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Positivity and Monotonicity Properties of C_0-Semigroups. II

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Abstract. If $S_t = \exp\{-tH\}$, $T_t = \exp\{-tK\}$, are self-adjoint positivity preserving semigroups on a Hilbert space $\mathscr{H} = L^2(X; d\mu)$ we write

$$T_t \succ 0 \tag{$*$}$$

if T_t is positivity improving and

$$S_t \succ T_t \tag{$**$}$$

if the difference $S_t - T_t$ is positivity improving. We derive a variety of characterizations of $(*)$ and $(**)$. In particular $(*)$ is valid for all $t > 0$ if, and only if, $T_t \cup L^\infty(X; d\mu)$ is irreducible for some $t > 0$. Similarly if the semigroups are ordered the strict order $(**)$ is valid if, and only if, $\{S_t - T_t\} \cup L^\infty(X; d\mu)$ is irreducible for some $t > 0$. These criteria are used to prove that if $(*)$ is valid for all $t > 0$ then

$$e^{-tf(K)} \succ 0, \quad t > 0,$$

and if $(**)$ is valid for all $t > 0$ then

$$e^{-tf(H)} \succ e^{-tf(K)}, \quad t > 0$$

for each non-constant f in the class characterized in the preceding paper.

We discuss the decomposition of positivity preserving semigroups in terms of positivity improving semigroups on subspaces. Various applications to monotonicity properties of Green's functions are given.

Introduction

A bounded operator A on the Hilbert space $\mathscr{H} = L^2(X; d\mu)$ is called positivity improving if

$$(\phi, A\psi) > 0$$

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for all non-negative ϕ, ψ, which are not identically zero and if this is the case we write

$$A \succ 0 .$$

More generally if B and $A-B$ are positivity improving we write

$$A \succ B \succ 0 .$$

These definitions give the strict ordering associated with the order relation $\succcurlyeq$ introduced in the preceding paper [1] (which we refer to as I).

We are interested in C_0-semigroups of self-adjoint positivity preserving operators which satisfy

$$e^{-tH} \succ e^{-tK} \succ 0$$

for all $t>0$, or for some $t>0$.

We derive two kinds of results. The first characterizes the strict ordering of the semigroups and the second derives stability of this order under the replacement of H and K by $f(H)$ and $f(K)$ where f is a non-constant function in the class characterized in I. For example we establish the following

Theorem D. *The following four conditions are equivalent*
1.(1′) $e^{-tK} \succ 0$ *for all* $t>0$ (*for some* $t>0$).
2.(2′) $\{e^{-tK}\}_{t>0} \cup L^\infty(X; d\mu)$ *is irreducible*
($e^{-tK} \cup L^\infty(X; d\mu)$ *is irreducible for some* $t>0$).

Moreover if

$$e^{-tH} \succcurlyeq e^{-tK} \succcurlyeq 0$$

the following four conditions are equivalent
1.(1′) $e^{-tH} \succ e^{-tK}$ *for all* $t>0$ (*for some* $t>0$).
2.(2′) $\{e^{-tH} - e^{-tK}\}_{t>0} \cup L^\infty(X; d\mu)$ *is irreducible*
($\{e^{-tH} - e^{-tK}\} \cup L^\infty(X; d\mu)$ *is irreducible for some* $t>0$).

Other characterizations of the strict order are given in terms of positivity improving relations or irreducibility criteria involving the resolvents of H and K.

The second kind of result is illustrated by the following

Theorem E. *Let f be a positive measurable function on* $[0, \infty)$ *such that*

$$e^{-tH} \succcurlyeq e^{-tK} \succcurlyeq 0, \quad t>0,$$

implies

$$e^{-tf(H)} \succcurlyeq e^{-tf(K)} \succcurlyeq 0, \quad t>0,$$

for all pairs of contraction semigroups and assume f is not constant.

It follows that

$$e^{-tH} \succ 0, \qquad t>0,$$

if, and only if

$$e^{-tf(H)} \succ 0, \qquad t>0,$$

and

$$e^{-tH} \succ e^{-tK}, \qquad t>0,$$

if, and only if,

$$e^{-tf(H)} \succ e^{-tf(K)}, \qquad t>0,$$

The class of functions described in Theorem E have been completely characterized in I.

1. Criteria for Positivity Improvement

We begin by deriving various criteria for a semigroup to be positivity improving (and in particular we prove the first half of Theorem D). Some of the implications in the following theorem are already known (see [2], Sect. XIII.12 for details and references). The principle new result appears to be the deduction of positivity improvement from an irreducibility criterion without any ancillary spectral assumptions.

Recall first that if T is a C_0-semigroup then there exist $M \geqq 1$ and $\beta \geqq 0$ such that

$$\|T_t\| \leqq M e^{\beta t}, \qquad t>0.$$

In the sequel the symbols M and β are used solely in this context.

Theorem 1. *Let $T_t = \exp\{-tH\}$ be a C_0-semigroup of self-adjoint positivity preserving operators on the Hilbert space $\mathscr{H} = L^2(X; d\mu)$.*

The following eight conditions are equivalent

1.(1′) $T_t \succ 0$ *for all* $t>0$ (*for some* $t>0$).
2.(2′) $\{T_t\}_{t>0} \cup L^\infty(X; d\mu)$ *is irreducible* ($T_t \cup L^\infty(X; d\mu)$ *is irreducible for some* $t>0$).
3.(3′) $(\lambda \mathbb{1} + H)^{-1} \succ 0$ *for all* $\lambda > \beta$ (*for some* $\lambda > \beta$).
4.(4′) $\{(\lambda \mathbb{1} + H)^{-1}\}_{\lambda > \beta} \cup L^\infty(X; d\mu)$ *is irreducible* ($(\lambda \mathbb{1} + H)^{-1} \cup L^\infty(X; d\mu)$ *is irreducible for some* $\lambda > \beta$).

Proof. Clearly $1 \Rightarrow 1'$, and $3 \Rightarrow 3'$. But $2 \Leftrightarrow 4$ by the standard semigroup relations

$$(\lambda \mathbb{1} + H)^{-1} = \int_0^\infty dt\, e^{-\lambda t} e^{-tH}$$

and

$$e^{-tH} = \lim_{n \to \infty} \left(\mathbb{1} + \frac{t}{n} H \right)^{-n}.$$

Also the equivalences of 2,2′, and 4,4′, are clear from the spectral theory of self-adjoint operators.

We will demonstrate that $2 \Rightarrow 1$, $1' \Rightarrow 3$ and $3' \Rightarrow 4$. This will complete the proof of equivalence.

First we need the following lemma which is essentially contained in [3].

Lemma 2. *If* $T_t = \exp\{-tH\}$ *is a* C_0*-semigroup of self-adjoint positivity preserving operators on* $\mathscr{H} = L^2(X; d\mu)$ *and* $\xi, \eta \geqq 0$ *with* $(\xi, \eta) > 0$ *then*

$$(\xi, T_t\eta) > 0 \quad \textit{for all} \quad t > 0.$$

Proof. If ϱ is defined by setting

$$\varrho(x) = \min(\xi(x), \eta(x))$$

then $\varrho \geqq 0$ and ϱ is not identically zero because $(\xi, \eta) > 0$. But

$$\begin{aligned}(\xi, T_t\eta) &\geqq (\xi, T_t\varrho)\\ &\geqq (\varrho, T_t\varrho) > 0.\end{aligned}$$

Now we return to the proof of Theorem 1.

$2 \Rightarrow 1$. Assume Condition 1 is false then there exist non-negative non-zero ϕ, ψ, and a $t_0 > 0$ such that

$$(\phi, T_{t_0}\psi) = 0.$$

Now suppose that

$$(\phi, T_s\psi) \neq 0$$

for some $s \in (0, t_0)$ then we can apply Lemma 2 with $\phi = \xi$, $T_s\psi = \eta$ and $t = t_0 - s$ to deduce that

$$(\phi, T_{t_0 - s}T_s\psi) = (\phi, T_{t_0}\psi) \neq 0$$

which is a contradiction. Therefore

$$(\phi, T_s\psi) = 0$$

for all $s \in (0, t_0]$.

But

$$z \in \mathbb{C} \to (\phi, e^{-zH}\psi)$$

is analytic in the right half plane and hence it must be identically zero.

Next let $\mathscr{K}$ be the closed linear span of

$$\{fT_t\psi; t > 0, f \in L^\infty(X; d\mu)\}.$$

We shall show that if $\chi \in \mathscr{K}$ then $T_t\chi \in \mathscr{K}$ for all $t \geqq 0$ and, moreover,

$$(\phi, \chi) = 0.$$

Thus $\mathscr{K}$ is a non-trivial closed subspace which is invariant under $\{T_t\}_{t \geqq 0} \cup L^\infty(X; d\mu)$ and hence Condition 2 is false.

It clearly suffices to prove the above properties for vectors of the form

$$\chi = \sum_{i=1}^{n} f_i T_{t_i}\psi,$$

where the f_i are real functions in $L^\infty(X; d\mu)$ and $t_i > 0$. But

$$\sum_{i=1}^{n} \|f_i\|_\infty T_{t_i}\psi \pm \chi \geqq 0$$

and hence

$$|T_t\chi| \leqq \sum_{i=1}^{n} \|f_i\|_\infty T_{t+t_i}\psi_i .$$

Now this inequality shows that the left hand vector $T_t\chi$ can be obtained from the right hand vector by multiplication with an L^∞-function. Thus $T_t\chi \in \mathscr{K}$. Finally

$$\begin{aligned} |(\phi, \chi)| &\leqq (\phi, |\chi|) \\ &\leqq \sum_{i=1}^{n} \|f_i\|_\infty (\phi, T_{t_i}\psi) = 0 . \end{aligned}$$

$1' \Rightarrow 3$. Suppose Condition 3 is false. Then there exist non-negative non-zero ϕ, ψ, and a $\lambda > \beta$, such that

$$0 = (\phi, (\lambda \mathbb{1} + H)^{-1}\psi) = \int_0^\infty dt e^{-\lambda t}(\phi, T_t\psi) .$$

Hence Condition $1'$ is false.

$3' \Rightarrow 4$. Suppose Condition 4 is false and let $D \subset \mathscr{H}$ be a closed subspace invariant under $\{(\lambda \mathbb{1} + H)^{-1}\}_{\lambda > \beta} \cup L^\infty(X; d\mu)$. Let $\psi \in D$ and set $\chi(x) = \bar{\psi}(x)/|\psi(x)|$ if $\psi(x) \neq 0$ and $\chi(x) = 0$ if $\psi(x) = 0$. Thus $\chi \in L^\infty(X; d\mu)$ and since $|\psi| = \chi\psi$ one has $|\chi| \in D$. Similarly if $\phi \in D^\perp$ then $|\phi| \in D^\perp$. But since D is invariant under $(\lambda \mathbb{1} + H)^{-1}$ for all $\lambda > \beta$ one has

$$(|\phi|, (\lambda \mathbb{1} + H)^{-1}|\psi|) = 0$$

even if ϕ, ψ, are not identically zero. Thus Condition $3'$ is false.

Theorem 1 allows us to extend a perturbation result of Segal [4] (see [2], Theorem XIII.45).

Corollary 3. *Let H and H_0 be self-adjoint lower semi-bounded operators on $L^2(X; d\mu)$ and suppose there exists a sequence of bounded self-adjoint multiplication operators V_n so that $H_0 + V_n$ converges to H in the strong resolvent sense and $H - V_n$ converges to H_0 in the same sense.*

It follows that $\exp\{-tH\}$ *is positivity improving if, and only if,* $\exp\{-tH_0\}$ *is positivity improving.*

Proof. By Segal's result, which follows from the Trotter product formula, the set $\exp\{-tH\} \cup L^\infty(X; d\mu)$ is irreducible if, and only if, $\exp\{-tH_0\} \cup L^\infty(X; d\mu)$ is irreducible so Corollary 3 follows from Theorem 1.

2. Comparison of Semigroups

Theorem 1 gives criteria for a C_0-semigroup to be positivity improving. Next we derive a generalization which gives criteria for the difference of two semigroups to be positivity improving. In particular we prove the second statement of Theorem D.

Theorem 4. *Let* $S_t = \exp\{-tH\}$, $T_t = \exp\{-tK\}$, *be two* C_0*-semigroups of self-adjoint positivity preserving operators on the Hilbert space* $\mathscr{H} = L^2(X; d\mu)$ *and suppose that*

$$S_t \succcurlyeq T_t$$

for all $t \geqq 0$.

The following conditions are equivalent

1.(1′) $S_t \succ T_t$ *for all* $t > 0$ (*for some* $t > 0$).
2.(2′) $\{S_t - T_t\}_{t>0} \cup L^\infty(X; d\mu)$ *is irreducible* ($\{S_t - T_t\} \cup L^\infty(X; d\mu)$ *is irreducible for some* $t > 0$).
3.(3′) $(\lambda\mathbb{1} + H)^{-1} \succ (\lambda\mathbb{1} + K)^{-1}$ *for all* $\lambda > \beta$ (*for some* $\lambda > \beta$).
4.(4′) $\{(\lambda\mathbb{1} + H)^{-1} - (\lambda\mathbb{1} + K)^{-1}\}_{\lambda > \beta} \cup L^\infty(X; d\mu)$ *is irreducible* ($\{(\lambda\mathbb{1} + H)^{-1} - (\lambda\mathbb{1} + K)^{-1}\} \cup L^\infty(X; d\mu)$ *is irreducible for some* $\lambda > \beta$).
5.(5′) $S_t K T_t - H S_t T_t \succ 0$ *for all* $t > 0$ (*for some* $t > 0$).

Proof. Clearly $1 \Rightarrow 1'$, $2' \Rightarrow 2$, $3 \Rightarrow 3'$, $4' \Rightarrow 4$, and $5 \Rightarrow 5'$.

The rest of the implications will be deduced from a series of lemmas which involve the following operators

$$A(s,t) = S_s K T_t - H S_s T_t.$$

Note that these operators are bounded if $s > 0$ and $t > 0$. Throughout these lemmas we adopt the assumptions of Theorem 4.

Lemma 5. *Suppose that*

$$(\phi, A(t_0, t_0)\psi) = 0$$

for some non-negative, non-zero, ϕ, ψ, *and a* $t_0 > 0$ *then*

$$(\phi, \{S_t - T_t\}\psi) = 0$$

for all $t > 0$ *and*

$$(\phi, A(s,t)\psi) = 0 \tag{$*$}$$

for all $s, t > 0$.

Proof. Suppose that there are $s_1, t_1 \in (0, t_0)$ such that

$$(\phi, A(s_1, t_1)\psi) \neq 0.$$

Since we are assuming $S_t \succcurlyeq T_t$ for all $t > 0$ it follows from Theorem A of I that

$$(\xi, A(s,t)\eta) \geqq 0$$

for all non-negative ξ, η, and all $s, t > 0$. In particular

$$A(s_1, t_1)\psi \geqq 0$$

and hence

$$(\phi, A(s_1, t_1)\psi) > 0.$$

Therefore

$$(\phi, S_{t_0 - s_1} A(s_1, t_1)\psi) > 0$$

by Lemma 2. Equivalently

$$(\phi, A(t_0, t_1)\psi) > 0.$$

Another application of Lemma 2 then gives

$$(\phi, A(t_0, t_1) T_{t_0 - t_1}\psi) > 0$$

or, equivalently,

$$(\phi, A(t_0, t_0)\psi) > 0$$

which is a contradiction.

Therefore

$$(\phi, A(s, t)\psi) = 0$$

for all $s, t \in (0, t_0]$. But

$$z_1, z_2 \in \mathbb{C} \to (\phi, A(z_1, z_2)\psi)$$

is analytic for $\operatorname{Re} z_1 > 0$, and $\operatorname{Re} z_2 > 0$, and hence

$$(\phi, A(s, t)\psi) = 0$$

for all $s, t > 0$. This is the second statement of the lemma. The first follows from the second by the Duhamel formula;

$$\begin{aligned}(\phi, \{S_t - T_t\}\psi) &= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1-\varepsilon} d\lambda \frac{d}{d\lambda}(\phi, S_{\lambda t} T_{(1-\lambda)t}\psi) \\ &= \lim_{\varepsilon \to 0} t \int_{\varepsilon}^{1-\varepsilon} d\lambda (\phi, A(\lambda t, (1-\lambda)t)\psi) \\ &= 0.\end{aligned}$$

Lemma 6. *Suppose that*

$$(\phi, \{S_{t_0} - T_{t_0}\}\psi) = 0$$

for some non-negative, non-zero, ϕ, ψ, and a $t_0 > 0$ then

$$(\phi, \{S_t - T_t\}\psi) = 0$$

for all $t > 0$ and

$$(\phi, A(s, t)\psi) = 0$$

for all $s, t > 0$.

Proof. Once again

$$(\phi, \{S_t - T_t\}\psi) = \lim_{\varepsilon \to 0} t \int_{\varepsilon}^{1-\varepsilon} d\lambda (\phi, A(\lambda t, (1-\lambda)t)\psi).$$

But since $S_t \succcurlyeq T_t$ the integrand is non-negative by Theorem A of I. Therefore

$$0 = (\phi, \{S_{t_0} - T_{t_0}\}\psi) \geqq t_0 \int_{\varepsilon}^{\delta} d\lambda (\phi, A(\lambda t_0, (1-\lambda)t_0)\psi) \geqq 0$$

for any $1 > \delta > \varepsilon > 0$. Since the integrand is non-negative and continuous this implies that

$$(\phi, A(\lambda t_0, (1-\lambda)t_0)\psi) = 0$$

for $0 < \lambda < 1$. Choosing $\lambda = 1/2$ one has

$$(\phi, A(t_0/2, t_0/2)\psi) = 0$$

and the desired result follows from Lemma 5.

Lemma 7. *Suppose that*

$$(\phi, A(s,t)\psi) = 0$$

for some non-negative, non-zero, ϕ, ψ, *and all* $s, t > 0$ *then*

$$(\phi, \{(\lambda \mathbb{1} + H)^{-1} - (\lambda \mathbb{1} + K)^{-1}\}\psi) = 0$$

for all $\lambda > \beta$.

Proof. This follows from the identities

$$\begin{aligned} &\int_0^\infty ds \int_0^\infty dt\, e^{-\lambda s - \lambda t} (\phi, A(s,t)\psi) \\ &\quad = ((\lambda \mathbb{1} + H)^{-1}\phi, K(\lambda \mathbb{1} + K)^{-1}\psi) - (H(\lambda \mathbb{1} + H)^{-1}\phi, (\lambda \mathbb{1} + K)^{-1}\psi) \\ &\quad = (\phi, \{(\lambda \mathbb{1} + H)^{-1} - (\lambda \mathbb{1} + K)^{-1}\}\psi). \end{aligned}$$

Lemma 8. *Suppose that*

$$(\phi, \{(\lambda_0 \mathbb{1} + H)^{-1} - (\lambda_0 \mathbb{1} + K)^{-1}\}\psi) = 0$$

for some non-negative, non-zero, ϕ, ψ, *and some* $\lambda_0 > \beta$ *then*

$$(\phi, \{(\lambda \mathbb{1} + H)^{-1} - (\lambda \mathbb{1} + K)^{-1}\}\psi) = 0$$

for all $\lambda > \beta$ *and*

$$(\phi, \{S_t - T_t\}\psi) = 0$$

for all $t > 0$.

Proof. Since $S_t \succcurlyeq T_t$ the operator $A(s,t)$ is positivity preserving by Theorem A of I. Hence the desired result follows from the identity preceding the lemma and Duhamel formula given before Lemma 6.

Now let us return to the proof of Theorem 4. If Condition 1 is false then Conditions 1′ and 5′ are false by Lemma 6 and Condition 3′ is false by Lemmas 6 and 7. Thus 1′⇒1, 5′⇒1, 3⇒1. If Condition 5 is false then Conditions 5′, 1′, and 3′ are false by Lemmas 5 and 7. Thus 1′⇒5, 5′⇒5, 3′⇒5. But 3′⇒3, and 1′⇒3 by Lemma 8. This completes the proof of the equivalence of Conditions 1, 3, 5 and their variants 1′, 3′, 5′.

1⇒2′. Suppose Condition 2′ is false and let $D\subset\mathscr{H}$ be a closed subspace left invariant by $\{S_t-T_t\}\cup L^\infty(X;d\mu)$. One argues as in the proof of 1⇒2′ in Theorem 1 that there exist non-negative, non-zero, ϕ, ψ, in $D^\perp$, and D, respectively, and by the invariance of D one has

$$(\phi,\{S_t-T_t\}\psi)=0,$$

i.e. Condition 1 is false.

3⇒4′. This is a repetition of the previous proof, 1⇒2′, with S_t-T_t replaced by $(\lambda\mathbb{1}+H)^{-1}-(\lambda\mathbb{1}+K)^{-1}$.

4⇒2. If Condition 2 is false then Condition 4 is false because of the Laplace transform relation

$$(\lambda\mathbb{1}+H)^{-1}-(\lambda\mathbb{1}+K)^{-1}=\int_0^\infty dt\, e^{-\lambda t}\{S_t-T_t\}.$$

2⇒1. Suppose Condition 1 is false. Then there exist non-negative, non-zero, ϕ, ψ, and a $t_0>0$ such that

$$(\phi,\{S_{t_0}-T_{t_0}\}\psi)=0$$

and hence, by Lemma 6,

$$(\phi,A(s,t)\psi)=0$$

for all $s,t>0$. But since

$$S_{s_0}A(s,t)=A(s+s_0,t)$$

one also has

$$(S_{s_0}\phi,A(s,t)\psi)=0$$

for all $s,s_0,t>0$. Now let $\mathscr{K}$ be the closed linear span of

$$\{fA(s,t)\psi\,;f\in L^\infty(X;d\mu),s,t>0\}.$$

If one repeats the argument used to establish 2⇒1 in Theorem 1 one concludes that $\mathscr{K}$ is invariant under $\{S_t\}_{t>0}\cup L^\infty(X;d\mu)$ and $\phi\in\mathscr{K}^\perp$. But if $\xi\in\mathscr{K}^\perp$, $\eta\in\mathscr{K}$, are non-negative

$$0=(\xi,S_t\eta)\geqq(\xi,T_t\eta)\geqq 0$$

for all $t>0$ and hence $\mathscr{K}$ is also invariant under $\{T_t\}_{t\geqq 0}$. Thus $\mathscr{K}$ is invariant under $\{S_t-T_t\}_{t\geqq 0}\cup L^\infty(X;d\mu)$ and Condition 2 is false unless $\mathscr{K}=\{0\}$ or $\mathscr{K}=\mathscr{H}$. But $\mathscr{K}\neq\mathscr{H}$ because $\phi\in\mathscr{K}^\perp$ and ϕ is non-zero. Thus it remains to discuss the case $\mathscr{K}=\{0\}$.

If $\mathscr{K}=\{0\}$ one must have

$$A(s,t)\psi=0$$

for all $s,t>0$. Hence for $\chi\in\mathscr{H}$

$$\begin{aligned}(\chi,\{S_s-T_s\}T_t\psi) &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} d\lambda\frac{d}{d\lambda}(S_{\lambda s}\chi, T_{(1-\lambda)s}T_t\psi)\\ &= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} d\lambda(\chi, A(\lambda s,(1-\lambda)s+t)\psi)\\ &=0.\end{aligned}$$

Therefore

$$\{S_s-T_s\}T_t\psi=0$$

for all $s,t>0$.

Next let $\mathscr{D}$ be the closed linear span of

$$\{fT_t\psi\,;t>0, f\in L^\infty(X\,;d\mu)\}.$$

We argue that $\mathscr{D}$ is left invariant by S_t, and T_t, and that

$$S_t|_{\mathscr{D}}=T_t|_{\mathscr{D}}, \qquad t>0.$$

Let

$$\chi=\sum_{i=1}^{n} f_iT_{t_i}\psi,$$

where the f_i are real. Since

$$\sum_{i=1}^{n}\|f_i\|_\infty T_{t_i}\psi\pm\chi\geqq 0$$

one has

$$|T_t\chi|\leqq\sum_{i=1}^{n}\|f_i\|_\infty T_{t+t_i}\psi$$

and

$$\pm(S_s-T_s)\chi\leqq\sum_{i=1}^{n}\|f_i\|_\infty(S_s-T_s)T_{t_i}\psi=0.$$

Thus $T_t\chi\in\mathscr{D}$ and $S_s\chi=T_s\chi\in\mathscr{D}$ for all $s,t>0$. In conclusion $\mathscr{D}$ is invariant under $\{S_s-T_s\}_{s>0}\cup L^\infty(X\,;d\mu)$ and

$$\{S_s-T_s\}_{s>0}\cup L^\infty(X\,;d\mu)|_{\mathscr{D}}=L^\infty(X\,;d\mu)|_{\mathscr{D}}\neq\{0\}.$$

Hence Condition 2 is again false.

The foregoing proof has one rather surprising corollary.

Corollary 9. *Let S and T be two self-adjoint C_0-semigroups on the Hilbert space $\mathscr{H} = L^2(X; d\mu)$ satisfying*

$$S_t \geqslant T_t \geqslant 0$$

for all $t > 0$. If

$$\text{either } S_t \succ 0 \quad \text{or} \quad T_t \succ 0 \text{ for some } t > 0$$

then

$$\text{either } S_t \succ T_t \quad \text{or} \quad S_t = T_t \text{ for all } t > 0.$$

Proof. Obviously $T_t \succ 0$ implies $S_t \succ 0$ and hence we need only consider this latter case. Now suppose that S_t is not strictly larger than T_t for all $t > 0$, i.e., suppose there exist non-negative, non-zero, ϕ, ψ, and a $t_0 > 0$ such that

$$(\phi, \{S_{t_0} - T_{t_0}\}\psi) = 0.$$

Repeating the beginning of the argument used to prove 2⇒1 in Theorem 4 we conclude that the closed linear span $\mathscr{K}$ of

$$\{f A(s, t)\psi ; f \in L^\infty(X; d\mu), s, t > 0\}$$

is invariant under $\{S_t\}_{t>0} \cup L^\infty(X; d\mu)$. But since $S_t \succ 0$ this latter set is irreducible and hence $\mathscr{K} = \{0\}$ or $\mathscr{K} = \mathscr{H}$. But $\phi \in \mathscr{K}^\perp$ and therefore the only possibility is $\mathscr{K} = \{0\}$, i.e., one must have

$$A(s, t)\psi = 0$$

for all $s, t > 0$.

But appealing once more to the proof of 2⇒1 in Theorem 4 we conclude that the closed linear span $\mathscr{D}$ of

$$\{f T_t \psi ; t > 0, f \in L^\infty(X; d\mu)\}$$

is left invariant by the irreducible set $\{S_t\}_{t>0} \cup L^\infty(X; d\mu)$ and moreover

$$S_t|_{\mathscr{D}} = T_t|_{\mathscr{D}}.$$

But $\mathscr{D} \neq \{0\}$ because it contains ψ. Hence $\mathscr{D} = \mathscr{H}$ and

$$S_t = T_t$$

for all $t > 0$.

This last conclusion can be extended to the comparison of semigroups on different spaces. In I we considered the situation of a space $\mathscr{H} = L^2(X; d\mu)$, a subspace $\mathscr{K} = L^2(Y; d\nu)$ where $Y \subset X$ and $\nu = \mu|_Y$, and two semigroups, S on $\mathscr{H}$, and T on $\mathscr{K}$. We then defined

$$S_t \geqslant T_t \geqslant 0$$

by demanding

$$(\phi, S_t \psi) \geqq (\phi, T_t \psi) \geqq 0$$

for all non-negative $\phi\in\mathscr{H}$, $\psi\in\mathscr{K}$ (or, equivalently, $\phi,\psi\in\mathscr{K}$). Similarly we may define

$$S_t \succ T_t$$

by demanding

$$(\phi, S_t\psi) > (\phi, T_t\psi)$$

for all non-negative, non-zero, $\phi\in\mathscr{H}$, $\psi\in\mathscr{K}$

Corollary 9′. *Let S and T be two self-adjoint C_0-semigroups on the Hilbert space $\mathscr{H}=L^2(X;d\mu)$ and $\mathscr{K}=L^2(Y;dv)$ respectively where $Y\subset X$ and $v=\mu|_Y$. Assume that*

$$S_t \succcurlyeq T_t \succcurlyeq 0$$

for all $t>0$. If

$$S_t \succ 0 \quad \textit{for some } t>0$$

then either

$$S_t \succ T_t$$

or

$$\mathscr{H}=\mathscr{K} \quad \textit{and} \quad S_t=T_t, \textit{ for all } t>0.$$

The proof is identical to the proof of Corollary 9 once one remarks that Lemmas 2,5, and 6–8 all have two space analogues, i.e. one can assume ξ, $\phi\in\mathscr{H}$ and $\eta,\psi\in\mathscr{K}$ and the conclusions remain valid.

The characterization of the ordering relation on the semigroups by irreducibility criteria opens the way for a decomposition theory of positivity preserving semigroups into positivity improving semigroups acting on invariant subspaces. This theory corresponds to the decomposition of the von Neumann algebra

$$\mathscr{M}_S=\{S_t\cup L^\infty(X;d\mu)\}''$$

and the Hilbert space $\mathscr{H}$. But if $S_t\succcurlyeq T_t\succcurlyeq 0$ for all $t>0$ this decomposition automatically gives a decomposition of $\mathscr{M}_T$ by the following.

Theorem 10. *Let S and T be two self-adjoint C_0-semigroups on the Hilbert space $\mathscr{H}=L^2(X;d\mu)$ satisfying*

$$S_t \succcurlyeq T_t \succcurlyeq 0$$

for all $t>0$.

It follows that

$$\{S_t\cup L^\infty(X;d\mu)\}''\supseteq\{T_t\cup L^\infty(X;d\mu)\}''.$$

Proof. It suffices to show that

$$\{S_t\cup L^\infty(X;d\mu)\}'\subseteq\{T_t\cup L^\infty(X;d\mu)\}'.$$

But each of the commutants is contained in $L^\infty(X;d\mu)$ thus it suffices to show that if $f\in L^\infty(X;d\mu)$ is a projection satisfying

$$fS_t(1-f)=0$$

then

$$fT_t(1-f)=0.$$

But if ϕ, ψ, are non-negative then $f\phi$, and $(1-f)\psi$, are non-negative and the ordering $S_t \succcurlyeq T_t \succcurlyeq 0$ gives

$$\begin{aligned}0&=(\phi, fS_t(1-f)\psi)\\ &\geqq(\phi, fT_t(1-f)\psi)\geqq 0\end{aligned}$$

which is the desired result.

It follows from Corollary 9 and Theorem 10 that if $\{S_t\cup L^\infty(X;d\mu)\}''$ is decomposed into irreducible components then the decomposition reduces T_t and in each component either $S_t \succ T_t$ or $S_t=T_t$ for all $t>0$. The conclusion of Corollary 9 could be re-expressed in an algebraic fashion; if $\{S_t\cup L^\infty(X;d\mu)\}''$ is irreducible then $\{(S_t-T_t)\cup L^\infty(X;d\mu)\}''$ is either irreducible or maximal abelian.

Theorem 10 also has an analogue for pairs of semigroups which satisfy the domination property discussed by Hess et al. [6] and Simon [7].

Theorem 10′. *Let S and T be two self-adjoint C_0-semigroups on the Hilbert space $\mathscr{H}=L^2(X;d\mu)$ and suppose*

$$S_t|\psi|\geqq|T_t\psi|$$

for all $\psi\in\mathscr{H}$ and all $t>0$.

It follows that

$$\{S_t\cup L^\infty(X;d\mu)\}''\supseteq\{T_t\cup L^\infty(X;d\mu)\}''.$$

Proof. The estimate in the proof of Theorem 10 is replaced by the calculation

$$\begin{aligned}|(\phi, fT_t(1-f)\psi)|&\leqq(\phi, f|T_t(1-f)\psi|)\\ &\leqq(\phi, fS_t|(1-f)\psi|)\\ &=(\phi, fS_t(1-f)\psi)=0\end{aligned}$$

for $\phi,\psi\geqq 0$.

This result is also valid for the class of non-self-adjoint T considered in [6].

3. Functions of Generators

Next we examine the contraction semigroups $\exp\{-tf(H)\}$ obtained from a C_0-semigroup of self-adjoint positivity preserving contractions $\exp\{-tH\}$ by replacing the generator H by $f(H)$ where f is one of the functions characterized in I. It is not strictly necessary to assume that $\exp\{-tH\}$ is a contraction semigroup. It would suffice to assume that H is lower semibounded, i.e. $H\geqq -c\mathbb{1}$ for some $c\geqq 0$. It is however straightforward to extend our results to this more general case. One simply examines the contraction semigroup $\exp\{-t(H+c\mathbb{1})\}$ and obtains statements about $\exp\{-tf(H+c\mathbb{1})\}$. But $\exp\{-tH\}$ and $\exp\{-t(H+c\mathbb{1})\}$ are simultaneously positivity improving and $\exp\{-tH\}\cup L^\infty(X; d\mu)$ and $\exp\{-t(H+c\mathbb{1})\}\cup L^\infty(X; d\mu)$ are simultaneously irreducible, etc.

In the statement of the next theorem, which is an elaboration of Theorem *E*, we give explicit conditions on f which ensure that it is in the class considered in I. Other characterizations are given in Theorem B of I.

Theorem 11. *Let $S_t=\exp\{-tH\}$, $T_t=\exp\{-tK\}$, be C_0-semigroups of self-adjoint positivity preserving contractions on the Hilbert space $\mathscr{H}=L^2(X; d\mu)$ and let $f\in C^\infty(0,\infty)$ be a non-negative, non-constant, function such that*

$$(-1)^n f^{(n+1)}(x)\geqq 0, x\in(0,\infty), n=0,1,2,\ldots$$

and

$$f(0)\leqq \lim_{x\to 0+} f(x).$$

Define S^f and T^f by

$$S_t^f=\exp\{-tf(H)\}, T_t^f=\exp\{-tf(K)\}.$$

The following four conditions are equivalent

1.(1′) $T_t \succ 0$ *for all* $t>0$ (*for some* $t>0$).
2.(2′) $T_t^f \succ 0$ *for all* $t>0$ (*for some* $t>0$).

Moreover if

$$S_t \succcurlyeq T_t$$

for all $t>0$ then the following four conditions are equivalent

1.(1′) $S_t \succ T_t$ *for all* $t>0$ (*for some* $t>0$).
2.(2′) $S_t^f \succ T_t^f$ *for all* $t>0$ (*for some* $t>0$).

Proof. Consider the first statement. By Theorem 1, Conditions 1 and 1′, are equivalent to irreducibility of $\{T_t\}_{t>0}\cup L^\infty(X; d\mu)$ and Conditions 2 and 2′, are equivalent to irreducibility of $\{T_t^f\}_{t>0}\cup L^\infty(X; d\mu)$. Thus we must show that these two sets are simultaneously irreducible and for this it suffices to show that they generate the same von Neumann algebra. But for this it suffices to show that $\{T_t\}_{t>0}$ and $\{T_t^f\}_{t>0}$ generate the same von Neumann algebra. This is straightforward.

Since f is not constant it follows from I that it is strictly increasing. Hence by spectral theory the von Neumann algebra generated by $\{T_t^f\}_{t>0}$ is the von Neumann algebra generated by the spectral family E_H associated with the self-adjoint operator H. This is of course the same as the algebra generated by $\{T_t\}_{t>0}$.

Now consider the second statement. Clearly $1\Rightarrow 1'$ and $2\Rightarrow 2'$. But $1'$ implies that $S_t\succ 0$ for some $t>0$ and hence $S_t^f\succ 0$ for all $t>0$ by the first part of the theorem. Hence $S_t^f\succ T_t^f$ or $S_t^f=T_t^f$ for all $t>0$ by Corollary 9. But the latter possibility implies $S_t=T_t$ for all $t>0$ which is a contradiction. Thus $1'\Rightarrow 2$. The proof of $2'\Rightarrow 1$ is similar.

4. Applications

One of the simplest self-adjoint positivity improving semigroups is the contraction semigroup associated with the heat equation. The generator H of this semigroup is the self-adjoint Laplacian $-\nabla^2$ on $L^2(\mathbb{R}^\nu)$ and

$$(e^{-tH}\psi)(x) = \int dy\, G_t(x-y)\,\psi(y),$$

where

$$G_t(x) = (4\pi t)^{-\nu/2} e^{-x^2/4t}.$$

The semigroup is positivity improving because $G_t > 0$. One can extend this result to semigroups with generators of the form $-\nabla^2 + V$ where V is a multiplication operator if the conditions of Corollary 3 are satisfied with $H_0 = -\nabla^2$ and $H = -\nabla^2 + V$. This is, for example, the case if V is a Rollnik potential (see [2], Chapters X and XIII, especially Theorem XIII.46). But then if V_1, and V_2, are two such operators and $V_2 \geqq V_1$, but $V_1 \neq V_2$, one has

$$e^{-t(-\nabla^2+V_1)} \succcurlyeq e^{-t(-\nabla^2+V_2)} \succ 0$$

by Theorem 5 of I and Corollary 3. But then it follows from Corollary 9 that in fact one has

$$e^{-t(-\nabla^2+V_1)} \succ e^{-t(-\nabla^2+V_2)} \succ 0.$$

Alternatively by applying Part 1 of Theorem 11 to $-\nabla^2$ and then repeating the above reasoning one then has

$$e^{-t(f(-\nabla^2)+V_1)} \succ e^{-t(f(-\nabla^2)+V_2)} \succ 0$$

for any f in the class characterized in I, e.g.

$$e^{-t((-\nabla^2+M^2)^\alpha+V_1)} \succ e^{-t((-\nabla^2+M^2)^\alpha+V_2)} \succ 0$$

for all $0 < \alpha \leqq 1$.

One can also draw conclusions for the Laplacian $-\nabla^2$ acting on $L^2(\Omega)$ where Ω is a connected bounded open such set of $\mathbb{R}^\nu$ with a suitably smooth boundary and we impose the classical boundary conditions $\partial\psi/\partial n = \sigma\psi$ with $\sigma \in C(\partial\Omega)$. If H_Ω^σ denotes the corresponding self-adjoint operator and H_Ω^∞ denotes the Dirichlet Laplacian then we argued in I that for $\sigma_2 \geqq \sigma_1$

$$e^{-tH_\Omega^{\sigma_1}} \succcurlyeq e^{-tH_\Omega^{\sigma_2}}$$

and if $\Omega_2 \supset \Omega_1$

$$e^{-tH_{\Omega_2}^\infty} \succcurlyeq e^{-tH_{\Omega_1}^\infty} \succcurlyeq 0.$$

But if $\sigma_n \in C(\partial\Omega)$ is a sequence which increases monotonically to $+\infty$ it is known that $\exp\{-tH_\Omega^{\sigma_n}\}$ converges strongly to $\exp\{-tH_\Omega^\infty\}$ and since this convergence is monotonic with respect to the order $\succcurlyeq$ one has

$$e^{-tH_\Omega^{\sigma_1}} \succcurlyeq e^{-tH_\Omega^{\sigma_2}} \succcurlyeq e^{-tH_\Omega^\infty} \succcurlyeq 0.$$

But one can also establish that $\exp\{-tH_\Omega^\infty\}$ is positivity improving either by explicit calculation of the kernel of $\exp\{-H_{\Omega_0}^\infty\}$ for sufficiently many $\Omega_0 \subset \Omega$, or by functional integration techniques. But if $\sigma_2 \geqq \sigma_1$ and $\sigma_2 \neq \sigma_1$ then $H_\Omega^{\sigma_2} \neq H_\Omega^{\sigma_1}$. Moreover if $\Omega_2 \supset \Omega_1$ but $\Omega_2 \neq \Omega_1$ then $H_{\Omega_2}^\infty \neq H_{\Omega_1}^\infty$. Thus by Corollaries 9 and 9'

$$e^{-tH_{\Omega_2}^{\sigma_1}} \succ e^{-tH_{\Omega_2}^{\sigma_2}} \succ e^{-tH_{\Omega_2}^\infty} \succ e^{-tH_{\Omega_1}^\infty} \succ 0.$$

These conclusions can again be extended to semigroups of the form $\exp\{-t(f(H_\Omega^\sigma)+V)\}$ by repetition of the arguments used above for $\mathbb{R}^\nu$.

Appendix

A Monotone Convergence Theorem

Let $\geqq$ denote the usual order on the bounded self-adjoint operators on $\mathscr{H}$. If A_α is a net of positive operators and A a bounded operator such that

$$0 \leqq A_\alpha \leqq A_\beta \leqq A$$

whenever $\alpha \leqq \beta$ then it is well known that A_α converges strongly. An analogous result is valid for the order relation but it is not necessary that the operators involved are self-adjoint.

Theorem. *If A_α and A are bounded positivity preserving operators on the Hilbert space $\mathscr{H}$ and*

$$0 \preccurlyeq A_\alpha \preccurlyeq A_\beta \preccurlyeq A,$$

whenever $\alpha \leqq \beta$, then A_α converges strongly.

Proof. Since each $\psi \in \mathscr{H}$ is a linear combination of four non-negative vectors it suffices to prove that $\|(A_\alpha - A_\beta)\psi\|$ has the Cauchy property for $\psi \geqq 0$. But if $\psi \geqq 0$ then

$$\begin{aligned}\|(A_\alpha - A_\beta)\psi\|^2 &= \|A_\beta\psi\|^2 - \|A_\alpha\psi\|^2 - ((A_\beta - A_\alpha)\psi, A_\alpha\psi) - (A_\alpha\psi, (A_\beta - A_\alpha)\psi) \\ &\leqq \|A_\beta\psi\|^2 - \|A_\alpha\psi\|^2\end{aligned}$$

whenever $\alpha \leqq \beta$. Thus $\|A_\alpha\psi\|$ is a monotone increasing net of real numbers and, since $\psi \geqq 0$,

$$\|A_\alpha\psi\|^2 = (A_\alpha\psi, A_\alpha\psi) \leqq (A\psi, A\psi) = \|A\psi\|^2.$$

Thus $\|A_\alpha\psi\|$ is bounded above. Consequently $\|A_\beta\psi\|^2 - \|A_\alpha\psi\|^2$ must have the Cauchy property, and hence A_α converges strongly.

A similar result is valid if A_α is monotone decreasing, i.e. if

$$0 \preccurlyeq A_\beta \preccurlyeq A_\alpha$$

whenever $\beta > \alpha$.

Corollary. *Let T^α and T be C_0-semigroups of positivity preserving semigroups on the Hilbert space $\mathscr{H}$ and suppose either*

$$0 \preccurlyeq T_t^\alpha \preccurlyeq T_t^\beta \preccurlyeq T_t$$

for all $t>0$, *whenever* $\beta \geqq \alpha$, *or*

$$0 \preccurlyeq T_t \preccurlyeq T_t^\beta \preccurlyeq T_t^\alpha$$

for all $t \geqq 0$, *whenever* $\beta \geqq \alpha$.

It follows that T^α *converges strongly to a* $C_0^{\cdot}$*-semigroup of positivity preserving operators.*

The semigroup properties of the limit are a consequence of the strong convergence.

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