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The Equations of Wilson's Renormalization Group and Analytic Renormalization

II. Solution of Wilson's Equations

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Abstract. Wilson's renormalization group equations are introduced and investigated in the framework of perturbation theory with respect to the deviation of the renormalization exponent from its bifurcation value. An exact solution of these equations is constructed using analytic renormalization of the projection hamiltonians introduced in Paper I.

1. Introduction

This paper is a continuation of [1] hereafter referred to as I. Reference to equations or statements in I is made as follows: Eq. (I.3.1), Proposition I.5.1, etc. We directly pass here to solving the renormalization group (RG) equations in the framework of perturbation theory. The set-up of the paper is the following. First, in Sect. 2 we define the chain of Wilson's equations and find a set of bifurcation values of the RG parameter a. In Sect. 3 we consider the analytic continuation in a of a class of projection hamiltonians introduced in I. In Sect. 4 theorems on the analytic renormalization of projection hamiltonians of a special form are given and in Sect. 5 the RG transformation for these hamiltonians is described. These results enable us to construct in Sect. 6 a solution of the chain of Wilson's equations. In Sect. 7 and Appendix some auxiliary results are proved.

2. Wilson's Equations

These equations arise when one seeks nontrivial fixed points of the renormalization transformation near the bifurcation points. Before giving precise definitions, we want to explain their meaning. We expect that, as usual in many problems of nonlinear analysis, for certain values of the parameter a, a new branch of non-Gaussian solutions bifurcates from the branch of Gaussian fixed points of the RG (see Proposition I.1.1). Typically, this new branch is unique, however several branches may arise in degenerate cases. One can try to construct non-Gaussian solutions on this new branch as power series in the deviation of the

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parameter a from the bifurcation value a_0. Due to the invariance of the hamiltonian under the action of the RG a chain of equations on the coefficients of this series arises which we call Wilson's (complete) chain of equations.

So, we have two problems:

(i) Evaluation of the bifurcation values of the parameter a.

(ii) Construction of the solutions of the Wilson equations for these bifurcation values.

In this section we discuss (i). Let us present first explicit definitions. Denote

$$\mathscr{F}_2^0 = \mathscr{F}^0 \bigotimes_{\mathbb{C}} \mathscr{F}, \mathscr{F}_2^0 = \left\{ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{nm} \delta^n \varepsilon^m \right\}$$

the space of formal power series in two variables, with no terms $a_{n0}\delta^n$. Let

$$\begin{aligned} &\tau : \mathscr{F}_2^0 \to \mathscr{F}^0 \\ &\tau : \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \delta^n \varepsilon^m \to \sum_{k=1}^{\infty} \left(\sum_{m+n=k} a_{mn} \right) \varepsilon^k \end{aligned} \tag{1.1}$$

be the operator of restriction to the diagonal, and let

$$\mathscr{F}_2 \mathscr{H}^\infty = \mathscr{F}_2^0 \otimes \mathscr{H}^\infty$$

be a space of formal hamiltonians depending on two variables.

Due to Proposition I.5.1 the renormalization operator is an entire function of the parameter a. Hence for $a = a_0 + \delta$

$$\mathscr{R}_{\chi,\lambda}^{(a)} = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \frac{d^n}{da^n} \mathscr{R}_{\chi,\lambda}^{(a_0)}, \tag{1.2}$$

where all the operators $\frac{d^n}{da^n} \mathscr{R}_{\chi,\lambda}^{(a_0)}$ are continuous in the space of formal hamiltonians $\mathscr{F}\mathscr{H}^\infty$ and the series converges for any δ.

Now we introduce a new operator

$$\mathscr{R}_{\chi,\lambda}^{(a_0,\delta)} : \mathscr{F}\mathscr{H}^\infty \to \mathscr{F}_2 \mathscr{H}^\infty, \tag{1.3}$$

$$\begin{aligned} \mathscr{R}_{\chi,\lambda}^{(a_0,\delta)} : \sum_{m=1}^{\infty} \varepsilon^m H_m &\to \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \frac{d^n}{da^n} \mathscr{R}_{\chi,\lambda}^{(a_0)} \left(\sum_{m=1}^{\infty} \varepsilon^m H_m \right) \\ &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \varepsilon^m H_m^{(n)} . \end{aligned} \tag{1.4}$$

In order to calculate $\mathscr{R}_{\chi,\lambda}^{(a_0,\delta)}(H)$ it is sufficient to calculate $\mathscr{R}_{\chi,\lambda}^{(a_0+\delta)}(H)$ and to expand the result as a series in δ.

Definition 1.1 $a_0 > 0$ is a bifurcation value, if there exists a formal hamiltonian $H \in \mathscr{F}\mathscr{H}^\infty$ such that

$$\tau \mathscr{R}_{\chi,\lambda}^{(a_0,\delta)}(H) = H, \tag{1.5}$$

where the operators τ and $\mathscr{R}_{\chi,\lambda}^{(a_0,\delta)}$ are defined by formulae (1.1)–(1.4). (1.5) is understood as equality of formal hamiltonians. If (1.5) holds, the hamiltonian H is called an effective hamiltonian.

Actually the equality (1.5) is reduced to a chain of nonlinear equations.

$$Q_j(H_1, \ldots, H_j) = H_j \tag{1.6}$$

for the coefficients $\{H_j, j=1,2,\ldots\}$ of the effective hamiltonian. These are named full Wilson's equations.

The first equation of the chain (1.6) corresponding to $j=1$, is as usual linear

$$\mathscr{D}^{(a_0)}_{\chi,\lambda} H_1 = H_1 \tag{1.7}$$

where $\mathscr{D}^{(a_0)}_{\chi,\lambda} = \langle \mathscr{R}^{(a_0)}_{\lambda} \cdot \rangle_{\Delta(\chi_\lambda - \chi)}$ [see (I.5.9)] is the differential of the renormalization transformation at zero with $a=a_0$. Let $H_1 = (h_1, h_2, \ldots, h_m, 0, 0, \ldots) \in \mathscr{H}^\infty$, $h_m \neq 0$. Then Eq. (1.7) is reduced to a "triangular" chain of functional linear equations for the functions $h_1, \ldots, h_m$. For the last function h_m this equation has the form

$$\lambda^{\frac{ma_0}{2} - md + d} h_m(\lambda^{-1}k_1, \ldots, \lambda^{-1}k_m) = h_m(k_1, \ldots, k_m). \tag{1.8}$$

Solutions of this equation are homogeneous functions of order $\left(\frac{ma_0}{2} - md + d\right)$.

The function h_m must satisfy the following four conditions:

(i) $h_m(k_1, \ldots, k_m) \in C^\infty(\mathbb{R}^{md})$ (since $H \in \mathscr{F}\mathscr{H}^\infty$).

(ii) $h_m(\mathscr{U}k_1, \ldots, \mathscr{U}k_m) = h_m(k_1, \ldots, k_m)$ for any orthogonal transformation $\mathscr{U}$ of the space $\mathbb{R}^d$ (isotropy).

(iii) $h_m(k_1, \ldots, k_m) = h_m(k_{\pi(1)}, \ldots, k_{\pi(m)})$, where $\pi:(1, \ldots, m) \to (\pi(1), \ldots, \pi(m))$ is any permutation (symmetry).

(iv) $h_m(k_1, \ldots, k_m) \equiv 0$, if m is odd (oddness of H in the spin variable).

Moreover $h_m(k_1, \ldots, k_m)$ and $g_m(k_1, \ldots, k_m)$ are considered identical, if they coincide on the subspace $k_1 + \ldots + k_m = 0$.

These conditions lead to the following solutions of Eq. (1.8) (modulo a constant factor)

(i) $m=2$, $\quad a_0 = d$, $\quad h_2(k_1, k_2) \equiv 1$

$a_0 = d+2$, $\quad h_2(k_1, k_2) = |k_1|^2 + |k_2|^2$

$a_0 = d+4$, $\quad h_2(k_1, k_2) = |k_1|^4 + |k_2|^4$

. . . .

(ii) $m=4$, $\quad a_0 = \frac{3}{2}d$, $\quad h_4(k_1, \ldots, k_4) \equiv 1$

$a_0 = \frac{3}{2}d + 1$, $\quad h_4(k_1, \ldots, k_4) = |k_1|^0 + \ldots + |k_4|^2$

. . . .

. . . .

(n) $m=2n$, $\quad a_0 = \left(2 - \frac{1}{n}\right)d$, $\quad h_m(k_1, \ldots, k_m) = 1$

$a_0 = \left(2 - \frac{1}{n}\right)d + \frac{2}{n}$, $\quad h_m(k_1, \ldots, k_m) = |k_1|^2 + \ldots + |k_m|^2$

. . . .

Therefore the equation $\mathscr{D}^{(a_0)}_{\chi,\lambda} H_1 = H_1$ has a solution only for certain values of a_0. Their general form is

$$a_0 = \left(2 - \frac{1}{n}\right) d + 2r, \tag{1.9}$$

where $n \geqq 1$ and $r \geqq 0$ are integer numers. Only these values may be bifurcation values of RG (see also Appendix of the paper [2] where similar considerations are made in a somewhat different situation).

Solutions with $n=1$ (i.e. $m=2$) are not interesting. In that case the corresponding hamiltonian is quadratic and coincides with H_0 modulo a constant factor. The solution with $m=4$, $a_0 = \frac{3}{2}d$

$$h_4(k_1, \dots, k_4) \equiv \text{const}$$

gives the first nontrivial example. In this paper we consider only this solution. It is easy to see from the general formula for the effective hamiltonian H (see below) that

$$H_1 = u_1 : \int \delta(k_1 + \dots + k_4) \sigma(k_1) \dots \sigma(k_4) dk :_{\Delta\chi}, \tag{1.10}$$

where $\Delta\chi(k) = |k|^{d-a}\chi(k)$ and $u_1 < 0$ is a numerical factor.

All our results hold (with appropriate modifications) for

$$n = 3, 4, \dots, h_{2n}(k_1, \dots, k_{2n}) \equiv \text{const}, \qquad a_0 = 2d\left(1 - \frac{1}{m}\right)$$

$$H_1 = u_1 : \int \delta(k_1 + \dots + k_{2n}) \sigma(k_1) \dots \sigma(k_{2n}) dk :_{\Delta\chi}.$$

Note, that solutions with $n=2$, $m=4$ appear in investigations of critical phenomena in systems with long range potential. Solutions with $m=6, 8, \dots$ arise in describing a multicritical behaviour in such systems (see in this connection [3]).

3. Analytic Continuation of Projection Hamiltonians

Now we turn to the construction of a solution of Eq. (1.5) with $a_0 = \frac{3}{2}d$. We shall seek a solution in the class of projection hamiltonians (see I, Sect. 7) and, moreover, in the form

$$H = H(\sigma, \varepsilon) = :\exp u(\varepsilon) \varphi^4 :^c_{-\Delta(1-\chi)}, \tag{2.1}$$

where

$$\varphi^4(\sigma) = \int \delta(k_1 + \dots + k_4) \sigma(k_1) \dots \sigma(k_4) dk,$$

$$\Delta(1-\chi)(k) = |k|^{d-a_0-\varepsilon}(1-\chi(k)) = |k|^{-\frac{d}{2}-\varepsilon}(1-\chi(k)).$$

The only quantity which is not defined here is $u(\varepsilon)$, which we assume to be a formal power series in ε:

$$u(\varepsilon) = \sum_{j=1}^{\infty} u_j \varepsilon^j.$$

In the following we shall see that the numbers $u_j, j=1,2,\dots$ are uniquely defined by Eq. (1.5). Moreover, using the "triangular" form of (1.6) one can prove the uniqueness of the solution of Eq. (1.5) in the space $\mathscr{F}\mathscr{H}^\infty$ (see [4], where the uniqueness was proved for $j=1,2$).

In the computation of a projection hamiltonian divergences appear. Namely, in φ_d^4-theory with the propagator $\Delta(1-\chi)(k)=|k|^{-d/2}(1-\chi(k))$ all diagrams with four and two external lines diverge: such a theory is renormalizable, but not superrenormalizable. In (2.1) we deal with the propagator

$$\Delta(1-\chi)(k)=|k|^{-\frac{d}{2}-\varepsilon}(1-\chi(k)).$$

In this case for $\varepsilon>0$ the φ_d^4-theory is already superrenormalizable: only a finite number of diagrams with two external lines diverge, but the number of such diagrams increases when ε tends to zero. If $\varepsilon<0$, then the φ_d^4-theory is nonrenormalizable. In both cases ($\varepsilon>0$ and $\varepsilon<0$), the diagrams with two and four external lines have poles when $\varepsilon\to 0$. Since we consider $H(\sigma,\varepsilon)$ in (2.1) as a formal series in ε, we are interested only in these poles, and not in the behaviour of various diagrams for small but nonzero ε.

To analyse these poles we use the analytic renormalization of Feynman diagrams which is based on the idea of analytic continuation of Feynman amplitudes in the power of the propagator.

The analytic continuation of the Feynman amplitude with propagator $\Delta(1-\chi)(\mathrm{k})=|\mathrm{k}|^{d-a}(1-\chi(k))$ was investigated in [5]. We give the corresponding result in a convenient form.

Theorem 2.1. *The projection hamiltonian* (2.1) *admits an analytic continuation from the domain* $\mathrm{Re}\,a>2d$ *to the whole complex plane as a meromorphic function of a. The poles of this continuation lie at the points*

$$a=\tfrac{3}{2}d+\frac{\left(1-\dfrac{m}{r}\right)d-2n}{r}, \tag{2.2}$$

where $m\geqq 1, n\geqq 0, r\geqq 1$ *are arbitrary integer numbers such that* $m+r\geqq 4$ *and* $m+r$ *is even. Outside these points* $H\in\mathscr{F}\mathscr{H}^\infty$ *and*

$$\begin{aligned}\mathscr{R}^{(a)}_{\chi,\lambda}H&=\mathscr{R}^{(a)}_{\chi,\lambda}(:\exp u\varphi^4:^c_{-\Delta(1-\chi)})\\&=:\exp\lambda^{2\varepsilon}u\varphi^4:^c_{-\Delta(1-\chi)}\,.\end{aligned} \tag{2.3}$$

Proof. According to Theorem 4 of the paper [5] the projection hamiltonian (2.1) can be analytically continued as a meromorphic function of a with simple poles at the points, defined by the equation:

$$\frac{(a-d)|L(H)|}{2}-\frac{|L(H)|-|V(H)|+1}{2}d=-n, \tag{2.4}$$

where H is an arbitrary 2-connected nonvacuum graph of the φ^4-theory, and n is an arbitrary natural number. $|L(H)|$ and $|V(H)|$ are the numbers of internal lines and of vertices of the graph H respectively. Let

$$r = |L(H)|, \quad m = 2|V(H)| - |L(H)|.$$

A simple arithmetical calculation shows that (2.2) and (2.4) are equivalent. If H is a graph of the φ^4-theory, then we have

$$\frac{|E(H)|}{2} = 2|V(H)| - |L(H)| = m, \tag{2.5}$$

where $|E(H)|$ is the number of external lines of the graph H. Therefore for nonvacuum graphs $m = \dfrac{|E(H)|}{2} \geqq 1$. The other restrictions on the numbers m, r are obvious. Equality (2.2) follows from Theorem I.6.1 due to the uniqueness of analytic continuation.

Note that equality (2.4) for a massive propagator $(|k|^2 + m^2)^{d-a}$, $m > 0$ is known since a long time (see, for example [6]). However, the standard proof of this fact, which uses the α-representation, is not suitable for the propagator $|k|^{d-a}(1-\chi(k))$. In [5] a special inductive reasoning is given which allows to reduce this case to the case of massive propagators.

Let us analyse (2.2). If $n=0$, $m=2$ then $a = \frac{3}{2}d$, i.e the most interesting for us bifurcation value $a_0 = \frac{3}{2}d$ is always a pole of the analytic continuation. According to (2.5) $|E(H| = 2m$, i.e. this pole is associated with the divergence of Feynman diagrams with four external lines. It is easy to see that $a < \frac{3}{2}d$, if $m \geqq 3$. Finally, if $m=1$, the equality $a = \frac{3}{2}d$ is possible only if $d = 4n$, i.e. if d is a multiple of 4.

Assume now that

$$d \textit{ is not a multiple of } 4. \tag{2.6}$$

This means, that $a = \frac{3}{2}d$ is not among the poles (2.2) with $m=1$. We note, that requirement (2.6) is essential. If the dimension of the space is a multiple of 4, then the situation is different. We shall consider the solvability of Eq. (1.5) with $d = 4n$ in another paper.

It is noteworthy that (2.6) can be also interpreted as the condition that the point $a_0 = \frac{3}{2}d$. is a bifurcation value of multiplicity one (see Sect. 2). Indeed all bifurcation values with $m=2$ are d, $d+2, \ldots$ and do not coincide with $\frac{3}{2}d$, if (2.6) holds. All bifurcation values with $m > 4$ are obviously larger than $\frac{3}{2}d$. Because of the existence of a pole at the point $a = \frac{3}{2}d$ the hamiltonian (2.1) contains both positive and negative powers of ε. In order to eliminate negative powers we need a procedure of analytic renormalization.

4. Theorems on the Additivity Property of Analytic Renormalization

Now we consider the analytic renormalization in the φ_d^4-theory with propagator $|k|^{d-a}(1-\chi(k))$ in the neighbourhood of $a = a_0 = \frac{3}{2}d$.

Note, that usually the renormalized amplitudes are only considered in one point: the pole itself (see [7]). To cover a neighbourhood of this point, we shall consider these amplitudes as power series in $\varepsilon = a - a_0 = a - \frac{3}{2}d$.

Let G be an arbitrary graph of the φ_d^4-theory and let $\mathscr{F}_G$ be a corresponding Feynman amplitude. As we have said in the previous section, $\mathscr{F}_G$ is a meromorphic function of ε. The renormalized amplitude is given by the formula

$$\mathrm{A.R.}\,\mathscr{F}_G = \sum_{\substack{\{H_1,\ldots,H_r\}\subset G \\ V(H_i)\cap V(H_J)=\emptyset}} \mathscr{F}_{G|\{H_1,\ldots,H_r\}} \prod_{j=1}^{r} O(H_j) \tag{3.1}$$

and is an analytic function of ε in the neighbourhood of zero. Here $\{H_1,\ldots,H_r\}$ is an arbitrary family of pairwise disjoint (by vertices) subgraphs such that $V(H_1)\cup\ldots\cup V(H_r)=V(G)$

$$O(H) = \sum_{n=1}^{|V(H)|-1} a_n^{(H)}\left(\frac{1}{\varepsilon}\right)^n \tag{3.2}$$

is a polynomial (without constant term) in $\left(\frac{1}{\varepsilon}\right)$ of degree $|V(H)|-1$, which is associated to every graph H. $O(H)$ depends only on H and not on G. There is only one exception: the trivial graph H with one vertex, for which $O(H)=1$. A set $\{H_1,\ldots,H_{|V(G)|}\}$, which consists of trivial subgraphs $H_1,\ldots,H_{|V(G)|}$ corresponds to the unrenormalized amplitude $\mathscr{F}_G$. All other terms can be considered as substraction of singularities in ε from $\mathscr{F}_G$.

Remember our supposition that d is not a multiple of 4. If d is a multiple of 4, then formula (3.1) has a more complicated structure. Actually the equality (3.1) is the object of a nontrivial theorem.

Theorem 3.1. *(Additivity for Feynman amplitudes.) To every connected graph H of the φ^4-theory a polynomial $O(H)=\sum_{n=1}^{|V(H)|-1} a_n^{(H)}\left(\frac{1}{\varepsilon}\right)^n$ can be associated in such a way that the renormalized amplitude* A.R. $\mathscr{F}_G$ *is an analytic function of ε in some neighbourhood of the origin.*

Remark. This theorem is generalized to arbitrary graphs G and arbitrary (complex) values of the dimension and the (multi-) parameter a (see [7]).

For the case of massive propagators $(|k|^2+m^2)^{d-a}$, $m>0$ the theorem has been proved in fact in [7]. Namely one can easily obtain it, repeating the proof of additivity of dimensional renormalization given in that paper. Reduction of the case of the Feynman amplitudes with propagator $\Delta(1-\chi)(k)$ to the massive one is done by the general inductive scheme introduced in [5]. We omit the details.

Here are some additional properties of the polynomials $O(H)$. We have

$$O(H)\equiv 0$$

if

(i) H is not one-particle irreducible or

(ii) the number of external lines of H is not equal to 4.

Moreover all the coefficients of the polynomial $O(H)$ except $a_1^{(H)}$ are expressed in terms of the coefficients $a_1^{(K)}$ of the subgraphs $K\subset H$, $K\neq H$ using the so-called scaling relations (see [9]).

Note, that from (i) and (ii) it follows, that in fact the summation in the basic formula (3.1) goes over families $\{H_1, \ldots, H_r\}$ such that all H_i, $i=1, \ldots, r$ are one-particle irreducible and $|E(H_i)|=4$.

Note also that the polynomial $O(H)$ can be defined as the principal part of the Laurent series of the function

$$\sum_{\{H_1, \ldots, H_r\}}' \mathscr{F}_{G|\{H, \ldots, H\}} .$$

The prime in $\sum'$ means that $\{H_1, \ldots, H_r\}$ is an arbitrary set of subgraphs with only one exception $\{H\}$. This allows to calculate all polynomials $O(H)$ with the help of recursion relations. A.R.$:(\varphi^4)^n:^c_{-\Delta(1-\chi)}$ is given by the formula

$$\text{A.R.}:(\varphi^4)^n:^c_{-\Delta(1-\chi)} = \sum_G{}^c \int \text{A.R.}\, \mathscr{F}_G(p) \prod_{e\in E(G)} \sigma_e(p)dp . \tag{3.3}$$

Summing in both sides of (3.1) over all connected graphs G, we come to an additivity property for A.R.$:(\varphi^4)^n:^c_{-\Delta(1-\chi)}$.

Theorem 3.2. *(Formula of additivity for* $:(\varphi^4)^n:^c_{-\Delta(1-\chi)}$.*) Let*

$$O_1 = 1$$

$$O_m = \sum_{G\in\mathscr{G}_m} O(G) = \sum_{n=1}^{m-1} \left(\frac{1}{\varepsilon}\right)^n \sum_{G\in\mathscr{G}_m} a_n^{(G)}, \qquad m \geqq 2, \tag{3.4}$$

where $\mathscr{G}_m$ *is the set of all one-particle irreducible graphs of the* φ^4*-theory with* m *vertices. Let*

$$q_{nr} = \sum_{k=1}^{n} \sum_{\substack{n_1+\ldots+kn_k=n\\ n_1+\ldots+n_k=r\\ n_1,\ldots,n_k\geqq 0}} \frac{n!}{n_1!\ldots n_k!} \left(\frac{O_1}{1!}\right)^{n_1} \ldots \left(\frac{O_k}{k!}\right)^{n_k} . \tag{3.5}$$

Then

$$\text{A.R.}:(\varphi^4)^n:^c_{-\Delta(1-\chi)} = \sum_{r=1}^{n} q_{nr} :(\varphi^4)^r:^c_{-\Delta(1-\chi)} . \tag{3.6}$$

Remark. It is easy to see from (3.5) that $q_{nn}=1$. We put $q_{nr}=0$ for $r>n$.

Proof. Let V be a set of n vertices and let $\varepsilon: V_1 \cup \ldots \cup V_r = V$, $V_i \cap V_j = \emptyset$ be a partition of this set. We have

$$\text{A.R.}:(\varphi^4)^n:^c_{-\Delta(1-\chi)} = \sum_G \int \text{A.R.}\, \mathscr{F}_G(p) \prod_{e\in E(G)} \sigma(p_e)dp$$

$$= \sum_G \sum_{\{H_1,\ldots,H_r\}\subset G} \prod_{j=1}^{r} (O(H_j)) \int \mathscr{F}_{G|\{H_1,\ldots,H_r\}}(p) \prod_{e\in E(G)} \sigma(p_e)dp . \tag{3.7}$$

Keep in $\sum_{\{H_1,\ldots,H_r\}\subset G}$ only such collections $\{H_1,\ldots,H_r\}$ which are subordinated to ε, what means, that after some renumeration $V(H_j)=V_{i_j}$, $j=1,\ldots,r$. Let us denote the corresponding sum by $\sum^{\varepsilon}_{\{H_1,\ldots,H_r\}\subset G}$. Note, that $\mathscr{F}_{G|\{H_1,\ldots,H_r\}}$ does not depend on $\{H_1,\ldots,H_r\}$ and therefore

$$\begin{aligned}&\sum_G \sum^{\varepsilon}_{\{H_1,\ldots,H_r\}\subset G} \prod_{j=1}^{r} (O(H_j)) \int \mathscr{F}_{G|\{H_1,\ldots,H_r\}}(p)\\
&\cdot \prod_{e\in E(G)} \sigma(p_e)dp = \sum^{\varepsilon}_{\{H_1,\ldots,H_r\}} \sum_{G:\{H_1,\ldots,H_r\}\subset G} \prod_{j=1}^{r} (O(H_j))\\
&\cdot \int \mathscr{F}_{G|\{H_1,\ldots,H_r\}}(p) \prod_{e\in E(G)} \sigma(p_e)dp = \sum^{\varepsilon}_{\{H_1,\ldots,H_r\}} \prod_{j=1}^{r} O(H_j)\\
&\cdot \sum_{G'} \int \mathscr{F}_{G'}(p) \prod_{e\in E(G')} \sigma(p_e)dp = \prod_{j=1}^{r} O_{|V_j|} :(\varphi^4)^r :^c_{-\Delta(1-\chi)}\,.\end{aligned} \tag{3.8}$$

Here we use the fact that any graph G such that $\{H_1,\ldots,H_r\}\subset G$ is uniquely determined by the reduced graph $G'=G|_{\{H_1,\ldots,H_r\}}$. So we can replace the summation $\sum_{G:\{H_1,\ldots,H_r\}\subset G}$ by $\sum_{G'}$. Note, that the latter sum extends over all graphs of the φ^4-theory, because all the contracted subgraphs $G|_{\{H_1,\ldots,H_r\}}$ have four external lines.

From (3.7), (3.8) we have

$$\text{A.R.} :(\varphi^4)^n :^c_{-\Delta(1-\chi)} = \sum_{\substack{\varepsilon: V_1\cup\ldots\cup V_r = V\\ V_i\cap V_j=\emptyset}} \prod_{j=1}^{r} O_{|V_j|} :(\varphi^4)^r :^c_{-\Delta(1-\chi)}\,. \tag{3.9}$$

Consider now such partitions ε, which contain r sets among which there are n_1 one-point sets, n_2 two-point sets, $\ldots$, n_k k-point sets, $n_1+\ldots+n_k=r$. The number of such partitions is

$$\frac{n!}{n_1!\ldots n_k!(1!)^{n_1}\ldots(k!)^{n_k}}\,.$$

The contribution to (3.9) of every such partition is

$$(O_1)^{n_1}\ldots(O_k)^{n_k} :(\varphi^4)^r :^c_{-\Delta(1-\chi)}\,,$$

i.e. the total contribution is

$$\frac{n!(O_1)^{n_1}\ldots(O_k)^{n_k}}{n_1!\ldots n_k!(1!)^{n_1}\ldots(k!)^{n_k}} :(\varphi^4)^r :^c_{-\Delta(1-\chi)}\,.$$

By summing over $r, k, n_1, \ldots, n_k$, we obtain the desired relation (3.6). The theorem is proved.

Now we can introduce the main object of our investigations, the renormalized projection hamiltonian

$$\text{A.R.} :\exp u\varphi^4 :^c_{-\Delta(1-\chi)} = \sum_{n=1}^{\infty} \frac{u^n}{n!} \text{A.R.} :(\varphi^4)^n :^c_{-\Delta(1-\chi)}\,. \tag{3.10}$$

Here u is a formal parameter. In the following u will be a formal series in ε.

Theorem 3.3. *(Introduction of counterterms.)*

$$\text{A.R.}:\exp u\varphi^4:^{c}_{-\Delta(1-\chi)}=:\exp\omega(u)\varphi^4:^{c}_{-\Delta(1-\chi)}, \tag{3.11}$$

where

$$\omega(u)=\sum_{n=1}^{\infty}O_n\frac{u^n}{n!} \tag{3.12}$$

and

$$:\exp\omega(u)\varphi^4:^{c}_{-\Delta(1-\chi)}=\sum_{n=1}^{\infty}\frac{\omega^n(u)}{n!}:(\varphi^4)^n:^{c}_{-\Delta(1-\chi)}.$$

The equality (3.12) *is understood in the sense of formal series in u.*

Remark. In the theory of R-operation analogous formulae were obtained in [8].

Proof. By the Theorem 3.2,

$$\begin{aligned}\text{A.R.}:\exp u\varphi^4:^{c}_{-\Delta(1-\chi)}&\equiv\sum_{n=1}^{\infty}\frac{u^n}{n!}\text{A.R.}:(\varphi^4)^n:^{c}_{-\Delta(1-\chi)}\\&=\sum_{n=1}^{\infty}\frac{u^n}{n!}\sum_{r=1}^{n}\sum_{k=1}^{n}\sum_{\substack{n_1+\ldots+kn_k=n\\ n_1+\ldots+n_k=r\\ n_1,\ldots,n_k\geqq 0}}\frac{n!(O_1)^{n_1}\ldots(O_k)^{n_k}}{n_1!\ldots n_k!(1!)^{n_1}\ldots(k!)^{n_k}}:(\varphi^4)^r:^{c}_{-\Delta(1-\chi)}\\&=\sum_{k=1}^{\infty}\sum_{\substack{n_1,\ldots,n_k\geqq 0\\ n_1+\ldots+n_k\geqq 1}}:\prod_{j=1}^{k}\frac{1}{n_j!}\left(\frac{O_ju^j}{j!}\varphi^4\right)^{n_j}:^{c}_{-\Delta(1-\chi)}\\&=:\exp\left(\sum_{j=1}^{\infty}\frac{O_j}{j!}u^j\varphi^4\right):^{c}_{-\Delta(1-\chi)}=:\exp(\omega(u)\varphi^4):^{c}_{-\Delta(1-\chi)}.\end{aligned}$$

Here we have used the formula

$$\exp\left(\sum_{j=1}^{\infty}a_j\right)=1+\sum_{k=1}^{\infty}\sum_{\substack{n_1,\ldots,n_k\geqq 0\\ n_1+\ldots+n_k\geqq 1}}\prod_{j=1}^{k}\frac{a_j^{n_j}}{n_j!}.$$

The theorem is proved.

5. Renormalization Group Transformations for Renormalized Projection Hamiltonians

The formal series

$$\varrho(u)=\varepsilon\frac{\omega(u)}{\omega'(u)}=\varepsilon\frac{\sum_{n=1}^{\infty}\frac{O_n}{n!}u^n}{\sum_{n=1}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}}=\sum_{n=1}^{\infty}c_nu^n \tag{4.1}$$

will play an important role in what follows. The coefficients O_n are polinomials in $\left(\frac{1}{\varepsilon}\right)$ of degree $(m-1)$ [see (3.4)] except $O_1=1$, therefore the coefficient c_n is represented as

$$c_n=\varepsilon\sum_{m=0}^{n-1} c_{nm}\left(\frac{1}{\varepsilon}\right)^m, \tag{4.2}$$

where the coefficients c_{nm} are determined in terms of the quantities $a_n^{(G)}$.

It is not difficult to see that

$$c_1=\varepsilon.$$

The following surprising result holds.

Theorem 4.1. *The coefficients c_n for $n\geqq 2$ do not depend on ε. In other words, all the coefficients c_{nm} in* (4.2) *for $n\geqq 2$, $m\neq 1$ are equal to zero.*

We shall give a proof of this theorem in Sect. 7. The proof presented there is indirect and use some properties of the renormalization transformation. There exists also a direct proof based on some explicit expressions for the polynomials $O(H)$ (see [9], where the so-called scaling relations for $O(H)$ are discussed). As the direct proof is longer we prefer to give here the indirect one.

In the following it will be convenient to use a new variable τ as a RG parameter,

$$\lambda=\exp(\tau/2).$$

The importance of the introduced series $\varrho(u)$ is explained by the following result.

Theorem 4.2

$$\begin{aligned}&\mathscr{R}_{\chi,e^{\tau/2}}\,\mathrm{A.R.}:\exp u\varphi^4:^c_{-\Delta(1-\chi)}\\&\quad=\left(\exp\tau\varrho\frac{d}{du}\right)\mathrm{A.R.}:\exp u\varphi^4:^c_{-\Delta(1-\chi)}.\end{aligned} \tag{4.3}$$

The equality has to be understood in the sense of formal series in u and τ.

Remark. As we shall see in Sect. 7 the series in τ converge and define an analytic function of τ in the whole complex plane.

Proof. From the Theorem 3.3 and equality (2.3) we have

$$\begin{aligned}&\mathscr{R}_{\chi,e^{\tau/2}}\mathrm{A.R.}:\exp u\varphi^4:^c_{-\Delta(1-\chi)}\\&\quad=\mathscr{R}_{\chi,e^{\tau/2}}:\exp\omega(u)\varphi^4:^c_{-\Delta(1-\chi)}\\&\quad=:\exp((\exp\varepsilon\tau)\omega(u)\varphi^4):^c_{-\Delta(1-\chi)}\\&\quad=\sum_{n=1}^{\infty}\frac{(\exp\varepsilon\tau)^n\omega^n(u)}{n!}:(\varphi^4)^n:^c_{-\Delta(1-\chi)}.\end{aligned} \tag{4.4}$$

We make use now of the easily verified identity

$$((\exp \varepsilon\tau)\omega(u))^n = \exp\left(\varepsilon\tau\omega\frac{d}{d\omega}\right)\omega^n(u).$$

Since $\frac{d}{d\omega} = \frac{1}{\omega'}\frac{d}{du}$,

$$((\exp \varepsilon\tau)\omega(u))^n = \exp\left(\varepsilon\tau\frac{\omega}{\omega'}\frac{d}{du}\right)\omega^n(u) = \exp\left(\tau\varrho(u)\frac{d}{du}\right)\omega^n(u).$$

Substituting this expression in (4.4), we obtain

$$\begin{aligned}&\mathscr{R}_{\chi, e^{\tau/2}}\,\text{A.R.}:\!\exp(u\varphi^4)\!:^c_{-\Delta(1-\chi)}\\ &\quad = \exp\left(\tau\varrho(u)\frac{d}{du}\right)\sum_{n=1}^{\infty}\frac{\omega^n(u)}{n!}:\!(\varphi^4)^n\!:^c_{-\Delta(1-\chi)}\\ &\quad = \exp\left(\tau\varrho(u)\frac{d}{du}\right)\text{A.R.}:\!\exp(u\varphi^4)\!:^c_{-\Delta(1-\chi)}.\end{aligned}$$

The theorem is proved.

6. Solution of Wilson's Equations

In this section we shall establish the main result of our paper: we shall prove the solvability of Wilson's equations.

As one can see from formula (4.3) of Theorem 4.1 the renormalized projection hamiltonian A.R.$:\!\exp(u(\varepsilon)\varphi^4)\!:^c_{-\Delta(1-\chi)}$, where $u(\varepsilon) = \sum_{j=1}^{\infty} u_j\varepsilon^j$ is a formal series in ε, is invariant under the action of the RG, if

$$\varrho(u) = 0. \tag{5.1}$$

This equation has to be solved in the formal series $u(\varepsilon) = \sum_{j=1}^{\infty} u_j\varepsilon^j$ in ε. By Theorem 4.1

$$\varrho(u) = \varepsilon u + \sum_{n=2}^{\infty} c_n u^n,$$

where c_n are real constants. Substituting $u(\varepsilon) = \sum_{j=1}^{\infty} u_j\varepsilon^j$ in this expression and equating to zero the coefficients of all powers in ε, we come to a chain of numerical recursion relations for the unknown constants u_j. Before writing this relations, we note that (5.1) splits into two equations:

$$u = 0,$$

$$\varepsilon + \sum_{n=2}^{\infty} c_n u^{n-1} = 0. \tag{5.2}$$

The solution $u=0$, $u_j=0$, $j=1,2,\ldots$ corresponds to a Gaussian fixed point. A nontrivial non-Gaussian solution is got from the solution of (5.2). We have from (5.2)

$$c_2u_1=-1\,,$$
$$c_2u_2=-c_3u_1^2$$
$$\cdots\cdots$$
$$\cdots\cdots$$

The general form of the recurrent equation is

$$c_2u_n=B_n(u_1,\ldots,u_{n-1})\,, \tag{5.3}$$

where $B_n(u_1,\ldots,u_{n-1})$, $n=3,4,\ldots$ are polynomials, whose coefficients are defined by c_3, c_4, ...

For the solvability of these equations it is necessary and sufficient that $c_2\neq 0$. In the appendix it is shown, that

$$c_2=18\pi^{\frac{d}{2}}\frac{1}{\Gamma\left(\frac{d}{2}\right)}\neq 0\,,$$

i.e. Eq. (5.3) are indeed solvable. So, we can formulate the next theorem.

Theorem 5.1 (Main theorem). *If d is not a multiple of 4, then the renormalized projection hamiltonian*

$$H=\text{A.R.}\,{:}\exp(u(\varepsilon)\varphi^4(\sigma)){:}^c_{-\Delta(1-\chi)}\,,$$

where $u(\varepsilon)=\sum_{j=1}^{\infty}u_j\varepsilon^j$ is found by solving the equation $\varrho(u)=0$ (see (5.1)–(5.3)),

$$\varphi^4(\sigma)=\int\delta(k_1+\ldots+k_4)\sigma(k_1)\ldots\sigma(k_4)dk$$

and

$$\Delta(1-\chi)(k)=|k|^{d-a}(1-\chi(k))$$

is (with any $\lambda>0$) a fixed point of the renormalization transformation $\tau\mathscr{R}^{(3/2d,\varepsilon)}_{\chi,\lambda}$. Moreover, $H\in\mathscr{F}\mathscr{H}^\infty$.

One can prove the uniqueness of the solution in the space $\mathscr{F}\mathscr{H}^\infty$ by using the methods of the paper [4].

7. Proof of Theorem 4.1

We have

$$\varrho(u)=\frac{\varepsilon\sum_{n=1}^{\infty}\frac{O_n}{n!}u^n}{1+\sum_{n=2}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}}=\varepsilon\left(u+\sum_{n=2}^{\infty}\frac{O_n}{n!}u^n\right)$$

$$\cdot\left[1-\sum_{n=2}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}+\left(\sum_{n=2}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}\right)^2-\ldots\right]=\sum_{n=1}^{\infty}c_nu^n\,.$$

Since O_n with $n \geqq 2$ are polynomials in $\left(\frac{1}{\varepsilon}\right)$ without constant terms, all c_n with $n \geqq 2$ are also polynomials in $\left(\frac{1}{\varepsilon}\right)$ (this follows from comparison of the coefficients in the last formula). Therefore, if we prove the analyticity of c_n in ε, we get immediately that all the coefficients c_n, $n \geqq 2$ are constants.

We shall prove the analyticity of the coefficients by induction. For $c_1 = \varepsilon$ the analyticity is obvious. If we suppose now that $c_2, \ldots, c_{n-1}$ are analytical functions of ε they are necessarily constants. Let us prove the analyticity of c_n. By Theorems 3.1–3.3 the renormalized projection hamiltonian

$$H = \text{A.R.} :\exp(u\varphi^4):^c_{-\Delta(1-\chi)}$$

is analytic in ε in the neighbourhood of the origin. Due to Proposition I.5.1 the renormgroup operator $\mathscr{R}^{(3/2d+\varepsilon)}_{\chi, e^{\tau/2}}$ is also analytic in ε. Therefore, the hamiltonian $\mathscr{R}^{(3/2d+\varepsilon)}_{\chi, e^{\tau/2}} H$ is analytic in ε in the neighbourhood of the origin. From this fact we shall deduce the analyticity of c_n. Due to Theorem 4.2

$$\mathscr{R}^{(3/2d+\varepsilon)}_{\chi, e^{\tau/2}} H = \exp\left(\tau\varrho \frac{d}{du}\right) \text{A.R.} :\exp(u\varphi^4):^c_{-\Delta(1-\chi)} . \tag{6.1}$$

Consider the operator

$$\mathscr{D} = \varrho(u) \frac{d}{du} \tag{6.2}$$

in the space of formal power series $a(u) = \sum_{j=1}^{\infty} a_j u^j$. This operator depends on ε as a parameter. The matrix of the operator $\mathscr{D}$ has a triangular form:

$$\begin{pmatrix} c_1 & 0 & 0 & \ldots \\ c_2 & 2c_1 & 0 & \ldots \\ c_3 & 2c_2 & 3c_1 & \ldots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix}$$

Hence we have immediately the convergence of the series

$$\exp(\tau\mathscr{D}) = 1 + \frac{\tau\mathscr{D}}{1!} + \frac{(\tau\mathscr{D})^2}{2!} + \ldots$$

for any $\tau \in \mathbb{C}$ and the analyticity in τ of all the elements of the matrix $\exp(\tau\mathscr{D})$. Let $\mathscr{D}_n$, $(\exp(\tau\mathscr{D}))_n$ be the principal minors defined by the first n rows and columns of the matrices $\mathscr{D}$ and $\exp(\tau\mathscr{D})$ respectively. Then from the triangular form of $\mathscr{D}$ it follows that

$$(\exp(\tau\mathscr{D}))_n = \exp(\tau\mathscr{D}_n) .$$

We establish now how the matrix $\exp(\tau\mathscr{D}_n)$ depends on c_n, $n \geqq 2$. Let

$$\mathscr{F}_n = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}}_{n} \left.\vphantom{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}\right\} n.$$

Then $\mathscr{D}_n = c_n\mathscr{F}_n + \mathscr{D}'_n$, where $\mathscr{D}'_n$ is a triangular matrix independent of c_n. We shall show, using induction, that the same analytic expression is true for $\mathscr{D}^j_n$ as well:

$$\mathscr{D}^j_n = \alpha_j c_n \mathscr{F}_n + E^{(j)}_n, \tag{6.3}$$

where the matrix $E^{(j)} = (e^{(j)}_{kl})^n_{k,l=1}$ has a triangular form and does not depend on c_n. It is directly checked that

$$\mathscr{F}_n\mathscr{D}'_n = c_1\tau\mathscr{F}_n, \qquad \mathscr{F}_n\mathscr{F}_n = 0,$$
$$E^{(j)}\mathscr{F}_n = e^{(j)}_{nn}\mathscr{F}_n.$$

Moreover, if $E = (e_{kl})^n_{k,l=1}$, $\mathscr{F} = (f_{kl})^n_{k,l=1}$, $E\mathscr{F} = (g_{kl})^n_{k,l=1}$ are triangular matrices, then $g_{nn} = e_{nn}f_{nn}$. Using these relations, we have

$$\mathscr{D}^{j+1}_n = (\alpha_j c_n\mathscr{F}_n + E^{(j)})(c_n\mathscr{F}_n + \mathscr{D}'_n)$$
$$= (\alpha_j c_1 + e^{(j)}_{nn})c_n\mathscr{F}_n + E^{(j)}\mathscr{D}'_n = \alpha_{j+1}c_n\mathscr{F}_n + E^{(j+1)},$$
$$E^{(j+1)} = E^{(j)}\mathscr{D}'_n,$$
$$e^{j+1}_{nn} = nc_1 e^{(j)}_{nn}, \tag{6.4}$$
$$\alpha_{j+1} = \alpha_j c_1 + e^{(j)}_{nn}. \tag{6.5}$$

So the relation (6.3) remains valid when we replace j by $(j+1)$, so it is valid for any $j \geqq 1$. Let us solve now the recursion Eqs. (6.4), (6.5) with initial conditions

$$e^{(1)}_{nn} = nc_1, \qquad \alpha_1 = 1.$$

From (6.4) we have

$$e^{(j)}_{nn} = (nc_1)^j$$

i.e.

$$\alpha_{j+1} = \alpha_j c_1 + (nc_1)^j.$$

Hence it follows

$$\alpha_j = (c_1)^{j-1}(1 + n + \dots + n^{j-1}) = (c_1)^{j-1}\frac{n^j - 1}{n-1},$$

i.e.

$$\mathscr{D}^j_n = (c_1)^{j-1}\frac{n^j-1}{n-1}c_n\mathscr{F}_n + E^{(j)}.$$

Thus

$$\exp(\tau\mathscr{D}_n)=\sum_{j=0}^{\infty}\frac{\tau^j\mathscr{D}_n^j}{j!}=\sum_{j=0}^{\infty}\frac{\tau^j(c_1)^{j-1}}{j!}\cdot\frac{n^j-1}{n-1}c_n\mathscr{F}_n$$
$$+E_0=\frac{\exp(\tau n\varepsilon)-\exp(\tau\varepsilon)}{(n-1)\varepsilon}\cdot c_n\mathscr{F}_n+E_0\,,\qquad (c_1=\varepsilon)\,,\tag{6.6}$$

where the matrix E_0 does not depend on c_n. Moreover, $\exp(\tau\mathscr{D}_n)$, and hence, E_0 do not depend on $c_{n+1}, c_{n+2}, \ldots$. We go back now to $H'=\mathscr{R}^{(3/2d+\varepsilon)}_{\chi, e^{\tau/2}}(H)$. According to (6.1), (6.2)

$$H'=\sum_{n=1}^{\infty}u^n\sum_{m=1}^{n}\frac{q_{nm}(\tau)}{m!}\,\mathrm{A.R.}:(\varphi^4)^m:^c_{-\Delta(1-\chi)}\,,$$

where

$$\exp(\tau\mathscr{D})=Q(\tau)=(q_{nm}(\tau))^{\infty}_{n,m=1}\,.$$

As we said earlier, the quantity $\mathrm{A.R.}:(\varphi^4)^n:^c_{-\Delta(1-\chi)}$ is analytic in ε as well as H'. This means the analyticity of the coefficient of u^n:

$$\sum_{m=1}^{n}\frac{q_{nm}(\tau)}{m!}\,\mathrm{A.R.}:(\varphi^4)^m:^c_{-\Delta(1-\chi)}\,.\tag{6.7}$$

Due to (6.6) all the elements $q_{nm}(\tau)$ except $q_{n1}(\tau)$ are expressed in terms of the $c_1, \ldots, c_{n-1}$ and therefore are analytic in ε by the inductive assumption. Further, due to (6.6)

$$q_{n1}(\tau)=\frac{\exp(n\tau\varepsilon)-\exp(\tau\varepsilon)}{(n-1)\varepsilon}c_n+q'_{n1}(\tau)\,,$$

where $q'_{n1}(\tau)$ also is expressed only in terms of $c_1, \ldots, c_{n-1}$ and, therefore analytic in ε. In this way, from the analyticity of the coefficient (6.7) it follows that the term

$$\frac{\exp(n\tau\varepsilon)-\exp(\tau\varepsilon)}{(n-1)\varepsilon}c_n\,\mathrm{A.R.}:\varphi^4:^c_{-\Delta(1-\chi)}$$

is analytic. As $\psi(\varepsilon)=\dfrac{\exp(n\tau\varepsilon)-\exp(\tau\varepsilon)}{(n-1)\varepsilon}$ is an entire function of ε and $\psi(0)=1$, the coefficient c_n is analytic, which was to be shown. The theorem is proved.

Appendix

Here we compute the second coefficient c_2 of the series $\varrho(u)$.

$$\varrho(u)=\varepsilon\frac{\omega(u)}{\omega'(u)}=\varepsilon\frac{\sum\limits_{n=1}^{\infty}\frac{O_n}{n!}u^n}{\sum\limits_{n=1}^{\infty}\frac{O_n}{(n-1)!}u^{n-1}}$$
$$=\varepsilon\left(u+\frac{O_2}{2!}u^2+\ldots\right)(1-O_2u+\ldots)$$
$$=\varepsilon\left(u-O_2u^2+\frac{O_2}{2!}u^2+\ldots\right)=\varepsilon u-\frac{\varepsilon O_2}{2}u^2+\ldots.$$

We have

$$c_2 = -\frac{\varepsilon O_2}{2},$$

where

$$O_2 = \sum_{G \in \mathscr{G}_2} O(G),$$

$\mathscr{G}_2$ being the set of all one-particle irreducible graphs of the φ^4-theory with $|E(G)|=4$ and $|V(G)|=2$. There is only one such graph, showed in Fig. 1. Taking into account a combinatorial factor we have

$$O_2 = 72\,O(G),$$

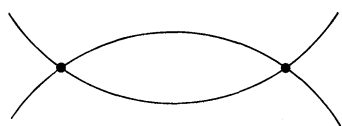

Fig. 1

where $O(G)$ is a meromorphic part in ε of the Feynman amplitude of the graph G. Now,

$$\begin{aligned}\mathscr{F}_G &= \int \delta(k_1+k_2+k_3+k_4)\delta(k_5+k_6+k_7+k_8)\\ &\quad \cdot \delta(k_3+k_5)\Delta(1-\chi)(k_3)\delta(k_4+k_6)\Delta(1-\chi)(k_4)dk_3dk_4dk_5dk_6\\ &= \delta(k_1+k_2+k_7+k_8)\int \Delta(1-\chi)(k_1+k_2+k_3)\Delta(1-\chi)(k_3)dk_3\\ &\quad \cdot (\Delta(1-\chi)*(\Delta(1-\chi)) = \Delta*\Delta - 2\Delta*(\Delta\chi) + (\Delta\chi)*(\Delta\chi),\end{aligned}$$

where

$$\Delta = |k|^{-\frac{d}{2}-\varepsilon}, \qquad \Delta\chi = |k|^{-\frac{d}{2}-\varepsilon}\chi(k).$$

It is clear, that $\Delta*(\Delta\chi)$ and $(\Delta\chi)*(\Delta\chi)$ are analytic in ε in the neighbourhood of 0. Next, we can calculate $\Delta*\Delta$ exactly:

$$\begin{aligned}\Delta*\Delta &= \mathscr{F}^{-1}_{x\to k}(\mathscr{F}_{k\to x}(|k|^{-\frac{d}{2}-\varepsilon}))^2\\ &= 2^{-\varepsilon}\pi^{\frac{d}{2}}\left(\frac{\Gamma\left(\frac{d}{4}-\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{d}{4}+\frac{\varepsilon}{2}\right)}\right)^2 \frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{d}{2}-\frac{\varepsilon}{2}\right)}|k|^{-\varepsilon},\end{aligned}$$

where $\mathscr{F}_{k\to x}$ is the Fourier transformation (see [10]). Therefore,

$$O(G) = -\frac{\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}\right)}\cdot\frac{1}{\varepsilon}.$$

Hence

$$c_2 = \frac{18\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \neq 0 .$$

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