let V^d_{ξ} be the $(\nu-1)$-dimensional, covariant lattice dipole potential, defined as follows: Let f be some M_{χ}-valued function defined on the links (nearest neighbor pairs) of $\delta\mathbb{Z}^{\nu-1}$ and h an arbitrary M_{χ}-valued function on $\mathbb{Z}^{\nu}$ of compact support. We define $\nabla^{(\delta)*}_{\xi} f$ by

$$(\nabla^{(\delta)*}_{\xi} f, h) \equiv (f, \nabla^{(\delta)}_{\xi} h) = \sum_{ij} \delta^{\nu-2}\,\mathrm{tr}(f^*_{ij}(h_i - \xi_{ij} h_j \xi^*_{ij})).$$

Let now f and g be arbitrary M_{χ}-valued functions on the links of $\delta\mathbb{Z}^{\nu-1}$, of compact support. Then V^d_{ξ} is defined by

$$(f, V^d_{\xi} g) = (\nabla^{(\delta)*}_{\xi} f, C^{(\delta)}_{0,\xi} \nabla^{(\delta)*}_{\xi} g) \tag{7.20}$$

By evaluating Gaussian integrals we get

$$\zeta^{(\delta)}(\beta h | \xi, B) = \det(-\Delta^{(\delta)} C^{(\delta)}_{0,\xi})^{1/2} \exp[(\beta h/2)(\delta^{-1} B, V^d_{\xi} \delta^{-1} B)]; \tag{7.21}$$

see [37]. (We thank E. Seiler for correcting a mistake in our original formula). Notice that the r.s. of (7.21) obeys the normalization condition, $\zeta^{(\delta)}(\beta h | \mathbb{1}, 0) = 1$.

For $h = \varepsilon$, $B = B^{(\varepsilon)}$ with $B^{(\varepsilon)}_{ij}(u) = \varepsilon^{-1}\{\xi_{ij}(u)\xi_{ij}(u+\varepsilon)^{-1} - \mathbb{1}\}$, the effect of the second factor on the r.s. of (7.21) is to modify the couplings between $\xi(u)$ and $\xi(u+\varepsilon)$, $u \in \varepsilon\mathbb{Z}$, in the measure $d\mu^{(\varepsilon)}_{\nu,\,\mathrm{rad.}}(\xi)$ (see (7.17)). We set

$$d\rho^{(\varepsilon)}_{\nu}(\xi) = Z^{(\varepsilon)-1}_{\nu} \exp[(\beta\varepsilon/2)(\delta^{-1} B^{(\varepsilon)}, V^d_{\xi} \delta^{-1} B^{(\varepsilon)})]\, d\mu^{(\varepsilon)}_{\nu,\,\mathrm{rad.}}(\xi), \tag{7.22}$$

where $Z^{(\varepsilon)}_{\nu}$ is a normalization factor chosen such that $d\rho^{(\varepsilon)}_{\nu}(\xi) = 1$. (In spite of the second factor on the r.s. of (7.21), the measure $d\rho^{(\varepsilon)}_{\nu}$ is well-defined, since $\exp[(\beta\varepsilon/2)(\delta^{-1} B^{(\varepsilon)}, V^d_{\xi} \delta^{-1} B^{(\varepsilon)})] \geqq \exp[(\beta\varepsilon/2)(\delta^{-1} B^{(\varepsilon)}, \delta^{-1} B^{(\varepsilon)})]$, which is compensated by the factor $\exp[-(\beta/\varepsilon)\,\mathrm{Re}(\xi(u), \xi(u+\varepsilon))]$ in $d\mu^{(\varepsilon)}_{\nu,\,\mathrm{rad.}}(\xi)$).

Notice that the formal action corresponding to $d\rho^{(\varepsilon)}_{\nu}$ is "non-polynomial" if G is non-abelian, even in the formal limits $\varepsilon = 0$, $\delta = 0$. Thus, our approach might be cumbersome for the discussion of ultraviolet renormalizations when one takes the limit $\delta \searrow 0$.

Next, we discuss the first factor on the r.s. of (7.21). Notice that $\det(-\Delta^{(\delta)} C^{(\delta)}_{0,\xi})^{1/2}$ is *independent* of βh. From [4, 37] we recall that

$$0 \leqq \det(-\Delta^{(\delta)} C^{(\delta)}_{0,\xi}) \leqq 1 \tag{7.23}$$

(diamagnetic inequality; see also Theorem 1.1)
For $\xi_{ij} = e^{i\delta \mathscr{B}^{\delta}_{ij}}$, $\mathscr{B}^{(\delta)}_{ij}$ the restriction of a continuum gauge field, $\mathscr{B}$, that is Hölder continuous of order $\alpha > 0$, $\nu - 1 = 2$, and O-Dirichlet data on the boundary of a bounded, open set in $\mathbb{R}^2$,

$$\lim_{\delta \searrow 0} \det(-\Delta^{(\delta)} C^{(\delta)}_{0,\xi})^{1/2} = \det(-\Delta C_{0,\mathscr{B}})^{1/2} \tag{7.24}$$

exists and is strictly positive; see [37]. (The methods of [37] suffice, in principle, to also handle the case $\nu - 1 = 3$, for smooth $\mathscr{B}$).

Thus, the results of [37], in particular (7.18)–(7.24), provide complete control over the $\delta \searrow 0$ limit of the Gaussian σ-model in an external gauge field with action given by (7.16), at arbitrary "inverse temperature" $0 < \beta h < \infty$, and $\nu - 1 = 2, (3)$.

Next, we study the two-point function

$$G^{(\delta)}_{\xi,B}(x,y)_{kl,mn} \equiv \langle (e^{h\Phi_x})_{kl}(e^{-h\Phi_y})_{mn}\rangle^{(\delta)}_{\beta h}(\xi,B), \tag{7.25}$$

in particular its cluster properties (related to confinement via (7.17)) and the existence of the limit $\delta \searrow 0$. For $\| G^{(\delta)}_{\xi,B}(x,y)\|$ to tend to 0, as $|x-y| \to \infty$, it is necessary that

$$\langle (e^{h\Phi_x})_{kl}\rangle^{(\delta)}_{\beta h}(\xi,B) = 0, \tag{7.26}$$

for all ξ, B and x (large enough).

Since $\langle - \rangle^{(\delta)}_{\beta h}(\xi,B)$ is Gaussian, with covariance $C^{(\delta)}_{0,\xi}$, (7.26) requires that $C^{(\delta)}_{0,\xi}(x,x)$ is *infinite*, with $2C^{(\delta)}_{0,\xi}(x,y) - C^{(\delta)}_{0,\xi}(x,x) - C^{(\delta)}_{0,\xi}(y,y)$ *finite*. This is possible for $\nu - 1 \leqq 2$, due to an *infrared divergence*: For $\xi \equiv \mathbb{1}$, $C^{(\delta)}(x,y) = C^{(\delta)}_{0,\mathbb{1}}(x,y)$ is the Fourier transform of $(\delta^2/2)\big[(\nu-1) - \sum_{j=1}^{\nu-1}\cos(\delta k^j)\big]^{-1}$, $(|k^j| \leqq \delta^{-1}\pi)$ which is linearly $(\nu-1=1)$, resp. logarithmically $(\nu-1=2)$ divergent at $k=0$. Moreover, $C^{(\delta)}(x) - C^{(\delta)}(0) \approx \frac{1}{2\pi}\log\frac{1}{|x|}$, for $\nu-1=2$. Thus, one might expect (7.26) to be valid and $\| G^{(\delta)}_{\xi,B}(x,y)\|$ to behave like

$$\exp\left[(h/2\pi\beta)\log\frac{1}{|x-y|}\right], \tag{7.27}$$

as $|x-y| \to \infty$, for arbitrary ξ, B, when $\nu-1=2$. By (7.17) this would yield permanent confinement of static quarks in three-dimensional Yang–Mills theory by a potential $\geqq \log|x|$, as $|x| \to \infty$. For $G = U(1)$, (7.26) and (7.27) are true, since $C^{(\delta)}_{0,\xi} = C^{(\delta)}$ and $G^{(\delta)}_{\xi,B}(x,y) = G^{(\delta)}_{\mathbb{1},B}(x,y)$ (independent of ξ), and if the center of G contains $U(1)$ the same conclusions hold, by the estimates of Sect. 3. Moreover, when $\nu-1=1$, ξ can be gauged away, for arbitrary G, so that (7.26) holds trivially, and

$$\| G^{(\delta)}_{\xi,B}(x,y)\| = \| G^{(\delta)}_{\mathbb{1},B}(x,y)\| \leqq \exp[-(h/2\beta)|x-y|],$$

for arbitrary G and all $\delta \geqq 0$.

However, for $G = \mathrm{SU}(2)$, $\nu-1=2$, $(\delta=0)$, there are choices of an external gauge field $\mathscr{B}$ such that $C_{\mathscr{B}}$ is a bounded operator with $\| C_{\mathscr{B}}(x,y)\| \leqq$ const., for $|x-y|$ large enough. In this case (7.27) is definitely violated. This is the result alluded to in Sect. 3: *For certain choices of* $\mathscr{B}$, *the clustering of* $G_{\mathscr{B},B}(x,y)$ is worse than that of $G_{0,0}(x,y)$. This is a consequence of *non-abelian Landau diamagnetism*. Some more details are given in the appendix to Sect. 7.

For $\nu \geqq 4$, (7.26) is always violated, and $\| G^{(\delta)}_{\xi,B}(x,y)\| \not\to 0$, as $|x-y| \to \infty$, as we are now going to demonstrate.

By (7.18),

$$\begin{aligned}\| C^{(\delta)}_{0,\xi}(x,y)\| &\leqq \| C^{(\delta)}(x-y)\| \\ &\leqq (2\pi)^{-(\nu-1)/2}(1/2)\int_{|k^j|\leqq \pi/\delta}\Big[(\nu-1) - \sum_{j=1}^{\nu-1}\cos(\delta k^j)\Big]^{-1} d^{\nu-1}k \\ &\equiv J^{(\delta)}(\nu-1) < \infty, \text{ for all } x, y.\end{aligned} \tag{7.28}$$

This shows that (7.26) is impossible.

Moreover $\| C^{(\delta)}_{0,\xi}(x, y) \| \leqq \| C^{(\delta)}(x - y) \| = 0(|x - y|^{-(\nu - 3)})$, as $|x - y| \to \infty$. Therefore $\| G^{(\delta)}_{\xi,B}(x, y) \| \not\to 0$, as $|x - y| \to \infty$, which proves our contention.

We conclude that presumably in the three-dimensional SU(2) Yang–Mills theory and certainly in all four-dimensional Yang–Mills theories confinement of static quarks can only arise as a consequence of cancellation of the random phase factors in $G^{(\delta)}_{\xi,B}(0, x)$ when integrating over ξ, $\left(\text{with } B = \xi\left(\frac{\partial}{\partial x^\nu}\xi^{-1}\right)\right)$. This is the the second mechanism emphasized in Sect. 6; see (6.11)–(6.15). A careful study of this mechanism in the limit $\varepsilon = 0$, (i.e. for the Gaussian σ-models) is beyond the scope of the present paper, but we recall that it has been shown in [14] that in all $\nu \geqq 4$ dimensional non-compact $U(1)$ theories there is no confinement.

In the present formalism, absence of confinement in the four-dimensional $U(1)$ theory can be understood as follows: For $G = U(1)$,

$$G^{(\delta)}_{\xi,B}(0, x) = \exp[-(\beta h/2)(C^{(\delta)}_{0,\xi}(0, 0) + C^{(\delta)}_{0,\xi}(x, x) - 2C^{(\delta)}_{0,\xi}(0, x))] \cdot \exp[\beta h\{\nabla^{(\delta)}_\xi \cdot C^{(\delta)}_{0,\xi} B^{(\delta)})(0) - (\nabla^{(\delta)}_\xi \cdot C^{(\delta)}_{0,\xi} B^{(\delta)})(x)\}] \tag{7.29}$$

Now, since the adjoint representation of $U(1)$ is the trivial one,

$$\nabla^{(\delta)}_\xi = \nabla^{(\delta)}_{\mathbb{1}} \equiv \partial^{(\delta)} \quad \text{and} \quad C^{(\delta)}_{0,\xi} = C^{(\delta)} \tag{7.30}$$

are *independent* of ξ (i.e. the same as for $\xi \equiv \mathbb{1}$). In particular, they are independent of the value, u, of x^ν. Furthermore, one can set

$$\xi_{ij}(u) = e^{iA^{(\delta)}_{ij}(u)}, \quad A^{(\delta)}_{ij}(u) \in [0, 2\pi),$$

for all $ij \subset \delta\mathbb{Z}^{\nu-1}$, $u \in \mathbb{R}$. Thus

$$B^{(\delta)}(u) = -i\left(\frac{\partial}{\partial u}A^{(\delta)}_{ij}\right)(u) \tag{7.31}$$

If we now insert (7.30) and (7.31) into (7.29) and, subsequently, (7.29) into formula (7.17) for the expectation of the Wilson loop we see that the random phase

$$\exp\left[-i\beta\int_{u=0}^{u=T} du\left[\left(\partial^{(\delta)} \cdot C^{(\delta)}\frac{\partial A^{(\delta)}}{\partial u}\right)(0, u) - \left(\partial^{(\delta)} \cdot C^{(\delta)}\frac{\partial A^{(\delta)}}{\partial u}\right)(x, u)\right]\right]$$
$$= \exp\left\{-i\beta[(\partial^{(\delta)} \cdot C^{(\delta)} A^{(\delta)})(0, u) - (\partial^{(\delta)} \cdot C^{(\delta)} A^{(\delta)})(x, u)]_0^T\right\} \tag{7.32}$$

reduces to a product of two random phase factors localized at $u = 0$, resp. $u = T$, i.e. to a *pure "surface term"*. Thus, using (7.29), (7.32) and (7.17)

$$\langle W^q(C)\rangle^{YM}_4 = \exp[-\beta T(C^{(\delta)}(0) - C^\delta(x))] \cdot \langle \exp\{-i\beta[(\partial^{(\delta)} \cdot C^{(\delta)} A^{(\delta)})(0, u) - (\partial^{(\delta)} \cdot C^{(\delta)} A^{(\delta)})(x, u)]_0^T\}\rangle^{YM}_4$$

(Since the second factor is a surface term, it cannot cause area decay, when $\delta \searrow 0$).

The basic difference between abelian and non-abelian theories is that, in the non-abelian case $\nabla^{(\delta)}_\xi$ and $C^{(\delta)}_{0,\xi}$ *do* depend on ξ in a non-trivial way, *so that the total random phase factor does not reduce to a pure surface term*, as can be checked by an explicit calculation. For this reason, four-dimensional non-abelian theories may still confine static quarks.

Finally we discuss the continuum limit ($\delta \searrow 0$) of the two-point functions $G^{(\delta)}_{\xi,B}0, x)$ and the related problem of how to "normal-order" the Wilson loops, $W^q(C)$, so as to be able to pass to the continuum limit. We concentrate on the discussion of $G = \mathrm{SU}(2)$, with $\chi^q = \chi$ the isospin 1/2 character; ($G = U(1)$ is very easy). For SU(2) $\Phi_x = i \sum_{j=1}^{3} \varphi^\alpha_x \sigma_\alpha$. Let $\langle - \rangle^{(\delta)}_0$ denote the Gaussian expectation with mean 0 and covariance $C^{(\delta)}_1 = (-\Delta^{(\delta)} + 1)^{-1}$, i.e.

$$\langle \varphi^\alpha_x \varphi^\beta_y \rangle^{(\delta)}_0 = C^{(\delta)}_1 (x - y) \delta^{\alpha\beta}.$$

Let $c^{(\delta)} = C^{(\delta)}_1(0)$. Then

$$\langle \exp(i\varphi^3_x \sigma_3)_{kl} \rangle^{(\delta)}_0 = \exp[-(1/8)c^{(\delta)}]\delta_{kl} \tag{7.33}$$

Moreover, for $\xi = \xi^{(\delta)}$ as in (7.24),

$$\lim_{\delta \searrow 0} (C^{(\delta)\alpha\beta}_{0,\xi^{(\delta)}} - c^{(\delta)}\delta^{\alpha\beta})(x, x) \quad \text{exists, for } \nu - 1 = 2,$$

(provided O-Dirichlet data are imposed on $C^{(\delta)}_{0,\xi^{(\delta)}}$ at the boundary of some bounded, open region, in order to eliminate infrared divergences). This is proven in [37]. Thus

$$\lim_{\delta \searrow 0} \langle \exp(ih\varphi^3_x \sigma^3)_{kl} \rangle^{(\delta)}_{\beta h} (\xi^{(\delta)}, B^{(\delta)}) \exp[(h/8\beta)c^{(\delta)}]\delta_{kl}$$

exists. In general we define

$$N^{(\delta)}(e^{h\Phi_x})_{kl} = (e^{h\Phi_x})_{kl}[\langle (e^{h\Phi_x})_{kl} \rangle^{(\delta)}_0]^{-1} \tag{7.34^4}$$

Then, for smooth $B' = \lim_{\delta \searrow 0} \delta^{-1} B^{(\delta)}$,

$$\lim_{\delta \searrow 0} \langle N^{(\delta)}(e^{h\Phi_x})_{kl} N^{(\delta)}(e^{-h\Phi_y})_{mn} \rangle^{(\delta)}_{\beta h} (\xi^{(\delta)}, B^{(\delta)})$$

exists, even in the thermodynamic limit; (there are *no* infrared divergences). This suggests replacing $W^q(C)$ (defined on the links of $\delta \mathbb{Z}^\nu$) by

$$N^{(\delta)}(W^q(C)) = \sum_{m} \prod_{xy \subset C} U^{\chi^q}(g_{xy})_{m_x m_y} [\langle (e^{\sqrt{(\delta/\beta)}\Phi_x})_{m_x m_y} \rangle^{(\delta)}_0]^{-1}.$$

This prescription ought to be appropriate for taking the limit $\delta \searrow 0$, at least in $\nu = 3$ dimensions. It suggests to formulate the renormalization conditions in a scheme of implicit renormalization for three-dimensional Yang–Mills theory in terms of δ-independent upper and lower bounds on

$$\langle N^{(\delta)}(W^q(C)) \rangle^{YM}_\nu,$$

for C a square loop with sides of length 1, for example.

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4 Another possibility is to choose a "unitary" gauge in which $\varphi^1 = \varphi^2 = 0$.

Appendix to Sect. 5

We show, in this appendix, the details of the calculations leading to the area decay of the expectation of the Wilson loop observable when $G = \mathrm{SU}(2)$. These are completely analogous to those performed in [29].

With notations as in Sect. 5 we get, by setting

$$\tilde{Z}^\sigma(b,t,n) = \int e^{(1/2)(S,\Delta_{b,t}S)} \prod_{j\in\Lambda^0} \int du_j \frac{u_j^{n-1}}{(n-1)!} e^{-u_j} \delta\left(|\vec{S}_j|^2 - \beta + \frac{u_j}{\nu-1}\right) d\vec{S}_j, \quad \text{(A1)}$$

and using a chessboard estimate, that

$$\int \prod_t \tilde{Z}^\sigma(g^h(t), g^h(t+1), \omega_t) d\mu_{\nu-1}(g^h(t))$$
$$\leqq \prod_{r,k} [\int \prod_t \tilde{Z}^\sigma(g^h(t), g^h(t+1), n_k(\omega_r)) d\mu_{\nu-1}(g^h(t))]^{1/|\Lambda|} \quad \text{(A2)}$$

By a second chessboard estimate we have

$$\beta^{-|\Lambda|} \int \prod_t \tilde{Z}^\sigma(g^h(t), g^h(t+1), n) d\mu_{\nu-1}(g^h(t))$$
$$= \beta^{-|\Lambda|} \int \Bigg[\prod_{t,j} \{du_{t,j} \frac{u_{t,j}^{n-1}}{(n-1)!} e^{-u_{t,j}}\} \int \prod_t \Bigg\{ d\mu_{\nu-1}(g^h(t)) e^{(1/2)(S_t, \Delta_{g^h(t), g^h(t+1)} S_t)}$$
$$\cdot \prod_{i\in\Lambda_t} \delta\left(|(\vec{S}_t)_i|^2 - \beta + \frac{u_{t,i}}{\nu-1}\right) d(\vec{S}_t)_i \Bigg\}\Bigg]$$
$$\leqq \beta^{-|\Lambda|} \int \prod_{t,j} du_{t,j} \frac{u_{t,j}^{n-1}}{(n-1)!} e^{-u_{t,j}} \prod_{q,l} \Bigg[\int \prod_t \Bigg\{ d\mu_{\nu-1}(g^h(t)) e^{(1/2)(S_t, \Delta_{g^h(t), g^h(t+1)} S_t)}$$
$$\cdot \prod_{i\in\Lambda_t} \delta\left(|(\vec{S}_t)_i|^2 - \beta + \frac{u_{q,l}}{\nu-1}\right) d(\vec{S}_t)_i \Bigg\}\Bigg]^{1/|\Lambda|}$$
$$= \int \prod_{t,j} du_{t,j} \frac{u_{t,j}^{n-1}}{(n-1)!} \mathrm{e}^{-u_{t,j}} \left(1 - \frac{u_{t,j}}{\beta(\nu-1)}\right) \prod_{q,l} \Bigg[\int \prod_t \Bigg\{ d\mu_{\nu-1}(g^h(t))$$
$$\cdot e^{(1/2)(\beta - u_{q,l}/(\nu-1))(S_t, \Delta_{g^h(t), g^h(t+1)} S_t)} \prod_{i\in\Lambda_t} \delta(|(\vec{S}_t)_i|^2 - 1) d(\vec{S}_t)_i \Bigg\}\Bigg]^{1/|\Lambda|} \quad \text{(A.3)}$$

Next we note by Jensen's inequality

$$\int e^{(1/2)\beta(S,\Delta_{b,t}S)} \prod_{j\in\Lambda^0} \delta(|\vec{S}_j|^2 - 1) d\vec{S}_j$$
$$= \int e^{(u/2(\nu-1))(S,\Delta_{b,t}S)} e^{(1/2)(\beta - u/(\nu-1))(S,\Delta_{b,t}S)} \prod_{j\in\Lambda^0} \delta(|\vec{S}_j|^2 - 1) d\vec{S}_j$$
$$\geqq e^{u/2(\nu-1)\langle (S,\Delta_{b,t}S)\rangle^\sigma_{\nu-1}(b,t)} \int e^{(1/2)(\beta - u/(\nu-1))(S,\Delta_{b,t}S)} \prod_{j\in\Lambda^0} \delta(|\vec{S}_j|^2 - 1) d\vec{S}_j \quad \text{(A.4)}$$

where the expectation $\langle \cdot \rangle^\sigma_{\nu-1}(b,t)$ is at inverse temperature $\beta - \dfrac{u}{\nu-1}$.

Now since

$$|(S,\Delta_{b,t}S)| = \sum_{ij\subset\Lambda^0} (\vec{S}_i - \mathrm{O}(b_{ij}, t_{ij})\vec{S}_j)^2 \leqq 4(\nu-1)|\Lambda^0|$$

when $|\vec{S}_j| = 1, \forall j \in \Lambda^0$, it follows from (A.3), (A.4) and the definition of Z, that

$$\beta^{-|\Lambda|}\int\prod_t \tilde{Z}^\sigma(g^h(t), g^h(t+1), n)d\mu_{v-1}(g^h(t))$$

$$\leqq \int\left\{\prod_{t,j} du_{t,j}\frac{u_{t,j}^{n-1}}{(n-1)!}e^{-u_{t,j}}\left(1-\frac{u_{t,j}}{\beta(v-1)}\right)e^{2u_{t,j}}\right\}\int\prod_t\{\beta^{-|\Lambda_t|}\tilde{Z}^\sigma_\beta(g^h(t),$$

$$\cdot g^h(t+1))d\mu_{v-1}(g^h(t))\} \leqq \left[\int du\frac{u^{n-1}}{(n-1)!}e^{-u(1/\beta(v-1)-1)}\right]^{|\Lambda|}\cdot Z$$

$$= \left(\frac{1}{\beta(v-1)}-1\right)^{-n|\Lambda|}\cdot Z \tag{A.5}$$

Formulas (A.2) and (A.5) then give

$$\beta^{-|\Lambda|}\int\prod_t\tilde{Z}^\sigma(g^h(t), g^h(t+1), \omega_t)d\mu_{v-1}(g^h(t)) \leqq \left[\prod_{t,j}\left(\frac{1}{\beta(v-1)}-1\right)^{-n_j(\omega_t)}\right]\cdot Z.$$

And finally this combined with (5.8) gives

$$|\langle W^q(C)\rangle_v^{YM}|$$

$$\leqq 4^T\beta^{-T}\sum_{\omega_0}\dots\sum_{\omega_{T-1}}(2(v-1))^{-\sum_{t,j}n_j(\omega_t)}\left(\left(\frac{1}{\beta(v-1)}\right)-1\right)^{-\sum_{t,j}n_j(\omega_t)}$$

$$= 4^T\beta^{-T}\left[\sum_{\substack{\omega_i \overset{\omega}{=} 0\\ \omega_f = x}}\left(2(v-1)\left(\frac{1}{\beta(v-1)}-1\right)\right)^{-|\omega|-1}\right]^T.$$

For $\beta < \frac{1}{2(v-1)}$ we have that $\frac{1}{\beta(v-1)} - 1 > 1$, and therefore there exists an $\varepsilon > 0$, independent of x, T and Λ, such that

$$|\langle W^q(C)\rangle_v^{YM}| \leqq 4^T\beta^{-T}\left[\sum_{\substack{\omega_i \overset{\omega}{=} 0\\ \omega_f = x}}(2(v-1)+\varepsilon)^{-|\omega|-1}\right]^T$$

$$= 4^T\beta^{-T}((-\Delta_\Lambda+\varepsilon)^{-1}(0,x))^T,$$

where Δ_Λ is the discrete Laplacean in $\Lambda \subset \mathbb{Z}^{v-1}$ with periodic boundary conditions.

Taking the thermodynamic limit and noticing that

$$(-\Delta+\varepsilon)^{-1}(0,x) \sim e^{-\varepsilon|x|}, \quad \text{as } |x| \to \infty,$$

completes the proof of confinement by a linear potential for $\beta < \frac{1}{2(v-1)}$. (A more refined estimate extends this result to $\beta < \frac{1}{v-1}$).

Appendix to Sect. 7

In this appendix we show that the clustering of certain two-point functions in a Gaussian SU(2) × SU(2)-σ-model in two dimensions is diminished when a suitable external gauge field is turned on.

As argued in Sect. 7 this indicates that for large β only the mechanism of strong cancellations of random phase factors can be expected to be responsible for confinement of static quarks in a SU(2)-gauge theory in 3 dimensions.

We consider a two-dimensional Gaussian σ-model, where the field X takes values in su(2) (the Lie-algebra of SU(2)), and an external gauge field $A_\mu \in \mathrm{su}(2)$ is acting. In other words we consider the model whose action $\mathscr{A}$ is given by

$$\mathscr{A} = \sum_{\mu=0}^{1} \mathrm{tr}((D_{A_\mu} X)^* (D_{A_\mu} X))$$

where

$$D_{A_\mu} X = \partial_\mu X + [A_\mu, X], \quad \mu = 0, 1,$$

and X and A_μ are su(2)-valued functions on $\mathbb{R}^2$. The measure of the model is thus the Gaussian measure with covariance

$$(-\Delta_A)^{-1} = (\sum_\mu D^*_{A_\mu} D_{A_\mu})^{-1}$$

defined on a suitable space of su(2)-valued functions.

To be able to analyze Δ_A we set

$$X = \sum_{\alpha=1}^{3} \varphi^\alpha \sigma_\alpha \quad \text{and} \quad A_\mu = \sum_{\alpha=1}^{3} A^\alpha_\mu \sigma_\alpha,$$

where the $\sigma'_\alpha s$ are the Pauli matrices, and $\varphi^\alpha, A^\alpha_\mu$ are realvalued functions on $\mathbb{R}^2$. Then

$$(D_{A_\mu})_{ij} = \delta_{ij} \partial_\mu + \varepsilon_{ijk} A^k_\mu \tag{B.1}$$

where ε_{ijk} is antisymmetric in i, j and k and $\varepsilon_{123} = 1$, acting on $\begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{pmatrix}$, which is an $\mathbb{R}^3$-valued function on $\mathbb{R}^2$.

Also we have that

$$\mathrm{tr}(X^* Y) = \sum_\alpha \varphi^\alpha \psi^\alpha$$

if $X = \sum_\alpha \varphi^\alpha \sigma_\alpha$ and $Y = \sum_\alpha \psi^\alpha \sigma_\alpha$.

From this we see that, expressed in the fields φ^α, the measure of the model is the Gaussian measure with covariance $C_A = (-\Delta_A)^{-1}$, where $-\Delta_A$ is given by

$$\begin{aligned} -\Delta_A &= \sum_\mu D^*_{A_\mu} D_{A_\mu} \\ &= \begin{pmatrix} -\Delta + (A^3)^2 + (A^2)^2 & -\{\nabla, A^3\} - A^2 \cdot A^1 & \{\nabla, A^2\} - A^3 \cdot A^1 \\ \{\nabla, A^3\} - A^2 \cdot A^1 & -\Delta + (A^3)^2 + (A^1)^2 & -\{\nabla, A^1\} - A^2 \cdot A^3 \\ -\{\nabla, A^2\} - A^3 \cdot A^1 & -\{\nabla, A^1\} - A^3 \cdot A^2 & -\Delta + (A^1)^2 + (A^2)^2 \end{pmatrix}, \end{aligned}$$

as a direct calculation shows by using (B.1). Here $A^i \cdot A^j = \sum_\mu A^i_\mu A^j_\mu$ and $\{\nabla, A^i\} = \sum_\mu (\partial_u A^i_\mu + A^i_\mu \partial_\mu)$.

To simplify this expression we set $A^1 = A^2 = 0$ and $A^3 = \vec{A} \neq 0$. Then (B.2) becomes

$$-\Delta_{\vec{A}} = \begin{pmatrix} -\Delta + \vec{A}^2 & -\{\nabla, \vec{A}\} & 0 \\ \{\nabla, \vec{A}\} & -\Delta + \vec{A}^2 & 0 \\ 0 & 0 & -\Delta \end{pmatrix}.$$

But this operator is unitary equivalent to the diagonal operator C with $C_{11} = -\Delta + i\{\nabla, \vec{A}\} + \vec{A}^2$, $C_{22} = -\Delta - i\{\nabla, \vec{A}\} + \vec{A}^2$, $C_{33} = -\Delta$, the equivalence being given by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next we choose $\vec{A}(x_0, x_1) = B(x_1, -x_0)$. With this choice of $\vec{A}$ it is well known (see e.g. [39]) that the operators $-\Delta \pm i\{\nabla, \vec{A}\} + \vec{A}^2$ have the same spectrum, and it is bounded below by a strictly positive number if $B \neq 0$ (Landau diamagnetism). Thus C_A, restricted to the subspace $\{(\varphi^1, \varphi^2, 0)\}$ is a bounded operator.

From this we conclude that if $\langle - \rangle$ denotes the expectation of the model, then

$$\begin{aligned} \langle e^{i\varphi^1(0)\sigma_1} e^{-i\varphi^1(x)\sigma_1} \rangle &= \langle e^{i(\varphi^1(0) - \varphi^1(x))\sigma_1} \rangle \\ &= \langle \cos(\varphi^1(0) - \varphi^1(x))\mathbb{1} + i \sin(\varphi^1(0) - \varphi^1(x))\sigma^1 \rangle \\ &= \langle \cos(\varphi^1(0) - \varphi^1(x)) \rangle \mathbb{1} \\ &= e^{-(C_{\vec{A}}(0,0) + C_{\vec{A}}(x,x) - 2C_{\vec{A}}(0,x))_{11}} \mathbb{1} \end{aligned} \tag{B.3}$$

does not converge to 0 as $|x| \to 0$ as a consequence of the boundedness of $C_{\vec{A}}$.

Finally, we remark that by fixing the gauge in the lattice theory such that $X = \varphi^1 \sigma_1$, only two-point functions of the form as in (B.3) will enter into the calculation of the expectation of the Wilson loop observable according to (1.34).

Thus, Landau diamagnetism may destroy clustering of the two-point function of the two-dimensional, Gaussian σ-model. This conclusion is not affected by the introduction of a two-dimensional, spatial lattice.

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