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The Equations of Wilson's Renormalization Group and Analytic Renormalization

I. General Results

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Abstract. In the present series of two papers we solve exactly Wilson's equations for a long-range effective hamiltonian. These equations arise when one seeks a fixed point of the Wilson's renormalization group transformations in the formulation of perturbation theory. The first paper has a general character. Wilson's renormalization transformation and its modifications are defined and the group property for them is established. Some topological aspects of the renormalization transformations are discussed. A space of "projection hamiltonians" is introduced and a theorem on the invariance of this space with respect to the renormalization transformations is proved.

1. Introduction

In the present work consisting of two papers we shall solve exactly the Wilson's renormalization group equations for an effective hamiltonian whose free part is defined by long-range potential $U(x) \sim -\frac{\text{const}}{|x|^a}$, $|x| \to \infty$. This hamiltonian is written as a formal series $H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots$, where $\varepsilon = a - \frac{3}{2}d$ and d is the dimensiality. Each of the H_i is an usual (not formal) finite-particle hamiltonian. H_0 is a free long-range quadratic hamiltonian. Under the Wilson's renormalization group transformation the hamiltonian H transforms into another one $H' = H'_0 + \varepsilon H'_1 + \varepsilon^2 H'_2 + \ldots$ (which is also a formal series) every coefficient H'_i of which is computed via the coefficients $H_0, H_1, \ldots, H_i$ of the original hamiltonian:

$$H'_i = R_i(H_0, H_1, \ldots, H_i).$$

The operators R_i have a rather complicated structure and are nonlinear in $H_0, H_1, \ldots, H_{i-1}$. By definition the effective hamiltonian is a fixed point of the renormalization group transformation and its coefficients satisfy the chain of equations

$$H_i = R_i(H_0, H_1, \ldots, H_i). \tag{1}$$

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The main results of our work is an exact construction of a non-trivial solution of this chain of equations. In particular the hamiltonians H_0, H_1 have in momentum space the form

$$H_0 = \int_{|k|<\Lambda} \tfrac{1}{2}|k|^{a-d}|\sigma(k)|^2 d^d k\,,$$

$$H_1 = u_1 \int_{|k_1|,\ldots,|k_4|<\Lambda} \delta(k_1+\ldots+k_4)\sigma(k_1)\ldots\sigma(k_4)d^d k_1\ldots d^d k_4\,,$$

where $u_1>0$ and : : is the Wick ordering with respect to free field with the hamiltonian H_0. The following hamiltonians H_i, $i=2,3,\ldots$, have a more complicated structure and are defined in the main part of the work.

The problem of finding solutions for the coefficients of the effective hamiltonian was first formulated and discussed in [1] (see also [2–4]).

Actually in [1] short range hamiltonians were considered and the expansion was carried out in the dimensiality parameter $\varepsilon'=4-d$. Undoubtedly there are many general features in the expansions in $\varepsilon=a-\frac{3}{2}d$ and $\varepsilon'=4-d$. The authors intend to consider ε'-expansion and the connection between ε- and ε'-expansions in subsequent papers. Moreover it is noteworthy that in the present work we deal only with the case when the dimensiality d is not divisible by 4. This restriction is essential and if d is a multiple of 4 or is close to such a number, the ε-expansion has a more complicated nature.

In [1] an iteration procedure was suggested for solving the chain of Eq. (1). The point is that the operator R_i can be written as

$$R_i(H_0,H_1,\ldots,H_i)=DH_i+T_i(H_0,H_1,\ldots,H_{i-1})\,,$$

where D is a linear operator and T_i does not depend on H_i. So one can rewrite the Eq. (1) in the form

$$(1-D)H_i=T_i(H_0,H_1,\ldots,H_{i-1})$$

and "solve" it:

$$H_i=(1-D)^{-1}T_i(H_0,H_1,\ldots,H_{i-1})=(1+D+D^2+\ldots)T_i(H_0,H_1,\ldots,H_{i-1})\,.$$

Some details of the inversion of the operator $(1-D)$ were analyzed in [1], but procedure described there seems too formal and in essence useless because it does not permit to investigate any property of the hamiltonians H_i (see also [2, 4]).

The above formulae for the coefficients H_0, H_1 are standard and well-known. An explicit expression for the coefficient H_2 was obtained in [3]. In this paper another renormalization group was used (Kadanoff's block RG), but this is unessential and an analogous expression can be obtained also for Wilson's renormalization group. The present work arises from the attempt to generalize the procedure used there in order to construct the hamiltonians $H_3, H_4, \ldots$. But direct generalization proved to be impossible in view of the fast increasing complexity of computations. All we could do in this way was to construct H_3. Therefore we went by another way and tried to guess the answer on the base of the explicit expressions for H_1, H_2, and H_3. After several unsuccessful attempts we managed to do it. In the meantime however we went so far from the paper [3] that in essence the methods used in present work have little in common with those of [3].

When one solves the chain of Eq. (1) a very important question is which are the properties of the hamiltonians H_i. Namely the solution of this chain of equations is not unique it no smoothness condition is required (see [3]). The smoothness condition is one of the three "analyticity postulates" formulated in [6] (other two postulates are connected with the absence in the effective hamiltonian of noninteger powers of the field $\sigma(k)$ and with the transversality of the intersection of the initial family of hamiltonians with the stable separatrix of the renormalization group transformation; the last condition is needed in the calculation of the critical exponents and does not concern us now). The essence of the smoothness condition is that any hamiltonian H_i, $i=1,2,\ldots$, is written as

$$H_i=\sum_m \int h_m^{(i)}(k_1,\ldots,k_m)\delta(k_1+\ldots+k_m)\sigma(k_1)\ldots\sigma(k_m)d^{md}k,$$

where $h_m^{(i)}(k_1,\ldots,k_m)$ are smooth functions of the arguments $k_1,\ldots,k_m\in\mathbb{R}^d$. This question is discussed in detail below.

The procedure used by us for the effective hamiltonian construction is closely related with analytic renormalization (see [7–10]). Namely the main formula for H which is proved in the present work is

$$H=\text{A.R.} :\exp(-u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)},$$

where by φ^4 we denote briefly the hamiltonian

$$\varphi^4=\int\sigma^4(x)d^dx=\int\sigma(k_1)\ldots\sigma(k_4)\delta(k_1+\ldots+k_4)d^{4d}k$$

and $:\ :^c_{-\Delta(1-\chi)}$ is the operation of connected Wick ordering with respect to the propagator

$$-\Delta(1-\chi)(k)=-|k|^{-a+d}(1-\chi_R(k)),$$

where $\chi_R(k)$ is the characteristic function of the ball $\{|k|<R\}$; $u(\varepsilon)=\sum_{j=1}^{\infty}u_j\varepsilon^j$ is a formal numerical series and A.R. is a variant of analytic renormalization. The expression

$$:\exp(-u(\varepsilon)\varphi^4):^c_{-\Delta(1-\chi)}=\sum_{n=1}^{\infty}\frac{(-u(\varepsilon))^n}{n!}:(\varphi^4)^n:^c_{-\Delta(1-\chi)}$$

to which the operation A.R. is applied is written as a series of Feynman integrals with the propagator $-\Delta(1-\chi)(k)$. When $\varepsilon>0$ the theory with this propagator is super-renormalizable and only a finite number of Feynman diagrams with two external legs diverge. However when $\varepsilon=0$ the theory is not super-renormalizable but only renormalizable and an infinite number of divergent diagrams with two and four external legs arises. For $\varepsilon\to 0$, $\varepsilon>0$, these diagrams (more precisely the corresponding Feynman amplitudes) are expanded in Laurent series in ε. The analytic renormalization A.R. is roughly speaking the substitution of negative terms of the Laurent series in the Feynman amplitudes. The precise definition of the A.R. is followed by a very important condition of additivity of this operation (see [7, 9] and below).

At first glance it is unexpected that, when solving the renormalization group equations in a finite region of momentum space, diagrams with ultraviolet divergencies appear. This fact becomes not so surprising if one takes the point of view that a scaling field in a finite region can be obtained by projection of a scaling field in the whole momentum space. The above formula for the effective hamiltonian realizes in some sense such projection.

For the coefficients $u_1, u_2, \ldots$ of the formal series $u(\varepsilon)$ we obtain a chain of numerical equations which permits us to find all these coefficients uniquely. Their concrete computation and the subsequent calculation of the critical exponents is a rather tedious work whose volume increases fast with the number of coefficients. In the present time there are many original papers and reviews devoted to computational problems of the ε-expansions (see, e.g. [11–16]). Long-range potentials were considered in [17–19]. Practically in all these works the critical exponents are sought avoiding the question of the existence of the effective hamiltonian, by means of the Callan-Symanzik equations. In this procedure the original dynamical Kadanoff-Wilson's picture of critical phenomena is put aside. Here we return to this original picture and as a first step of its realization we construct explicitly the effective hamiltonian (for long-range models). The second step is the construction of eigenvectors and eigenvalues of the linearized transformation. In a paper in preparation one of us (M.D.M.) constructs the so-called relevant eigenvector and eigenfunction.

We want to emphasize that the effective hamiltonian which we construct here is a formal series in ε. Apparently this series diverges. A very attracting but apparently very difficult problem is the construction of a scaling translation invariant random field with a hamiltonian $H = H(\varepsilon)$ such that the hamiltonians $H_0, H_1, H_2, \ldots$ found by us, are the coefficients of the expansion of the hamiltonian $H(\varepsilon)$ in asymptotique series in ε. A similar problem has been solved in essence for hierarchical models (see [20, 21]).

The set-up of the paper is the following. First we give definitions and some general results for scaling random fields. Next, in Sect. 3, we define the main object of our paper, the space of formal hamiltonians, and introduce Wilson's renormalization transformation as a map in this space. For convenience standard "physical" (non-rigorous) arguments are given which elucidate the definition of this transformation. After that, in Sect. 4 we define some generalizations of Wilson's renormalization transformation and in particular we introduce a smoothed transformation which preserves the smoothness properties of the hamiltonians. In Sect. 5 the fact that the renormalization transformations form a one-parameter group is proved. The infinitesimal operator and some topological aspects of the renormalization transformations are considered briefly in Sect. 6. At last in Sect. 7 we introduce a space of so-called "projection hamiltonians" and derive a surprisingly simple formula for renormalization transformation in this space. It is noteworthy that the effective hamiltonian, which will be constructed in the second part of the work, is obtained by the procedure of analytic continuation of an projection hamiltonian.

2. The Wilson's Renormalization Group for Random Fields

Let a generalized random field $P(\sigma)$ in the d-dimensional ball $\Omega = \{k \| k| < R\}$ be given, i.e. a system of probability distributions $P\{(\sigma, \varphi_1), \ldots, (\sigma, \varphi_m)\}$ with the usual conditions of accordance (see [22]). Here $\varphi_1 = \varphi_1(k), \ldots, \varphi_m = \varphi_m(k)$ are arbitrary test functions in the space $C_0^\infty(\Omega)$ of infinitely differentiable finite functions.

Let us introduce the following operations on random fields. If $P(\sigma)$ is a generalized random field in the ball $\lambda\Omega = \{k \| k| < \lambda R\}$, $\lambda \geqq 1$, we denote by $S_{\Omega,\lambda}$ the operator of restriction on the ball Ω,

$$S_{\Omega,\lambda} P = P|_\Omega . \tag{1.1}$$

If $P(\sigma)$ is a random field in Ω and $a > 0$ is a positive real number we define the scaling operator

$$R_\lambda^{(a)} P(\sigma) = P(\lambda^{-\frac{a}{2}} \sigma_{\lambda^{-1}}), \tag{1.2}$$

where $(\sigma_{\lambda^{-1}}, \varphi) = (\sigma(\lambda^{-1} k), \varphi(k)) = \lambda^d(\sigma(k), \varphi(\lambda k))$ and $P(\lambda^{-(a/2)}\sigma_{\lambda^{-1}})$ is a generalized random field with probability distributions

$$P\{(\lambda^{-(a/2)}\sigma_{\lambda^{-1}}, \varphi_1), \ldots, (\lambda^{-(a/2)}\sigma_{\lambda^{-1}}, \varphi_m)\} .$$

It is clear that $R_\lambda^{(a)} P(\sigma)$ is a generalized random field in the ball $\lambda\Omega$.

Definition 1.1. The Wilson's renormalization transformation $R_{\Omega,\lambda}^{(a)}$, $\lambda \geqq 1$, is a composition of the transformations $R_\lambda^{(a)}$ and $S_{\Omega,\lambda}$,

$$R_{\Omega,\lambda}^{(a)} = S_{\Omega,\lambda} R_\lambda^{(a)} . \tag{1.3}$$

It is easy to see that

$$R_{\Omega,\lambda}^{(a)} = S_{\Omega,\lambda} R_\lambda^{(a)} = R_\lambda^{(a)} S_{\lambda^{-1}\Omega,\lambda} ,$$

where $\lambda^{-1}\Omega = \{k \| k| < \lambda^{-1} R\}$. Moreover

$$\begin{aligned} R_{\Omega,\lambda}^{(a)} R_{\Omega,\mu}^{(a)} &= S_{\Omega,\lambda} R_\lambda^{(a)} S_{\Omega,\mu} R_\mu^{(a)} \\ &= S_{\Omega,\lambda} S_{\lambda\Omega,\mu} R_\lambda^{(a)} R_\mu^{(a)} = S_{\Omega,\lambda\mu} R_{\lambda\mu}^{(a)} = R_{\Omega,\lambda\mu}^{(a)} , \end{aligned}$$

so that the transformations $\{R_{\Omega,\lambda}^{(a)}; \lambda \geqq 1\}$ form a one-parameter commutative semigroup of transformations. This semigroup was considered first by Wilson (see [1]). For $\lambda < 1$ the transformations $S_{\Omega,\lambda}$ and $R_{\Omega,\lambda}^{(a)}$ are not defined. But for $R = \infty$, i.e. when $\Omega = \mathbb{R}^d$, one can consider also $\lambda < 1$. In this case $S_{\Omega,\lambda}$ is the identity operator and the transformations $R_{\Omega,\lambda}^{(a)} = R_\lambda^{(a)}$ form a group (see [23] where rigorous definitions are given and some limit distributions for $\lambda \to \infty$ and $\lambda \to 0$ are investigated).

Definition 1.2. A generalized random field is scaling invariant in Ω if $R_{\Omega,\lambda}^{(a)} P(\sigma) = P(\sigma)$ for all $\lambda \geqq 1$.

Proposition 1.1. a) *If P is scaling invariant in $\mathbb{R}^d$ then $P|_\Omega$ is scaling invariant in Ω.*

b) *If P is scaling invariant in Ω then $R_\lambda^{(a)} P$ is scaling invariant in $\lambda\Omega$ and its restriction on Ω coincides with P (if $\lambda \geqq 1$).*

c) *If P is scaling invariant in Ω then $\lim_{\lambda \to \infty} R_\lambda^{(a)} P$ is scaling invariant in $\mathbb{R}^d$ (over $\mathscr{D}' = (C_0^\infty(\mathbb{R}^d))'$).*

All these statements are easily verified. Thus there is a natural one-to-one correspondence between scaling invariant fields in $\mathbb{R}^d$ and in any finite ball Ω.

Let us introduce the Fourier transform of the translation operator,

$$t_\alpha : \sigma(k) \to e^{i\alpha k}\sigma(k), \qquad \alpha \in \mathbb{R}^d,$$

the orthogonal transformation operator

$$u_\beta : \sigma(k) \to \sigma(\beta k), \qquad \beta \in O(d),$$

and the parity operator

$$i : \sigma(k) \to -\sigma(k).$$

Denote the conjugated operators in the space of generalized random field by T_α, U_β, and I respectively. A random field $P(\sigma)$ is called translation invariant if $T_\alpha P(\sigma) = P(\sigma)$ for any $\alpha \in \mathbb{R}^d$, isotropic if $U_\beta P(\sigma) = P(\sigma)$ for any $\beta \in O(d)$ and even if $IP(\sigma) = P(\sigma)$. In this paper we are interested in translation invariant, isotropic, even, scaling invariant random fields.

If is easy to describe all Gaussian fields possessing such properties.

Proposition 1.2 [1, 23]. *A generalized Gaussian random field with zero mean and binary correlation function* $\langle \sigma(k)\sigma(k')\rangle = C\delta(k+k')|k|^{-a+d}\chi_\Omega(k)$, *where* $\chi_\Omega(k)$ *is the characteristic function of the ball* Ω, *is the unique translation invariant, isotropic, even, scaling invariant Gaussian field.*

3. The Wilson's Renormalization Group for Formal Hamiltonians

Now we give another definition of the renormalization transformation. In this new "diagram" definition the renormalization transformation will be defined not on the space of random fields but on the space of formal hamiltonians. As a matter of fact this definition is always used in physical words (see [1, 4, 6] and others).

A hamiltonian in the ball $\Omega = \{k \mid |k| < R\}$ is an expression of the form

$$H(\sigma) = \sum_{m=1}^{\infty} \int_{\Omega^m} h_m(k_1, \ldots, k_m)\delta(k_1 + \ldots + k_m)\sigma(k_1)\ldots\sigma(k_m)d^{md}k. \tag{2.1}$$

To give a hamiltonian is the same as to give the sequence of its coefficient functions

$$h = (h_1(k_1), h_2(k_1, k_2), \ldots)$$

on the subspaces $\sum_{i=1}^{m} k_i = 0$, i.e. two sequences $(h_1^{(i)}(k_1), h_2^{(i)}(k_1, k_2), \ldots)$, $i = 1, 2$, define the same hamiltonian if

$$h_m^{(1)}(k_1, \ldots, k_m)\big|_{k_1 + \ldots + k_m = 0} = h_m^{(2)}(k_1, \ldots, k_m)\big|_{k_1 + \ldots + k_m = 0}, \qquad m = 1, 2, \ldots .$$

If all $h_i \equiv 0$ for $i \neq m$, the hamiltonian $H(\sigma)$ is called m-particle. If for some $n, h_i \equiv 0$, when $i > n$, the hamiltonian $H(\sigma)$ is called finite-particle.

By $\mathscr{H}^n$, $n = 0, 1, 2, \ldots, \infty$, we denote the space of finite-particle hamiltonians with coefficient functions $h_m(k_1, \ldots, k_m) \in C^n(\Omega^m)$. A formal hamiltonian is a formal series in ε,

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots \tag{2.2}$$

whose coefficients are finite-particle hamiltonians (see [3]). In what follows the coefficient H_0 will be fixed,

$$H_0 = \int_{|k|<R} \tfrac{1}{2}|k|^{a-d}|\sigma(k)|^2 d^d k. \tag{2.3}$$

The hamiltonian H_0 corresponds to the Gaussian scaling invariant field with the propagator

$$\langle \sigma(k)\sigma(k')\rangle = \delta(k+k')|k|^{-a+d}\chi_R(k) \tag{2.4}$$

(see Proposition 1.1).

The space of formal hamiltonians

$$H' = \varepsilon H_1 + \varepsilon^2 H_2 + \dots$$

with $H_i \in \mathscr{H}^n$ will be denoted by $\mathscr{F}\mathscr{H}^n$,

$$\mathscr{F}\mathscr{H}^n = \mathscr{F}^0 \bigotimes \mathscr{H}^n, \tag{2.5}$$

where $\mathscr{F}^0$ is the space of the complex-valued formal series with zero free term.

Wilson's renormalization transformation in the form in which we consider it, acts in the space of formal hamiltonians. As before it is a composition of two transformations, the scaling $\mathscr{R}_\lambda^{(a)}$ and the restriction $\mathscr{S}_{\Omega,\lambda}$. Let us introduce first the operator $\mathscr{R}_\lambda^{(a)}$ (otherwise named the operator of multiplicative renormalization, see [24]).

Definition 2.1. Let a m-particle hamiltonian

$$H = \int_{|k_i|<R} h(k_1, \dots, k_m)\delta(k_1 + \dots + k_m)\sigma(k_1)\dots\sigma(k_m)d^{md}k$$

be given. Then

$$\begin{aligned}\mathscr{R}_\lambda^{(a)} H = &\int_{|k_i|<R} h(k_1, \dots, k_m)\delta(k_1 + \dots + k_m) \\ &\cdot(\lambda^{a/2}\sigma(\lambda k_1))\dots(\lambda^{a/2}\sigma(\lambda k_m))d^{md}k \\ = \lambda^{\frac{am}{2}-md+d} &\int_{|k_i|<\lambda R} h(\lambda^{-1}k_1, \dots, \lambda^{-1}k_m)\delta(k_1 + \dots + k_m)\sigma(k_1)\dots\sigma(k_m)d^{md}k.\end{aligned}$$

In other words, $\mathscr{R}_\lambda^{(a)}$ changes $h(k_1, \dots, k_m)$ into $\lambda^{(am/2)-md+d}h(\lambda^{-1}k_1, \dots, \lambda^{-1}k_m)$. To the whole space of hamiltonians and to the space of formal hamiltonians the operator $\mathscr{R}_\lambda^{(a)}$ is extended by linearity.

In contrast to $\mathscr{R}_\lambda^{(a)}$ the restriction operator $\mathscr{S}_{\Omega,\lambda}$ has a rather complicated structure and is defined with the help of a summation on Feynman graphs. Before giving a rigorous definition we represent a "physical" deduction of the formulae which are used in this definition (see [1, 4]).

Suppose P is a random Gibbsian field in the ball $\lambda\Omega$ with hamiltonian $H_0 + H'$ and let P_0 be a free field defined by the hamiltonian H_0. The restriction operation consists in the computation of unconditional probability distributions in a subvolume. In other words we fix a configuration $\sigma_0(k)$ in the subvolume and compute the density of the probability distribution for this configuration given by the Gibbsian measure in the whole volume. The density of the distribution "is represented" as the ratio of two partition functions, the conventional one with the fixed configuration $\sigma_0(k)$ and the unconventional one:

$$p(\sigma_0)=\frac{Z(\sigma_0)}{\Xi}.$$

Writing the density in Gibbsian form, $p(\sigma_0)=\exp(H'_\lambda(\sigma_0))$, we have

$$H'_\lambda(\sigma_0)=\ln Z(\sigma_0)-\ln \Xi .$$

For computation of the quantity $\ln Z(\sigma_0)$ one can use the well known formulae of the expansion in cumulants:

$$\ln Z(\sigma_0)=\sum_{n=1}^{\infty}\frac{1}{n!}\langle H',\ldots,H'\rangle^c_{\text{cond. measure}},$$

where the cumulants are taken with respect to the conditional free measure $P_0(\cdot\,|\sigma_0)$ with the fixed configuration $\sigma_0(k)$ in the subvolume. The quantity $\ln\Xi$ does not depend on $\sigma_0(k)$ and can be excluded from the hamiltonian H'_λ. Thus

$$H'_\lambda=\sum_{n=1}^{\infty}\frac{1}{n!}\langle H',\ldots,H'\rangle^c_{\text{cond. measure}}. \tag{2.6}$$

This formula is taken as a definition of the restriction operator in the space of formal hamiltonians. Remark that in a somewhat different but close situation (lattice spin models) this formula is proved rigorously in the high-temperature region (see [25]).

In the particular case under consideration a Gaussian scaling invariant field with the hamiltonian H_0 is taken as a free field and all the cumulants $\langle H',\ldots,H'\rangle^c_{\text{cond. measure}}$ can be represented as sums on connected Feynman graphs with the propagators $|k|^{-a+d}(\chi_{\lambda R}(k)-\chi_R(k))$ (see [1, 4]). For $H'=\sum_{m=1}^{\infty}\varepsilon^m H_m$ we set by definition (for sake of brevity in this and in the subsequent formulae the words "cond. measure" are omitted)

$$\left\langle\sum_{m=1}^{\infty}\varepsilon^m H_m,\ldots,\sum_{m=1}^{\infty}\varepsilon^m H_m\right\rangle^c$$
$$=\sum_{m=1}^{\infty}\varepsilon^m\sum_{m_1+\ldots+m_n=m}\langle H_{m_1},\ldots,H_{m_n}\rangle^c, \tag{2.7}$$

$$\langle H_{m_1},\ldots,H_{m_n}\rangle^c=\sum_G{}^c\int\mathscr{F}_G(p)\prod_{l\in E(G)}\sigma(p_l)\,dp, \tag{2.8}$$

where the summation goes over connected Feynman graphs G and $\mathscr{F}_G(p)$ is the Feynman amplitude corresponding the graph G. In the present work we adhere to the system of notations of the Feynman diagram theory adopted at the school on renormalization theory (see [8]). For a more precise definition of the Feynman amplitudes $\mathscr{F}_G(p)$ let us expand H_m in a sum of homogeneous hamiltonians:

$$H_m=\sum_r H_{mr},$$
$$H_{mr}=\int h_{mr}(k_1,\ldots,k_r)\,\delta(k_1+\ldots+k_r)\,\sigma(k_1)\ldots\sigma(k_r)\,d^{rd}k.$$

Then

$$\langle H_{m_1}, \ldots, H_{m_n}\rangle^c = \sum_{r_1, \ldots, r_n} \langle H_{m_1 r_1}, \ldots, H_{m_n r_n}\rangle^c, \tag{2.9}$$

where

$$\langle H_{m_1 r_1}, \ldots, H_{m_n r_n}\rangle^c = \sum_G^c \int \mathscr{F}_G(p) \prod_{l\in E(G)} (\sigma(p_l))\,dp, \tag{2.10}$$

the sum $\sum_G^c$ going over the set $\mathscr{G}^c(r_1, \ldots, r_n)$ of all connected graphs G with n vertices such that exactly r_j lines come from the j-th vertex[1]. The Feynman amplitude of a graph G is defined by the integral

$$\mathscr{F}_G(p) = \int \left(\prod_{j=1}^{n} h_{m_j r_j}(k^{(j)})\, \delta\left(\sum_{i=1}^{r_j} k_i^{(j)} \right)\right)$$
$$\cdot \prod_{l\in L(G)} (\Delta(\chi_{\lambda R} - \chi_R)(k_l)\delta(k_l + k_l')\, d^d k_l\, d^d k_l'), \tag{2.11}$$

where the propagator

$$\Delta(\chi_{\lambda R} - \chi_R)(k) = |k|^{-a+d}(\chi_{\lambda R}(k) - \chi_R(k)) \tag{2.12}$$

and each of the variables $k_i^{(j)}$ is redenoted as p_l if it corresponds to an external line $l\in E(G)$ of the graph G and as k_l (or k_l') if it corresponds to an end (an origin) of an internal line $l\in L(G)$. Performing the integration on the variables k_l' in (2.11) we come to the usual expression

$$\mathscr{F}_G(p) = \int \left[\prod_{j=1}^{n} h_{m_j r_j}(k^{(j)}) \right]_G \prod_{l\in L(G)} \Delta(\chi_{\lambda R} - \chi_R)(k_l)$$
$$\cdot \prod_{v\in V(G)} \delta\left(\sum_{l\in \mathrm{St}_E v} p_l + \sum_{l\in \mathrm{St}_L v} k_l \right) dk, \tag{2.13}$$

where $\left[\prod_{j=1}^{n} h_{m_j r_j}(k_1^{(j)}, \ldots, k_{r_j}^{(j)}) \right]_G$ means the identification of the variables $(-k_l')$ and k_l, $l\in L(G)$, in the product $\prod_{j=1}^{n} h_{m_j r_j}(k^{(j)})$. The multiplier $\prod_{v\in V(G)} \delta\left(\sum_{l\in \mathrm{St}_E v} p_l + \sum_{l\in \mathrm{St}_L v} \varepsilon_{vl} k_l \right)$ assures the equality to zero of the full momentum at any vertex v of the graph G. The quantities ε_{vl}, $v\in V(G)$, $l\in L(G)$, define the incidence matrix of the graph G; $\varepsilon_{vl} = \pm 1$ if a line l comes in (from) a vertex v, $\varepsilon_{vl} = 0$ otherwise. Finally $\mathrm{St}_E v$ and $\mathrm{St}_L v$ are the external and internal parts of the star $\mathrm{St}\, v$ of a vertex $v\in V(G)$. $\mathrm{St}_E v(\mathrm{St}_L v)$ is the set of external (internal) lines which are incident to a given vertex v.

The quantity $\mathscr{F}_G(p)$ is not defined for vacuum graphs G i.e. those with no external lines, due to redundant number of δ-functions in the diagram integral. For

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1 In order to avoid writing combinatorial multipliers we consider Feynman graphs with enumerated vertices and ends of lines

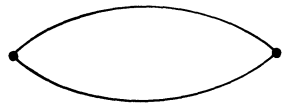

Fig. 1

example the graph shown in Fig. 1 leads to the integral

$$\iint f(k_1)f(k_2)\delta(k_1+k_2)\delta(k_1+k_2)dk_1dk_2$$

having no sense. To avoid such graphs we introduce a rule of "vacuum forbidding". It consists in the fact that vacuum graphs are excluded from the expansion (2.8). Note that this rule is very natural because vacuum diagrams give only constant inputs in the hamiltonian H'_λ. For a nonvacuum connected graph G we have

$$\mathscr{F}_G(p)=\delta\Big(\sum_{l\in E(G)} p_l\Big)F_G(p),$$

where $F_G(p)$ is a piecewise continuous function. Let $C^*(\mathbb{R}^{md})$ be the smallest extension of the space of continuous functions $C(\mathbb{R}^{md})$ with the same topology which contains all the piecewise continuous functions and $\mathscr{H}^*$ be the corresponding space of hamiltonians. Then the Feynman amplitude $\mathscr{F}_G(p)$ of any connected nonvacuum Feynman graph G defines a hamiltonian

$$H_G=\int \mathscr{F}_G(p)\prod_{l\in E(G)}\sigma(p_l)dp\in\mathscr{H}^*.$$

Now we can introduce the restriction operator $\mathscr{S}_{\Omega,\lambda}$.

Definition 2.2. The action of the restriction operator $\mathscr{S}_{\Omega,\lambda}$ on a formal hamiltonian $H'=\varepsilon H_1+\varepsilon^2H_2+\dots$ *is defined by the formula*

$$\begin{aligned}\mathscr{S}_{\Omega,\lambda}(H')&=\langle\exp H'\rangle^c\\&=\sum_{n=1}^{\infty}\frac{1}{n!}\langle\underbrace{H',\dots,H'}_{n}\rangle^c,\end{aligned}$$

where $\langle\underbrace{H',\dots,H'}_{n}\rangle^c$ is a formal hamiltonian which is computed by the formulae (2.7)–(2.13), the summation in (2.8) and (2.10) going over nonvacuum connected graphs G. The operator $\mathscr{S}_{\Omega,\lambda}$ maps $\mathscr{F}\mathscr{H}^n$ into

$$\mathscr{F}\mathscr{H}^*=\mathscr{F}^0\otimes\mathscr{H}^*,\qquad n=0,1,2,\dots,\infty,*.$$

For the following calculations we need a proposition which in quantum field theory is named "theorem on exponent".

Proposition 2.1. ("Theorem on exponent", see e.g. [4].)

$$\ln\langle\exp H\rangle=\langle\exp H\rangle^c$$

or in diagram writing,

$$\begin{aligned}&\ln\sum_G\frac{1}{|V(G)|!}\int\mathscr{F}_G(p)\prod_{l\in E(G)}\sigma(p_l)dp\\&=\sum_G^c\frac{1}{|V(G)|!}\int\mathscr{F}_G(p)\prod_{l\in E(G)}\sigma(p_l)dp\end{aligned}$$

where in the LHS *the summation goes over the set of all Feynman graphs with no vacuum connectivity components, including empty graph, whereas in the* RHS *the summation goes only over the set of connected nonvacuum graphs (the empty graph does not enter in this sum). Remark that* $|V(G)|$ *is the number of vertices of the graph* G.

Now we give the main definition of this section.

Definition 2.3. The Wilson's renormalization transformation in the space of formal hamiltonians is a composition of the scaling and restriction operators,

$$\mathscr{R}^{(a)}_{\Omega;\lambda}=\mathscr{S}_{\Omega,\lambda}\mathscr{R}^{(a)}_{\lambda}. \tag{2.14}$$

Remark. It is easy to see that

$$\mathscr{R}^{(a)}_{\Omega;\lambda}=\mathscr{S}_{\Omega,\lambda}\mathscr{R}^{(a)}_{\lambda}=\mathscr{R}^{(a)}_{\lambda}\mathscr{S}_{\lambda^{-1}\Omega,\lambda}. \tag{2.15}$$

Moreover let us introduce the operators $\mathscr{T}_\alpha$, $\mathscr{U}_\beta$, $\mathscr{I}$. Let $h(k_1,\dots,k_m)$ be the coefficient function of an m-particle hamiltonian. Then

$$\mathscr{T}_\alpha: h(k_1,\dots,k_m)\to e^{i(k_1+\dots+k_m)\alpha}h(k_1,\dots,k_m),\qquad \alpha\in\mathbb{R}^d,$$
$$\mathscr{U}_\beta: h(k_1,\dots,k_m)\to h(\beta^{-1}k_1,\dots,\beta^{-1}k_m),\qquad \beta\in O(d),$$
$$\mathscr{I}: h(k_1,\dots,k_m)\to(-1)^m h(k_1,\dots,k_m).$$

$\mathscr{T}_\alpha$ is a translation operator. Due to

$$e^{i(k_1+\dots+k_m)\alpha}h(k_1,\dots,k_m)|_{k_1+\dots+k_m=0}=h(k_1,\dots,k_m)|_{k_1+\dots+k_m=0}.$$

$\mathscr{T}_\alpha$ coincides with the indentity operator. $\mathscr{U}_\beta$ is an operator of orthogonal transformation in the space of hamiltonians and $\mathscr{I}$ is the parity operator. To the spaces of hamiltonians $\mathscr{H}^n$ and to the spaces of formal hamiltonians $\mathscr{F}\mathscr{H}^n$ the operators $\mathscr{T}_\alpha$, $\mathscr{U}_\beta$, $\mathscr{I}$ are extended by linearity.

Accordingly a (formal) hamiltonian H is called isotropic if $\mathscr{U}_\beta H=H$ for any $\beta\in O(d)$ and even if $\mathscr{I}H=H$. The last condition is equivalent to $h_m(k_1,\dots,k_m)\equiv 0$ for odd m.

4. Modifications of the Renormalization Transformation

Now we should like to make three essential remarks to the definition of the Wilson's renormalization transformations. The first one is concerned with the domain of definition of the coefficient functions of the hamiltonians under consideration. It is assumed usually that the arguments of the coefficient functions of an initial hamiltonian H' vary in the ball $\Omega=\{k\,|\,|k|<R\}$. However in the construction of the effective hamiltonian it is convenient to think the whole space $\mathbb{R}^d$ as the domain of the coefficient functions. It is noteworthy that both approaches agree. Namely, if one restricts first the domain of definition of the coefficient functions from the whole space $\mathbb{R}^d$ to the ball Ω and applies then the renormalization transformation or, the other way round, applies first the renormalization transformation and restricts then the domain of definition, the result will be the same. Indeed, in the process of computing the diagram integrals

(2.11), the integration goes in fact over such a domain that each variable k_i varies in the ring $R<k_i<\lambda R$ due to the fact that the propagator $\Delta(\chi_{\lambda R}-\chi_R)(\mathrm{k})$ is equal to zero outside of this ring. Therefore the values of the coefficient functions outside of the ball $\Omega=\{k \mid |k|<R\}$ have no influence on the values of the diagram integrals.

The second remark is connected with the introduction of a smoothed renormalization transformation. The point is that the propagator $\Delta(\chi_{\lambda R}-\chi_R)(\mathrm{k})$ contains the characteristic functions $\chi_{\lambda R}(k)$, $\chi_R(k)$ and is not a smooth function. As a result the image of a smooth hamiltonian $H'\in\mathscr{F}\mathscr{H}^\infty$ under the renormalization transformation is not a smooth hamiltonian. This leads in the process of construction of the effective hamiltonian to the rise of irrelevant singularities which destroy the Wilson's analyticity postulate (see [6] and the Introduction). To get rid of these singularities we introduce a smoothed renormalization transformation.

Consider a test function $\chi(k)\in C_0^\infty(\mathbb{R}^d)$ depending only on $|k|$, $\chi(k)=\chi_0(|k|)$, such that

$$\chi(k)\begin{cases} =0, & |k|\geqq R_1, \\ >0, <1, & R_1>|k|>R_0, \\ =1, & R_0\geqq |k|, \end{cases}$$

where $R_1>R_0>0$ are some real numbers, with the additional requirement that $\chi_0(|k|)$ is a nonincreasing function. Denote

$$\Delta(\chi_\lambda-\chi)(k)=|k|^{-a+d}(\chi(k/\lambda)-\chi(k)). \tag{3.1}$$

This function is a smoothing of the propagator $\Delta(\chi_{\lambda R}-\chi_R)$. When R_1, $R_0\to R$, $\Delta(\chi_\lambda-\chi)(k)\to\Delta(\chi_{\lambda R}-\chi_R)(k)$ (in various senses).

For convenience of notations we introduce an operator $\mathscr{S}_\psi$ which is a generalization of the operator in Definition 2.2:

$$\begin{aligned}\mathscr{S}_\psi(H')&=\langle\exp H'\rangle_\psi^c \\ &=\sum_{n=1}^{\infty}\frac{1}{n!}\langle\underbrace{H',\dots,H'}_{n}\rangle_\psi^c,\end{aligned} \tag{3.2}$$

where $\langle H',\dots,H'\rangle_\psi^c$ is defined by the formulae (2.7)–(2.13) with the only difference that instead of the propagator $\Delta(\chi_{\lambda R}-\chi_R)(k)$ a given function $\psi(k)$ is used.

It is noteworthy that the formulae (2.7)–(2.11) have sense for non-positive functions $\psi(k)$ too so the operator $\mathscr{S}_\psi$ is defined for an arbitrary function $\psi(k)\in C_0^\infty(\mathbb{R}^d)$.

Definition 3.1. A smoothed renormalization transformation is defined by the formula

$$\mathscr{R}^{(a)}_{\chi,\lambda}=\mathscr{S}_{\Delta(\chi_\lambda-\chi)}\mathscr{R}^{(a)}_\lambda,$$

where the propagator $\Delta(\chi_\lambda-\chi)$ is given in (3.1).

The smoothed renormalization transformation maps a hamiltonian H' into another one $H'_\lambda=\mathscr{R}^{(a)}_{\chi,\lambda}(H')$. The coefficient functions of the hamiltonian H'_λ are computed in terms of the coefficient functions of the hamiltonian H' with the help of the diagram integrals (2.13) with the smoothed propagator $\Delta(\chi_\lambda - \chi)(k)$. So the smoothed renormalization transformation $\mathscr{R}^{(a)}_{\chi,\lambda}$ preserves the smoothness of the coefficient functions. Thus

$$\mathscr{R}^{(a)}_{\chi,\lambda} : \mathscr{F}\mathscr{H}^\infty \to \mathscr{F}\mathscr{H}^\infty . \tag{3.3}$$

In the following we shall construct a smooth formal hamiltonian $H^* \in \mathscr{F}\mathscr{H}^\infty$ which is a fixed point of the smoothed renormalization transformation (see the Paper II). From the exact formulae for H^* it will be seen that one can go to the limit $\chi(k) \to \chi_R(k)$ and obtain a fixed point of the nonsmoothed renormalization transformation $\mathscr{R}^{(a)}_{\Omega;\lambda}$. However the limit hamiltonian will be only piecewise continuous and its coefficient functions will have singularities. These singularities are connected only with the jump of the characteristic function $\chi_R(k)$ and have no special meaning.

Another smoothed renormalization transformation was considered in [26]. In this paper Gaussian fixed points of the smoothed transformation were investigated.

The third remark we want to make is about the group character of the renormalization transformations. The fact, that the renormalization transformations of random fields form a one-parameter semi-group, follows almost immediately from the definition. Since we do not yet have relations between the renormalization transformations of random fields and those of formal hamiltonians the fact, that the renormalization transformations satisfy the group property, needs a proof.

Theorem 3.1. $\mathscr{R}^{(a)}_{\chi,\lambda} \mathscr{R}^{(a)}_{\chi,\mu} = \mathscr{R}^{(a)}_{\chi,\lambda\mu}$.

We shall give the proof of the theorem in the next section. Remark that as we have pointed out above the operator $\mathscr{S}_\psi$ is defined for arbitrary (in general not positive) function ψ. So the renormalization transformation $\mathscr{R}^{(a)}_{\chi,\lambda} = = \mathscr{S}_{\Delta(\chi_\lambda - \chi)} \mathscr{R}^{(a)}_\lambda$ is defined for both $\lambda \geqq 1$ and $\lambda < 1$. Thus in contrast to the renormalization transformations of random fields, the $\mathscr{R}^{(a)}_{\chi,\lambda}$, $0 < \lambda < \infty$, form a one-parameter group (and not only semi-group) of transformations.

5. The Wick Operation : : and the Renormalization Transformation

Let a Gaussian field be given with zero mean and a binary correlation function $G(k,k')$. The Wick operation with respect to this Gaussian field transfers a monomial $\sigma(k_1)\ldots\sigma(k_m)$ into the Wick polynomial

$$:\sigma(k_1)\ldots\sigma(k_m): = \sum_S \sum_{\pi(\bar{S})} \prod_r (-G(k_{i_r}, k_{j_r})) \prod_{i \in S} \sigma(k_i) , \tag{4.1}$$

where $\sum\limits_S$ means summation over all subsets $S \subset \{1, \ldots, m\}$ and $\sum\limits_{\pi(\bar{S})}$ means summation over all partitions π of the set $\bar{S} = \{1, \ldots, m\} \setminus S$ into pairs (i_1, j_1), (i_2, j_2), ... Instead of $\sum\limits_S \sum\limits_{\pi(S)}$ one can write as usual the sum $\sum\limits_G$ over all diagrams with single vertex. It is convenient for us to modify a little the Wick operation. Namely in what follows we shall assume that in the $\sum\limits_S$ the summation goes over all non-empty subsets $S \subset \{1, \ldots, m\}$. In other words we throw away the free term from the Wick polynomial.

For an m-particle hamiltonian

$$H(\sigma) = \int h(k_1, \ldots, k_m)\delta(k_1 + \ldots k_m)\sigma(k_1)\ldots\sigma(k_m)d^{md}k$$

and an integrable propagator $\Delta(k)$ we set by definition

$$\begin{aligned} :H(\sigma):_\Delta &= \int h(k_1, \ldots, k_m)\delta(k_1 + \ldots + k_m) :\sigma(k_1)\ldots\sigma(k_m): d^{md}k \\ &= \int h(k_1, \ldots, k_m)\delta(k_1 + \ldots + k_m)\sum_S \sum_{\pi(\bar{S})} \prod_r (-\Delta(k_{i_r})\delta(k_{i_r} + k_{j_r})) \\ &\quad \cdot \prod_{l\in S} \sigma(k_l)d^{md}k = \sum_G \int [h(k_1, \ldots, k_m)]_G\, \delta\Big(\sum_{l\in E(G)} p_l\Big) \\ &\quad \cdot \prod_{l\in L(G)} (-\Delta(k_l)d^dk_l) \prod_{l\in E(G)} (\sigma(p_l)d^dp_l), \end{aligned}$$

where in $\sum_G$ the summation goes over all nonvacuum graphs G with single vertex. As a result $:H(\sigma):_\Delta$ is a finite-particle hamiltonian. Similarly for a finite product of hamiltonians

$$H_j(\sigma) = \int h_j(k_1, \ldots, k_{m_j})\delta(k_1 + \ldots + k_{m_j})\sigma(k_1)\ldots\sigma(k_{m_j})d^{m_jd}k, \qquad j = 1, \ldots, n,$$

we define the Wick operation

$$\begin{aligned} :H_1(\sigma)\ldots H_n(\sigma):_\Delta = \sum_G \int \Bigg[\prod_{j=1}^n h_j(k_1^{(j)}, \ldots, k_{m_j}^{(j)})\Bigg]_G \\ \cdot \prod_{v\in V(G)} \delta\Big(\sum_{l\in \mathrm{St}_E v} p_l + \sum_{l\in \mathrm{St}_L v} \varepsilon_{vl}k_l\Big) \\ \cdot \prod_{l\in L(G)} (-\Delta(k_l)) \prod_{l\in E(G)} \sigma(p_l)dk\,dp \end{aligned} \tag{4.2}$$

and the connected Wick operation

$$\begin{aligned} :H_1(\sigma)\ldots H_n(\sigma):_\Delta^c = \sum_G^c \int \Bigg[\prod_{j=1}^n h_j(k_1^{(j)}, \ldots, k_{m_j}^{(j)})\Bigg]_G \\ \cdot \prod_{v\in V(G)} \delta\Big(\sum_{l\in \mathrm{St}_E v} p_l + \sum_{l\in \mathrm{St}_L v} \varepsilon_{vl}k_l\Big) \\ \cdot \prod_{l\in L(G)} (-\Delta(k_l)) \prod_{l\in E(G)} \sigma(p_l)dk\,dp\,. \end{aligned} \tag{4.3}$$

Here the same notations as in (2.13) are used. The difference between $:\ :_\Delta$ and $:\ :_\Delta^c$ lies in the sets over which the summations $\sum_G$ and $\sum_G^c$ go. The first set $\mathscr{G}(m_1, \ldots, m_n)$ contains all the graphs G with no vacuum connectivity components such that exactly m_j lines come from the j-th vertex, $j = 1, \ldots, n$, the second one $\mathscr{G}^c(m_1, \ldots, m_n)$ contains all the graphs from $\mathscr{G}(m_1, \ldots, m_n)$ which have a single connectivity component.

The connected Wick polynomial $:H_1(\sigma)\ldots H_n(\sigma):_\Delta^c$ is a finite-particle hamiltonian, the Wick polynomial $:H_1(\sigma)\ldots H_n(\sigma):_\Delta$ is a sum of products of finite-particle polynomials corresponding to the connectivity components of the graphs G. By linearity (with respect to each multiplier) the Wick operations are extended to products of finite-particle and formal hamiltonians and, moreover, to finite sums of the form $\sum_r H_{i_1}\ldots H_{i_{n_r}}$ where all $H_i \in \mathscr{H}^n$ or all $H_i \in \mathscr{F}\mathscr{H}^n$.

For the Wick operation a simple formula of composition takes place (see e.g. [27, 28]):

$$: \; : \; :_{\Delta_1} :_{\Delta_2} = : \; :_{\Delta_1+\Delta_2} . \tag{4.4}$$

A propagator $\Delta(k)$ of a Gaussian field has to satisfy the positivity property, but the formulae (4.1)–(4.3) have sense also for $\Delta(k)$ not satisfying this property. The rule of composition of Wick operations remains the same. Indeed one can consider two families

$$\Delta_i^t = t\Delta_i + (1-t)\Delta_i', \qquad i=1,2,$$

of propagators such that Δ_i', $i=1,2$, are strictly positive as well Δ_i^t, $i=1,2$, for small $t>0$ (one can put $\Delta_i' = |\Delta_i| + 1$, $i=1,2$). Then the composition rule (4.4) is valid for small t and by analyticity it is continued to $t=1$, i.e. to Δ_i, $i=1,2$, which was to be shown.

For $\Delta(k) = -\psi(k)$ the RHSs of (4.2), (4.3) are nothing else than $\langle \cdot \rangle_\psi^c$, $\langle \cdot \rangle_\psi$:

$$:H_1(\sigma)\ldots H_n(\sigma):_{-\psi}^c = \langle H_1(\sigma), \ldots, H_n(\sigma)\rangle_\psi^c , \tag{4.5}$$

$$:H_1(\sigma)\ldots H_n(\sigma):_{-\psi} = \langle H_1(\sigma), \ldots, H_n(\sigma)\rangle_\psi . \tag{4.6}$$

These equalities permit to write the renormalization transformation $\mathscr{R}_{\chi,\lambda}^{(a)}$ with the help of Wick operations:

$$\mathscr{R}_{\chi,\lambda}^{(a)}(H) = :\exp(\mathscr{R}_\lambda^{(a)}\mathrm{H}):_{-\Delta(\chi_\lambda-\chi)}^c , \tag{4.7}$$

$$\mathscr{R}_{\chi,\lambda}^{(a)}(H) = \ln :\exp(\mathscr{R}_\lambda^{(a)}\mathrm{H}):_{-\Delta(\chi_\lambda-\chi)} \tag{4.8}$$

(Proposition 2.1 is used).

Let us compute $\mathscr{R}_{\chi,\lambda}^{(a)}\mathscr{R}_{\chi,\mu}^{(a)}$. We have

$$\mathscr{S}_\psi(H) = :\exp H:_{-\psi}^c = \ln :\exp H:_{-\psi} ,$$

so that

$$\begin{aligned}\mathscr{S}_{\psi_1}\mathscr{S}_{\psi_2}(H) &= \ln : \; :\exp H:_{-\psi_2}:_{-\psi_1} \\ &= \ln :\exp H:_{-\psi_1-\psi_2} = \mathscr{S}_{\psi_1+\psi_2}(H) ,\end{aligned}$$

i.e.

$$\mathscr{S}_{\psi_1}\mathscr{S}_{\psi_2} = \mathscr{S}_{\psi_1+\psi_2} . \tag{4.9}$$

Moreover we have the equalities

$$\mathscr{R}_\lambda^{(a)}\mathscr{R}_\mu^{(a)} = \mathscr{R}_{\lambda\mu}^{(a)} , \tag{4.10}$$

$$\mathscr{S}_\psi\mathscr{R}_\lambda^{(a)} = \mathscr{R}_\lambda^{(a)}\mathscr{S}_{\lambda^{a-d}\psi_\lambda} \qquad (\psi_\lambda(k) \equiv \psi(\lambda k)) , \tag{4.11}$$

which follow directly from the definitions of the operators $\mathscr{S}_\psi$ and $\mathscr{R}_\lambda^{(a)}$. Hence

$$\begin{aligned}\mathscr{R}_{\chi,\lambda}^{(a)}\mathscr{R}_{\chi,\mu}^{(a)} &= \mathscr{S}_{\Delta(\chi_\lambda-\chi)}\mathscr{R}_\lambda^{(a)}\mathscr{S}_{\Delta(\chi_\mu-\chi)}\mathscr{R}_\mu^{(a)} \\ &= \mathscr{S}_{\Delta(\chi_\lambda-\chi)}\mathscr{S}_{\Delta(\chi_{\lambda\mu}-\chi_\lambda)}\mathscr{R}_\lambda^{(a)}\mathscr{R}_\mu^{(a)} = \mathscr{S}_{\Delta(\chi_{\lambda\mu}-\chi)}\mathscr{R}_{\lambda\mu}^{(a)} = \mathscr{R}_{\chi,\lambda\mu}^{(a)} .\end{aligned}$$

Thus the renormalization transformations $\mathcal{R}_{\chi,\lambda}^{(a)}$ form a one-parameter group. The Theorem 3.1 is proved.

6. The Infinitesimal Operator of the Renormalization Group

In this section we shall consider briefly some topological questions connected with the renormalization group $\{\mathcal{R}_{\chi,\lambda}^{(a)}\}$. Let us introduce a notion of convergence in the space of hamiltonians $\mathcal{H}^{\infty}$. As $\mathcal{H}^{\infty}$ is a linear space it is enough to define the convergence to zero. Namely we put $h^{(n)}=(h_1^{(n)}, h_2^{(n)}, \ldots)\to 0=(0,0,\ldots)$, if

(i) there exists $N>0$ such that $h_m^{(n)}\equiv 0$ for $m\geqq N$;

(ii) $h_m^{(n)}\to 0$ in the C^{∞}-topology.

The notion of convergence in $\mathcal{H}^{\infty}$ implies that in $\mathcal{F}\mathcal{H}^{\infty}$.

Proposition 5.1. *For any $\lambda>0$ the operator $\mathcal{R}_{\chi,\lambda}^{(a)}$ is a (nonlinear) continuous infinitely differentiable in the sense of Gâteaux (differentiability along any direction) mapping from $\mathcal{F}\mathcal{H}^{\infty}$ to $\mathcal{F}\mathcal{H}^{\infty}$. Moreover the operator $\mathcal{R}_{\chi,\lambda}^{(a)}$ is an entire function of the parameter a.*

Proof. We have

$$\mathcal{R}_{\chi,\lambda}^{(a)}=\mathcal{S}_{\Delta(\chi_\lambda-\chi)}\mathcal{R}_{\lambda}^{(a)}.$$

The operator $\mathcal{S}_{\Delta(\chi_\lambda-\chi)}$ is determined by a chain of finite-dimensional nonlinear integral operators with kernels, acting on the coefficient functions $h_n(k_1, \ldots, k_n)$ so $\mathcal{S}_{\Delta(\chi_\lambda-\chi)}$ is a continuous infinitely differentiable in the sense of Gâteaux mapping from $\mathcal{F}\mathcal{H}^{\infty}$ to $\mathcal{F}\mathcal{H}^{\infty}$. The operator $\mathcal{R}_{\lambda}^{(a)}$ is linear,

$$\mathcal{R}_{\lambda}^{(a)}=\mathcal{M}_{\lambda}^{(a)}\mathcal{H}_{\lambda}=\mathcal{H}_{\lambda}\mathcal{M}_{\lambda}^{(a)}, \tag{5.1}$$

where

$$\mathcal{H}_{\lambda}: h(k_1, \ldots, k_n)\to\lambda^{(1-n)d}h(\lambda^{-1}k_1, \ldots, \lambda^{-1}k_n) \tag{5.2}$$

is a homothety operator and

$$\mathcal{M}_{\lambda}^{(a)}: h(k_1, \ldots, k_n)\to\lambda^{an}h(k_1, \ldots, k_n) \tag{5.3}$$

is an operator of renormalization of spin variables. Both operators, $\mathcal{H}_{\lambda}$, $\mathcal{M}_{\lambda}^{(a)}$ are continuous and infinitely differentiable in the sense of Gâteaux in $\mathcal{F}\mathcal{H}^{\infty}$. Thus the operator

$$\mathcal{R}_{\chi,\lambda}^{(a)}=\mathcal{S}_{\Delta(\chi_\lambda-\chi)}\mathcal{M}_{\lambda}^{(a)}\mathcal{H}_{\lambda} \tag{5.4}$$

is continuous and infinitely differentiable in $\mathcal{F}\mathcal{H}^{\infty}$. Next the propagator $\Delta(\chi_\lambda-\chi)(k)=|k|^{-a+d}(\chi(k/\lambda)-\chi(k))$ is defined for all complex values $a\in\mathbb{C}$ and is an entire function of a. Differentiation by a of $\mathcal{S}_{\Delta(\chi_\lambda-\chi)}(H)$ is reduced to sums of differentiations of propagators on the lines of Feynman graphs. Hence $\mathcal{S}_{\Delta(\chi_\lambda-\chi)}$ is differentiable in a in the whole complex plane, so that the operator $\mathcal{S}_{\Delta(\chi_\lambda-\chi)}$ depends analytically on $a\in\mathbb{C}$. The operator $\mathcal{H}_{\lambda}$ does not depend on a and $\mathcal{M}_{\lambda}^{(a)}$ is evidently analytic in a. So $\mathcal{R}_{\chi,\lambda}^{(a)}$ is analytic in a. The proposition is proved.

The infinitesimal operator $\mathscr{W}$ of the renormalization group $\{\mathscr{R}^{(a)}_{\chi,\lambda}\}$ is defined as

$$\mathscr{W} = \lim_{\lambda\to 1} \frac{\mathscr{R}^{(a)}_{\chi,\lambda} - I}{\lambda - 1}, \tag{5.5}$$

where I is the identity operator. It follows easily from (5.4), that the limit (5.5) exists in the space $\mathscr{F}\mathscr{H}^{\infty}$ and the operator $\mathscr{W}$ is the sum of infinitesimal operators of the transformations $\mathscr{S}_{\Delta(\chi_\lambda - \chi)}$, $\mathscr{M}^{(a)}_{\lambda}$ and $\mathscr{H}_{\lambda}$,

$$\begin{aligned} &\mathscr{W} = \mathscr{W}^S + \mathscr{W}^M + \mathscr{W}^H, \\ &\mathscr{W}^S : H = (h_1, h_2, \ldots) \to \langle \exp H \rangle^{c'}_{\Delta\chi'}, \\ &\mathscr{W}^M : h_n(k_1, \ldots, k_n) \to n a h_n(k_1, \ldots, k_n), \\ &\mathscr{W}^H : h_n(k_1, \ldots, k_n) \to \sum_{i=1}^{n} (k_i, \nabla_{k_i}) h_n(k_1, \ldots, k_n), \end{aligned} \tag{5.6}$$

where $\langle \cdot \rangle^{c'}_{\Delta\chi'}$ is defined by a summation on connected Feyman graphs with only one internal line to which the propagator $\Delta\chi'(k) = |k|^{-a+d+1}\chi'_0(|k|)$ corresponds. The infinitesimal operator $\mathscr{W}$ was considered before in [29, 1] and in other papers. In force of the explicit formula (5.6) the operator $\mathscr{W}$ is continuous and infinitely differentiable in the sense of Gâteaux in $\mathscr{F}\mathscr{H}^{\infty}$.

Let $H^{(0)} = \varepsilon H^{(0)}_1 + \varepsilon^2 H^{(0)}_2 + \ldots \in \mathscr{F}\mathscr{H}^{\infty}$. By Proposition 5.1 the operator $\mathscr{R}^{(a)}_{\chi,\lambda}$ is infinitely differentiable in $\mathscr{F}\mathscr{H}^{\infty}$. Denote its differential at the point $H^{(0)}$ by $D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}$. The operator $D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}$ is computed easily from the definition of the transformation $\mathscr{R}^{(a)}_{\chi,\lambda}$:

$$D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda} = (D_{\mathscr{R}^{(a)}_{\lambda} H^{(0)}} \mathscr{S}_{\Delta(\chi_\lambda - \chi)}) \mathscr{R}^{(a)}_{\lambda}, \tag{5.7}$$

where

$$(D_{H^{(1)}}\mathscr{S}_{\Delta(\chi_\lambda - \chi)}) H = \sum_{n=1}^{\infty} \frac{n}{n!} \langle \underbrace{H^{(1)}, \ldots, H^{(1)}}_{n}, H \rangle^{c}_{\Delta(\chi_\lambda - \chi)}. \tag{5.8}$$

For sake of brevity denote $D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}$ with $H^{(0)} = 0$ by $\mathscr{D}^{(a)}_{\chi,\lambda}$. Then

$$\mathscr{D}^{(a)}_{\chi,\lambda} H = \langle \mathscr{R}^{(a)}_{\lambda} H \rangle_{\Delta(\chi_\lambda - \chi)}. \tag{5.9}$$

Similarly one can compute the differential of the infinitesimal operator:

$$D_{H^{(0)}}\mathscr{W} = \mathscr{W}^M + \mathscr{W}^H + \langle \cdot \rangle^{c'}_{\Delta\chi'} + \langle H^{(0)}, \cdot \rangle^{c'}_{\Delta\chi'}.$$

In the general case the operators $D_{H(0)}\mathscr{R}^{(a)}_{\chi,\lambda}$ do not satisfy a group property (in this connection see [30]). But if $H^{(0)}$ is a fixed point of the renormalization transformations then the operators $D_{H^{(0)}}\mathscr{R}^{(a)}_{\chi,\lambda}$ form a group of linear operators with infinitesimal operator $D_{H^{(0)}}\mathscr{W}$. In particular this takes place for $H^{(0)} = 0$: the operators $\mathscr{D}^{(a)}_{\chi,\lambda}$ form a group with the infinitesimal operator $D_0\mathscr{W}$.

7. The Renormalization Transformation for Projection Hamiltonians

The renormalization transformation is defined by rather complicated formulae. These formulae are considerably simplified for a class of hamiltonians which we call "projection hamiltonians". The essence of the matter is clarified in the theorem stated below. This theorem enables us to introduce the following definition.

Let

$$C_b^\infty(\mathbb{R}^d)=\{f(x)\in C^\infty(\mathbb{R}^d)|\forall N\geqq 0,$$
$$\sup_{x\in\mathbb{R}^d}\sum_{|\beta|\leqq N}|D^\beta f(x)|<\infty\}.$$

be the space of the functions which are bounded at infinity together with all their derivatives. Let

$$\mathscr{H}_b^\infty=\{H=(h_1,h_2,\ldots)\in\mathscr{H}^\infty|h_i\in C_b^\infty,\ i=1,2,\ldots\},$$
$$\mathscr{F}\mathscr{H}_b^\infty=\{H=\varepsilon H_1+\varepsilon^2 H_2+\ldots\in\mathscr{F}\mathscr{H}^\infty|H_i\in\mathscr{H}_b^\infty,\ i=1,2,\ldots\},$$
$$\mathscr{F}\mathscr{H}_b^\infty=\mathscr{F}^0\bigotimes_{\mathbb{C}}\mathscr{H}_b^\infty$$

be the corresponding spaces of finite-particle and formal hamiltonians.

Definition 6.1. A hamiltonian $H\in\mathscr{F}\mathscr{H}_b^\infty$ of the form

$$H=:\exp\mathscr{L}:^c_{-\Delta(1-\chi)}=\langle\exp\mathscr{L}\rangle^c_{\Delta(1-\chi)}$$
$$=\mathscr{S}_{\Delta(1-\chi)}(\mathscr{L})=\sum_{n=1}^\infty\frac{1}{n!}\sum_{G:|V(G)|=n}^c\int\mathscr{F}_G(p)\prod_{l\in E(G)}\sigma(p_l)dp,$$

where $\Delta(1-\chi)(k)=|k|^{-a+d}(1-\chi(k))$ and $\mathscr{L}\in\mathscr{F}\mathscr{H}_b^\infty$ is called a projection hamiltonian with generating hamiltonian $\mathscr{L}$.

Remark that by Theorem 6.1 the projection hamiltonian is defined when $\mathrm{Re}\,a>2d$. In Paper II we shall extend the domain of allowed values of a with the help of analytic continuation in a.

Theorem 6.1. *Let* $\mathrm{Re}\,a>2d$. *Then*

(i) *all diagram integrals in*

$$H=:\exp\mathscr{L}:^c_{-\Delta(1-\chi)}=\langle\exp\mathscr{L}\rangle^c_{\Delta(1-\chi)}=\mathscr{S}_{\Delta(1-\chi)}(\mathscr{L})$$

are finite:

(ii) $H\in\mathscr{F}\mathscr{H}_b^\infty$;

(iii) H *depends analytically on the parameters* a *(which appears in the propagator* $\Delta(1-\chi)$*)*;

(iv) $\mathscr{R}^{(a)}_{\chi,\lambda}(H)=:\exp\mathscr{R}^{(a)}_\lambda\mathscr{L}:^c_{-\Delta(1-\chi)}$
$$=\mathscr{S}_{\Delta(1-\chi)}(\mathscr{R}^{(a)}_\lambda\mathscr{L}).$$

Remark. This theorem shows that in terms of $\mathscr{L}$ the renormalization transformation for projection hamiltonians is reduced to the application of the operator $\mathscr{R}^{(a)}_\lambda$.

Proof. Let $\mathscr{L}=\sum_{n=1}^\infty\varepsilon^n L_n$, $L_n=(L_{n1},L_{n2},\ldots)$; consider a diagram integral in $H=:\exp\mathscr{L}:^c_{-\Delta(1-\chi)}$. It has the form

$$\mathscr{F}_{njG}(p)=\int\left[\prod_{m=1}^M L_{n_m j_m}(k^{(j_m)})\right]_G$$
$$\cdot\prod_{v\in V(G)}\delta(\textstyle\sum p_l+\sum\varepsilon_{vl}k_l)\displaystyle\prod_{l\in L(G)}\Delta(1-\chi)(k_l)dk,$$

where $n=(n_1, \ldots, n_M)$, $j=(j_1, \ldots, j_M)$, G is a connected nonvacuum graph. The functions $L_{n_m j_m}(k^{(j_m)}) \in C_b^\infty$, so that they are bounded. Therefore we have to verify the convergence of the integral only in the case when $L_{n_m j_m}(k^{(j_m)}) \equiv \text{const}$. By the "power counting theorem" (see [31]) for the convergence of the digram integral it suffices that the index

$$\operatorname{ind} H = (\operatorname{Re} a - d)|L(H)| - d(|L(H)| - |V(H)| + 1)$$

of any subgraph $H \subset G$ be positive. But for $\operatorname{Re} a > 2d$,

$$\operatorname{ind} H > d(|V(H)| - 1) > 0$$

so the positivity condition is valid and the digram integral converges. Moreover with the help of the α-representation it is easy to show that the digram integral is uniformly bounded in p (see [7]). More precisely due to the inequality

$$|k|^{-a+d}(1-\chi(k)) < C(|k|^2+1)^{(-a+d)/2}$$

for some $C > 0$, we can estimate $\mathscr{F}_{njG}(p)$ by the corresponding Feynman amplitude with massive propagators $(|k|^2+1)^{(-a+d)/2}$ and after that estimate the latter amplitude with the help of the α-representation.

Under differentiation by k the functions $L_{n_m j_m}(k^{(j_m)})$ remain bounded and the propagator $\Delta(1-\chi)(k)$ decreases at infinity somewhat faster. Thus the diagram integral remains finite after the differentiation by the variables p_l and $\mathscr{F}_{njG}(p) \in C_b^\infty$ which was to be proved in (i), (ii).

Now the differentiation by a of an amplitude $\mathscr{F}_{njG}(p)$ is reduced to that of the propagators on the lines of the graph G. Under differentiation, the propagator $\Delta(1-\chi)(k) = |k|^{-a+d}(1-\chi(k))$ changes only for a logarithmic multiplier, so by the same "power counting theorem" the Feynman amplitude remains finite for $\operatorname{Re} a > 2d$ after differentiation by a. Hence in this domain it is an analytic function of a, which was to be shown in (iii).

We prove now (iv). We have

$$\begin{aligned}
\mathscr{R}^{(a)}_{\chi,\lambda}(H) &= \mathscr{S}_{\Delta(\chi_\lambda - \chi)} \mathscr{R}^{(a)}_\lambda \mathscr{S}_{\Delta(1-\chi)}(\mathscr{L}) \\
&= \mathscr{S}_{\Delta(\chi_\lambda - \chi)} \mathscr{S}_{\Delta(1-\chi_\lambda)} \mathscr{R}^{(a)}_\lambda(\mathscr{L}) \\
&= \mathscr{S}_{\Delta(1-\chi)} \mathscr{R}^{(a)}_\lambda(\mathscr{L}),
\end{aligned}$$

which was to be shown. Here we have used the commutation relation (4.11) and the composition formula (4.9).

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