

The Navier-Stokes Equations on a Bounded Domain*

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Abstract. Suppose U is an open bounded subset of 3-space such that the boundary of U has Lebesgue measure zero. Then for any initial condition with finite kinetic energy we can find a global (i.e. for all time) weak solution u to the time dependent Navier-Stokes equations of incompressible fluid flow in U such that the curl of u is continuous outside a locally closed set whose $5/3$ dimensional Hausdorff measure is finite.

1. Introduction

Definition 1.1. Suppose f is a C^∞ function defined on an open subset V of $R^3 \times R$. If $i \in \{1, 2, 3\}$ then $D_i f$ is the partial derivative of f with respect to the i component of R^3 . The partial derivative of f with respect to the R component of $R^3 \times R$ is denoted by $D_t f$. The letter t is used because the second component of $R^3 \times R$ represents time. The vector function $(D_1 f, D_2 f, D_3 f)$ is written Df . The function Δf is defined on the set V by $(\Delta f)(x, t) = \sum_{i=1}^3 D_i(D_i f)(x, t)$. When the range of f is R^3 we define the functions $\text{div}(f): V \rightarrow R$ and $\text{curl}(f): V \rightarrow R^3$ by

$$(\text{div}(f))(x, t) = \sum_{i=1}^3 D_i f_i(x, t)$$

and

$$(\text{curl}(f))(x, t) = ((D_2 f_3 - D_3 f_2)(x, t), (D_3 f_1 - D_1 f_3)(x, t), (D_1 f_2 - D_2 f_1)(x, t)).$$

We extend these definitions in the obvious way to the case where f is a distribution. Hausdorff measure is defined in Definition 6.5, R^+ is the set $\{t \in R : t > 0\}$, L^p is the Lebesgue space of p -integrable functions with norm $\| \cdot \|_p$, and the summation convention for repeated indices is used. If A and B are sets then $A \sim B = \{x \in A : x \notin B\}$.

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The purpose of this paper is to prove the following:

Theorem 1.2. *Suppose U is an open bounded subset of \mathbb{R}^3 such that the boundary of U has Lebesgue measure zero. Suppose that $w^0: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an L^2 function such that $w^0(x)=0$ for almost every $x \notin U$ and $\int w_i^0(x) D_i f(x) dx = 0$ for every C^∞ function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ with compact support. Then there exists a set $S \subset U \times \mathbb{R}^+$ and there exists an $L^{10/3}$ function $u: \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ such that Du is an L^2 function and the following properties hold:*

$$(1) \quad u(x, t) = 0 \text{ for almost every } (x, t) \notin U \times \mathbb{R}^+,$$

$$(2) \quad \int_0^\infty \int_{\mathbb{R}^3} u_i(x, t) D_i f(x, t) dx dt = 0 \text{ if } f: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a } C^\infty \text{ function with compact support,}$$

$$(3) \quad - \int_{\mathbb{R}^3} w_i^0(x) f_i(x, 0) dx = \int_0^\infty \int_{\mathbb{R}^3} [u_i(x, t)] [D_t f_i(x, t) + \Delta f_i(x, t)] dx dt \\ + \int_0^\infty \int_{\mathbb{R}^3} u_j(x, t) u_i(x, t) D_j f_i(x, t) dx dt$$

if $f: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ is a C^∞ function with compact support such that $\operatorname{div}(f) = 0$ and the support of f is contained in $U \times \mathbb{R}$,

$$(4) \quad \text{if } K \text{ is a compact subset of } U \times \mathbb{R}^+ \text{ then } K \cap S \text{ is compact,}$$

(5) the function $\operatorname{curl}(u)$ can be modified on a set of Lebesgue measure zero so that it becomes continuous on $(U \times \mathbb{R}^+) \sim S$,

$$(6) \quad \text{the } 5/3 \text{ dimensional Hausdorff measure of } S \text{ is finite,}$$

(7) the one dimensional Hausdorff measure of $S \cap (\mathbb{R}^3 \times \{t\})$ is a bounded function of $t \in \mathbb{R}^+$.

A few informal comments will clarify the statement of this theorem. The two factor spaces in $\mathbb{R}^3 \times \mathbb{R}^+$ represent space and time, respectively. The function w^0 is a given velocity vector field with finite kinetic energy which represents an incompressible flow inside U . This function should be thought of as being defined on U . Making w^0 equal to zero outside of U helps to simplify the exposition. Parts (1)–(3) state that u is a time dependent weak solution to the Navier-Stokes equations of incompressible fluid flow in U with initial condition w^0 . Again, we make u equal to zero outside of $U \times \mathbb{R}^+$ to simplify certain statements. Part (1) and $Du \in L^2$ are a weak way of saying that u is zero on the boundary of U . The function $\operatorname{curl}(u)$ is an element of L^2 (since $Du \in L^2$). Hence $\operatorname{curl}(u)$ is an equivalence class of functions which differ only on sets of Lebesgue measure zero. Parts (4), (5) state that one of the functions in this equivalence class is continuous on the open set $(U \times \mathbb{R}^+) \sim S$. Parts (6), (7) say that S is a very small set.

Definition 1.3. If $-\infty \leq a < b \leq \infty$ then

$$(a, b) = \{x: a < x < b\}, \quad [a, b] = \{x: a \leq x \leq b\},$$

$$[a, b) = \{x: a \leq x < b\}, \quad \text{and } (a, b] = \{x: a < x \leq b\}.$$

Euclidean norms will be denoted by $|\cdot|$. If f is a function defined on a subset of \mathbb{R}^3 and $i \in \{1, 2, 3\}$ then $D_i f$ is the partial derivative of f with respect to the i component of \mathbb{R}^3 and $Df = (D_1 f, D_2 f, D_3 f)$. If $i, j, k \in \{1, 2, 3\}$ then $D_{ij} f = D_i(D_j f)$

and $D_{ijk}f = D_i(D_j(D_k f))$. The collection of C^∞ functions with compact support from X into Y will be written $C_0^\infty(X, Y)$. An *absolute constant* is a positive constant that does not depend on any of the parameters in this paper. Since there are several hundred absolute constants in this paper, we will denote them by the letter C without bothering to distinguish most of them with subscripts. Thus the various C 's below represent different constants. If a constant depends only on a parameter H we write it as $C(H)$.

This research was inspired by the work of Almgren [1], Leray [3], and Mandelbrot [4]. There are other results of this nature in [5, 6] and [7], where it was assumed that U is all of euclidean space. In [5] it was shown that u is continuous except for a locally closed subset of $R^3 \times R^+$ whose Hausdorff dimension is at most 2. In [6] we had a similar result for flows in 5 dimensional space-time (with $R^3 \times R^+$ replaced by $R^4 \times R^+$) where the Hausdorff dimension of the singular set was shown to be at most 3. The proofs in [5] and [6] are very different from the proof in this paper, which is based on the method of [7]. Paper [7] was a primitive version of this type of result in which the conclusion resembled statement (7) of Theorem 1.2.

2. Technical Preparation

Throughout this section we assume that we are given a continuous function $u: R^3 \times R^+ \rightarrow R^3$ such that

$$\left\{ \int_{R^3} |u(x, t)|^2 dx : t > 0 \right\} \quad (2.1)$$

is a bounded set of real numbers,

$$Du \text{ is an } L^2 \text{ function.} \quad (2.2)$$

We also assume that $\{a, c\} \subset R^3$, $\{b, d\} \subset R$, $\{m, p, n, q\}$ is a set of integers, and the following six properties hold:

$$b > 2^{-2m}, \quad (2.3)$$

$$|a - c| < 2^{-m}, \quad (2.4)$$

$$b - 2^{-2m} < d \leq b, \quad (2.5)$$

$$2^{-(n+1)} < 2^{-m} - |a - c| \leq 2^{-n}, \quad (2.6)$$

$$2^{-2(p+1)} < d - (b - 2^{-2m}) \leq 2^{-2p}, \quad (2.7)$$

$$q = \text{maximum of } n \text{ and } p. \quad (2.8)$$

Frequent use will be made of the following definition:

Definition 2.1. If $A \subset R^3 \times R$ and f is a function defined on A we use the abbreviations $I(f, A) = \int_A f$ and $M(f, A) =$ the supremum of $\{|f(x, t)| : (x, t) \in A\}$. If

$x \in \mathbb{R}^3$, $t \in \mathbb{R}$, $r > 0$, $s > 0$, and $h > 0$ then

$$\begin{aligned} B(x, r) &= \{y \in \mathbb{R}^3 : |y - x| \leq r\}, \\ K(x, t, r, s) &= \{(y, w) \in \mathbb{R}^3 \times \mathbb{R} : |y - x| \leq r \text{ and } t - s \leq w \leq t\}, \\ A(x, r, h) &= \{y \in \mathbb{R}^3 : r - h \leq |y - x| \leq r + h\}, \\ T(x, t, r, s, h) &= \{(y, w) \in \mathbb{R}^3 \times \mathbb{R} : r - h \leq |y - x| \leq r + h \text{ and } t - s \leq w \leq t\}, \\ D(t) &= \{(y, w) \in \mathbb{R}^3 \times \mathbb{R} : w \leq t\}. \end{aligned}$$

Lemma 2.2. *If $B(y, r/4) \subset B(x, r)$ and $t > s > 0$ then*

$$\begin{aligned} I(|u|^2, K(x, t, r, s)) \\ \leq Cr^3 s(M(|u|^2, K(y, t, r/4, s))) + Cr^2(I(|Du|^2, K(x, t, r, s))). \end{aligned}$$

Proof. Using the continuity of u , the assumption $Du \in L^2$, approximation by smooth functions, and Lemma 1.2 of [7] we obtain

$$\begin{aligned} \int_{B(x, r)} |u(z, w)|^2 dz &\leq Cr^3 (\supremum \{|u(z, w)|^2 : z \in B(y, r/4)\}) \\ &\quad + Cr^2 \left(\int_{B(x, r)} |Du(z, w)|^2 dz \right) \end{aligned}$$

for almost every $w > 0$. Now integrate over $t - s \leq w \leq t$.

Lemma 2.3. *We have $n \geq m$ and $p \geq m$.*

Proof. From (2.6) we obtain $2^{-(n+1)} < 2^{-m} - |a - c| \leq 2^{-m}$. Since n and m are integers, we conclude $n \geq m$. From (2.7) and (2.5) we obtain

$$2^{-2(p+1)} < d - (b - 2^{-2m}) \leq b - (b - 2^{-2m}) = 2^{-2m}.$$

Again, the fact that p and m are integers implies $p \geq m$.

We will use the abbreviation

$$G(i) = K(a, b, 2^{-m} - 2^{-i}, 2^{-2m} - 2^{-2i}) \quad \text{if } i > m. \quad (2.9)$$

Lemma 2.4. *If k is an integer, $k \geq m + 1$, and $2^{-(m+1)} < |a - c|$ then*

(a) *if $k \leq n + 1$ and $k \leq p + 1$ then*

$$\begin{aligned} 2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2(p+2)})) &\leq C2^k 2^{-2p} M(|u|^2, D(d) \cap G(p+2)) \\ &\quad + C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})), \end{aligned}$$

(b) *if $n > p$, $p + 2 \leq k$, and $k \leq n + 1$ then*

$$\begin{aligned} 2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2k})) &\leq C2^{-k} M(|u|^2, D(d) \cap G(k+1)) \\ &\quad + C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})), \end{aligned}$$

(c) *if $n \leq p$, $n + 2 \leq k$, and $k \leq p + 1$ then*

$$2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2(p+2)})) \leq C2^k 2^{-2p} M(|u|^2, D(d) \cap G(p+2)),$$

(d) *if $k \geq n + 2$ and $k \geq p + 2$ then*

$$2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2k})) \leq C2^{-k} M(|u|^2, D(d) \cap G(q+2)).$$

Proof. From the hypotheses we obtain $|a-c| > 2^{-(m+1)} \geq 2^{-k}$.

This allows us to make the following definition: e_k is the point on the line segment joining a and c that satisfies $|e_k - c| = (3/4)2^{-k}$.

We have

$$B(e_k, (1/4)2^{-k}) \subset B(c, 2^{-k}). \quad (2.10)$$

We will now prove

$$B(e_k, (1/4)2^{-k}) \subset B(a, 2^{-m} - 2^{-(k+1)}). \quad (2.11)$$

If $|x - e_k| \leq (1/4)2^{-k}$ then the definition of e_k and (2.4) yield

$$|x - a| \leq |x - e_k| + |e_k - a| \leq (1/4)2^{-k} + (|a - c| - (3/4)2^{-k}) < 2^{-m} - 2^{-(k+1)}.$$

The above proves (2.11). Now we will prove

$$B(c, 2^{-k}) \subset A(a, 2^{-m}, 2^{-k+2}) \quad \text{if } k \leq n+1. \quad (2.12)$$

Suppose $|x - c| \leq 2^{-k}$. Then (2.4) yields

$$|x - a| \leq |c - a| + |x - c| < 2^{-m} + 2^{-k} < 2^{-m} + 2^{-k+2}.$$

In addition, (2.6) and $k \leq n+1$ yield

$$\begin{aligned} |x - a| &\geq |c - a| - |x - c| \\ &\geq 2^{-m} - 2^{-n} - 2^{-k} \geq 2^{-m} - 2^{-k+1} - 2^{-k} > 2^{-m} - 2^{-k+2}. \end{aligned}$$

The above proves (2.12). Now we show

$$B(c, 2^{-k}) \subset B(a, 2^{-m} - 2^{-(n+2)}) \quad \text{if } k \geq n+2. \quad (2.13)$$

If $|x - c| \leq 2^{-k}$ and $k \geq n+2$ then (2.6) yields

$$\begin{aligned} |x - a| &\leq |c - a| + |x - c| < 2^{-m} - 2^{-(n+1)} + 2^{-k} \\ &\leq 2^{-m} - 2^{-(n+1)} + 2^{-(n+2)} = 2^{-m} - 2^{-(n+2)}. \end{aligned}$$

The above proves (2.13). From (2.7) we obtain

$$d - b + 2^{-2m} > 2^{-2(p+1)} > 2^{-2(p+2)} + 2^{-2(p+2)}.$$

The above and (2.3) yield

$$d - 2^{-2(p+2)} > b - (2^{-2m} - 2^{-2(p+2)}) > 0. \quad (2.14)$$

Now we are ready to prove (a). We assume

$$k \leq n+1 \quad \text{and} \quad k \leq p+1. \quad (2.15)$$

From (2.11) and (2.15) we obtain

$$B(e_k, (1/4)2^{-k}) \subset B(a, 2^{-m} - 2^{-(k+1)}) \subset B(a, 2^{-m} - 2^{-(p+2)}). \quad (2.16)$$

Now (2.16), (2.5), (2.14), and (2.9) yield

$$\begin{aligned} &K(e_k, d, (1/4)2^{-k}, 2^{-2(p+2)}) \\ &\subset D(d) \cap K(a, b, 2^{-m} - 2^{-(p+2)}, 2^{-2m} - 2^{-2(p+2)}) = D(d) \cap G(p+2). \end{aligned} \quad (2.17)$$

Observe that the proof of the inclusion in (2.17) forces $2^{-m} - 2^{-(p+2)} > 0$ and $2^{-2m} - 2^{-2(p+2)} > 0$. These conditions are required in Definition 2.1 [and in (2.9)]. This sort of thing will come up several times. Alternatively, we can use Lemma 2.3.

From (2.15), (2.12), (2.5), and (2.14) we obtain

$$K(c, d, 2^{-k}, 2^{-2(p+2)}) \subset T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2}). \quad (2.18)$$

From (2.10), (2.14), Lemma 2.2, (2.17), and (2.18) we obtain

$$\begin{aligned} & 2^{4k}I(|u|^2, K(c, d, 2^{-k}, 2^{-2(p+2)})) \\ & \leq 2^{4k}C2^{-3k}2^{-2(p+2)}M(|u|^2, K(e_k, d, (1/4)2^{-k}, 2^{-2(p+2)})) \\ & \quad + 2^{4k}C2^{-2k}I(|Du|^2, K(c, d, 2^{-k}, 2^{-2(p+2)})) \\ & \leq C2^k2^{-2p}M(|u|^2, D(d) \cap G(p+2)) \\ & \quad + C2^{2k}I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})). \end{aligned}$$

Properties (2.14) and (2.3) imply that u is defined on the sets that appear above. This proves (a).

Now we prove (b). We suppose

$$n > p, \quad p+2 \leq k, \quad \text{and} \quad k \leq n+1. \quad (2.19)$$

Using (2.19), (2.14), (2.19) again, and (2.3) we obtain

$$\begin{aligned} d - 2^{-2k} & \geq d - 2^{-2(p+2)} > b - (2^{-2m} - 2^{-2(p+2)}) \\ & \geq b - (2^{-2m} - 2^{-2k}) > b - (2^{-2m} - 2^{-2(k+1)}) > 0. \end{aligned} \quad (2.20)$$

Now (2.11), (2.5), (2.20), and (2.9) yield

$$\begin{aligned} & K(e_k, d, (1/4)2^{-k}, 2^{-2k}) \\ & \quad \subset D(d) \cap K(a, b, 2^{-m} - 2^{-(k+1)}, 2^{-2m} - 2^{-2(k+1)}) \\ & \quad = D(d) \cap G(k+1). \end{aligned} \quad (2.21)$$

From (2.19), (2.12), (2.5), and (2.20) we obtain

$$K(c, d, 2^{-k}, 2^{-2k}) \subset T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2}). \quad (2.22)$$

Now (2.10), (2.20), Lemma 2.2, (2.21), and (2.22) yield

$$\begin{aligned} & 2^{4k}I(|u|^2, K(c, d, 2^{-k}, 2^{-2k})) \\ & \leq 2^{4k}C2^{-3k}2^{-2k}M(|u|^2, K(e_k, d, (1/4)2^{-k}, 2^{-2k})) \\ & \quad + 2^{4k}C2^{-2k}I(|Du|^2, K(c, d, 2^{-k}, 2^{-2k})) \\ & \leq C2^{-k}M(|u|^2, D(d) \cap G(k+1)) \\ & \quad + C2^{2k}I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})). \end{aligned}$$

Properties (2.20) and (2.3) imply that u is defined on the sets that appear above. This proves (b).

Now we prove (c). We suppose

$$n \leq p, \quad n+2 \leq k, \quad \text{and} \quad k \leq p+1. \quad (2.23)$$

Using (2.23), (2.13), and (2.23) again we obtain

$$B(c, 2^{-k}) \subset B(a, 2^{-m} - 2^{-(n+2)}) \subset B(a, 2^{-m} - 2^{-(p+2)}). \quad (2.24)$$

Now (2.24), (2.5), (2.14), and (2.9) yield

$$\begin{aligned} & K(c, d, 2^{-k}, 2^{-2(p+2)}) \\ & \subset D(d) \cap K(a, b, 2^{-m} - 2^{-(p+2)}, 2^{-2m} - 2^{-2(p+2)}) = D(d) \cap G(p+2). \end{aligned} \quad (2.25)$$

From (2.25) we obtain

$$\begin{aligned} & 2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2(p+2)})) \\ & \leq 2^{4k} C 2^{-3k} 2^{-2(p+2)} M(|u|^2, D(d) \cap G(p+2)). \end{aligned}$$

Property (2.14) implies that u is defined on the sets that appear above. This proves (c).

Finally, we prove (d). We assume

$$k \geq n+2 \quad \text{and} \quad k \geq p+2. \quad (2.26)$$

From (2.26), (2.13), and (2.8) we get

$$B(c, 2^{-k}) \subset B(a, 2^{-m} - 2^{-(n+2)}) \subset B(a, 2^{-m} - 2^{-(q+2)}). \quad (2.27)$$

From (2.26), (2.14), (2.8), and (2.3) we obtain

$$\begin{aligned} d - 2^{-2k} & \geq d - 2^{-2(p+2)} > b - (2^{-2m} - 2^{-2(p+2)}) \\ & \geq b - (2^{-2m} - 2^{-2(q+2)}) > 0. \end{aligned} \quad (2.28)$$

From (2.27), (2.5), (2.28), and (2.9) we obtain

$$\begin{aligned} & K(c, d, 2^{-k}, 2^{-2k}) \\ & \subset D(d) \cap K(a, b, 2^{-m} - 2^{-(q+2)}, 2^{-2m} - 2^{-2(q+2)}) = D(d) \cap G(q+2). \end{aligned} \quad (2.29)$$

From (2.29) we obtain

$$2^{4k} I(|u|^2, K(c, d, 2^{-k}, 2^{-2k})) \leq 2^{4k} C 2^{-3k} 2^{-2k} M(|u|^2, D(d) \cap G(q+2)).$$

Property (2.28) implies that u is defined on the sets that appear above. This proves (d). The lemma is proved.

Lemma 2.5. *There is an absolute constant C_1 such that (see (2.9))*

$$\begin{aligned} & \int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-c| + (d-t)^{1/2})^{-4} dx dt \\ & \leq C_1 \left(\int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \right) \\ & \quad + \sum_{k=m+1}^{q+1} C_1 2^{-k} M(|u|^2, D(d) \cap G(k+1)) \\ & \quad + \sum_{k=m+1}^{q+1} C_1 2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})). \end{aligned}$$

Proof. For every integer k we define $E(k)$ by

$$E(k) = K(c, d, 2^{-k}, 2^{-2(p+2)}) \sim K(c, d, 2^{-(k+1)}, 2^{-2(p+2)}) \quad (2.30)$$

if $k \leq p+1$, and

$$E(k) = K(c, d, 2^{-k}, 2^{-2k}) \sim K(c, d, 2^{-(k+1)}, 2^{-2(k+1)}) \quad (2.31)$$

if $k \geq p+2$. We have

$$\{(x, t) : d - 2^{-2(p+2)} \leq t \leq d\} \sim \{(c, d)\} = \bigcup_{k=-\infty}^{\infty} E(k). \quad (2.32)$$

Now we will prove

$$(|x-c| + (d-t)^{1/2})^{-4} < 2^{4(k+1)} \quad \text{if } (x, t) \in E(k). \quad (2.33)$$

If $k \leq p+1$ and $(x, t) \in E(k)$ then $|x-c| > 2^{-(k+1)}$ and (2.33) follows. If $k \geq p+2$ and $(x, t) \in E(k)$ then either $|x-c| > 2^{-(k+1)}$ or $d-t > 2^{-2(k+1)}$. In either case, (2.33) follows. From (2.32) and (2.33) we obtain

$$\begin{aligned} & \int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-c| + (d-t)^{1/2})^{-4} dx dt \\ & \leq \sum_{k=-\infty}^{\infty} 2^{4(k+1)} I(|u|^2, E(k)). \end{aligned} \quad (2.34)$$

Next we prove

$$\begin{aligned} & \sum_{k=-\infty}^{m+1} 2^{4(k+1)} I(|u|^2, E(k)) \\ & \leq C \left(\int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \right). \end{aligned} \quad (2.35)$$

Since the $E(k)$ are disjoint and (2.32) holds, it suffices to show

$$2^{4(k+1)} \leq C(|x-a| + 2^{-m})^{-4} \quad \text{if } k \leq m+1 \quad \text{and } (x, t) \in E(k). \quad (2.36)$$

Suppose $k \leq m+1$ and $(x, t) \in E(k)$. Then Lemma 2.3 and (2.30) yield $|x-c| \leq 2^{-k}$. Hence (2.4) implies

$$\begin{aligned} |x-a| + 2^{-m} & \leq |x-c| + |c-a| + 2^{-m} < 2^{-k} + 2^{-m} + 2^{-m} \\ & \leq 2^{-k} + 2^{-k+2} < 2^{-k+3}. \end{aligned}$$

Hence (2.36) holds, and this implies (2.35).

There are two possibilities: Either $|a-c| \leq 2^{-(m+1)}$ or $2^{-(m+1)} < |a-c|$. First we examine the case

$$|a-c| \leq 2^{-(m+1)}. \quad (2.37)$$

We will prove

$$B(c, 2^{-k}) \subset B(a, 2^{-m} - 2^{-(p+2)}) \quad \text{if } k \geq m+2. \quad (2.38)$$

If $k \geq m+2$ then Lemma 2.3 implies

$$2^{-k} + 2^{-(p+2)} \leq 2^{-(m+2)} + 2^{-(m+2)} = 2^{-(m+1)}.$$

Hence (2.37) implies that every $x \in B(c, 2^{-k})$ satisfies

$$\begin{aligned} |x-a| &\leq |x-c| + |c-a| \leq 2^{-k} + 2^{-(m+1)} \\ &= 2^{-k} + 2^{-m} - 2^{-(m+1)} \leq 2^{-m} - 2^{-(p+2)}. \end{aligned}$$

Hence (2.38) holds. The argument that gave us (2.14) still yields

$$d - 2^{-2(p+2)} > b - (2^{-2m} - 2^{-2(p+2)}) > 0 \quad (2.39)$$

since the hypotheses of Lemma 2.4 were not used in the proof of (2.14).

Now (2.5), (2.30)–(2.32), (2.38), (2.39), and (2.9) imply

$$E(k) \subset D(d) \cap K(a, b, 2^{-m} - 2^{-(p+2)}, 2^{-2m} - 2^{-2(p+2)}) = D(d) \cap G(p+2) \quad (2.40)$$

if $k \geq m+2$. From Lemma 2.3, (2.30), (2.31), and (2.40) we obtain

$$\begin{aligned} &\sum_{k=m+2}^{\infty} 2^{4k} I(|u|^2, E(k)) \\ &= \sum_{k=m+2}^{p+1} 2^{4k} I(|u|^2, E(k)) + \sum_{k=p+2}^{\infty} 2^{4k} I(|u|^2, E(k)) \\ &\leq \sum_{k=m+2}^{p+1} 2^{4k} C 2^{-3k} 2^{-2(p+2)} M(|u|^2, D(d) \cap G(p+2)) \\ &\quad + \sum_{k=p+2}^{\infty} 2^{4k} C 2^{-3k} 2^{-2k} M(|u|^2, D(d) \cap G(p+2)) \\ &\leq C 2^{-p} M(|u|^2, D(d) \cap G(p+2)). \end{aligned} \quad (2.41)$$

From Lemma 2.3 and (2.8) we get $m+1 \leq p+1 \leq q+1$. Hence (2.34), (2.35), and (2.41) yield the conclusion of the lemma if possibility (2.37) holds.

The second possibility we must examine is

$$2^{-(m+1)} < |a-c|. \quad (2.42)$$

Now we have two cases: Either $n \leq p$ or $n > p$. Assume first that $n \leq p$ holds. Then (2.42), (2.30), (2.31), parts (a), (c), (d) of Lemma 2.4, (2.8), and $p=q$ [which follows from $n \leq p$ and (2.8)] yield

$$\begin{aligned} &\sum_{k=m+1}^{\infty} 2^{4k} I(|u|^2, E(k)) \\ &= \sum_{k=m+1}^{n+1} 2^{4k} I(|u|^2, E(k)) + \sum_{k=n+2}^{p+1} 2^{4k} I(|u|^2, E(k)) \\ &\quad + \sum_{k=p+2}^{\infty} 2^{4k} I(|u|^2, E(k)) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=m+1}^{n+1} C2^k 2^{-2p} M(|u|^2, D(d) \cap G(p+2)) \\
&\quad + \sum_{k=m+1}^{n+1} C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\quad + \sum_{k=n+2}^{p+1} C2^k 2^{-2p} M(|u|^2, D(d) \cap G(p+2)) \\
&\quad + \sum_{k=p+2}^{\infty} C2^{-k} M(|u|^2, D(d) \cap G(q+2)) \\
&\leq C2^{p2} 2^{-2p} M(|u|^2, D(d) \cap G(p+2)) \\
&\quad + \sum_{k=m+1}^{q+1} C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\quad + C2^{-p} M(|u|^2, D(d) \cap G(q+2)) \\
&= C2^{-q} M(|u|^2, D(d) \cap G(q+2)) \\
&\quad + \sum_{k=m+1}^{q+1} C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})). \tag{2.43}
\end{aligned}$$

From (2.8) and Lemma 2.3 we get $m+1 \leq q+1$. Hence (2.34), (2.35), and (2.43) yield the conclusion of the lemma if possibilities (2.42) and $n \leq p$ hold.

Now we consider the case where (2.42) and $n > p$ hold. Then (2.42), (2.30), (2.31), parts (a), (b), (d) of Lemma 2.4, Lemma 2.3, $q = n$ [which follows from $n > p$ and (2.8)], and $m+1 \leq p+1 \leq q+1$ [which follows from Lemma 2.3 and (2.8)] imply

$$\begin{aligned}
&\sum_{k=m+1}^{\infty} 2^{4k} I(|u|^2, E(k)) \\
&= \sum_{k=m+1}^{p+1} 2^{4k} I(|u|^2, E(k)) + \sum_{k=p+2}^{n+1} 2^{4k} I(|u|^2, E(k)) \\
&\quad + \sum_{k=n+2}^{\infty} 2^{4k} I(|u|^2, E(k)) \\
&\leq \sum_{k=m+1}^{p+1} C2^k 2^{-2p} M(|u|^2, D(d) \cap G(p+2)) \\
&\quad + \sum_{k=m+1}^{p+1} C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\quad + \sum_{k=p+2}^{n+1} C2^{-k} M(|u|^2, D(d) \cap G(k+1)) \\
&\quad + \sum_{k=p+2}^{n+1} C2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\quad + \sum_{k=n+2}^{\infty} C2^{-k} M(|u|^2, D(d) \cap G(q+2))
\end{aligned}$$

$$\begin{aligned}
&\leq C2^{p2}2^{-2p}M(|u|^2, D(d)\cap G(p+2)) \\
&\quad + \sum_{k=m+1}^{n+1} C2^{-k}M(|u|^2, D(d)\cap G(k+1)) \\
&\quad + \sum_{k=m+1}^{n+1} C2^{2k}I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\quad + C2^{-n}M(|u|^2, D(d)\cap G(q+2)) \\
&\leq C2^{-(p+1)}M(|u|^2, D(d)\cap G(p+2)) \\
&\quad + \sum_{k=m+1}^{q+1} C2^{-k}M(|u|^2, D(d)\cap G(k+1)) \\
&\quad + C2^{-(q+1)}M(|u|^2, D(d)\cap G(q+2)) \\
&\quad + \sum_{k=m+1}^{q+1} C2^{2k}I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
&\leq \sum_{k=m+1}^{q+1} C2^{-k}M(|u|^2, D(d)\cap G(k+1)) \\
&\quad + \sum_{k=m+1}^{q+1} C2^{2k}I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})). \tag{2.44}
\end{aligned}$$

The conclusion of the lemma in this case follows from (2.34), (2.35), and (2.44). Lemma 2.5 has been proved.

Lemma 2.6. *If $f: R^3 \rightarrow R$, $f \in L^2$, and $Df \in L^2$ then*

$$\begin{aligned}
&\left(\int_{B(x, r)} |f(y)|^3 dy \right)^{1/3} \\
&\leq Cr^{-1/2} \left(\int_{B(x, 2r)} |f(y)|^2 dy \right)^{1/2} + Cr^{1/2} \left(\int_{B(x, 2r)} |Df(y)|^2 dy \right)^{1/2}.
\end{aligned}$$

Proof. Let $g: R^3 \rightarrow [0, 1]$ be a smooth function such that $g(y) = 1$ for $y \in B(x, r)$, $g(y) = 0$ for $y \notin B(x, 2r)$, and $\|Dg\|_\infty \leq Cr^{-1}$. Set $h(y) = g(y)f(y)$. Using the argument in Lemma 3.1 of [5] and Young's inequality we obtain

$$\begin{aligned}
&\left(\int_{B(x, r)} |f(y)|^3 dy \right)^{1/3} \leq \left(\int_{R^3} |h(y)|^3 dy \right)^{1/3} \\
&\leq C \left(\int_{R^3} |h(y)|^2 dy \right)^{1/4} \left(\int_{R^3} |Dh(y)|^2 dy \right)^{1/4} \\
&\leq Cr^{-1/2} \left(\int_{R^3} |h(y)|^2 dy \right)^{1/2} + Cr^{1/2} \left(\int_{R^3} |Dh(y)|^2 dy \right)^{1/2} \\
&\leq Cr^{-1/2} \left(\int_{B(x, 2r)} |f(y)|^2 dy \right)^{1/2} + Cr^{1/2} \left(\int_{B(x, 2r)} |Dg(y)|^2 |f(y)|^2 dy \right)^{1/2} \\
&\quad + Cr^{1/2} \left(\int_{B(x, 2r)} |g(y)|^2 |Df(y)|^2 dy \right)^{1/2} \\
&\leq Cr^{-1/2} \left(\int_{B(x, 2r)} |f(y)|^2 dy \right)^{1/2} + Cr^{1/2} \left(\int_{B(x, 2r)} |Df(y)|^2 dy \right)^{1/2}.
\end{aligned}$$

Lemma 2.7. *If $0 < h \leq 2^{-m}$ then for almost every $t > 0$ we have*

$$\begin{aligned} & \int_{\mathbb{R}^3} |u(x, t)|(|x - c| + h)^{-4} h^{-1} dx \\ & \leq C 2^{-3m/2} h^{-3} \left(\int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx \right)^{1/2} \\ & \quad + C 2^{-m/2} h^{-3} \left(\int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx \right)^{1/2}. \end{aligned}$$

Proof. From Hölder's inequality, the conditions on h , (2.1), (2.2), Lemma 2.6, and (2.4) we obtain the following for almost every $t > 0$:

$$\begin{aligned} & \int_{B(c, 2^{-m})} |u(x, t)|(|x - c| + h)^{-4} h^{-1} dx \\ & \leq \left(\int_{B(c, 2^{-m})} |u(x, t)|^3 dx \right)^{1/3} \left(\int_{B(c, 2^{-m})} (|x - c| + h)^{-4} h^{-1})^{3/2} dx \right)^{2/3} \\ & \leq C \left(\int_{B(c, 2^{-m})} |u(x, t)|^3 dx \right)^{1/3} h^{-3} \\ & \leq C 2^{m/2} \left(\int_{B(c, 2^{-m+1})} |u(x, t)|^2 dx \right)^{1/2} h^{-3} \\ & \quad + C 2^{-m/2} \left(\int_{B(c, 2^{-m+1})} |Du(x, t)|^2 dx \right)^{1/2} h^{-3} \\ & \leq C 2^{m/2} h^{-3} \left(\int_{B(a, 2^{-m+2})} |u(x, t)|^2 dx \right)^{1/2} \\ & \quad + C 2^{-m/2} h^{-3} \left(\int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx \right)^{1/2}. \end{aligned} \tag{2.45}$$

Let $E = \{x \in \mathbb{R}^3 : x \notin B(c, 2^{-m})\}$. If $x \in E$ then (2.4) yields

$$\begin{aligned} |x - a| + 2^{-m} & < (|x - c| + |c - a|) + |x - c| \\ & < |x - c| + 2^{-m} + |x - c| < 3|x - c| < 3(|x - c| + h). \end{aligned}$$

Hence the Schwarz inequality and $0 < h \leq 2^{-m}$ yield

$$\begin{aligned} & \int_E |u(x, t)|(|x - c| + h)^{-4} h^{-1} dx \\ & \leq Ch^{-1} \left(\int_E |u(x, t)|(|x - a| + 2^{-m})^{-4} dx \right) \\ & \leq Ch^{-1} \left(\int_{\mathbb{R}^3} (|u(x, t)|(|x - a| + 2^{-m})^{-2}) (|x - a| + 2^{-m})^{-2} dx \right) \\ & \leq Ch^{-1} \left(\int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx \right)^{1/2} \left(\int_{\mathbb{R}^3} (|x - a| + 2^{-m})^{-4} dx \right)^{1/2} \\ & \leq Ch^{-1} 2^{m/2} \left(\int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx \right)^{1/2} \\ & \leq Ch^{-3} 2^{-3m/2} \left(\int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx \right)^{1/2}. \end{aligned} \tag{2.46}$$

If $x \in B(a, 2^{-m+2})$ then

$$(|x - a| + 2^{-m})^{-4} 2^{-4m} \geq (2^{-m+2} + 2^{-m})^{-4} 2^{-4m} > 2^{-12}.$$

Hence

$$\begin{aligned}
& \left(\int_{B(a, 2^{-m+2})} |u(x, t)|^2 dx \right)^{1/2} \\
& \leq C \left(\int_{B(a, 2^{-m+2})} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} 2^{-4m} dx \right)^{1/2} \\
& = 2^{-2m} C \left(\int_{R^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx \right)^{1/2}. \tag{2.47}
\end{aligned}$$

The conclusion follows from (2.45), (2.46), and (2.47).

Lemma 2.8. *There exists an absolute constant C_2 such that the following holds: If $h = 2^{-(p+2)}$ then*

$$\begin{aligned}
& \int_{d-h^2}^d \int_{R^3} |u(x, t)| (|x-c| + h)^{-4} h^{-1} dx dt \\
& \leq C_2 2^{-3m/2} h^{-2} \left(\int_{d-h^2}^d \int_{R^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \right)^{1/2} \\
& + C_2 2^{-m/2} h^{-2} \left(\int_{d-h^2}^d \int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx dt \right)^{1/2}.
\end{aligned}$$

Proof. The hypothesis of Lemma 2.7 is satisfied because of Lemma 2.3. Now use Lemma 2.7, (2.14), and the estimate

$$\int_{d-h^2}^d |f(t)|^{1/2} dt \leq h \left(\int_{d-h^2}^d |f(t)| dt \right)^{1/2},$$

which follows from the Schwarz inequality.

3.

We continue to use the notation introduced in Definition 2.1 and (2.9) and we assume that $u : R^3 \times R^+ \rightarrow R^3$ is a continuous function satisfying (2.1) and (2.2).

Lemma 3.1. *For every $H > 0$ there exist positive constants $C_3(H)$ and $C_4(H)$, which depend only on H , such that the following is satisfied. Suppose we have the following conditions: $a \in R^3$, $b \in R^+$, m is an integer, $b > 2^{-2m}$, $\Gamma \in R^+$, the inequality*

$$\begin{aligned}
H^{-1} |u(c, d)| & \leq \int_{d-h^2}^d \int_{R^3} |u(x, t)|^2 (|x-c| + (d-t)^{1/2})^{-4} dx dt \\
& + \int_{d-h^2}^d \int_{R^3} |u(x, t)| (|x-c| + h)^{-4} h^{-1} dx dt + \Gamma \tag{3.1}
\end{aligned}$$

holds whenever $(c, d) \in K(a, b, 2^{-m}, 2^{-2m})$, $0 < h \leq 2^{-m}$, and $d > h^2$,

$$I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, r)) \leq C_3(H)r \quad \text{if } 0 < r \leq 2^{-m+1}, \tag{3.2}$$

$$I(|Du|^2, K(a, b - 2^{-2m} + s, 2^{-m+2}, s)) \leq C_3(H)2^m s \quad \text{if } 0 < s \leq 2^{-2m}, \tag{3.3}$$

$$\int_{b-2^{-2m}}^{b-2^{-2m}+s} \int_{R^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \leq C_3(H)2^{3m}s \tag{3.4}$$

holds whenever $0 < s \leq 2^{-2m}$, and

$$\Gamma \leq C_3(H)2^m. \quad (3.5)$$

Then we can conclude

$$|u(x, t)| \leq C_4(H)2^{m+1} \quad \text{if } (x, t) \in K(a, b, 2^{-(m+1)}, 2^{-2m}(3/4)). \quad (3.6)$$

Proof. We choose $C_4(H) > 0$ so that (3.7) holds (see Lemma 2.5):

$$2^4 C_1(C_4(H))^2 \leq (1/4)H^{-1}C_4(H). \quad (3.7)$$

Then we choose $C_3(H) > 0$ so that (3.8) holds (see Lemmas 2.5 and 2.8):

$$\begin{aligned} C_1 C_3(H) + 2^4 C_1 C_3(H) + 2^5 C_2(C_3(H))^{1/2} &\leq (1/4)H^{-1}C_4(H), C_3(H) \\ &\leq (1/4)H^{-1}C_4(H). \end{aligned} \quad (3.8)$$

For every integer i satisfying $i \geq m+1$ we define $Z(i)$ to be the union of

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x - a| = 2^{-m} - 2^{-i}, b - 2^{-2m} + 2^{-2i} \leq t \leq b\}$$

and

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x - a| \leq 2^{-m} - 2^{-i}, t = b - 2^{-2m} + 2^{-2i}\}.$$

If i is an integer and $i \geq m+2$ we define $Y(i)$ to be the closure of $G(i) \sim G(i-1)$ [see (2.9)]. We also set $Y(m+1) = G(m+1)$. We obtain that $Y(i) \cap Y(j)$ is empty if $|i-j| \geq 2$. We also obtain $Y(i) \cap Y(i+1) = Z(i)$. Set $P = \bigcup_{i=m+1}^{\infty} Y(i)$. We define a function $f: P \rightarrow \mathbb{R}^+$ as follows: If $(x, t) \in Z(i)$ then $f(x, t) = C_4(H)2^i$. If $i \geq m+2$ we use the Urysohn theorem to extend this definition of f continuously to $Y(i)$ so that $C_4(H)2^i \geq f(x, t) \geq C_4(H)2^{i-1}$ whenever $(x, t) \in Y(i)$. Finally, we set $f(x, t) = C_4(H)2^{m+1}$ if $(x, t) \in Y(m+1)$. We have the following properties for every $(x, t) \in P$ and any integer $i \geq m+1$:

$$\text{If } (x, t) \in G(i) \text{ then } f(x, t) \leq C_4(H)2^i, \quad (3.9)$$

$$\text{if } |x - a| \geq 2^{-m} - 2^{-i} \text{ or } t \leq b - 2^{-2m} + 2^{-2i} \text{ then } f(x, t) \geq C_4(H)2^i. \quad (3.10)$$

We intend to show

$$|u(x, t)| \leq f(x, t) \quad \text{for all } (x, t) \in P. \quad (3.11)$$

If (3.11) is false then the set $S = \{(x, t) \in P : |u(x, t)| \geq f(x, t)\}$ is nonempty. Since u is continuous, u is bounded on the compact set $K(a, b, 2^{-m}, 2^{-2m})$. Hence there exists an integer $i \geq m+1$ such that $|u(x, t)| < C_4(H)2^i$ for all $(x, t) \in P$. Hence (3.10) yields $S \subset G(i)$. Since $G(i)$ is compact and the functions u and f are continuous, we conclude that S is compact. Hence there exists $(c, d) \in P$ such that

$$|u(c, d)| \geq f(c, d), \quad (3.12)$$

$$|u(x, t)| \leq f(x, t) \quad \text{if } (x, t) \in P \text{ and } t \leq d. \quad (3.13)$$

From the hypotheses we obtain that (2.1)–(2.3) are satisfied. From $(c, d) \in P$ we conclude that (2.4), (2.5) hold. We define the integers n and p by (2.6), (2.7) [this determines n and p uniquely in view of (2.4), (2.5)]. We also define q by (2.8). Hence all the hypotheses of Sect. 2 are satisfied. Using $h = 2^{-(p+2)}$ and observing that $d > h^2$ follows from (2.3) and (2.7), and that $h \leq 2^{-m}$ follows from Lemma 2.3, we obtain (3.14) from (3.1), Lemma 2.5, and Lemma 2.8:

$$\begin{aligned}
& H^{-1}|u(c, d)| \\
& \leq C_1 \left(\int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx dt \right) \\
& \quad + \sum_{k=m+1}^{q+1} C_1 2^{-k} M(|u|^2, D(d) \cap G(k+1)) \\
& \quad + \sum_{k=m+1}^{q+1} C_1 2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
& \quad + C_2 2^{-3m/2} 2^{2(p+2)} \left(\int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx dt \right)^{1/2} \\
& \quad + C_2 2^{-m/2} 2^{2(p+2)} \left(\int_{d-2^{-2(p+2)}}^d \int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx dt \right)^{1/2} + \Gamma. \tag{3.14}
\end{aligned}$$

From (2.6) we obtain $|a-c| \geq 2^{-m} - 2^{-n}$. From (2.7) we obtain $d \leq b - 2^{-2m} + 2^{-2p}$. Hence (2.8) implies that either $|a-c| \geq 2^{-m} - 2^{-q}$ or $d \leq b - 2^{-2m} + 2^{-2q}$ must hold. If $q \geq m+1$ then (3.10) yields $f(c, d) \geq C_4(H)2^q$. If $q = m$ then $f(c, d) \geq C_4(H)2^q$ still follows because $f(x, t) \geq C_4(H)2^{m+1}$ holds for all $(x, t) \in P$. Since Lemma 2.3 and (2.8) imply $q \geq m$, we can use (3.12) to conclude

$$C_4(H)2^q \leq f(c, d) \leq |u(c, d)|. \tag{3.15}$$

From (3.9) and (3.13) we conclude

$$\begin{aligned}
& \sum_{k=m+1}^{q+1} C_1 2^{-k} M(|u|^2, D(d) \cap G(k+1)) \\
& \leq \sum_{k=m+1}^{q+1} C_1 2^{-k} (C_4(H)2^{k+1})^2 < 2^{q+4} C_1 (C_4(H))^2. \tag{3.16}
\end{aligned}$$

From (3.2) we obtain

$$\begin{aligned}
& \sum_{k=m+1}^{q+1} C_1 2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})) \\
& \leq \sum_{k=m+1}^{q+1} C_1 2^{2k} C_3(H) 2^{-k+2} < 2^{q+4} C_1 C_3(H). \tag{3.17}
\end{aligned}$$

Setting $s = d - (b - 2^{-2m})$, we use (2.5), (2.7), and Lemma 2.3 to obtain $0 < s \leq 2^{-2p} \leq 2^{-2m}$, $d - 2^{-2(p+2)} > b - 2^{-2m}$. Hence (3.4) and (3.3) yield

$$\begin{aligned} & \int_{d-2^{-2(p+2)}}^d \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \\ & \leq \int_{b-2^{-2m}}^{b-2^{-2m+s}} \int_{\mathbb{R}^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \\ & \leq C_3(H) 2^{3m} s \leq C_3(H) 2^{3m} 2^{-2p}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} & \int_{d-2^{-2(p+2)}}^d \int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx dt \\ & \leq \int_{b-2^{-2m}}^{b-2^{-2m+s}} \int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx dt \\ & = I(|Du|^2, K(a, b - 2^{-2m} + s, 2^{-m+2}, s)) \leq C_3(H) 2^m s \leq C_3(H) 2^m 2^{-2p}. \end{aligned} \quad (3.19)$$

Combining (3.14)–(3.19), (3.5), and $q \geq m$ [which follows from (2.8) and Lemma 2.3] we conclude

$$\begin{aligned} H^{-1} C_4(H) 2^q & \leq C_1 C_3(H) 2^{3m} 2^{-2p} + 2^{q+4} C_1 (C_4(H))^2 + 2^{q+4} C_1 C_3(H) \\ & \quad + C_2 2^{-3m/2} 2^{2(p+2)} (C_3(H) 2^{3m} 2^{-2p})^{1/2} \\ & \quad + C_2 2^{-m/2} 2^{2(p+2)} (C_3(H) 2^m 2^{-2p})^{1/2} + C_3(H) 2^q. \end{aligned} \quad (3.20)$$

From Lemma 2.3 and (2.8) we get $2^{3m} 2^{-2p} \leq 2^p \leq 2^q$, $p \leq q$. Using this in (3.20) and dividing by 2^q we get

$$\begin{aligned} H^{-1} C_4(H) & \leq C_1 C_3(H) + 2^4 C_1 (C_4(H))^2 + 2^4 C_1 C_3(H) + 2^4 C_2 (C_3(H))^{1/2} \\ & \quad + 2^4 C_2 (C_3(H))^{1/2} + C_3(H). \end{aligned} \quad (3.21)$$

However, (3.21) contradicts (3.7) and (3.8). This contradiction was obtained by assuming that (3.11) is false. Now (3.6) follows from (3.11) and the substitution of $i = m + 1$ in (3.9).

4.

As in Sect. 3, we assume that $u : \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ is a continuous function satisfying (2.1) and (2.2). The notation comes from Sect. 2.

Lemma 4.1. *Suppose that $a \in \mathbb{R}^3$, $b \in \mathbb{R}^+$, m is an integer, and $b > 2^{-2m}$. Then there exists a number e such that $1/2 < e < 1$ and the following holds: If $v : \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ is defined by $v(x, t) = e(u(ex, e^2t))$, $a' = e^{-1}a$, and $b' = e^{-2}b$ then*

$$\begin{aligned} & I(|Dv|^2, T(a', b', 2^{-m}, 2^{-2m}, r)) \\ & \leq (80) 2^m r (I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m}))) \quad \text{if } 0 < r \leq 2^{-m+1}, \\ & I(|Dv|^2, K(a', b' - 2^{-2m} + s, 2^{-m+2}, s)) \\ & \leq (80) 2^{2m} s (I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m}))) \quad \text{if } 0 < s \leq 2^{-2m}, \end{aligned}$$

and

$$\begin{aligned} & \int_{b'-2^{-2m}}^{b'-2^{-2m+s}} \int_{R^3} |v(x, t)|^2 (|x-a'|+2^{-m})^{-4} dx dt \\ & \leq (160)2^{2ms} \left(\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx dt \right) \quad \text{if } 0 < s \leq 2^{-2m}. \end{aligned}$$

Proof. Define $g_1 : R \rightarrow R$ as follows: If $h \leq 0$ or $h \geq 2^{-m+2}$ then $g_1(h) = 0$. If $0 < h < 2^{-m+2}$ then $g_1(h)$ is the integral of $|Du(x, t)|^2$ over the set

$$\{(x, t) : |x-a| = h, b-2^{-2m} \leq t \leq b\}$$

with respect to the 3 dimensional measure on that set. We obtain

$$\|g_1\|_1 = I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m})), \quad (4.1)$$

$$\int_{h-r}^{h+r} g_1(y) dy = I(|Du|^2, T(a, b, h, 2^{-2m}, r)) \quad (4.2)$$

if $0 < h \leq 2^{-m}$ and $0 < r \leq 2^{-m+1}$. For any integrable function $f : R \rightarrow R^+$ we define

$$Mf(h) = \sup_{r>0} (2r)^{-1} \left(\int_{h-r}^{h+r} f(y) dy \right).$$

If S is a measurable subset of R we define $m(S)$ to be the Lebesgue measure of S . The Hardy-Littlewood theorem ([9], p. 55) says

$$m\{h \in R : Mf(h) > 1/c\} \leq 5c \|f\|_1. \quad (4.3)$$

Hence we obtain $m\{h \in R : Mg_1(h) > 2^{m+3}(5)\|g_1\|_1\} \leq 2^{-m-3}$ and, consequently,

$$m\{e \in R : Mg_1(2^{-m}e) > 2^{m+3}(5)\|g_1\|_1\} \leq 1/8. \quad (4.4)$$

Now define $g_2 : R \rightarrow R$ and $g_3 : R \rightarrow R$ as follows: If $t < b-2^{-2m}$ or $t > b$ then $g_2(t) = g_3(t) = 0$. If $b-2^{-2m} \leq t \leq b$ then

$$g_2(t) = \int_{R^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx, \quad (4.5)$$

$$g_3(t) = \int_{B(a, 2^{-m+2})} |Du(x, t)|^2 dx. \quad (4.6)$$

Using (4.3) we obtain

$$m\{t \in R : Mg_i(t) > (40)2^{2m}\|g_i\|_1\} \leq (1/8)2^{-2m} \quad \text{if } i=2, 3. \quad (4.7)$$

Let $k : R \rightarrow R$ be the function defined by $k(e) = b-2^{-2m}e^2$. Then $k'(e) < -2^{-2m}$ if $1/2 < e < 1$. Hence (4.7) yields

$$m\{e \in R : 1/2 < e < 1 \text{ and } Mg_i(b-2^{-2m}e^2) > (40)2^{2m}\|g_i\|_1\} \leq 1/8 \quad (4.8)$$

if $i=2, 3$. From (4.4) and (4.8) we conclude that there exists e such that $1/2 < e < 1$ and (4.9) holds:

$$\begin{aligned} Mg_1(2^{-m}e) &\leq (40)2^m \|g_1\|_1, \\ Mg_2(b - 2^{-2m}e^2) &\leq (40)2^{2m} \|g_2\|_1, \\ Mg_3(b - 2^{-2m}e^2) &\leq (40)2^{2m} \|g_3\|_1. \end{aligned} \quad (4.9)$$

Now define v, a', b' as in the statement of the lemma using this particular e . If $0 < r \leq 2^{-m+1}$ then (4.2), (4.9), and (4.1) yield

$$\begin{aligned} &I(|Dv|^2, T(a', b', 2^{-m}, 2^{-2m}, r)) \\ &= (e^{-1})I(|Du|^2, T(a, b, 2^{-m}e, 2^{-2m}e^2, re)) \\ &\leq (e^{-1})I(|Du|^2, T(a, b, 2^{-m}e, 2^{-2m}, re)) \\ &\leq (e^{-1})(2re)Mg_1(2^{-m}e) \\ &\leq (80)2^{mr} \|g_1\|_1 \\ &= (80)2^{mr} I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m})). \end{aligned} \quad (4.10)$$

If $0 < s \leq 2^{-2m}$ then (4.6) and (4.9) yield

$$\begin{aligned} &I(|Dv|^2, K(a', b' - 2^{-2m} + s, 2^{-m+2}, s)) \\ &= (e^{-1})I(|Du|^2, K(a, b - 2^{-2m}e^2 + se^2, 2^{-m+2}e, se^2)) \\ &\leq (e^{-1})I(|Du|^2, K(a, b - 2^{m-2}e^2 + se^2, 2^{-m+2}, se^2)) \\ &\leq (e^{-1})(2se^2)Mg_3(b - 2^{-2m}e^2) \\ &\leq (80)2^{2ms} \|g_3\|_1 \\ &= (80)2^{2ms} I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m})). \end{aligned} \quad (4.11)$$

If $0 < s \leq 2^{-2m}$ then (4.5) and (4.9) yield

$$\begin{aligned} &\int_{b' - 2^{-2m}}^{b' - 2^{-2m} + s} \int_{\mathbb{R}^3} |v(x, t)|^2 (|x - a'| + 2^{-m})^{-4} dx dt \\ &= (e^{-3}) \left(\int_{b - 2^{-2m}e^2}^{b - 2^{-2m}e^2 + se^2} \int_{\mathbb{R}^3} |u(x, t)|^2 (|e^{-1}(x - a)| + 2^{-m})^{-4} dx dt \right) \\ &\leq (e^{-3}) \left(\int_{b - 2^{-2m}e^2}^{b - 2^{-2m}e^2 + se^2} \int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx dt \right) \\ &\leq (e^{-3})(2se^2)Mg_2(b - 2^{-2m}e^2) \leq (160)2^{2ms} \|g_2\|_1 \\ &= (160)2^{2ms} \left(\int_{b - 2^{-2m}}^b \int_{\mathbb{R}^3} |u(x, t)|^2 (|x - a| + 2^{-m})^{-4} dx dt \right). \end{aligned} \quad (4.12)$$

The conclusion of the lemma follows from (4.10), (4.11), (4.12).

The purpose of Sects. 2–4 is to prove the following theorem:

Theorem 4.2. *For every positive number H there exist positive constants $C_5(H)$ and $C_6(H)$, which depend only on H , such that the following is satisfied. Suppose we have the following conditions: $a \in \mathbb{R}^3$, $b \in \mathbb{R}^+$, m is an integer, $b > 2^{-2m}$, $\Gamma \in \mathbb{R}^+$,*

$u : R^3 \times R^+ \rightarrow R^3$ is a continuous function satisfying (2.1) and (2.2), the conditions

$$I(|Du|^2, K(a, b, 2^{-m+2}, 2^{-2m})) \leq C_5(H)2^{-m}, \quad (4.13)$$

$$\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx dt \leq C_5(H)2^m, \quad (4.14)$$

$$\Gamma \leq C_5(H)2^m \quad (4.15)$$

are satisfied, and inequality (3.1) holds whenever c, d, h satisfy the properties $(c, d) \in K(a, b, 2^{-m}, 2^{-2m})$, $0 < h \leq 2^{-m}$, and $d > h^2$. Then we can conclude that (4.16) holds:

$$|u(x, t)| \leq C_6(H)2^m \quad \text{if } (x, t) \in K(a, b, 2^{-m-2}, 2^{-2m-3}). \quad (4.16)$$

Proof. We set $C_5(H) = (160)^{-1}C_3(H)$ and $C_6(H) = (4)C_4(H)$ (see Lemma 3.1). We use Lemma 4.1 to find e and define v, a', b' . Then the hypotheses (4.13) and (4.14) and Lemma 4.1 yield

$$\begin{aligned} & I(|Dv|^2, T(a', b', 2^{-m}, 2^{-2m}, r)) \\ & \leq C_3(H)r \quad \text{if } 0 < r \leq 2^{-m+1}, \\ & I(|Dv|^2, K(a', b' - 2^{-2m} + s, 2^{-m+2}, s)) \\ & \leq C_3(H)2^m s \quad \text{if } 0 < s \leq 2^{-2m}, \\ & \int_{b'-2^{-2m}}^{b'-2^{-2m}+s} \int_{R^3} |v(x, t)|^2 (|x-a'|+2^{-m})^{-4} dx dt \\ & \leq C_3(H)2^{3m} s \quad \text{if } 0 < s \leq 2^{-2m}. \end{aligned}$$

Our objective is to show that the hypotheses of Lemma 3.1 hold when a, b, Γ, u are replaced by $a', b', e\Gamma, v$. The above shows that we already have the analogue of (3.2)–(3.4). The analogue of (3.5) and $b' > 2^{-2m}$ follow from $1/2 < e < 1$ and the hypotheses. We proceed to prove the analogue of (3.1). We assume $(c', d') \in K(a', b', 2^{-m}, 2^{-2m})$, $0 < h' \leq 2^{-m}$, $d' > (h')^2$. Setting $c = ec'$, $d = e^2 d'$, $h = eh'$ we obtain $(c, d) \in K(a, b, 2^{-m}, 2^{-2m})$ and $0 < h \leq 2^{-m}$ from $1/2 < e < 1$. We also conclude $d > h^2$. The hypotheses of the theorem yield (3.1) (as written), which can be expressed as

$$\begin{aligned} & H^{-1}|v(c', d')| \\ & \leq \int_{a'-(h')^2}^{a'} \int_{R^3} |v(y, s)|^2 (|y-c'|+(d'-s)^{1/2})^{-4} dy ds \\ & \quad + \int_{a'-(h')^2}^{a'} \int_{R^3} |v(y, s)| (|y-c'|+h')^{-4} (h')^{-1} dy ds + e\Gamma. \end{aligned}$$

This is the analogue of (3.1). Lemma 3.1 yields $|v(x, t)| \leq C_4(H)2^{m+1}$ if $(x, t) \in K(a', b', 2^{-(m+1)}, 2^{-2m}(3/4))$. Combining this with $1/2 < e < 1$ we obtain (4.16).

5. Properties of Approximate Solutions

Definition 5.1. We fix, once and for all, a C^∞ function $\theta : R^3 \rightarrow [0, 1]$ such that $\theta(x) = 0$ if $|x| \geq 1$, $\theta(x) = 1$ if $|x| \leq 1/2$, and $\theta(x) = \theta(-x)$. We define $\theta' \in C_0^\infty(R^3, R)$ by $\theta'(x) = \theta(x)/\|\theta\|_1$ and for every $\varepsilon > 0$ we define $\theta'_\varepsilon \in C_0^\infty(R^3, R)$ by $\theta'_\varepsilon(x) = \varepsilon^{-3}\theta'(\varepsilon^{-1}x)$. We have $\|\theta'_\varepsilon\|_1 = 1$ and the estimates $\|\theta'_\varepsilon\|_\infty \leq C\varepsilon^{-3}$, $\|D_i\theta'_\varepsilon\|_\infty \leq C\varepsilon^{-4}$, $\|D_{ij}\theta'_\varepsilon\|_\infty \leq C\varepsilon^{-5}$, and $\|D_{ijk}\theta'_\varepsilon\|_\infty \leq C\varepsilon^{-6}$.

For the remainder of this section we fix $0 < \alpha < \varepsilon$ such that

$$\|\Delta(\theta'_\varepsilon) - \Delta(\theta'_\varepsilon * \theta'_\alpha * \theta'_\alpha)\|_2 \leq 1. \quad (5.1)$$

This means, of course, that α is much smaller than ε when ε is small.

Definition 5.2. We set $\Phi = \theta'_\varepsilon$, $\Omega = \theta'_\alpha$, and $\Psi = \Phi * \Omega$. Observe that we have $\|\Psi\|_1 = 1$, $\Psi(x) = 0$ if $|x| \geq 2\varepsilon$, $\|\Psi\|_\infty \leq \|\Phi\|_\infty \|\Omega\|_1 \leq C\varepsilon^{-3}$, $\|D_i\Psi\|_\infty \leq \|D_i\Phi\|_\infty \|\Omega\|_1 \leq C\varepsilon^{-4}$, $\|D_{ij}\Psi\|_\infty \leq C\varepsilon^{-5}$, and $\|D_{ijk}\Psi\|_\infty \leq C\varepsilon^{-6}$. Let X be the Hilbert space of all square integrable functions $f : R^3 \rightarrow R^3$ with the inner product $(f, g) = \int f_i(x)g_i(x)dx$. Recall that U is the open subset of R^3 given in Sect. 1. We define W to be the closed linear subspace of X consisting of all w such that (1) $w(x) = 0$ for almost every $x \notin U$, (2) $\int w_i(x)D_jg(x)dx = 0$ for every $g \in C_0^\infty(R^3, R)$. The orthogonal projection of X onto W will be denoted by P . The function $S : W \rightarrow X$ is defined by

$$S(w)_i = -((w_j * \Psi)D_j(w_i * \Psi)) * \Psi + \Delta(w_i * \Omega * \Omega). \quad (5.2)$$

Lemma 5.3. *There exists a number $M < \infty$ (which depends on Φ and Ω) such that*

$$\|P(S(w)) - P(S(w'))\|_2 \leq M\|w - w'\|_2(\|w\|_2 + \|w'\|_2 + 1) \quad \text{if } w, w' \in W.$$

Proof. Using Young's inequality and the Schwarz inequality we obtain

$$\begin{aligned} & \|S(w)_i - S(w')_i\|_2 \\ & \leq \|((w_j * \Psi)((w_i - w'_i) * D_j\Psi)) * \Psi + (((w_j - w'_j) * \Psi)(w'_i * D_j\Psi)) * \Psi\|_2 \\ & \quad + \|(w_i - w'_i) * \Delta(\Omega * \Omega)\|_2 \\ & \leq \|(w_j * \Psi)((w_i - w'_i) * D_j\Psi)\|_2 \|\Psi\|_1 + \|((w_j - w'_j) * \Psi)(w'_i * D_j\Psi)\|_2 \|\Psi\|_1 \\ & \quad + \|w_i - w'_i\|_2 \|\Delta(\Omega * \Omega)\|_1 \\ & \leq \|w_j * \Psi\|_\infty \|(w_i - w'_i) * D_j\Psi\|_2 + \|(w_j - w'_j) * \Psi\|_2 \|w'_i * D_j\Psi\|_\infty \\ & \quad + \|w_i - w'_i\|_2 \|\Delta(\Omega * \Omega)\|_1 \\ & \leq \|w_j\|_2 \|\Psi\|_2 \|w_i - w'_i\|_2 \|D_j\Psi\|_1 + \|w_j - w'_j\|_2 \|\Psi\|_1 \|w'_i\|_2 \|D_j\Psi\|_2 \\ & \quad + \|w_i - w'_i\|_2 \|\Delta(\Omega * \Omega)\|_1. \end{aligned}$$

Finally, we have $\|P(S(w) - S(w'))\|_2 \leq \|S(w) - S(w')\|_2$ because P is an orthogonal projection.

Lemma 5.4. *If $w \in W$ then $(P(S(w)), w) = -\int |D(w * \Omega)(x)|^2 dx$.*

Proof. The definition of W , the symmetry of Ψ and Ω , and approximation by C^∞ functions yields

$$\begin{aligned} (P(S(w)), w) &= (S(w), w) \\ &= -\int (((w_j * \Psi) D_j (w_i * \Psi)) * \Psi) w_i + \int (\Delta (w_i * \Omega * \Omega)) w_i \\ &= -\int ((w_j * \Psi) D_j (w_i * \Psi)) (w_i * \Psi) + \int (\Delta (w_i * \Omega)) (w_i * \Omega) \\ &= -(1/2) (\int (w_j * \Psi) D_j (|w * \Psi|^2)) - \int |D(w * \Omega)|^2 = -\int |D(w * \Omega)|^2. \end{aligned}$$

Definition 5.5. If f is a function from an open interval of R into X , s is an element of the interval, and $(f(s+h) - f(s))/h$ converges in norm as h approaches zero, then the limit is denoted by $D_t f(s)$.

Recall that w^0 is the initial condition given in Sect. 1.

Lemma 5.6. *There exists a norm continuous function $w: [0, \infty) \rightarrow W$ such that $w(0) = w^0$, $D_t w$ exists and is a norm continuous function from $(0, \infty)$ into W ,*

$$(D_t w)(s) = P(S(w(s))), \quad (5.3)$$

$$\|w(s)\|_2 \leq \|w^0\|_2 \quad \text{if } s \geq 0, \quad (5.4)$$

and

$$\int_0^\infty \int_{R^3} |D(w(s) * \Psi)(x)|^2 dx ds \leq (1/2) \|w^0\|_2^2. \quad (5.5)$$

Proof. The result in Lemma 5.3 and the standard local existence proof for ordinary differential equations yield $\delta > 0$ with the following property: If $s \in R$, $w' \in W$, and $\|w'\|_2 \leq \|w^0\|_2$ then (5.3) can be solved on the interval $[s, s + \delta]$ with initial condition w' at s . Since Lemma 5.4 yields $(d/dt) \|w(t)\|_2^2 \leq 0$ for any solution of (5.3), the local solutions can be pieced together into a global solution with initial condition w^0 at 0 that satisfies (5.4). From Lemma 5.4 we obtain

$$\int_0^\infty \int_{R^3} |D(w(s) * \Omega)(x)|^2 dx ds \leq (1/2) \|w^0\|_2^2.$$

In addition, Young's inequality yields

$$\begin{aligned} \int_{R^3} |D(w(s) * \Psi)(x)|^2 dx &= \int_{R^3} |D(w(s) * \Omega * \Phi)(x)|^2 dx \\ &\leq \left(\int_{R^3} |D(w(s) * \Omega)(x)|^2 dx \right) \|\Phi\|_1^2. \end{aligned}$$

Now (5.5) follows from $\|\Phi\|_1 = 1$.

Lemma 5.7. *If $f: R^3 \sim \{0\} \rightarrow R$ is a C^∞ function, if i, j, k are elements of $\{1, 2, 3\}$, $M \geq 0$, $|f(x)| \leq M|x|^{-1}$, $|D_{ijk} f(x)| \leq M|x|^{-4}$, $g \in C_0^\infty(R^3, R)$, $r > 0$, and $g(x) = 0$ whenever $|x| \geq r$ then*

$$\begin{aligned} |(f * g)(x)| &\leq CM(\|g\|_1 + r^3 \|g\|_\infty) |x|^{-1}, \\ |D_{ijk}(f * g)(x)| &\leq CM(\|g\|_1 + r^6 \|D_{ijk} g\|_\infty) |x|^{-4}, \\ \|D_{ijk}(f * g)\|_\infty &\leq CM(\|g\|_1 + r^6 \|D_{ijk} g\|_\infty) r^{-4}. \end{aligned}$$

Proof. If $|x| \geq 2r$ then

$$\begin{aligned} |(f*g)(x)| &= \left| \int_{|x-y| \leq |x|/2} f(y)g(x-y)dy \right| \\ &\leq M(|x|/2)^{-1} \left(\int_{|x-y| \leq |x|/2} |g(x-y)|dy \right) = 2M|x|^{-1} \|g\|_1. \end{aligned}$$

If $|x| \geq 2r$ then a similar argument yields

$$|D_{ijk}(f*g)(x)| = |(D_{ijk}f*g)(x)| \leq 16M|x|^{-4} \|g\|_1.$$

If $|x| < 2r$ then

$$\begin{aligned} |(f*g)(x)| &= \left| \int_{|y| \leq 3r} f(y)g(x-y)dy \right| \\ &\leq \left(\int_{|y| \leq 3r} |f(y)|dy \right) \|g\|_\infty \leq CMr^2 \|g\|_\infty = CMr^{-1} (r^3 \|g\|_\infty). \end{aligned}$$

Once again, if $|x| < 2r$ then the same argument yields

$$|D_{ijk}(f*g)(x)| = |(f*D_{ijk}g)(x)| \leq CMr^2 \|D_{ijk}g\|_\infty = CMr^{-4} (r^6 \|D_{ijk}g\|_\infty).$$

The conclusion follows by combining these estimates.

Definition 5.8. The functions $J: R^3 \setminus \{0\} \rightarrow R$ and $Q_t: R^3 \rightarrow R$ for $t > 0$ are given by

$$J(x) = -(4\pi|x|)^{-1}, \quad (5.6)$$

$$Q_t(x) = (2\sqrt{\pi})^{-3} t^{-3/2} \exp(-|x|^2/(4t)). \quad (5.7)$$

Lemma 5.9. *If $t > 0$, $R > 2(t^{1/2})$, $\lambda(x) = \theta((2R)^{-1}x)$, $J'(x) = \lambda(x)J(x)$, and $g(x) = \lambda(x)Q_t(x)$ then*

$$|D_{ijk}(J'*g*\Psi*\Psi)(x)| \leq C(|x| + t^{1/2})^{-4} \quad \text{for all } x \in R^3.$$

Proof. We fix $i, j, k \in \{1, 2, 3\}$. The integer n is defined by the condition $2^{-n}R \geq t^{1/2} > 2^{-n-1}R$. From $R > 2(t^{1/2})$ we conclude $n \geq 1$. For any integer m satisfying $0 \leq m \leq n$ we set $\lambda_m(x) = \lambda(2^m x)$. We also define $g_m(x) = (\lambda_m - \lambda_{m+1})(x)Q_t(x)$ for $0 \leq m < n$, and $g_n(x) = \lambda_n(x)Q_t(x)$. We have the following properties:

$$g(x) = \sum_{m=0}^n g_m(x), \quad g_m(x) = 0 \quad \text{if } |x| \geq 2^{1-m}R, \quad \sum_{m=0}^n \|g_m\|_1 = \|g\|_1 \leq 1,$$

$$\sum_{m=0}^n (2^{1-m}R)^3 \|g_m\|_\infty \leq C, \quad \sum_{m=0}^n (2^{1-m}R)^6 \|D_{ijk}g_m\|_\infty \leq C.$$

We also have $|J'(x)| \leq C|x|^{-1}$, $|D_{ijk}J'(x)| \leq C|x|^{-4}$. Now we can use Lemma 5.7 to conclude

$$\begin{aligned} |(J'*g)(x)| &\leq \sum_{m=0}^n |(J'*g_m)(x)| \\ &\leq \sum_{m=0}^n C(\|g_m\|_1 + (2^{1-m}R)^3 \|g_m\|_\infty) |x|^{-1} \leq C|x|^{-1}, \end{aligned} \quad (5.8)$$

$$\begin{aligned} |D_{ijk}(J'*g)(x)| &\leq \sum_{m=0}^n |D_{ijk}(J'*g_m)(x)| \\ &\leq \sum_{m=0}^n C(\|g_m\|_1 + (2^{1-m}R)^6 \|D_{ijk}g_m\|_\infty) |x|^{-4} \leq C|x|^{-4}, \end{aligned} \quad (5.9)$$

$$\begin{aligned} \|D_{ijk}(J'*g)\|_\infty &\leq \sum_{m=0}^n \|D_{ijk}(J'*g_m)\|_\infty \\ &\leq \sum_{m=0}^n C(\|g_m\|_1 + (2^{1-m}R)^6 \|D_{ijk}g_m\|_\infty) (2^{1-m}R)^{-4} \\ &\leq \sum_{m=0}^n C(\|g_m\|_1 + (2^{1-m}R)^6 \|D_{ijk}g_m\|_\infty) (t^{1/2})^{-4} \leq C(t^{1/2})^{-4}. \end{aligned} \quad (5.10)$$

We also have

$$\|\Psi*\Psi\|_1 = \|\Psi\|_1 \|\Psi\|_1 = 1$$

and

$$\|D_{ijk}(\Psi*\Psi)\|_\infty \leq \|D_{ijk}\Psi\|_\infty \|\Psi\|_1 \leq C\varepsilon^{-6}.$$

Hence (5.8), (5.9), the fact that $(\Psi*\Psi)(x) = 0$ whenever $|x| \geq 4\varepsilon$, and Lemma 5.7 yield

$$|D_{ijk}(J'*g*\Psi*\Psi)(x)| \leq C(\|\Psi*\Psi\|_1 + (4\varepsilon)^6 \|D_{ijk}(\Psi*\Psi)\|_\infty) |x|^{-4} \leq C|x|^{-4}. \quad (5.11)$$

From (5.10) we obtain

$$\begin{aligned} \|D_{ijk}(J'*g*\Psi*\Psi)\|_\infty &\leq \|D_{ijk}(J'*g)\|_\infty \|\Psi*\Psi\|_1 \\ &= \|D_{ijk}(J'*g)\|_\infty \leq C(t^{1/2})^{-4}. \end{aligned} \quad (5.12)$$

Now (5.11) and (5.12) yield the conclusion of the lemma by considering separately the cases $|x| > t^{1/2}$ and $|x| \leq t^{1/2}$.

Definition 5.10. The i components of $w(s)$ and $(D_t w)(s)$ will be written $w_i(s)$ and $D_t w_i(s)$ (see Lemma 5.6). We will also use the notation $w'_i(s) = w_i(s)*\Psi$ and $w'(s) = w(s)*\Psi$.

Lemma 5.11. *If the hypotheses of Lemma 5.9 are satisfied, $c \in U$,*

$$\{x \in R^3 : |x - c| \leq 4R + 2\varepsilon\} \subset U,$$

and $s > 0$ then

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} [(D_t w'_i(s) - \Delta(w'_i(s)))(x)] [\Delta(g*J')(x-c)] dx \right| \\ & \leq C \left(\int_{\mathbb{R}^3} |(w'(s))(x)|^2 (|x-c| + t^{1/2})^{-4} dx \right) + \|w^0\|_2 \|\Delta(g*J')\|_1. \end{aligned}$$

Proof. We fix $i \in \{1, 2, 3\}$ and define $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $f_i = g$, $f_j = 0$ if $j \neq i$. We have

$$\Delta(f*J'*\Psi) = -\text{curl}(\text{curl}(f*J'*\Psi)) + \text{grad}(\text{div}(f*J'*\Psi)).$$

Hence we can use $D_t w(s) \in W$, the fact that the function

$$h(x) = \text{curl}(\text{curl}(f*J'*\Psi))(x-c)$$

is an element of W , (5.3), (5.2), and $w(s) \in W$ to conclude

$$\begin{aligned} & \int [(D_t w'_i(s))(x)] [\Delta(g*J')(x-c)] dx \\ & = \int [(D_t w'_n(s))(x)] [\Delta(f_n*J')(x-c)] dx \\ & = \int [(D_t w_n(s))(x)] [\Delta(f_n*J'*\Psi)(x-c)] dx \\ & = - \int [(D_t w_n(s))(x)] [\text{curl}(\text{curl}(f*J'*\Psi))(x-c)]_n dx \\ & = \int [(w'_j(s) D_j(w'_n(s))*\Psi)(x)] [\text{curl}(\text{curl}(f*J'*\Psi))(x-c)]_n dx \\ & \quad - \int [\Delta(w_n(s)*\Omega*\Omega)(x)] [\text{curl}(\text{curl}(f*J'*\Psi))(x-c)]_n dx \\ & = - \int [(w'_j(s))(x)] [(w'_n(s))(x)] [D_j(\text{curl}(\text{curl}(f*J'*\Psi*Psi)))(x-c)]_n dx \\ & \quad + \int [\Delta(w_n(s)*\Omega*\Omega)(x)] [\Delta(f_n*J'*\Psi)(x-c)] dx. \end{aligned} \tag{5.13}$$

The Schwarz inequality, (5.4), Young's inequality, (5.1), and Definitions 5.1 and 5.2 yield

$$\begin{aligned} & \left| \int [\Delta(w'_i(s))(x)] [\Delta(g*J')(x-c)] dx \right| \\ & \quad - \left| \int [\Delta(w_n(s)*\Omega*\Omega)(x)] [\Delta(f_n*J'*\Psi)(x-c)] dx \right| \\ & = \left| \int [(w'_i(s)*\Delta\Psi)(x)] [\Delta(g*J')(x-c)] dx \right| \\ & \quad - \left| \int [(w'_i(s)*\Delta(\Psi*\Omega*\Omega))(x)] [\Delta(g*J')(x-c)] dx \right| \\ & \leq \|w(s)*(\Delta(\Phi*\Omega) - \Delta(\Phi*\Omega*\Omega))\|_\infty \|\Delta(g*J')\|_1 \\ & \leq \|w(s)\|_2 \|(\Delta(\Phi) - \Delta(\Phi*\Omega*\Omega))*\Omega\|_2 \|\Delta(g*J')\|_1 \\ & \leq \|w^0\|_2 \|\Delta(\Phi) - \Delta(\Phi*\Omega*\Omega)\|_2 \|\Omega\|_1 \|\Delta(g*J')\|_1 \leq \|w^0\|_2 \|\Delta(g*J')\|_1. \end{aligned} \tag{5.14}$$

Finally, (5.13), (5.14), and Lemma 5.9 yield

$$\begin{aligned} & \left| \int [(D_t w'_i(s) - \Delta(w'_i(s)))(x)] [\Delta(g*J')(x-c)] dx \right| \\ & \leq \left| \int [(w'_j(s))(x)] [(w'_n(s))(x)] [D_j(\text{curl}(\text{curl}(f*J'*\Psi*Psi)))(x-c)]_n dx \right| \\ & \quad + \|w^0\|_2 \|\Delta(g*J')\|_1 \\ & < C \left(\int |(w'(s))(x)|^2 (|x-c| + t^{1/2})^{-4} dx \right) + \|w^0\|_2 \|\Delta(g*J')\|_1. \end{aligned}$$

Theorem 5.12. *There exists an absolute constant $C_7 > 0$ with the following property: If $c \in U$, $0 < h < R/2$, $d > h^2$, and $\{x \in \mathbb{R}^3 : |x - c| \leq 4R + 4\epsilon\} \subset U$ then*

$$\begin{aligned} & C_7^{-1} |(w'(d))(c)| \\ & \leq \int_{d-h^2}^d \int_{\mathbb{R}^3} |(w'(s))(x)|^2 (|x-c| + (d-s)^{1/2})^{-4} dx ds \\ & \quad + \int_{d-h^2}^d \int_{\mathbb{R}^3} |(w'(s))(x)| (|x-c| + h)^{-4} h^{-1} dx ds \\ & \quad + h^2 \|w^0\|_2 + h^2 \|w^0\|_2 R^{-7/2} + \|w^0\|_2 R^{-3/2}. \end{aligned}$$

Proof. We fix $i \in \{1, 2, 3\}$, define λ and J' as in Lemma 5.9, set $J''(x) = (1 - \lambda(x))J(x)$, and define the function g and f from $[0, d]$ into $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ by

$$(g(s))(x) = \lambda(x)Q_{d-s}(x), \quad f(s) = \Delta(g(s)*J').$$

We have

$$f(s) = \Delta(g(s)*J) - \Delta(g(s)*J'') = g(s) - g(s)*\Delta J''. \quad (5.15)$$

Since $J''(x) = J(x)$ for $|x| > 2R$, we obtain $\Delta J''(x) = 0$ for $|x| > 2R$. We also have $J''(x) = 0$ for $|x| < R$. Combining this with $\|\lambda\|_\infty \leq 1$, $\|D_i \lambda\|_\infty \leq CR^{-1}$, and $\|\Delta \lambda\|_\infty \leq CR^{-2}$ we obtain

$$\|\Delta J''\|_1 \leq C, \quad \|\Delta J''\|_2 \leq CR^{-3/2}. \quad (5.16)$$

From (5.15) and (5.16) we obtain

$$\|f(s)\|_1 \leq \|g(s)\|_1 + \|g(s)\|_1 \|\Delta J''\|_1 \leq C. \quad (5.17)$$

If we set $Y(x, s) = Q_{d-s}(x)$ for $s < d$ then we obtain $D_i Y + \Delta Y = 0$. Hence we conclude

$$(D_i g(s) + \Delta(g(s)))(x) = \Delta \lambda(x)Q_{d-s}(x) + 2(D_i \lambda(x)D_i Q_{d-s}(x)). \quad (5.18)$$

This shows

$$(D_i g(s) + \Delta(g(s)))(x) = 0 \quad \text{if } |x| < R \quad \text{or} \quad |x| > 2R. \quad (5.19)$$

We also have the estimates $|Q_{d-s}(x)| \leq C|x|^{-3}$, $|D_i Q_{d-s}(x)| \leq C|x|^{-4}$. If $|x| > R$ then (5.18) yields

$$|(D_i g(s) + \Delta(g(s)))(x)| \leq CR^{-2}|x|^{-3} + CR^{-1}|x|^{-4} \leq CR^{-5} \quad \text{if } |x| > R. \quad (5.20)$$

Now (5.19) and (5.20) yield $\|D_i g(s) + \Delta(g(s))\|_2 \leq CR^{-7/2}$.

Combining this with (5.15) and (5.16) we obtain

$$\|D_i f(s) + \Delta(f(s))\|_2 \leq CR^{-7/2} + \|D_i g(s) + \Delta(g(s))\|_2 \|\Delta J''\|_1 \leq CR^{-7/2}. \quad (5.21)$$

From Definition 5.10 and (5.4) we obtain

$$\|w'(s)\|_2 \leq \|w(s)\|_2 \|\Psi\|_1 \leq \|w(s)\|_2 \leq \|w^0\|_2. \quad (5.22)$$

Now suppose $d - h^2 \leq s' \leq d - h^2/2 < s'' < d$. From Lemmas 5.6 and 5.11, the Schwarz inequality, (5.21), (5.17), and (5.22) we obtain

$$\begin{aligned}
& \left| \int_{R^3} [(w'_i(s''))(x)] [(f(s''))(x-c)] dx \right. \\
& \quad \left. - \int_{R^3} [(w'_i(s'))(x)] [(f(s'))(x-c)] dx \right| \\
&= \left| \int_{s'}^{s''} \int_{R^3} [(D_t w'_i(s))(x)] [(f(s))(x-c)] dx ds \right. \\
& \quad \left. + \int_{s'}^{s''} \int_{R^3} [(w'_i(s))(x)] [(D_t f(s))(x-c)] dx ds \right| \\
&\leq \left| \int_{s'}^{s''} \int_{R^3} [(D_t w'_i(s))(x)] [(f(s))(x-c)] dx ds \right. \\
& \quad \left. - \int_{s'}^{s''} \int_{R^3} [(w'_i(s))(x)] [\Delta(f(s))(x-c)] dx ds \right| \\
& \quad + \left| \int_{s'}^{s''} \int_{R^3} [(w'_i(s))(x)] [(D_t f(s) + \Delta(f(s)))(x-c)] dx ds \right| \\
&\leq C \left(\int_{s'}^{s''} \int_{R^3} |(w'(s))(x)|^2 (|x-c| + (d-s)^{1/2})^{-4} dx ds \right) \\
& \quad + \int_{s'}^{s''} \|w^0\|_2 \|f(s)\|_1 ds + C \left(\int_{s'}^{s''} \|w'(s)\|_2 R^{-7/2} ds \right) \\
&\leq C \left(\int_{s'}^{s''} \int_{R^3} |(w'(s))(x)|^2 (|x-c| + (d-s)^{1/2})^{-4} dx ds \right) \\
& \quad + C(s'' - s') \|w^0\|_2 + C(s'' - s') \|w^0\|_2 R^{-7/2}. \tag{5.23}
\end{aligned}$$

In addition, (5.15), (5.22), and (5.16) yield

$$\begin{aligned}
& \left| \int_{R^3} [(w'_i(s))(x)] [(f(s))(x-c)] dx \right. \\
& \quad \left. - \int_{R^3} [(w'_i(s))(x)] [(g(s))(x-c)] dx \right| \\
&\leq \|w'(s)\|_2 \|g(s) * \Delta J''\|_2 \leq \|w^0\|_2 \|g(s)\|_1 \|\Delta J''\|_2 \leq C \|w^0\|_2 R^{-3/2} \tag{5.24}
\end{aligned}$$

when $s = s'$ or $s = s''$. From (5.23), (5.24), and the properties of s' , s'' we get

$$\begin{aligned}
& \left| \int_{R^3} [(w'_i(s''))(x)] [(g(s''))(x-c)] dx \right. \\
& \quad \left. - \int_{R^3} [(w'_i(s'))(x)] [(g(s'))(x-c)] dx \right| \\
&\leq C \left(\int_{d-h^2}^d \int_{R^3} |(w'(s))(x)|^2 (|x-c| + (d-s)^{1/2})^{-4} dx ds \right) \\
& \quad + Ch^2 \|w^0\|_2 + Ch^2 \|w^0\|_2 R^{-7/2} + C \|w^0\|_2 R^{-3/2}. \tag{5.25}
\end{aligned}$$

From Lemma 5.6 we obtain that the function $u(x, s) = (w'(s))(x)$ is bounded and continuous. This implies

$$\lim_{s'' \rightarrow d} \int_{\mathbb{R}^3} [(w'_i(s''))(x)] [(g(s''))(x - c)] dx = (w'_i(d))(c). \quad (5.26)$$

Since $d - h^2 \leq s' \leq d - h^2/2$ we have

$$|(g(s'))(x - c)| \leq |Q_{d-s'}(x - c)| \leq C(|x - c| + h)^{-4}h. \quad (5.27)$$

Combining (5.25)–(5.27) we obtain

$$\begin{aligned} |(w'(d))(c)| &\leq C \left(\int_{d-h^2}^d \int_{\mathbb{R}^3} |(w'(s))(x)|^2 (|x - c| + (d - s)^{1/2})^{-4} dx ds \right) \\ &\quad + C \left(\int_{\mathbb{R}^3} |(w'(s'))(x)| (|x - c| + h)^{-4} h dx \right) \\ &\quad + Ch^2 \|w^0\|_2 + Ch^2 \|w^0\|_2 R^{-7/2} + C \|w^0\|_2 R^{-3/2} \end{aligned} \quad (5.28)$$

for $d - h^2 \leq s' \leq d - h^2/2$. The conclusion follows by averaging (5.28) over s' .

Lemma 5.13. *If $s > 0$, $f \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$, the distance between $\text{spt}(f)$ and the complement of U exceeds 2ε , $i \in \{1, 2, 3\}$, the function $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $g_i = f$, $g_j = 0$ whenever $j \neq i$, and $z: [0, \infty) \rightarrow X$ is defined by $z(s) = \text{curl}(w'(s))$, then*

$$\begin{aligned} &\int [(D_t z_i(s))(x)] [f(x)] dx \\ &= \int [(w'_j(s))(x)] [(z_i(s))(x)] [D_j(f * \Psi * \Psi)(x)] dx \\ &\quad - \int [(z_j(s))(x)] [(w'_i(s))(x)] [D_j(f * \Psi * \Psi)(x)] dx \\ &\quad + \int [(z_i(s))(x)] [\Delta f(x)] dx \\ &\quad + \int [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi))(x)] [\text{curl}(g)(x)]_n dx. \end{aligned}$$

Proof. We fix $i \in \{1, 2, 3\}$. The fact $w(s) \in W$ implies $\text{div}(w'(s)) = 0$. Hence we obtain

$$\text{curl}(w'_j(s) D_j(w'(s))) = w'_j(s) D_j(z(s)) - z_j(s) D_j(w'(s)).$$

Hence the property $\text{curl}(g) * \Psi \in W$, Lemma 5.6, and the properties $\text{div}(w'(s)) = 0$, $\text{div}(z(s)) = 0$ yield

$$\begin{aligned} &\int [(D_t z_i(s))(x)] [f(x)] dx = \int [(D_t z_n(s))(x)] [g_n(x)] dx \\ &= \int [(D_t w'_n(s))(x)] [\text{curl}(g)(x)]_n dx \\ &= \int [(D_t w'_n(s))(x)] [(\text{curl}(g) * \Psi)(x)]_n dx \\ &= - \int [((w'_j(s) D_j(w'_n(s))) * \Psi)(x)] [(\text{curl}(g) * \Psi)(x)]_n dx \\ &\quad + \int [\Delta(w_n(s) * \Omega * \Omega)(x)] [(\text{curl}(g) * \Psi)(x)]_n dx \\ &= - \int [(w'_j(s) D_j(z_n(s)) - z_j(s) D_j(w'_n(s)))(x)] [(g_n * \Psi * \Psi)(x)] dx \\ &\quad + \int [\Delta(w'_n(s))(x)] [\text{curl}(g)(x)]_n dx \\ &\quad + \int [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi))(x)] [\text{curl}(g)(x)]_n dx \\ &= \int [(w'_j(s) z_n(s) - z_j(s) w'_n(s))(x)] [D_j(g_n * \Psi * \Psi)(x)] dx \\ &\quad + \int [(z_n(s))(x)] [\Delta g_n(x)] dx \\ &\quad + \int [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi))(x)] [\text{curl}(g)(x)]_n dx. \end{aligned}$$

The conclusion of the lemma now follows.

Definition 5.14. The functions u and z from $R^3 \times [0, \infty)$ into R^3 are defined by

$$u(x, s) = (w'(s))(x) \quad \text{and} \quad z(x, s) = (\text{curl}(w'(s)))(x).$$

Note that

$$|u(x, s)| \leq \|w(s)\|_2 \|\Psi\|_2 \leq \|w^0\|_2 \|\Psi\|_2$$

and

$$|Du(x, s)| \leq \|w(s)\|_2 \|D\Psi\|_2 \leq \|w^0\|_2 \|D\Psi\|_2$$

(see Lemma 5.6). Hence u and z are bounded functions.

Lemma 5.15. *If $a \in U$, $b \in R$, and $h > 0$ such that*

$$b > h^2, \quad \{x \in R^3 : |x - a| \leq h\} \subset U,$$

and $\varepsilon < h/64$ then there exist integrable functions

$$F: R^3 \times (-\infty, 0) \rightarrow R,$$

$$G: R^3 \times (-\infty, 0) \rightarrow R^3,$$

and

$$H_i: R^3 \times (-\infty, 0) \rightarrow R^3$$

for $i \in \{1, 2, 3\}$ such that

(a) if either $|x| \geq h/8$ or $s \leq -h^2/8$ then $F(x, s) = 0$ and $G(x, s) = H_i(x, s) = 0$,

(b) $\|F\|_1 \leq C$, $\|F\|_{6/5} \leq Ch^{-5/6}$, $\|G\|_1 \leq Ch$, $\|G\|_{6/5} \leq Ch^{1/6}$, $\|H_i\|_1 \leq Ch$,

(c) if $c \in R^3$, $|c - a| \leq 7h/8$, and $b - 7h^2/8 \leq d \leq b$ then

$$\begin{aligned} z_i(c, d) &= \int_0^d \int_{R^3} [u_j(x, s)z_i(x, s) - z_j(x, s)u_i(x, s)] G_j(x - c, s - d) dx ds \\ &\quad + \int_0^d \int_{R^3} z_i(x, s) F(x - c, s - d) dx ds \\ &\quad + \int_0^d \int_{R^3} [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta\Psi))(x)] [H_i(x - c, s - d)]_n dx ds. \end{aligned}$$

Proof. We can find a C^∞ function $\beta: R \rightarrow [0, 1]$ such that $\beta(s) = 0$ if

$$s \leq -h^2/8, \quad \beta(s) = 1 \quad \text{if} \quad s \geq -h^2/16,$$

and $\|D_t \beta\|_\infty \leq 32h^{-2}$. We also define

$$\eta: R^3 \rightarrow [0, 1] \quad \text{by} \quad \eta(x) = \theta(16h^{-1}x)$$

(see Definition 5.1). We fix $i \in \{1, 2, 3\}$ and define the functions

$$f: R^3 \times (-\infty, 0) \rightarrow R \quad \text{and} \quad g: R^3 \times (-\infty, 0) \rightarrow R^3$$

as follows:

$$f(x, s) = \beta(s)\eta(x)Q_{-s}(x), \quad g_i(x, s) = f(x, s),$$

and

$$g_j(x, s) = 0 \quad \text{if } j \neq i.$$

We set

$$F = D_i f + \Delta f, \quad G_j = D_j f * \Psi * \Psi, \quad \text{and} \quad H_i = \text{curl}(g).$$

If c and d are as in part (c) and $d - h^2/16 \leq d' < d$ then Lemma 5.13 yields

$$\begin{aligned} & \int_{\mathbb{R}^3} z_i(x, d') \eta(x - c) Q_{d-d'}(x - c) dx = \int_{\mathbb{R}^3} z_i(x, d') f(x - c, d' - d) dx \\ &= \int_0^{d'} \int_{\mathbb{R}^3} D_i z_i(x, s) f(x - c, s - d) dx ds \\ & \quad + \int_0^{d'} \int_{\mathbb{R}^3} z_i(x, s) D_i f(x - c, s - d) dx ds \\ &= \int_0^{d'} \int_{\mathbb{R}^3} [u_j(x, s) z_i(x, s) - z_j(x, s) u_i(x, s)] G_j(x - c, s - d) dx ds \\ & \quad + \int_0^{d'} \int_{\mathbb{R}^3} z_i(x, s) F(x - c, s - d) dx ds \\ & \quad + \int_0^{d'} \int_{\mathbb{R}^3} [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi))(x)] [H_i(x - c, s - d)]_n dx ds. \end{aligned} \quad (5.29)$$

Part (a) follows from $\varepsilon < h/64$, Definition 5.1, and Definition 5.2. We have $F(x, s) = 0$ if $|x| \leq h/32$ and $-h^2/16 \leq s < 0$. This yields the estimates on F in (b). We also have

$$\|G_j\|_p \leq \|D_j f\|_p \|\Psi * \Psi\|_1 = \|D_j f\|_p \|\Psi\|_1 \|\Psi\|_1 = \|D_j f\|_p$$

for $p=1$ and $p=6/5$. This and the definition of f yield the rest of (b). Since z is bounded and continuous (see Lemma 5.6), the left hand side of (5.29) converges to $z_i(c, d)$ as $d' \rightarrow d$. Since u , z , and $w(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi)$ are bounded and G , F , H_i are integrable, we conclude that the integrals in (c) exist and that the right side of (5.29) converges to the right hand side of (c) as $d' \rightarrow d$.

Theorem 5.16. *If $a \in U$, $b \in R$, $h > 0$, $b > h^2$, $\{x \in \mathbb{R}^3 : |x - a| \leq h\} \subset U$, $\varepsilon < h/64$,*

$$K_m = \{(x, s) \in \mathbb{R}^3 \times \mathbb{R} : |x - a| \leq (4 + m)h/8, b - (4 + m)h^2/8 \leq s \leq b\}, \quad (5.30)$$

$M = \max \{|u(x, s)| : (x, s) \in K_4\}$, and $M' = \int_{K_4} |Du|^2$ then there exist a function $\gamma : [0, \infty) \rightarrow [0, \infty)$ and a number $A < \infty$ such that

(a) γ and A depend only on h , M , M' , and $\|w^0\|_2$,

(b) $\gamma(0) = 0$ and $\lim_{\delta \rightarrow 0} \gamma(\delta) = 0$,

(c) $|z(c, d)| \leq A$ if $(c, d) \in K_1$,

(d) $|z(c, d) - z(c', d')| \leq \gamma(|c - c'| + |d - d'|)$ if (c, d) and (c', d') are elements of K_0 .

Proof. If $(c, d) \in K_3$ we define (see Lemma 5.15)

$$\begin{aligned} z'_i(c, d) &= \int_0^d \int_{\mathbb{R}^3} [u_{j_f}(x, s)z_i(x, s) - z_{j_f}(x, s)u_i(x, s)]G_{j_f}(x - c, s - d)dx ds \\ &\quad + \int_0^d \int_{\mathbb{R}^3} z_i(x, s)F(x - c, s - d)dx ds, \\ z''_i(c, d) &= \int_0^d \int_{\mathbb{R}^3} [(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta\Psi))(x)] [H_i(x - c, s - d)]_n dx ds. \end{aligned}$$

We have $z = z' + z''$. Using Lemma 5.6, Definition 5.1, Definition 5.2, Lemma 5.15, and (5.1) we obtain

$$\begin{aligned} |z''_i(c, d)| &\leq \int_0^d (\|w(s)\|_2 \|\Delta(\Psi * \Omega * \Omega) - \Delta\Psi\|_2) \left(\int_{\mathbb{R}^3} |H_i(x - c, s - d)| dx \right) ds \\ &\leq C \|w^0\|_2 \|\Delta(\Phi * \Omega * \Omega) - \Delta\Phi * \Omega\|_2 h \\ &\leq C \|w^0\|_2 \|\Delta(\Phi * \Omega * \Omega) - \Delta\Phi\|_2 \|\Omega\|_1 h \leq C \|w^0\|_2 h. \end{aligned} \quad (5.31)$$

Using Young's inequality $\|f * g\|_3 \leq \|f\|_2 \|g\|_{6/5}$ (see [8, p. 271]) and Lemma 5.15 yields

$$\left(\int_{K_3} |z'|^3 \right)^{1/3} \leq C \left(\int_{K_4} (|u||z|)^2 \right)^{1/2} \|G\|_{6/5} + C \left(\int_{K_4} |z|^2 \right)^{1/2} \|F\|_{6/5}.$$

Since (5.31) yields $\left(\int_{K_3} |z''|^3 \right)^{1/3} \leq C \|w^0\|_2 h^{8/3}$, we can use Lemma 5.15 and the above to obtain

$$\begin{aligned} \left(\int_{K_3} |z|^3 \right)^{1/3} &\leq C \left(\int_{K_4} (|u||z|)^2 \right)^{1/2} h^{1/6} + C \left(\int_{K_4} |z|^2 \right)^{1/2} h^{-5/6} + C \|w^0\|_2 h^{8/3} \\ &\leq C(M + h^{-1}) \left(\int_{K_4} |z|^2 \right)^{1/2} h^{1/6} + C \|w^0\|_2 h^{8/3}. \end{aligned} \quad (5.32)$$

Once again we can use Young's inequality $\|f * g\|_6 \leq \|f\|_3 \|g\|_{6/5}$ and Lemma 5.15 to obtain

$$\left(\int_{K_2} |z'|^6 \right)^{1/6} \leq C \left(\int_{K_3} (|u||z|)^3 \right)^{1/3} \|G\|_{6/5} + C \left(\int_{K_3} |z|^3 \right)^{1/3} \|F\|_{6/5}.$$

From (5.31) we obtain $\left(\int_{K_2} |z''|^6 \right)^{1/6} \leq C \|w^0\|_2 h^{11/6}$. Hence Lemma 5.15 and the above yield

$$\begin{aligned} \left(\int_{K_2} |z|^6 \right)^{1/6} &\leq C \left(\int_{K_3} (|u||z|)^3 \right)^{1/3} h^{1/6} + C \left(\int_{K_3} |z|^3 \right)^{1/3} h^{-5/6} + C \|w^0\|_2 h^{11/6} \\ &\leq C(M + h^{-1}) \left(\int_{K_3} |z|^3 \right)^{1/3} h^{1/6} + C \|w^0\|_2 h^{11/6}. \end{aligned} \quad (5.33)$$

If $(c, d) \in K_1$ then Hölder's inequality, Lemma 5.15, and (5.31) yield

$$\begin{aligned} |z(c, d)| &\leq |z'(c, d)| + |z''(c, d)| \\ &\leq C \left(\int_{K_2} (|u||z|)^6 \right)^{1/6} \|G\|_{6/5} + C \left(\int_{K_2} |z|^6 \right)^{1/6} \|F\|_{6/5} + C \|w^0\|_2 h \\ &\leq C(M + h^{-1}) \left(\int_{K_2} |z|^6 \right)^{1/6} h^{1/6} + C \|w^0\|_2 h. \end{aligned} \quad (5.34)$$

Combining (5.32)–(5.34), and the estimates $\left(\int_{K_4} |z|^2 \right)^{1/2} \leq C \left(\int_{K_4} |Du|^2 \right)^{1/2} = C(M')^{1/2}$ we obtain that $|z|$ is bounded on K_1 by a number that depends only on h, M, M' , and $\|w^0\|_2$. We conclude from this and from the argument in (5.31) that the functions

$$|u_j(x, s)z_i(x, s)|, |z_i(x, s)|, \quad \text{and} \quad |(w_n(s) * (\Delta(\Psi * \Omega * \Omega) - \Delta\Psi))(x)|$$

are bounded on K_1 by a number that depends only on h, M, M' , and $\|w^0\|_2$. The rest follows from Lemma 5.15 and the definitions of G_j, F , and H_i .

Lemma 5.17. *There exist absolute constants C_8 and C_9 such that*

$$\int_0^\infty \int_{R^3} |u(x, t)|^{10/3} dx dt \leq C_8 \|w^0\|_2^{10/3}$$

and

$$\begin{aligned} &\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \\ &\leq C_9 \left(\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^{10/3} (|x-a| + 2^{-m})^{-4} dx dt \right)^{3/5} 2^{-2m/5} \end{aligned}$$

whenever $b > 2^{-2m}$ and $a \in R^3$.

Proof. Using Hölder's inequality, the first inequality in Line 9, p. 127 of [8], Definition 5.10, Definition 5.14, Young's inequality, and Lemma 5.6 we obtain

$$\begin{aligned} &\int_0^\infty \int_{R^3} |u(x, t)|^{10/3} dx dt \\ &= \int_0^\infty \int_{R^3} |u(x, t)|^2 |u(x, t)|^{4/3} dx dt \\ &\leq \int_0^\infty \left(\int_{R^3} |u(x, t)|^6 dx \right)^{1/3} \left(\int_{R^3} |u(x, t)|^2 dx \right)^{2/3} dt \\ &\leq \int_0^\infty C \left(\int_{R^3} |Du(x, t)|^2 dx \right) \left(\int_{R^3} |u(x, t)|^2 dx \right)^{2/3} dt \\ &\leq \int_0^\infty C \left(\int_{R^3} |Du(x, t)|^2 dx \right) \left(\int_{R^3} |(w(t))(x)|^2 dx \right)^{2/3} \|\Psi\|_1^{4/3} dt \\ &\leq \int_0^\infty C \left(\int_{R^3} |D(w(t) * \Psi)(x)|^2 dx \right) \|w^0\|_2^{4/3} dt \leq C((1/2)\|w^0\|_2^2 \|w^0\|_2^{4/3}). \end{aligned}$$

If $b > 2^{-2m}$ and $a \in R^3$ then Hölder's inequality yields

$$\begin{aligned}
& \int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^2 (|x-a|+2^{-m})^{-4} dx dt \\
&= \int_{b-2^{-2m}}^b \int_{R^3} (|u(x, t)|^2 (|x-a|+2^{-m})^{-12/5}) (|x-a|+2^{-m})^{-8/5} dx dt \\
&\leq \left(\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^{10/3} (|x-a|+2^{-m})^{-4} dx dt \right)^{3/5} \\
&\quad \cdot \left(\int_{b-2^{-2m}}^b \int_{R^3} (|x-a|+2^{-m})^{-4} dx dt \right)^{2/5} \\
&\leq C_9 \left(\int_{b-2^{-2m}}^b \int_{R^3} |u(x, t)|^{10/3} (|x-a|+2^{-m})^{-4} dx dt \right)^{3/5} 2^{-2m/5}.
\end{aligned}$$

Definition 5.18. If f is a function defined on $R^3 \times R$ and g is a function on R^3 then we set

$$(f * g)(x, t) = \int_{R^3} f(y, t) g(x-y) dy$$

whenever the integral makes sense.

Lemma 5.19. *If f is a C^∞ function with compact support from $R^3 \times R$ into R^3 such that $\operatorname{div}(f) = 0$ and the distance between the support of f and the complement of $U \times R$ exceeds 2ε then there exist real numbers $a < b$ and a C^∞ function $g : R^3 \times R \rightarrow R^3$ such that $\operatorname{curl}(g) = f$, $g(x, s) = 0$ if $s \notin [a, b]$, and*

$$\begin{aligned}
& - \int_{R^3} [w_i^0(x)] [(f_i * \Psi)(x, 0)] dx \\
&= \int_0^\infty \int_{R^3} [u_i(x, s)] [D_t f_i(x, s) + \Delta(f_i * \Omega * \Omega)(x, s)] dx ds \\
&\quad + \int_0^\infty \int_{R^3} [u_j(x, s) z_i(x, s) - z_j(x, s) u_i(x, s)] [D_j (g_i * \Psi * \Psi)(x, s)] dx ds.
\end{aligned}$$

Proof. We can find $a < b$ such that $f(x, s) = 0$ if $s \notin [a, b]$. Recalling Definition 5.8 we set $g = -\operatorname{curl}(f * J)$ and obtain

$$\begin{aligned}
\operatorname{curl}(g) &= -\operatorname{curl}(\operatorname{curl}(f * J)) = \Delta(f * J) - \operatorname{grad}(\operatorname{div}(f * J)) \\
&= f - \operatorname{grad}((\operatorname{div}(f)) * J) = f.
\end{aligned}$$

If $s > 0$ then Definition 5.10, Definition 5.14, the fact that the function $h(x) = (f * \Psi)(x, s)$ is an element of W , and the argument in the proof of Lemma 5.13

yield

$$\begin{aligned}
& \int [(D_t w'_i(s))(x)] [f_i(x, s)] dx \\
&= \int [(D_t w_i(s))(x)] [(f_i * \Psi)(x, s)] dx \\
&= - \int [((w'_j(s) D_j (w'_i(s))) * \Psi)(x)] [(f_i * \Psi)(x, s)] dx \\
&\quad + \int [\Delta (w_i(s) * \Omega * \Omega)(x)] [(f_i * \Psi)(x, s)] dx \\
&= - \int [u_j(x, s) D_j z_i(x, s) - z_j(x, s) D_j u_i(x, s)] [(g_i * \Psi * \Psi)(x, s)] dx \\
&\quad + \int [\Delta u_i(x, s)] [(f_i * \Omega * \Omega)(x, s)] dx \\
&= \int [u_j(x, s) z_i(x, s) - z_j(x, s) u_i(x, s)] [D_j (g_i * \Psi * \Psi)(x, s)] dx \\
&\quad + \int [u_i(x, s)] [\Delta (f_i * \Omega * \Omega)(x, s)] dx.
\end{aligned}$$

The conclusion follows by integrating over s and using Lemma 5.6.

6.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$ be an infinite sequence of positive numbers with limit zero. We can find an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ of positive numbers such that $\alpha_n < \varepsilon_n$ and (5.1) is satisfied whenever $(\alpha, \varepsilon) = (\alpha_n, \varepsilon_n)$. For each n we can set $\alpha = \alpha_n$ and $\varepsilon = \varepsilon_n$ and use all of the definitions and results of Sect. 5. The corresponding functions u and z introduced in Definition 5.14 will be denoted by u^n and z^n [so that $z^n = \text{curl}(u^n)$]. From (5.4), Young's inequality $\|w(s) * \Psi\|_2 \leq \|w(s)\|_2 \|\Psi\|_1$, Lemma 5.17, and (5.5) we obtain

$$\int_{R^3} |u^n(x, t)|^2 dx \leq \|w^0\|_2^2 \quad \text{if } t \geq 0, \quad (6.1)$$

$$\int_0^\infty \int_{R^3} |u^n(x, t)|^{10/3} dx dt \leq C_8 \|w^0\|_2^{10/3}, \quad (6.2)$$

$$\int_0^\infty \int_{R^3} |Du^n(x, t)|^2 dx dt \leq (1/2) \|w^0\|_2^2. \quad (6.3)$$

By passing to a subsequence, we may assume that u^n converges weakly in $L^{10/3}$ to a function u and that Du^n converges weakly in L^2 to Du . Hence $\text{curl}(u^n)$ converges weakly in L^2 to $\text{curl}(u)$.

Definition 6.1. For every integer m we fix a countable set $A(m) \subset R^3$ with the following properties: (a) Every $x \in R^3$ is contained in the interior of $B(a, 2^{-m-3})$ for some $a \in A(m)$; (b) $\{B(a, 2^{-m-4}) : a \in A(m)\}$ is a collection of disjointed sets. We also set $B(m) = \{2^{-2m} + 2^{-2m-6}i : i \text{ is a positive integer}\}$. Recalling Theorems 4.2 and 5.12, we define $D(m) = \{(a, b) \in A(m) \times B(m) : \text{there exists an integer } N \text{ such that (6.4) is false for all } n \geq N\}$.

$$\begin{aligned}
& 2^m I(|Du^n|^2, K(a, b, 2^{-m+2}, 2^{-2m})) \\
& + 2^{-m} \left(\int_{b-2^{-2m}}^b \int_{R^3} |u^n(x, t)|^2 (|x-a| + 2^{-m})^{-4} dx dt \right) \leq C_5(C_7). \quad (6.4)
\end{aligned}$$

The proof of the next lemma is left to the reader.

Lemma 6.2. *There exists an absolute constant C_{10} with the following properties :*

$$\begin{aligned} \sum_{a \in A(m)} (|x-a| + 2^{-m})^{-4} &\leq C_{10} 2^{4m} \quad \text{for all } x \in R^3, \\ \sum_{(a,b) \in A(m) \times B(m)} \left(\int_{b-2^{-2m}}^b \int_{R^3} f(x,t) (|x-a| + 2^{-m})^{-4} dx dt \right) \\ &\leq C_{10} 2^{4m} I(f, R^3 \times R^+), \\ \sum_{(a,b) \in A(m) \times B(m)} I(f, K(a,b, 2^{-m+2}, 2^{-2m})) &\leq C_{10} I(f, R^3 \times R^+) \end{aligned}$$

for any integrable function $f \geq 0$.

Lemma 6.3. *If $m \geq 0$ then the number of elements in the set $D(m)$ is less than*

$$M = C_{11}^{-1} 2^{5m/3} C_{10} ((1/2) \|w^0\|_2^2 + C_9^{5/3} C_8 \|w^0\|_2^{10/3}),$$

where C_{11} is the minimum of $(1/2)C_5(C_7)$ and $((1/2)C_5(C_7))^{5/3}$.

Proof. If this is false then we can find k distinct elements in $D(m)$, where k is the smallest integer greater than or equal to M . We denote these elements by (a_i, b_i) , $i \in \{1, 2, \dots, k\}$. There exists an integer N such that (6.4) is false whenever $n \geq N$ and (a, b) is one of the (a_i, b_i) . This implies that we have either

$$2^m I(|Du^n|^2, K(a_i, b_i, 2^{-m+2}, 2^{-2m})) > (1/2)C_5(C_7) \quad (6.5)$$

or

$$2^{-m} \left(\int_{b_i-2^{-2m}}^{b_i} \int_{R^3} |u^n(x,t)|^2 (|x-a_i| + 2^{-m})^{-4} dx dt \right) > (1/2)C_5(C_7) \quad (6.6)$$

if $n \geq N$ and $1 \leq i \leq k$. Since (6.6) is equivalent to

$$\begin{aligned} 2^{-5m/3} \left(\int_{b_i-2^{-2m}}^{b_i} \int_{R^3} |u^n(x,t)|^2 (|x-a_i| + 2^{-m})^{-4} dx dt \right)^{5/3} \\ > ((1/2)C_5(C_7))^{5/3}, \end{aligned} \quad (6.7)$$

the sum of the left hand sides of (6.5) and (6.7) must exceed C_{11} for every $n \geq N$ and $1 \leq i \leq k$. Now Lemma 6.2, Lemma 5.17, (6.3), Lemma 6.2 again, $m \geq 0$, and (6.2) yield

$$\begin{aligned} C_{11} k &< \sum_{i=1}^k 2^m I(|Du^N|^2, K(a_i, b_i, 2^{-m+2}, 2^{-2m})) \\ &\quad + \sum_{i=1}^k 2^{-5m/3} \left(\int_{b_i-2^{-2m}}^{b_i} \int_{R^3} |u^N(x,t)|^2 (|x-a_i| + 2^{-m})^{-4} dx dt \right)^{5/3} \\ &\leq 2^m C_{10} I(|Du^N|^2, R^3 \times R^+) \\ &\quad + \sum_{i=1}^k 2^{-7m/3} C_9^{5/3} \left(\int_{b_i-2^{-2m}}^{b_i} \int_{R^3} |u^N(x,t)|^{10/3} (|x-a_i| + 2^{-m})^{-4} dx dt \right) \\ &\leq 2^m C_{10} (1/2) \|w^0\|_2^2 + 2^{-7m/3} C_9^{5/3} C_{10} 2^{4m} \left(\int_0^\infty \int_{R^3} |u^N(x,t)|^{10/3} dx dt \right) \\ &\leq 2^{5m/3} C_{10} (1/2) \|w^0\|_2^2 + 2^{5m/3} C_9^{5/3} C_{10} C_8 \|w^0\|_2^{10/3}. \end{aligned}$$

This contradicts the definition of k .

Lemma 6.4. *If $b \in B(m)$ then the number of elements in the set*

$$A = \{a \in A(m) : (a, b) \in D(m)\}$$

is less than

$$M' = (C_5(C_7))^{-1} 2^m C_{10} \|w^0\|_2^2 (3/2).$$

Proof. If this is false then we can find k distinct elements of A , where k is the smallest integer greater than or equal to M' . We denote these elements by a_i , $i \in \{1, 2, \dots, k\}$. There exists an integer N such that (6.4) is false whenever $n \geq N$, a is one of the a_i , and b in the hypothesis. Hence the first and third inequalities in Lemma 6.2, (6.3), and (6.1) yield

$$\begin{aligned} C_5(C_7)k &< \sum_{i=1}^k 2^m I(|Du^N|^2, K(a_i, b, 2^{-m+2}, 2^{-2m})) \\ &+ \sum_{i=1}^k 2^{-m} \left(\int_{b-2^{-2m}}^b \int_{R^3} |u^N(x, t)|^2 (|x - a_i| + 2^{-m})^{-4} dx dt \right) \\ &\leq 2^m C_{10} I(|Du^N|^2, R^3 \times R^+) + 2^{-m} C_{10} 2^{4m} \left(\int_{b-2^{-2m}}^b \int_{R^3} |u^N(x, t)|^2 dx dt \right) \\ &\leq 2^m C_{10} (1/2) \|w^0\|_2^2 + 2^{-m} C_{10} 2^{4m} 2^{-2m} \|w^0\|_2^2 = 2^m C_{10} \|w^0\|_2^2 (3/2). \end{aligned}$$

As before, this contradicts the definition of k .

Definition 6.5. For any nonempty subset B of $R^3 \times R$ we define

$$\text{diam}(B) = \sup \{(|a - c|^2 + |b - d|^2)^{1/2} : (a, b) \in B \text{ and } (c, d) \in B\}$$

and we set $\text{diam}(\phi) = 0$. Suppose $0 < d \leq 4$ and A is a subset of $R^3 \times R$. Then for every $\delta > 0$ we define $\phi_\delta^d(A)$ to be the infimum of all numbers of the form

$$\sum_{i=1}^{\infty} (\Gamma(1/2)^d / \Gamma((d/2) + 1)) 2^{-d} (\text{diam}(A_i))^d,$$

where A_i is a subset of $R^3 \times R$, $A \subset \bigcup_{i=1}^{\infty} A_i$, and $\text{diam}(A_i) \leq \delta$. Observe that $\phi_\delta^d(A) \geq \phi_\eta^d(A)$ if $\delta \leq \eta$. This allows us to define $H^d(A) = \lim_{\delta \rightarrow 0} \phi_\delta^d(A)$. $H^d(A)$ is called the d dimensional Hausdorff measure of A . For additional details, see [2].

Lemma 6.6. *There exist absolute constants C_{12} , C_{13} , and C_{14} with the following property: For every integer $k \geq 0$ there exists a compact set $S(k) \subset R^3 \times R^+$ such that $S(k) \subset S(k+1)$,*

- (a) $H^{5/3}(S(k)) \leq C_{12} \|w^0\|_2^2 + C_{13} \|w^0\|_2^{10/3}$,
- (b) $H^1(S(k) \cap (R^3 \times \{t\})) \leq C_{14} \|w^0\|_2^2$ if $t > 0$,
- (c) if $(c, d) \in (R^3 \times [2^{-2k}, \infty)) \sim S(k)$ then there exist an integer $m \geq k$ and $(a, b) \in A(m) \times B(m)$ such that (c, d) is in the interior of $K(a, b, 2^{-m-3}, 2^{-2m-5})$ and (6.4) is true for infinitely many n .

Proof. For every integer $m \geq 0$ we let $T(m) \subset R^3 \times R^+$ be the union of all the sets $K(a, b, 2^{-m-3}, 2^{-2m-5})$ such that $(a, b) \in D(m)$. Then we let $S(k)$ be the intersection of all $T(m)$ such that $m \geq k$. Lemma 6.3 implies that $D(m)$ is finite. Hence every $T(m)$ is compact, and this implies that $S(k)$ is compact. If $(c, d) \notin S(k)$ then $(c, d) \notin T(m)$ for some $m \geq k$. If, in addition, $(c, d) \in R^3 \times [2^{-2k}, \infty)$, then $d \geq 2^{-2k} \geq 2^{-2m}$ and hence there exists $(a, b) \in A(m) \times B(m)$ such that (c, d) is in the interior of

$K(a, b, 2^{-m-3}, 2^{-2m-5})$. Since $(c, d) \notin T(m)$, we must have $(a, b) \notin D(m)$. This proves (c). We have $\text{diam}(K(a, b, 2^{-m-3}, 2^{-2m-5})) < 2^{-m}$ when $m \geq 0$. If k is given then, for every $m \geq k$, $S(k)$ is covered by the sets $K(a, b, 2^{-m-3}, 2^{-2m-5})$ where $(a, b) \in D(m)$. Hence Lemma 6.3 yields

$$\begin{aligned} \phi_\delta^{5/3}(S(k)) &\leq \sum_{(a, b) \in D(m)} C(\text{diam}(K(a, b, 2^{-m-3}, 2^{-2m-5})))^{5/3} \\ &\leq C_{12} \|w^0\|_2^2 + C_{13} \|w^0\|_2^{10/3} \end{aligned} \quad (6.8)$$

whenever $\delta \geq 2^{-m}$. Taking the limit as $m \rightarrow \infty$, we obtain (a). Now suppose that $k \geq 0$ is an integer and $t > 0$. For every $m \geq k$, $T(m) \cap (R^3 \times \{t\})$ is covered by a collection of sets $K(a, b, 2^{-m-3}, 2^{-2m-5})$ such that $(a, b) \in D(m)$ and b takes at most three distinct values [since we can discard those sets $K(a, b, 2^{-m-3}, 2^{-2m-5})$ which do not intersect $R^3 \times \{t\}$]. Hence we can use the argument of (6.8) and Lemma 6.4 to conclude

$$\phi_\delta^1(S(k) \cap (R^3 \times \{t\})) \leq \phi_\delta^1(T(m) \cap (R^3 \times \{t\})) \leq C_{14} \|w^0\|_2^2$$

when $\delta \geq 2^{-m}$. We obtain (b) by taking the limit as $m \rightarrow \infty$.

Lemma 6.7. *Suppose $\beta > 0$ and V is an open subset of R^3 such that the closure of V is compact and the closure of V is contained in U . Then there exists an integer $k \geq 0$ and a function $z : (V \times [2^{-2k}, \infty)) \sim S(k) \rightarrow R^3$ such that*

$$2^{-2k} \leq \beta, \quad z(x, t) = (\text{curl}(u))(x, t)$$

for almost every (x, t) in the domain of z , and z is a continuous function.

Proof. We let δ be the distance between the closure of V and the complement of U . The number δ is positive because the closure of V is compact. There exists $R > 0$ such that $4R < \delta$. Now we can find an integer k large enough so that $k \geq 0$, $2^{-2k} \leq \beta$, $2^{-k} < R/2$, $4R + 2^{-k+1} < \delta$, and

$$2^{-2k} \|w^0\|_2 + 2^{-2k} \|w^0\|_2 R^{-7/2} + \|w^0\|_2 R^{-3/2} \leq C_5(C_7)2^k. \quad (6.9)$$

Suppose $(x, t) \in (V \times [2^{-2k}, \infty)) \sim S(k)$. Lemma 6.6 yields an integer $m \geq k$ and $(a, b) \in A(m) \times B(m)$ such that (x, t) is in the interior of $K(a, b, 2^{-m-3}, 2^{-2m-5})$ and (6.4) is true for infinitely many n . We can find an infinite sequence $n_1 < n_2 < n_3 < \dots$ such that (6.4) is true for every $n = n_i$ and, in addition, $4R + 2^{-k+1} + 4\epsilon_n < \delta$ for every $n = n_i$. We now establish the following claim: If n is one of the n_i , $(c, d) \in K(a, b, 2^{-m}, 2^{-2m})$, $0 < h \leq 2^{-m}$, and $d > h^2$ then

$$\begin{aligned} &C_7^{-1} |u^n(c, d)| \\ &\leq \int_{d-h^2}^d \int_{R^3} |u^n(y, s)|^2 (|y-c| + (d-s)^{1/2})^{-4} dy ds \\ &\quad + \int_{d-h^2}^d \int_{R^3} |u^n(y, s)| (|y-c| + h)^{-4} h^{-1} dy ds + C_5(C_7)2^m. \end{aligned} \quad (6.10)$$

If $|y - c| \leq 4R + 4\varepsilon_{n_i}$, then

$$\begin{aligned} |y - x| &\leq |y - c| + |c - a| + |a - x| \leq 4R + 4\varepsilon_{n_i} + 2^{-m} + 2^{-m-3} \\ &< 4R + 4\varepsilon_{n_i} + 2^{-k+1} < \delta. \end{aligned}$$

Since $x \in V$, we conclude $y \in U$. In particular, we have $c \in U$. In addition, we have $0 < h \leq 2^{-m} \leq 2^{-k} < R/2$. Now we can use $k \leq m$ and Theorem 5.12 to conclude (6.10). We use (6.1), (6.3), and Theorem 4.2 with $H = C_7$ and $\Gamma = C_5(C_7)2^m$ to conclude

$$|u^n(c, d)| \leq C_6(C_7)2^m \text{ if } (c, d) \in K(a, b, 2^{-m-2}, 2^{-2m-4}) \text{ and } n = n_i \text{ for some } i. \quad (6.11)$$

Since (6.4) is true for all $n = n_i$, the collection of numbers

$$\{I(|Du^n|^2, K(a, b, 2^{-m-2}, 2^{-2m-4})) : n = n_i \text{ for some } i\}$$

is bounded. Combining this with (6.11) and Theorem 5.16 we conclude that the collection of functions $\{\text{curl}(u^n) : n = n_i \text{ for some } i \text{ and } \varepsilon_n < (2^{-m-2})/64\}$ is uniformly bounded and equicontinuous on $K(a, b, 2^{-m-3}, 2^{-2m-5})$. By Ascoli's theorem, a subsequence converges uniformly to a continuous function. This limit function must coincide almost everywhere with the weak limit $\text{curl}(u)$ of the original sequence $\text{curl}(u^n)$. This means that we can make $\text{curl}(u)$ continuous on the interior of $K(a, b, 2^{-m-3}, 2^{-2m-5})$ by redefining it on a set of Lebesgue measure zero. The conclusion of the lemma follows easily.

Theorem 6.8. *There exists a set $S \subset U \times \mathbb{R}^+$ such that*

- (a) *if K is a compact subset of $U \times \mathbb{R}^+$ then $K \cap S$ is compact,*
- (b) *the function $\text{curl}(u)$ can be modified on a set of Lebesgue measure zero so that it becomes continuous on $(U \times \mathbb{R}^+) \sim S$,*
- (c) $H^{5/3}(S) \leq C_{12} \|w^0\|_2^2 + C_{13} \|w^0\|_2^{10/3}$,
- (d) $H^1(S \cap (\mathbb{R}^3 \times \{t\})) \leq C_{14} \|w^0\|_2^2$ if $t > 0$.

Proof. We construct an infinite sequence V_1, V_2, V_3, \dots of open subsets of U such that $\text{closure}(V_n) \subset V_{n+1}$, $\text{closure}(V_n)$ is compact, and the union of the V_n is equal to U . We will use the abbreviation $2^{-2k} = b(k)$. Using Lemma 6.7 we obtain an increasing sequence $k_1 < k_2 < k_3 < \dots$ of nonnegative integers such that the restriction of $\text{curl}(u)$ to $(V_n \times [b(k_n), \infty)) \sim S(k_n)$ is equal almost everywhere to a continuous function. We define

$$S = S(k_1) \cup \left(\bigcup_{n=1}^{\infty} (S(k_{n+1}) \sim (V_n \times (b(k_n), \infty))) \right).$$

If K is a compact subset of $U \times \mathbb{R}^+$ then there exists an integer N such that $K \subset V_n \times (b(k_n), \infty)$ for all $n > N$. Hence

$$S \cap K = (S(k_1) \cap K) \cup \left(\bigcup_{n=1}^N ((S(k_{n+1}) \sim (V_n \times (b(k_n), \infty))) \cap K) \right)$$

which is a union of finitely many compact sets. This proves (a). Now suppose $(a, b) \in (U \times \mathbb{R}^+) \sim S$. We have two possibilities: (1) $(a, b) \in V_1 \times (b(k_1), \infty)$; (2) there exists n such that $(a, b) \in V_{n+1} \times (b(k_{n+1}), \infty)$ but $(a, b) \notin V_n \times (b(k_n), \infty)$. If (1) holds then $(a, b) \in (V_1 \times (b(k_1), \infty)) \sim S(k_1)$. If (2) holds then $(a, b) \in (V_{n+1} \times (b(k_{n+1}), \infty)) \sim S(k_{n+1})$ because $(a, b) \notin S$ implies $(a, b) \notin S(k_{n+1}) \sim (V_n \times (b(k_n), \infty))$, and combining this with (2) yields $(a, b) \notin S(k_{n+1})$. In either case, (a, b) is contained in an open set on which $\text{curl}(u)$ coincides almost everywhere with a continuous function. This proves (b). Since Hausdorff measure is a Borel measure and $S(k) \subset S(k+1)$, we obtain (c) and (d) from Lemma 6.6 and the following:

$$\begin{aligned} H^{5/3}(S) &\leq H^{5/3}\left(\bigcup_{n=1}^{\infty} S(k_n)\right) = \lim_{n \rightarrow \infty} H^{5/3}(S(k_n)) \\ &\leq C_{12} \|w^0\|_2^2 + C_{13} \|w^0\|_2^{10/3}, \\ H^1(S \cap (\mathbb{R}^3 \times \{t\})) &\leq H^1\left(\left(\bigcup_{n=1}^{\infty} S(k_n)\right) \cap (\mathbb{R}^3 \times \{t\})\right) \\ &= \lim_{n \rightarrow \infty} H^1(S(k_n) \cap (\mathbb{R}^3 \times \{t\})) \leq C_{14} \|w^0\|_2^2 \quad \text{if } t > 0. \end{aligned}$$

Definition 6.9. For every integer m and every positive integer n we set $E(m, n) = \{b \in B(m) : (6.4) \text{ is false for some } a \in A(m)\}$.

Lemma 6.10. *The number of elements in the set $E(m, n)$ is less than the number M described in Lemma 6.3.*

Proof. If this is false then we can find k distinct elements in $E(m, n)$, where k is the smallest integer greater than or equal to M . We denote these elements by b_i , $i \in \{1, 2, \dots, k\}$. For each i , there exists $a_i \in A(m)$ such that (6.4) is false when (a, b) is replaced by (a_i, b_i) . Carrying out the argument in the proof of Lemma 6.3 (the only difference being that we are now dealing with a single n , which only simplifies things), we obtain a contradiction.

Lemma 6.11. *If V is an open subset of \mathbb{R}^3 such that the closure of V is compact and the closure of V is contained in U , $\eta > 0$, and $0 < \beta < T < \infty$ then there exists a closed set $F \subset \mathbb{R}^+$ such that the Lebesgue measure of F is at most η and a subsequence of the sequence $\text{curl}(u^n)$ converges uniformly on $V \times ((\beta, T) \sim F)$.*

Proof. We let $\delta > 0$ be the distance between the closure of V and the complement of U . We find $R > 0$ such that $4R < \delta$ and we find an integer m large enough so that $m \geq 0$, $2^{-2m} \leq \beta$, $2^{-m} < R/2$, $4R + 2^{-m+1} < \delta$,

$$2^{-2m} \|w^0\|_2 + 2^{-2m} \|w^0\|_2 R^{-7/2} + \|w^0\|_2 R^{-3/2} \leq C_5 (C_7)^{2m},$$

and (see Lemma 6.3)

$$2^{-2m-5} C_{11}^{-1} 2^{5m/3} C_{10} ((1/2) \|w^0\|_2^2 + C_9^{5/3} C_8 \|w^0\|_2^{10/3}) \leq \eta. \quad (6.12)$$

We let $E'(m, n) = \{b \in E(m, n) : b \leq T + 2^{-2m}\}$. Since $\{b \in B(m) : b \leq T + 2^{-2m}\}$ is a finite set which contains $E'(m, n)$ for every n , we conclude that, as n varies, $E'(m, n)$ can only be one of a finite number of different sets. Hence there exists an infinite

sequence $n_1 < n_2 < n_3 < \dots$ of positive integers such that

$$E'(m, n_1) = E'(m, n_2) = E'(m, n_3) = \dots$$

Setting $E' = E'(m, n_i)$, we define F to be the union of the intervals $[b - 2^{-2m-5}, b]$ for $b \in E'$. From Lemma 6.10 and (6.12) we conclude that the Lebesgue measure of F is at most η . If $(x, t) \in V \times ((\beta, T) \sim F)$ then (x, t) is contained in the interior of $K(a, b, 2^{-m-3}, 2^{-2m-5})$ for some $(a, b) \in A(m) \times B(m)$. If $b \in E'$ then $t \in [b - 2^{-2m-5}, b] \subset F$, which contradicts the choice of (x, t) . Hence we conclude $b \notin E'$. From $b - 2^{-2m-5} < t < T$ we obtain $b < T + 2^{-2m}$. This implies that $b \notin E(m, n_i)$ for all i and, in particular, that (6.4) is true for every $n = n_i$. Since $V \times ((\beta, T) \sim F)$ is bounded, we obtain that $V \times ((\beta, T) \sim F)$ is covered by the interiors of a finite collection of sets of the form $K(a, b, 2^{-m-3}, 2^{-2m-5})$ such that $(a, b) \in A(m) \times B(m)$ and (6.4) is true for every $n = n_i$. The rest follows from the argument in the proof of Lemma 6.7.

Lemma 6.12. *If f is a C^∞ function with compact support from $R^3 \times R$ into R^3 such that $\operatorname{div}(f) = 0$ and the support of f is contained in $U \times R$ then there exist real numbers $a < b$ and a C^∞ function $g: R^3 \times R \rightarrow R^3$ such that $\operatorname{curl}(g) = f$, $g(x, s) = 0$ if $s \notin [a, b]$, and*

$$\begin{aligned} & - \int_{R^3} w_i^0(x) f_i(x, 0) dx \\ & = \int_0^\infty \int_{R^3} [u_i(x, s)] [D_t f_i(x, s) + \Delta f_i(x, s)] dx ds \\ & \quad + \int_0^\infty \int_{R^3} [u_j(x, s)] [(\operatorname{curl}(u))_i(x, s)] [D_j g_i(x, s)] dx ds \\ & \quad + \int_0^\infty \int_{R^3} -[(\operatorname{curl}(u))_j(x, s)] [u_i(x, s)] [D_j g_i(x, s)] dx ds. \end{aligned} \quad (6.13)$$

Proof. The a, b, g found in the proof of Lemma 5.19 depend only on f . Therefore we can use Lemma 5.19, Definition 5.1, and Definition 5.2 to find real numbers $a < b$ and a C^∞ function $g: R^3 \times R \rightarrow R^3$ such that $\operatorname{curl}(g) = f$, $g(x, s) = 0$ if $s \notin [a, b]$, and

$$\begin{aligned} & - \int_{R^3} [w_i^0(x)] [(f_i * \theta'_{\varepsilon_n} * \theta'_{\alpha_n})(x, 0)] dx \\ & = \int_0^\infty \int_{R^3} [u_i^n(x, s)] [D_t f_i(x, s) + \Delta (f_i * \theta'_{\alpha_n} * \theta'_{\varepsilon_n})(x, s)] dx ds \\ & \quad + \int_0^\infty \int_{R^3} [u_j^n(x, s) z_i^n(x, s)] [D_j (g_i * \theta'_{\varepsilon_n} * \theta'_{\alpha_n} * \theta'_{\varepsilon_n} * \theta'_{\alpha_n})(x, s)] dx ds \\ & \quad + \int_0^\infty \int_{R^3} -[z_j^n(x, s) u_i^n(x, s)] [D_j (g_i * \theta'_{\varepsilon_n} * \theta'_{\alpha_n} * \theta'_{\varepsilon_n} * \theta'_{\alpha_n})(x, s)] dx ds \end{aligned} \quad (6.14)$$

is satisfied for sufficiently large n . Since f is C^∞ with compact support and u^n converges to u weakly in $L^{1/3}$, the first two integrals in (6.14) converge to the first

two integrals in (6.13) as $n \rightarrow \infty$. To complete the proof we have to show that for every $\varepsilon > 0$ and every positive integer N there exists an integer $n \geq N$ such that the sum of the last two integrals of (6.14) is within ε of the sum of the last two integrals of (6.13). Suppose ε and N are given. We set $\Phi_n = \theta'_{\varepsilon_n} * \theta'_{\alpha_n} * \theta'_{\varepsilon_n} * \theta'_{\alpha_n}$ and use the boundedness of U and the fact that $g(x, s) = 0$ for $s \notin [a, b]$ to find a positive number M such that $|D_j(g_i * \Phi_n)(x, s)| \leq M$ and $|D_j g_i(x, s)| \leq M$ if $x \in U$. We may as well assume that $b > 0$ [otherwise the integrals in (6.13) are all zero]. Letting $m(U)$ be the Lebesgue measure of U , we find $\eta > 0$ such that

$$(36) M C_8^{3/10} \|w^0\|_2^2 (\eta b + 2\eta(m(U)))^{1/5} (1/2)^{1/2} < \varepsilon/4. \quad (6.15)$$

Since U is bounded, we can find an open subset V of R^3 such that the closure of V is compact, the closure of V is contained in U , and the Lebesgue measure of $U \sim V$ is at most η . Let β be a positive number such that $\beta < \eta$ and $\beta < b$. Setting $T = b$, we can use Lemma 6.11 to find a closed set $F \subset R^+$ such that the Lebesgue measure of F is at most η and there is an infinite sequence $n_1 < n_2 < n_3 < \dots$ of positive integers such that $N \leq n_1$ and the subsequence

$$(z^n = \text{curl}(u^n) : n = n_1, n_2, n_3, \dots)$$

converges uniformly on $V \times ((\beta, T) \sim F)$. We set $E = V \times ((\beta, T) \sim F)$ and let $h: R^3 \times R^+ \rightarrow R$ be the function defined by $h(x, s) = 1$ if $(x, s) \in E$ and $h(x, s) = 0$ otherwise. Setting $z = \text{curl}(u)$, we conclude from the boundedness of E that the subsequences

$$(h(x, s)z_i^n(x, s)D_j(g_i * \Phi_n)(x, s) : n = n_1, n_2, n_3, \dots)$$

and

$$(h(x, s)z_j^n(x, s)D_j(g_i * \Phi_n)(x, s) : n = n_1, n_2, n_3, \dots)$$

converge in $L^{10/7}$ norm to $h(x, s)z_i(x, s)D_j g_i(x, s)$ and $h(x, s)z_j(x, s)D_j g_i(x, s)$. Now the weak convergence in $L^{10/3}$ of u^n to u yields that a subsequence of

$$\int_E [u_j^n(x, s)z_i^n(x, s) - z_j^n(x, s)u_i^n(x, s)] [D_j(g_i * \Phi_n)(x, s)] dx ds \quad (6.16)$$

converges to

$$\int_E [u_j(x, s)z_i(x, s) - z_j(x, s)u_i(x, s)] [D_j g_i(x, s)] dx ds. \quad (6.17)$$

In particular, we can find arbitrarily large n such that (6.16) and (6.17) differ by no more than $\varepsilon/2$. The proof will be completed by setting $G = (R^3 \times [0, T]) \sim E$ and showing that

$$\left| \int_G [u_j^n(x, s)z_i^n(x, s) - z_j^n(x, s)u_i^n(x, s)] [D_j(g_i * \Phi_n)(x, s)] dx ds \right| \leq \varepsilon/4, \quad (6.18)$$

$$\left| \int_G [u_j(x, s)z_i(x, s) - z_j(x, s)u_i(x, s)] [D_j g_i(x, s)] dx ds \right| \leq \varepsilon/4 \quad (6.19)$$

hold for sufficiently large n . Since $T = b$, the Lebesgue measure of $(U \times [0, T]) \sim E$ is at most the sum of the Lebesgue measure of $(U \sim V) \times [0, b]$ and the Lebesgue measure of $U \times ([0, \beta] \cup F)$. Hence the Lebesgue measure of $(U \times [0, T]) \sim E$ is at most $\eta b + 2\eta(m(U))$. Since, by hypothesis, the boundary of U has Lebesgue measure zero, we conclude that the Lebesgue measure of $(\text{closure}(U) \times [0, T]) \sim E$ is at most $\eta b + 2\eta(m(U))$. Let K_n be the collection of all $x \in R^3$ such that the distance between x and $\text{closure}(U)$ is at most $2\varepsilon_n$, set $G_n = (K_n \times [0, T]) \sim E$, and let a_n be the Lebesgue measure of G_n . The above and the boundedness of U yield

$\lim_{n \rightarrow \infty} a_n \leq \eta b + 2\eta(m(U))$. Hence we can use (6.15) and find a positive integer N' such that

$$(36)MC_8^{3/10} \|w^0\|_2^2 (a_n)^{1/5} (1/2)^{1/2} < \varepsilon/4 \quad \text{if } n \geq N'. \tag{6.20}$$

Using Hölder's inequality, (6.2), and (6.3) we obtain

$$\begin{aligned} & \int_{G_n} |u_j^n(x, s) z_i^n(x, s)| dx ds \\ & \leq \left(\int_{G_n} |u_j^n(x, s)|^2 dx ds \right)^{1/2} \left(\int_{G_n} |z_i^n(x, s)|^2 dx ds \right)^{1/2} \\ & \leq \left(\int_{G_n} |u_j^n(x, s)|^{10/3} dx ds \right)^{3/10} (a_n)^{1/5} (2) \left(\int_{G_n} |Du^n(x, s)|^2 dx ds \right)^{1/2} \\ & \leq C_8^{3/10} \|w^0\|_2 (a_n)^{1/5} (2) (1/2)^{1/2} \|w^0\|_2. \end{aligned}$$

The same argument and the weak convergence of u^n and Du^n yield

$$\int_{G_n} |u_j(x, s) z_i(x, s)| dx ds \leq C_8^{3/10} \|w^0\|_2^2 (a_n)^{1/5} (2) (1/2)^{1/2}.$$

If n is sufficiently large then (6.18) and (6.19) follow from the above, (6.20), and the fact that u^n and z^n are zero outside of $K_n \times R^+$. This completes the proof of the lemma.

Now we can tie these results together and obtain the conclusion of Theorem 1.2. The construction of u yields that $u(x, t) = 0$ for almost every $(x, t) \notin \text{closure}(U) \times R^+$. Part (1) follows because we assume that the boundary of U has Lebesgue measure zero. Part (2) also follows easily from the construction. Part (3) follows from Lemma 6.12 and integration by parts in the last two integrals of (6.13) [using the fact $\text{div}(u) = 0$]. The remainder follows from Theorem 6.8, (6.2), (6.3), and the definition of u .

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