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Surface Tension and Phase Transition for Lattice Systems

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Abstract. We introduce the surface tension for arbitrary spin systems and study its general properties. In particular we show that for a large class of systems, the surface tension is zero at high temperature. We also derive a geometrical condition for the surface tension to be zero at all temperature. For discrete spin systems this condition becomes a criterion to establish the existence of a phase transition associated with surface tension. This criterion is illustrated on several examples.

1. Introduction

The phenomenon of "phase transition" is one of the important problems of statistical mechanics because of its physical and mathematical interest. As is well known several definitions have been introduced to discuss the existence of phase transitions and the equivalence of these definitions has not always been established. It seems possible however to classify all phase transitions into two classes, those which occur with a spontaneous breakdown of the symmetry group of the system (coexistence of several phases, existence of local order parameters) and those which occur without any symmetry breakdown[1].

One of the standard methods to prove the existence of a phase transition for lattice systems is the "Peierls argument"; its generality relies on the fact that it takes explicitly into account the underlying group structure of the system [1, 7]. However, it is well adapted for systems which have a complete breakdown of the internal symmetry group at low temperature and does not apply as readily to describe phase transition associated with partial symmetry breakdown.

In this article, we propose to introduce *the surface tension as definition of phase transition*, i.e., we shall say that "there exists a phase transition associated with a

1 In such cases we know that there exist phase transitions associated with the coexistence of several phases and the existence of a local order parameter; however it is *not* always so: there exist models for which the Gibbs state is unique at all temperatures and which do exhibit a phase transition; there exist also models which show a phase transition without any local order parameter

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surface tension $\tau^{(12)}$ between the phases ω^1 and ω^2" if there exists a temperature T_c such that

$$\tau^{(12)}=0 \quad \text{if} \quad T>T_c$$

and

$$\tau^{(12)}\neq 0, \quad \text{or not defined, if} \quad T<T_c.$$

We therefore study general properties of the surface tension and show that for a large class of lattice systems of arbitrary spin the surface tension is zero at high temperature; we obtain also a condition for the surface tension to be zero at all temperatures; it is then shown that there exists a phase transition associated with surface tension if this condition is not satisfied. We have thus a criterion to prove the existence or absence of such phase transition and we shall illustrate this criterion on several examples.

It should be noticed that using a duality transformation [1] the surface tension is related to the expectation value of a product of characters on the dual model; therefore our definition of phase transition appears as the "dual" of the definition introduced in Gauge theory [2].

Let us also recall that since the original work of Onsager [3] the surface tension has been extensively studied especially for the case of the 2-dimensional Ising model [4, 5]; more recently the existence of the surface tension has been established for general ferromagnetic spin$\frac{1}{2}$ systems [6].

In Sect. 2 we recall the standard definitions of arbitrary spin systems and introduce the groups necessary for the High and Low Temperature expansions.

In Sect. 3 we define the surface tension $\tau^{(12)}$ between two phases ω^1 and ω^2 and we give bounds valid for any ω^1, ω^2; these bounds yield then a condition for the surface tension to be zero at all temperatures; furthermore the surface tension is non negative for ferromagnetic systems. In the cases where ω^1 and ω^2 are related by internal symmetries the surface tension appears as the expectation value of an observable associated with the surface of separation and we briefly discuss the problem of the existence of surface tension; it follows in particular that for ferromagnetic spin$\frac{1}{2}$ systems the conditions introduced in [6] to prove the existence of the surface tension are not necessary since this theorem can be obtained without duality transformation.

In Sects. 4 and 5 we study the surface tension respectively in the High and Low Temperature domains; it is shown that for a large class of systems the surface tension is zero at high temperature; in the low temperature domain we consider only discrete spin systems and we derive a geometric criterion for the existence of a phase transition associated with surface tension.

This criterion is illustrated in Sect. 6 on several examples; these examples indicate that a surface tension might appear in all cases where there is a coexistence of different phases.

2. Lattice Systems: Definitions and Expansions

2.1. Definitions and Notations

A "classical spin system" on a lattice can always be defined by the following structure [1] (see examples of Sect. 6):

Let $\mathscr{L}$ be countable set of points in $\mathbb{R}^\nu$, called „*sites*"; with every site x in $\mathscr{L}$ is associated a variable θ_x which belongs to a *Locally Compact Abelian Group* $\mathscr{G}_x$ and $d\theta_x$ will denote the Haar measure on $\mathscr{G}_x$. The configuration group $\mathscr{G}_{\mathscr{L}}$ is defined by

$$\mathscr{G}_{\mathscr{L}} = \prod_{x\in\mathscr{L}} \mathscr{G}_x = \{\boldsymbol{\theta} = (\theta_x); \theta_x \in \mathscr{G}_x\}. \tag{1}$$

For any $M \subset \mathscr{L}$ and $\boldsymbol{\theta} \in \mathscr{G}_{\mathscr{L}}$ we introduce the notation:

$$\boldsymbol{\theta}_M = \begin{cases} \theta_x & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases} \tag{2}$$

$$M^c = \mathscr{L} \backslash M$$

$$|M| = \text{cardinality of } M.$$

The interactions are described by means of a countable set $\mathscr{B}$ of indices called "*bonds*"; with every bond b in $\mathscr{B}$ is associated a variable ϑ_b which belongs to a locally compact abelian group[2] $\mathscr{G}_b^\wedge$, a continuous homeomorphism $\gamma_b : \boldsymbol{\theta} \mapsto \gamma_b(\boldsymbol{\theta})$ of $\mathscr{G}_{\mathscr{L}}$ onto $\mathscr{G}_b^\wedge$, and a real function $V_b = V_b[\vartheta_b]$ on $\mathscr{G}_b^\wedge$.

We shall say that the interactions have "*finite range*" if the homeomorphisms γ_b are such

$$\sup_b |\{x; \gamma_b(\theta_x) \neq 0\}| < \infty$$

and

$$\sup_x |\{b; \gamma_b(\theta_x) \neq 0\}| < \infty.$$

The system is a "*discrete spin system*", respectively a "*finite spin system*", if $\mathscr{G}_x$ is discrete, respectively finite, for all x in $\mathscr{L}$. The system has " "*finite density of lattice site*" if $\sup_{V \subset \mathbb{R}^\nu} \frac{1}{|V|} |\mathscr{L} \cap V| < \infty$.

We associate with every bond b in $\mathscr{B}$, the subset B_b of $\mathscr{L}$ defined by:

$$B_b = \{x \in \mathscr{L}; \exists \boldsymbol{\theta} \in \mathscr{G}_{\mathscr{L}} \text{ s.t. } \gamma_b(\boldsymbol{\theta}_x) \neq 0\}. \tag{3}$$

Note that B_b is the base of the cylindrical function on $\mathscr{G}_{\mathscr{L}}$ defined by

$$V_b(\boldsymbol{\theta}) = V_b[\gamma_b(\boldsymbol{\theta})] \tag{4}$$

which describes an interaction between the "spins" situated on B_b.

With the mapping $b \mapsto B_b$ the set $\mathscr{B}$ of bonds becomes a graph: two bonds b_1 and b_2 are connected if $B_{b_1} \cap B_{b_2} \neq \phi$.

For any $\Lambda \subsetneqq \mathscr{L}$ we introduce:

The "configuration group for Λ": $\mathscr{G}_\Lambda = \prod_{x\in\Lambda} \mathscr{G}_x = \{\boldsymbol{\theta}_\Lambda = (\theta_x)_{x\in\Lambda}\}$.

The "bonds intersecting Λ": $\mathscr{B}_\Lambda = \{b \in \mathscr{B}; B_b \cap \Lambda \neq \phi\}$.

The "group of graphs for Λ": $\mathscr{G}_{\mathscr{B}_\Lambda} = \prod_{b\in\mathscr{B}_\Lambda} \mathscr{G}_b = \{\mathbf{l} = (l_b)_{b\in\mathscr{B}_\Lambda}\}$ where $\mathscr{G}_b$ denotes the dual group[3] of $\mathscr{G}_b^\wedge$.

2 In most application $\mathscr{G}_b^\wedge$ is isomorphic to a subgroup of $\prod_{x\in B_b} \mathscr{G}_x$ where $B_b \subset \mathscr{L}$ is defined by Eq. (3).

3 If $\mathscr{G}_x$ is compact for all x in $\mathscr{L}$, then $\mathscr{G}_b$ is isomorphic to $\mathbb{Z}_{\alpha_b}$ the group of integers modulo $\alpha_b (\alpha_b \leqq \infty)$. Since we shall discuss mostly compact groups we shall write $\sum_{l_b\in\mathscr{G}_b}$ instead of $\int_{\mathscr{G}_b} dl_b$

$\gamma:\boldsymbol{\theta}\mapsto\gamma(\boldsymbol{\theta})$ homeomorphism of $\mathscr{G}_{\mathscr{L}}$ into $\mathscr{G}_{\mathscr{B}}^{\hat{}}=\prod_{b\in\mathscr{B}}\mathscr{G}_b^{\hat{}}$ defined by $[\gamma(\boldsymbol{\theta})]_b=\gamma_b(\boldsymbol{\theta})$.

For the "*finite system* $\Lambda\subset\mathscr{L}$ *with boundary condition* $\boldsymbol{\theta}^0$ *in* $\mathscr{G}_{\mathscr{L}}$", the Hamiltonian $H_{\Lambda,\boldsymbol{\theta}^0}$ is the function on $\mathscr{G}_\Lambda$ defined by:

$$-\frac{1}{kT}H_{\Lambda,\boldsymbol{\theta}^0}(\boldsymbol{\theta}_\Lambda)=\sum_{b\in\mathscr{B}_\Lambda}V_b[\gamma_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^0_{\Lambda^c})]=\sum_{b\in\mathscr{B}_\Lambda}V_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^0_{\Lambda^c}),$$

where T=temperature and the Partition Function $Z(\Lambda;\boldsymbol{\theta}^0)$ is defined by:

$$Z(\Lambda;\boldsymbol{\theta}^0)=\int_{\mathscr{G}_\Lambda}d\theta_\Lambda\prod_{b\in\mathscr{B}_\Lambda}\exp V_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^0_{\Lambda^c}). \tag{5}$$

2.2. *Expansions*

The "*High-Temperature (H.T.) Expansion*" is an expansion expressed in terms of the subgroup $\mathscr{K}_\Lambda$ of $\mathscr{G}_{\mathscr{B}_\Lambda}$ which is the "*group of closed graphs*" (or H.T. group), [1]:

$$\mathscr{K}_\Lambda=\left\{\mathbf{k}=(k_b)\in\mathscr{G}_{\mathscr{B}_\Lambda};\ \prod_b\langle\gamma_b(\boldsymbol{\theta});k_b\rangle=1\ \forall\boldsymbol{\theta}\in\mathscr{G}_\Lambda\right\} \tag{6}$$

and we have [1],

$$Z(\Lambda;\boldsymbol{\theta}^0)=\sum_{\mathbf{k}\in\mathscr{K}_\Lambda}\prod_{b\in\mathscr{B}_\Lambda}f(b;k_b)\langle\gamma_b(\boldsymbol{\theta}^0);k_b\rangle, \tag{7}$$

where:

$$f(b;l_b)=\int_{\mathscr{G}_b^{\hat{}}}d\vartheta_b\langle l_b;-\vartheta_b\rangle e^{V_b[\vartheta_b]}$$

i.e.

$$f(b;\cdot)=\mathscr{F}(e^{V_b[\cdot]})\qquad(=\text{Fourier Transform}) \tag{8}$$

and:

$$e^{V_b[\vartheta_b]}=\sum_{l_b\in\mathscr{G}_b}\langle l_b;\vartheta_b\rangle f(b;l_b).$$

In the following we shall assume that the interactions are such that $e^{V_b[\vartheta_b]}\in\mathscr{L}^1(\mathscr{G}_b^{\hat{}})$ and $f(b;l_b)\in\mathscr{L}^1(\mathscr{G}_b)$.

Another expansion which we also need in the high temperature domain is the "*Cluster Expansion*" defined in the following manner:

Let

$$\mathscr{B}^1_\Lambda=\{B=B_b\cap\Lambda;\ b\in\mathscr{B}_\Lambda\}\subset\mathscr{L} \tag{9}$$

and

$$H_{\Lambda,\boldsymbol{\theta}^0}(\boldsymbol{\theta}_\Lambda)=-\sum_{B\in\mathscr{B}^1_\Lambda}\phi_B(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^0_{\Lambda^c}),$$

where

$$\phi_B(\boldsymbol{\theta})=kT\sum_{\substack{b\in\mathscr{B}_\Lambda\\ B_b\cap\Lambda=B}}V_b(\boldsymbol{\theta})$$

then

$$Z(\Lambda;\boldsymbol{\theta}^0)=\sum_{\substack{\{\beta_1,\ldots,\beta_q\}\\ \beta_i\,\text{connected}\\ \beta_i\subset\mathscr{B}^1_\Lambda\\ [\beta_i]\cap[\beta_j]=\phi}}\prod_i \psi_{(\Lambda;\boldsymbol{\theta}^0)}(\beta_i), \tag{10}$$

where

$$[\beta]=\bigcup_{B\in\beta} B\subset\mathscr{L} \qquad (=\text{set of sites covered by } \beta)$$

and

$$\psi_{(\Lambda;\boldsymbol{\theta}^0)}(\beta)=\int_{\mathscr{G}_\Lambda}\prod_{x\in\Lambda} d\tilde{v}_x\prod_{B\in\beta}\left[e^{\frac{1}{kT}\phi_B(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^0_{\Lambda^c})}-1\right]. \tag{11}$$

In this approach $\mathscr{G}_x$ could be *any* measurable space (not necessarily a group) and $d\tilde{v}_x$ *any* probability measure on $\mathscr{G}_x$ (which could also depend on some physical parameter such as the temperature).

3. Surface Tension: Definition and General Properties

3.1. Definition

In thermodynamics the surface tension $\tau^{(12)}$ between the pure phases (1) and (2) is introduced as

$$F=F^{(1)}+F^{(2)}+A\tau^{(12)},$$

where $F^{(1)}, F^{(2)}$ are the free energies of the phases (1) and (2), F is the free energy of the mixture and A is the area of the boundary between the two phases.

By analogy we consider the following definition of the surface tension for lattice systems; which it is the direct generalization of the surface tension, introduced for spin$\frac{1}{2}$ [4]:

Let ω^1 and ω^2 be two states defined respectively by the boundary conditions $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2$ in $\mathscr{G}_{\mathscr{L}}$; to define the "mixture ω^{12}" of these two states we decompose $\mathscr{L}$ into $\mathscr{L}^u$ and $\mathscr{L}^d$

$$\mathscr{L}^u=\{x\in\mathscr{L};\, x_\nu>0\}$$
$$\mathscr{L}^d=\mathscr{L}\backslash\mathscr{L}^u$$

and we consider the finite system Λ with boundary condition $\boldsymbol{\theta}^{12}\in\mathscr{G}_{\mathscr{L}}$ where Λ is a parallelipiped with sides $(L_1, \ldots, L_{\nu-1}, 2M)$, symmetric with respect to the plane $x_\nu=0$, and $\boldsymbol{\theta}^{12}$ is the configuration which coincide with $\boldsymbol{\theta}^1$ in the upper half of $\mathscr{L}$ and coincide with $\boldsymbol{\theta}^2$ in the lower half, i.e.

$$\boldsymbol{\theta}^{12}=\begin{cases}\theta^1_x & \text{if} \quad x\in\mathscr{L}^u\\ \theta^2_x & \text{if} \quad x\in\mathscr{L}^d.\end{cases}$$

The *surface tension* τ^{12} between the states ω^1 and ω^2 is then defined as:

$$\tau^{12}=\lim_{L_i\to\infty}\lim_{M\to\infty}\frac{(-1)}{\prod L_i}\frac{1}{2}\left\{\mathrm{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}+\mathrm{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^2)}\right\}. \tag{12}$$

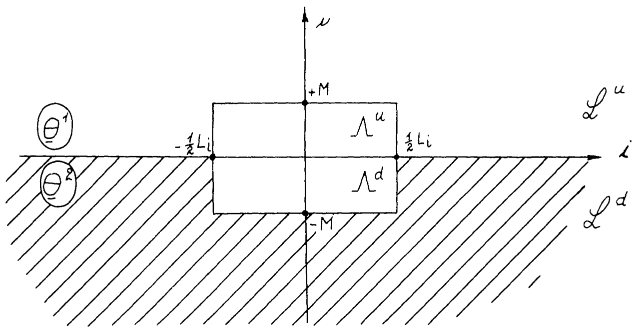

Fig. 1. Definition of surface tension

From Eqs. (4) and (5) it follows that

$$Z(\Lambda;\boldsymbol{\theta}^{12}) = \int_{\mathscr{G}_\Lambda} d\boldsymbol{\theta}_\Lambda \prod_{b\in\mathscr{B}_\Lambda} e^{V_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^i_{\Lambda^c})} \prod_{b\in\mathscr{B}_\Lambda}$$

$$\cdot \exp\{V_b[\gamma_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^i_{\Lambda^c})+\gamma_b(\boldsymbol{\theta}^{12}-\boldsymbol{\theta}^i_{\Lambda^c})] - V_b[\gamma_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^i_{\Lambda^c})]\}, \qquad i=1,2.$$

Let us then introduce, as it is usual in the study of symmetric states, the function $\mu_{(\Lambda;\boldsymbol{\theta}^i)}$ on $\mathscr{G}^{\wedge}_{\mathscr{B}_\Lambda} = \prod_{b\in\mathscr{B}_\Lambda} \mathscr{G}^{\wedge}_b$ defined by:

$$\mu_{(\Lambda;\boldsymbol{\theta}^i)}(\boldsymbol{\vartheta}_\Lambda) = \left\langle \prod_{b\in\mathscr{B}_\Lambda} e^{W_{b;\vartheta_b}} \right\rangle_{(\Lambda;\boldsymbol{\theta}^i)}, \tag{13}$$

where

$$W_{b;\vartheta_b}(\boldsymbol{\theta}_\Lambda) = V_b[\gamma_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^i_{\Lambda^c})+\vartheta_b] - V_b[\gamma_b(\boldsymbol{\theta}_\Lambda+\boldsymbol{\theta}^i_{\Lambda^c})].$$

This function satisfies the following important and useful identity[4]:

$$\mu_{(\Lambda;\boldsymbol{\theta}^i)}(\boldsymbol{\vartheta}_\Lambda) = \mu_{(\Lambda;\boldsymbol{\theta}^i)}(\boldsymbol{\vartheta}_\Lambda+\gamma(\boldsymbol{\theta}_\Lambda)) \qquad \forall \boldsymbol{\theta}_\Lambda \in \mathscr{G}_\Lambda. \tag{14}$$

We obtain:

$$\begin{cases} \dfrac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^i)} = \mu_{(\Lambda;\boldsymbol{\theta}^i)}(\boldsymbol{\varrho}^i_\Lambda) \qquad i=1,2 \\ \boldsymbol{\varrho}^i_\Lambda = (\gamma_b(\boldsymbol{\theta}^{12}-\boldsymbol{\theta}^i))_{b\in\mathscr{B}_\Lambda} \quad \text{i.e.} \quad \varrho^1_b = \gamma_b(\boldsymbol{\theta}^2_{\mathscr{L}^d}-\boldsymbol{\theta}^1_{\mathscr{L}^d}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \varrho^2_b = \gamma_b(\boldsymbol{\theta}^1_{\mathscr{L}^u}-\boldsymbol{\theta}^2_{\mathscr{L}^u}) \end{cases}$$

and therefore

$$\tau^{12} = \lim_{L_i\to\infty} \lim_{M\to\infty} \frac{(-1)}{\prod L_i} \frac{1}{2} \{\mathrm{Log}\,\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}^1_\Lambda) + \mathrm{Log}\,\mu_{(\Lambda;\boldsymbol{\theta}^2)}(\boldsymbol{\varrho}^2_\Lambda)\}. \tag{15}$$

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4 It was shown for spin$\frac{1}{2}$ systems that this identity yields a definition of "Symmetric Equilibrium States" [6]

To state our first result we introduce the "*length* $|\boldsymbol{\vartheta}|$" of a graph $\boldsymbol{\vartheta} \in \prod_{b\in\mathscr{B}} \mathscr{G}_b^{\wedge}$ as

$$|\boldsymbol{\vartheta}| = \text{cardinality of the set } \beta_{\boldsymbol{\vartheta}} = \{b \in \mathscr{B}; \vartheta_b \neq 0\}$$

and

$$\boldsymbol{\vartheta} \cap \mathscr{B}_\Lambda = \{\vartheta_b\}_{b\in\mathscr{B}_\Lambda}.$$

Theorem 1. *If the surface tension* τ^{12} *exists then it satisfied the following bound*

$$|\tau^{12}| \leqq 2\bar{K} C^{\blacktriangledown}_{12},$$

where

$$\bar{K} = \sup_{b\in\mathscr{B}} \left[\sup_{\vartheta_b \in \mathscr{G}_b^{\wedge}} |V_b[\vartheta_b]|\right], \tag{16}$$

$$C^{\blacktriangledown}_{12} = \limsup_{\Lambda\to\infty} \frac{1}{\prod L_i} C_\Lambda, \tag{17}$$

and

$$C_\Lambda = \sup_{i=1,2} \left[\min_{\boldsymbol{\theta}_\Lambda \in \mathscr{G}_\Lambda} |\gamma(\boldsymbol{\theta}_\Lambda) + \boldsymbol{\varrho}^i_\Lambda|\right]. \tag{18}$$

This result follows from the fundamental identity Eq. (14), together with the inequality [obtained from the definition Eq. (13)]:

$$\prod_{b:\vartheta_b \neq 0} e^{-2\sup|V_b(\theta)|} \leqq \mu_{(\Lambda;\boldsymbol{\theta}^i)}(\boldsymbol{\vartheta}_\Lambda) \leqq \prod_{b:\vartheta_b\neq 0} e^{2\sup|V_b(\theta)|}.$$

The interest of this theorem is given by the following corollary which will be illustrated on some examples in Sect. 6.

Corollary 1. *If for any* Λ *there exists some* $\boldsymbol{\theta}_\Lambda \in \mathscr{G}_\Lambda$ *such that*

$$\boldsymbol{\varrho}^i_\Lambda + \gamma(\boldsymbol{\theta}_\Lambda) = \boldsymbol{\vartheta}^i_\Lambda$$

with $\lim_{\Lambda\to\infty} \frac{|\vartheta^i_\Lambda|}{\prod L_j} = 0$, *then the surface tension exists and is zero at all temperature.*

Remarks. 1. As we shall show in Sect. 5 (for discrete systems) if the condition of the corollary is not satisfied then the surface tension is non zero at low temperature (if it exists).

2. The constant C_Λ is most easily evaluated using duality transformation; indeed by definition of duality, any graph $\gamma(\boldsymbol{\theta})$ becomes a closed graph on the dual model; therefore C_Λ is the minimum length of the graphs $\mathbf{l}^*$ on the dual, such that $\mathbf{l}^* + \boldsymbol{\varrho}^*_\Lambda$ is a closed graph (see Sects. 5 and 6).

3. Let us also note that by duality transformation $\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}$ is the expectation value of a product of characters on the dual model. This remark show that the surface tension is the dual of the definition introduced in Gauge Theory for phase transition [2].

In the following we shall only consider boundary condition $\boldsymbol{\theta}^1$ *and* $\boldsymbol{\theta}^2$ which *are related by "Internal Symmetry"*, i.e.

$$\theta^2_x = \tau_x \theta^1_x, \quad \boldsymbol{\theta}^2 = \tau \boldsymbol{\theta}^1,$$

where τ_x is a measure preserving transformation of $\mathcal{G}_x$ onto $\mathcal{G}_x$ such that $V_b(\tau\boldsymbol{\theta}) = V_b(\boldsymbol{\theta})$ for all b in $\mathcal{B}$ and $\boldsymbol{\theta} \in \mathcal{G}_{\mathcal{L}}$; we denote by τ_d the restriction of τ to $\mathcal{L}^d$, i.e.

$$\tau_d \boldsymbol{\theta} = \begin{cases} \theta_x & \text{if } x \in \mathcal{L}^u \\ \tau_x \theta_x & \text{if } x \in \mathcal{L}^d . \end{cases} \tag{19}$$

For example the "*Internal Symmetry Groups*" defined by [1]:

$$\mathcal{S} = \{\boldsymbol{\theta}^{(s)} \in \mathcal{G}_{\mathcal{L}}; \gamma_b(\boldsymbol{\theta}^s) = 0 \quad \forall b \in \mathcal{B}\} \tag{20}$$

yields a group of internal symmetries for the transformation

$$\tau^{(s)} : \boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \boldsymbol{\theta}^{(s)}$$

(if $d\tilde{v}_x$ is invariant under "translation"). In fact for spin$\frac{1}{2}$ systems all internal symmetries are given by the group $\mathcal{S}$.

For general spin systems there may exist internal symmetries of the type:

$$\tau_x \theta_x = (-1)^{n_x} \theta_x .$$

In particular if $V_b(\boldsymbol{\theta}) = V_b(-\boldsymbol{\theta})$ for all b in $\mathcal{B}$ and if $d\tilde{v}_x(\theta_x) = d\tilde{v}_x(-\theta_x)$ then the "*Inversion* τ^I" defined by $\tau_x \theta_x = -\theta_x$ is an internal symmetry.

For boundary conditions which are related by *internal symmetry* we have $Z(\Lambda; \boldsymbol{\theta}^1) = Z(\Lambda; \boldsymbol{\theta}^2)$ and then

$$\begin{aligned} \tau^{12} &= \lim_{L_i \to \infty} \lim_{M \to \infty} \frac{(-1)}{\prod L_i} \operatorname{Log} \frac{Z(\Lambda; \boldsymbol{\theta}^{12})}{Z(\Lambda; \boldsymbol{\theta}^1)} \\ &= \lim_{L_i \to \infty} \lim_{M \to \infty} \frac{(-1)}{\prod L_i} \operatorname{Log} \mu_{(\Lambda; \boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_\Lambda) , \end{aligned}$$

$$\boldsymbol{\varrho}_\Lambda = \{\gamma_b(\boldsymbol{\theta}^2_{\mathcal{L}^d} - \boldsymbol{\theta}^1_{\mathcal{L}^d})\}_{b \in \mathcal{B}_\Lambda} .$$

Property 1 (Ferromagnetic systems). *Let τ be an internal symmetry; if the interactions V_b are such that $f_b = \mathcal{F} e^{V_b} \geqq 0$, then the surface tension between the states $\boldsymbol{\theta}^1 = 0$ and $\boldsymbol{\theta}^2 = \tau \boldsymbol{\theta}^1$ is non negative (assuming it exists).*

Indeed, since $f(b; l) = \overline{f(b; -l)}$, the condition that $f(b; l)$ is non negative implies that for $\boldsymbol{\theta}^1 = 0$[5]

$$\mu_{(\Lambda; +)}(\boldsymbol{\vartheta}_\Lambda) = \frac{\sum_{\mathbf{k} \in \mathcal{K}_\Lambda} \left[\prod_{b \in \mathcal{B}_\Lambda} f(b; k_b) \right] \operatorname{Re} \prod_{b \in \mathcal{B}_\Lambda} \langle \vartheta_b ; k_b \rangle}{\sum_{\mathbf{k} \in \mathcal{K}_\Lambda} \left[\prod_{b \in \mathcal{B}_\Lambda} f(b; k_b) \right]} \leqq 1 .$$

Property 2. *Let $\boldsymbol{\theta}^2 = \tau \boldsymbol{\theta}^1$ where τ is any internal symmetry, then*

i) $$\frac{Z(\Lambda; \boldsymbol{\theta}^{12})}{Z(\Lambda; \boldsymbol{\theta}^1)} = \left\langle \prod_{b \in \mathcal{B}_\Lambda(S)} e^{W_b} \right\rangle_{(\Lambda; \boldsymbol{\theta}^1)} ,$$

where

$$\mathcal{B}_\Lambda(S) = \{b \in \mathcal{B}_\Lambda; \mathcal{B}_b \cap \mathcal{L}^u \neq \emptyset \quad \text{and} \quad \mathcal{B}_b \cap \mathcal{L}^d \neq \emptyset\} , \tag{21}$$

5 We shall always denote by $\omega^{(+)}$ the state defined by the boundary condition $\boldsymbol{\theta}^1 = \{0\}$

and

$$W_b(\boldsymbol{\theta}_\Lambda) = V_b(\tau_d(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c})) - \mathbf{V}_b(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c}). \tag{22}$$

$$\text{ii)} \quad \frac{1}{\prod L_i}\left|\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}\right| \leqq 2\bar{K}C,$$

where

$$\bar{K} = \sup_{b\in\mathscr{B}} \sup_{\vartheta_b \in \hat{\mathscr{G}_b}} |V_b[\vartheta_b]|$$

and

$$C = \sup_\Lambda \frac{|\mathscr{B}_\Lambda(S)|}{\prod L_i} = \textit{maximum density of bonds which cross the surface.}$$

Proof.

$$Z(\Lambda;\boldsymbol{\theta}^{12}) = \int_{\mathscr{G}_\Lambda} d\boldsymbol{\theta}_\Lambda \prod_{b\in\mathscr{B}_\Lambda(S)} e^{V_b(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^{12}_{\Lambda^c})} \prod_{\substack{b\notin\mathscr{B}_\Lambda(S)\\ b\in\mathscr{B}_\Lambda}} e^{V_b(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^{12}_{\Lambda^c})}.$$

But using the fact that $\boldsymbol{\theta}^{12} = \tau_d\boldsymbol{\theta}^1$ with τ an internal symmetry, a change of variables yields:

$$Z(\Lambda;\boldsymbol{\theta}^{12}) = \int_{\mathscr{G}_\Lambda} d\boldsymbol{\theta}_\Lambda \prod_{b\in\mathscr{B}_\Lambda(S)} e^{V_b(\tau_d(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c}))} \prod_{\substack{b\notin\mathscr{B}_\Lambda(S)\\ b\in\mathscr{B}_\Lambda}} e^{V_b(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c})}$$

which concludes the proof of i).

The inequality ii) follows then from

$$\prod_{b\in\mathscr{B}_\Lambda(S)} e^{-2\sup|V_b(\boldsymbol{\theta})|} \leqq \left\langle \prod_{b\in\mathscr{B}_\Lambda(S)} e^{W_b} \right\rangle_{(\Lambda;\boldsymbol{\theta}^1)} \leqq \prod_{b\in\mathscr{B}_\Lambda} e^{2\sup|V_b(\boldsymbol{\theta})|}.$$

Remarks. 1. Property 2 shows that if $\boldsymbol{\theta}^2 = \tau\boldsymbol{\theta}^1$, the *surface tension between the states ω^1 and ω^2 is related to the incremental free energy of a lattice with the interaction "crossing" the surface $x_\nu = 0$, $V_b(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c})$, replaced by $V_b(\tau_d(\boldsymbol{\theta}_\Lambda + \boldsymbol{\theta}^1_{\Lambda^c}))$*; it is thus the natural generalization of the definition given by Fisher and Ferdinand for the 2-dimensional Ising Model [4].

2. This property also shows that the "*surface tension*" *is the expectation value of a non local observable associated with the dividing surface $x_\nu = 0$.*

3. Furthermore, for systems with *finite range interactions, the surface tension appears as the expectation value of an observable which does not depend on M as soon as M is large enough;* in this case C is a finite geometrical constant as soon as the system has some translation invariance property, independent of the boundary conditions.

Example. "Spin $\frac{1}{2}$" (see [6]).

$$\mathscr{G}_x = \{1, -1\} = \hat{\mathscr{G}_b} \cong \mathscr{G}_b = \{0, 1\} = \mathbb{Z}_2,$$

$$\mathscr{B} \subset \mathscr{P}_f(\mathscr{L}) \quad \text{and} \quad B_b = b,$$

$$\gamma_b : \boldsymbol{\theta} \mapsto \prod_{x\in b} \theta_x,$$

$$\mathbf{V}_b(\boldsymbol{\theta}) = K_b\gamma_b(\boldsymbol{\theta}), \qquad f(b;l) = \tfrac{1}{2}\sum_{\vartheta=-1}^{1} \vartheta^l e^{K_b\vartheta},$$

i.e.

$$f(b;1)=\sinh K_b, \qquad f(b;0)=\cosh K_b.$$

Let $\tau\boldsymbol{\theta}=\boldsymbol{\theta}+\boldsymbol{\theta}^{(s)}$ with $\theta^{(s)}\in\mathscr{S}$ and $\boldsymbol{\theta}^1=\{1\}$ then

$$W_b(\boldsymbol{\theta}_\Lambda)=K_b\gamma_b(\boldsymbol{\theta}_\Lambda)\,[\gamma_b(\boldsymbol{\theta}^{(s)}_{\mathscr{L}^d})-1]$$

and

$$\begin{cases}\dfrac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;+)}=\Big\langle\prod\limits_{b\in\beta_\Lambda(S)}e^{-2K_b\gamma_b}\Big\rangle_{(\Lambda;+)}\\ \qquad\beta_\Lambda(S)=\{b\in\mathscr{B};\ b\cap\Lambda\neq\emptyset,\ \gamma_b(\boldsymbol{\theta}^{(s)}_{\mathscr{L}^d})=-1\}.\end{cases}$$

3.2. *General Properties for $\tau^{(s)}$ in $\mathscr{S}$*

Let us first remark that for $\boldsymbol{\theta}^2=\boldsymbol{\theta}^1+\boldsymbol{\theta}^{(s)}$ with $\boldsymbol{\theta}^{(s)}\in\mathscr{S}$ then for all $b\notin\mathscr{B}(S)$ [Eq. (21)]

$$\varrho_b=\gamma_b(\boldsymbol{\theta}^2_{\mathscr{L}^d}-\boldsymbol{\theta}^1_{\mathscr{L}^d})=\gamma_b(\boldsymbol{\theta}^{(s)}_{\mathscr{L}^d})=0. \tag{23}$$

Theorem 2. *If the following inequalities* (I) *and* (II) *hold then for finite range interaction the surface tension $\tau^{(12)}$ between the states defined by $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2=\boldsymbol{\theta}^1+\boldsymbol{\theta}^{(s)}$, $\boldsymbol{\theta}^{(s)}\in\mathscr{S}$, exists and satisfies*

$$|\tau^{(12)}|\leqq 2\bar{K}C$$

(I) $\mu_{(\Lambda_1;\boldsymbol{\theta}^1)}(\boldsymbol{\vartheta})\leqq\mu_{(\Lambda_2;\boldsymbol{\theta}^1)}(\boldsymbol{\vartheta})$ for all $\Lambda_2\supset\Lambda_1$, $\boldsymbol{\vartheta}\in\mathscr{G}^{\wedge}_{\mathscr{B}_{\Lambda_1}}$,

(II) $\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\vartheta}')\cdot\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\vartheta}'')\leqq\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\vartheta}'+\boldsymbol{\vartheta}'')$.

Proof. Let $\Lambda=\Lambda'\cup\Lambda''$ where $\Lambda',\Lambda'',\Lambda$ are parallelipiped defined by

$$(L_1,\ldots,L_i',\ldots,L_{\nu-1},2M),\quad(L_1,\ldots,L_i'',\ldots,L_{\nu-1},2M),\quad(L_1,\ldots,L_i'+L_i'',\ldots,2M)$$

$$\boldsymbol{\varrho}_\Lambda\equiv(\varrho_b)_{b\in\mathscr{B}_\Lambda(S)}=\boldsymbol{\varrho}_{\Lambda'}+\boldsymbol{\varrho}_{\Lambda''}+\delta\boldsymbol{\varrho}_{\Lambda',\Lambda''},$$

where

$$\delta\boldsymbol{\varrho}_{\Lambda',\Lambda''}=(-\varrho_b)_{b\in\mathscr{B}_{\Lambda'}(S)\cap\mathscr{B}_{\Lambda''}(S)}.$$

It thus follows from the inequalities (I) and (II) that

$$\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_\Lambda)\geqq\mu_{(\Lambda';\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_{\Lambda'})\mu_{(\Lambda'';\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_{\Lambda''})\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\delta\boldsymbol{\varrho})$$

furthermore:

$$\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\delta\boldsymbol{\varrho}_{\Lambda',\Lambda''})\geqq e^{-2\bar{K}\delta},$$

where

$$\delta=\sup|\mathscr{B}_{\Lambda'}(S)\cap\mathscr{B}_{\Lambda''}(S)|=\tilde{C}\frac{L_1,\ldots,L_{\nu-1}}{L_i}.$$

Therefore:

$$\operatorname{Log}\mu_{(\Lambda;\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_\Lambda)\geqq\operatorname{Log}\mu_{(\Lambda';\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_{\Lambda'})+\operatorname{Log}\mu_{(\Lambda'';\boldsymbol{\theta}^1)}(\boldsymbol{\varrho}_{\Lambda''})-2\bar{K}\delta.$$

Using the subadditivity of the function,

$$g(L_1, \ldots, L_\nu) = -\operatorname{Log} \mu_{(\Lambda;\,\theta^1)}(\varrho_\Lambda) - 2\tilde{C}\bar{K} \prod_{i=1}^{\nu-1} L_i \left(\sum_j \frac{1}{L_j} \right)$$

we conclude the proof using the same argument as in [6].

Remark. For spin $\frac{1}{2}$ ferromagnetic systems it is easily seen that the inequalities (I), and (II) are satisfied for $\boldsymbol{\theta}^1 = (+)$ using the fact that

$$\frac{f(b\,;l)}{f(b\,;0)} = (\tanh K_b)^l \qquad l \in \{0, 1\}$$

and

$$\mu_{(\Lambda;\,+)}(\boldsymbol{\vartheta}) = \frac{\sum\limits_{\mathbf{k} \in \mathscr{K}_\Lambda} \prod\limits_{b \in \mathscr{B}_\Lambda} \vartheta_b^{k_b} t_b^{k_b}}{\sum\limits_{\mathbf{k} \in \mathscr{K}_\Lambda} \prod\limits_{b \in \mathscr{B}_\Lambda} t_b^{k_b}},$$

$$t_b = \tan K_b \in [0, 1].$$

However, we shall not discuss the validity of these inequalities for general system in the present article.

4. Surface Tension at High Temperature

In the high temperature domain one can use either the "H.T. expansion" Eq. (7) or the "Cluster expansion" Eq. (10). We shall first study compact groups using the H.T. expansion and then extend the results to arbitrary groups using the cluster expansion. Let us remark that it is interesting to discuss both methods since for a given interaction only one of the methods may be applicable; furthermore the bounds one can obtain may be better with one or the other approach depending on the interaction. We should also insist on the fact that in the cluster expansion method the group structure plays no role.

4.1. Compact Groups (H.T. expansion)

Let us first consider the "H.T. expansion" Eq. (7). Since any closed graph $\mathbf{k}$ in $\mathscr{K}_\Lambda$ can be uniquely decomposed as union of "connected closed graphs" where two graphs $\mathbf{k}'$ and $\mathbf{k}''$ are said to be disconnected[6] if $\beta_{\mathbf{k}'}$ and $\beta_{\mathbf{k}''}$ are disconnected where

$$\beta_{\mathbf{k}} = \{b \in \mathscr{B}\,; k_b = 0\} \subset \mathscr{B}\,; \qquad [\mathbf{k}] = [\beta_{\mathbf{k}}] = \bigcup_{b \in \beta_{\mathbf{k}}} B_b \subset \mathscr{L}\,.$$

The H.T. expansion can thus be written as

$$Z(\Lambda\,;\boldsymbol{\theta}^0) = \sum_{q=0}^{\infty} \sum_{\substack{\{\mathbf{k}^1, \ldots, \mathbf{k}^q\} \subset \mathscr{K}_\Lambda \\ \mathbf{k}^i \text{ connected}}} \prod_{i=1}^{q} \Phi^0(\mathbf{k}^i)\, G(\beta_{\mathbf{k}^1}, \ldots, \beta_{\mathbf{k}^q}), \tag{24}$$

6 For all $\mathbf{k} \in \mathscr{K}_\Lambda$ such that $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$ with $\beta_{\mathbf{k}'}$ not connected to $\beta_{\mathbf{k}''}$ then $\mathbf{k}'$ and $\mathbf{k}''$ are also in $\mathscr{K}_\Lambda$: indeed for all $\boldsymbol{\theta} \in \mathscr{G}_\Lambda \prod\limits_b \langle \gamma_b(\theta)\,; k_b' \rangle = \prod\limits_{b \in \beta_{k'}} \langle \gamma_b(\boldsymbol{\theta})\,; k_b' \rangle = \prod\limits_b \langle \gamma_b(\boldsymbol{\theta}_{[\beta_{k'}]})\,; k_b \rangle = 1$

where

$$\Phi^0(\mathbf{k})=\prod_{b\in\mathscr{B}} f(b;k_b)\langle\gamma_b(\boldsymbol{\theta}^0);k_b\rangle,$$

$$G(\beta^1,\ldots,\beta^q)=\prod_{(ij)\subset\{1\ldots q\}} g(\beta^i,\beta^j),$$

$$g(\beta^i,\beta^j)=\begin{cases}0 & \text{if } \beta^i \text{ is connected to } \beta^j\\ 1 & \text{if } \beta^i \text{ is not connected to } \beta^j.\end{cases}$$

The expansion can thus be written as:

$$Z(\Lambda;\boldsymbol{\theta}^0)=\sum_{q=0}^{\infty}\sum_{\substack{\{\beta^1,\ldots,\beta^q\}\\ \beta^i\,\text{connected}\subset\mathscr{B}_\Lambda}}\prod_{i=1}^{q}\phi^0(\beta^i)\,G(\beta^1,\ldots,\beta^q)$$

with

$$\phi^0(\beta)=\sum_{\substack{\mathbf{k}\in\mathscr{K}_\Lambda\\ \beta_{\mathbf{k}}=\beta}}\Phi^0(\mathbf{k}).$$

.Notice that for any $\beta\subset\mathscr{B}_\Lambda$ with $|\beta|=1$ there exists no $\mathbf{k}$ in $\mathscr{K}_\Lambda$ with $\beta_{\mathbf{k}}=\beta$.

Let us introduce as usual [5] the space $\mathscr{X}_\Lambda$ of finite (non ordered) subsets with repetition of the set of all connected graphs $\beta\subset\mathscr{B}_\Lambda$ with $|\beta|\geqq 2$ and V the space of functions on $\mathscr{X}$ together with the *-product:

$$(F*G)(X)=\sum_{X=X_1\cup X_2}F(X_1)\,G(X_2),$$

$$\Gamma G_T=\sum_{n=0}^{\infty}\frac{1}{n!}G_T^{*n}=\mathbb{1}+G_+,$$

$$G_T=\Gamma^{-1}(\mathbb{1}+G_+)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}G_+^{*n},$$

we obtain for the "H.T. expansion":

$$Z(\Lambda;\boldsymbol{\theta}^0)=\exp\left[\sum_{X\in\mathscr{X}_\Lambda}\prod_{\beta\in X}\phi^0(\beta)\,G_T(X)\right],$$

where:

$$G=\Gamma G_T.$$

Therefore

$$-\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}=\sum_{X\in\mathscr{X}_\Lambda}\left[\prod_{\beta\in X}\phi^1(\beta)-\prod_{\beta\in X}\phi^{12}(\beta)\right]G_T(X).$$

Let $\boldsymbol{\theta}^2=\tau\boldsymbol{\theta}^1$ where τ is any internal symmetry which induces a measure preserving transformation $\hat{\tau}$ of $\mathscr{G}_b^{\hat{}}$ onto $\mathscr{G}_b^{\hat{}}$ by the relation

$$\hat{\tau}\gamma_b(\boldsymbol{\theta})=\gamma_b(\tau^{-1}\boldsymbol{\theta}).$$

(This will be in particular the case for the internal symmetry $\tau^{(s)}:\boldsymbol{\theta}\mapsto\boldsymbol{\theta}+\boldsymbol{\theta}^{(s)}$ in which case $\hat{\tau}=\mathbb{1}$; it will also be the case if τ is the inversion in which case $\hat{\tau}l_b=-l_b$.)

The transformation $\hat{\tau}$ induces an automorphism τ' of $\mathscr{G}_b$ onto $\mathscr{G}_b$ such that

$$\langle \gamma_b(\boldsymbol{\theta}); \tau' l_b \rangle = \langle \gamma_b(\tau\boldsymbol{\theta}); l_b \rangle .$$

This automorphism τ' has the property to leave the group $\mathscr{K}_\Lambda$ invariant and is such that

$$f(b; \tau' l_b) = f(b; l_b) .$$

In then follows that

$$\begin{aligned} \Phi^{12}(\mathbf{k}) = \Phi^1(\mathbf{k}) \quad &\text{if} \quad [\mathbf{k}] \subset \Lambda \\ &\text{or if} \quad [\mathbf{k}] \subset \mathscr{L}^u \end{aligned}$$

and

$$\Phi^{12}(\mathbf{k}) = \prod_{b \in \mathscr{B}} f(b; k_b) \langle \gamma_b(\boldsymbol{\theta}^1); \tau' k_b \rangle \quad \text{if} \quad [\mathbf{k}] \subset \mathscr{L}^d .$$

But $\tau'\mathbf{k} \in \mathscr{K}_\Lambda$, $\beta_{\mathbf{k}} = \beta_{\tau'\mathbf{k}}$ and $f(b; \tau' k_b) = f(b; k_b)$ implies that:

$\phi^{12}(\beta) = \phi^1(\beta)$ either if for any b in β, B_b is a subset of $\mathscr{L}^u$ or $\mathscr{L}^d$

or if for any b in β, B_b is a subset of Λ.

We have thus established the following result:

Lemma 1. *Let $\boldsymbol{\theta}^2 = \tau\boldsymbol{\theta}^1$ where τ is any internal symmetry which induces a measure preserving transformations of $\mathscr{G}_b^{\hat{}}$ onto $\mathscr{G}_b^{\hat{}}$.*

Then

$$-\operatorname{Log} \frac{Z(\Lambda; \boldsymbol{\theta}^{12})}{Z(\Lambda; \boldsymbol{\theta}^1)} = \sum_{\substack{X \in \mathscr{X}_\Lambda \\ X \cap \mathscr{B}_\Lambda(s) \neq \emptyset \\ X \cap \partial\Lambda \neq \emptyset}} \left[\prod_{\beta \in X} \phi^1(\beta) - \prod_{\beta \in X} \phi^{12}(\beta) \right] G_T(X) ,$$

where: $X \cap \mathscr{B}_\Lambda(S) \neq \phi$ means that at least one β in X has some b such that B_b intersects the upper and lower half of $\mathscr{L}$; $X \cap \partial\Lambda \neq \phi$ means that at least one β in X has some b such that B_b intersects Λ^c.

Lemma 2. *There exists $z_0 \in]0, 1]$ such that*

$$\sup_{x \in \mathscr{L}} \sum_{\substack{X \in \mathscr{X}_\Lambda \\ [X] \ni x, \operatorname{diam}[X] > d}} z_0^{|X|} |G_T(x)| \leqq (c_1 + c_2 d^\nu) e^{-c_3 d^{1/2}} ,$$

where: $|X| = \sum_{\beta \in X} |\beta| \qquad |\beta| = $ *cardinality of β in $\mathscr{P}(\mathscr{B})$*

$$[X] = \bigcup_{\beta \in X} \bigcup_{b \in \beta} B_b .$$

The proof of this lemma is given in the Appendix 1.

Theorem 3. *Let $\{\mathscr{L}, \mathscr{B}, \gamma, V\}$ be a compact spin system with finite density of lattice sites and finite range interactions such that*

$$\sum_{l_b \neq 0} |\mathscr{F}[e^{\lambda V_b}](l_b)|$$

tends to zero uniformly in b as $\lambda \to 0$.

For $\boldsymbol{\theta}^2 = \tau \boldsymbol{\theta}^1$ with τ any internal symmetry satisfying the condition of Lemma 1, there exists a temperature T_0 such that the surface tension $\tau^{(12)}$ is zero for all temperature above T_0.

Proof. Using Lemmas 1 and 2, we have:

$$\left|\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}\right| \leqq 2 \sum_{\substack{X \in \mathscr{X}_\Lambda \\ X \cap \mathscr{B}_\Lambda(s) \neq \emptyset \\ X \cap \partial\Lambda \neq \emptyset}} \prod_{\beta \in X} |\phi(\beta)|\, |G_T(X)|$$

$$\leqq 2 \sum_{x \in \Lambda_c} \sum_{\substack{X \in \mathscr{X}_\Lambda \\ X \cap \mathscr{B}_\Lambda(s) \neq \emptyset \\ [X] \ni x}} \prod_{\beta \in X} |\phi(\beta)| |G_T(X)|$$

$$\leqq 2 \sum_{d=0}^{\infty} \sum_{\substack{x \in \Lambda_c \\ |x_\nu| = d}} \sum_{\substack{X \in \mathscr{X}_\Lambda \\ [X] \ni x \\ \operatorname{diam}[X] \geqq d}} \prod_{\beta \in X} |\phi(\beta)| |G_T(x)|$$

therefore for

$$|\phi(\beta)| < z_0^{|\beta|}$$

$$\left|\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)}\right| \leqq 2 \sum_{d=0}^{\infty} \sum_{\substack{x \in \Lambda_c \\ |x_\nu| = d}} (c_1 + c_2 d^\nu) e^{-c_3 d^{1/2}}$$

$$\leqq 2 \left(\sum_{i=1}^{\nu-1} \prod_{j \neq i} L_j \right) \mathscr{C} + \left(\prod_{j=1}^{\nu-1} L_j \right) (c_1 + c_2 M^\nu) e^{-c_3 M^\nu}$$

and thus $\tau^{(12)} = 0$ for any T such that $|\phi(\beta)| < z_0^{|\beta|}$.

But for any $\beta = (b_1, \ldots, b_q)$

$$|\phi(\beta)| \leqq \sum_{\substack{\mathbf{k} \in \mathscr{K}_\Lambda \\ \beta_{\mathbf{k}} = \beta}} \prod_{b \in \beta} |f(b; k_b)| \leqq \prod_{b \in \beta} \left(\sum_{l_b \neq 0} |f(b; l_b)| \right)$$

which concludes the proof by the assumption on the potentials V_b

4.2. Arbitrary Groups

Starting with the cluster expansion Eq. (10), we have

$$Z(\Lambda; \boldsymbol{\theta}^0) = \sum_q \sum_{\substack{\{\beta^1 \ldots \beta^q\} \\ \beta^i \text{ connected} \subset \mathscr{B}_\Lambda^1}} \prod_{i=1}^{q} \Psi_{(\Lambda;\boldsymbol{\theta}^0)}(\beta^i) G(\beta^1 \ldots \beta^q).$$

Again we consider $\mathscr{X}_\Lambda$, the space of (non ordered) subsets with repetition of the set of all "connected graphs $\beta \subset \mathscr{B}^1_\Lambda$" to obtain:

$$-\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)} = \sum_{X\in\mathscr{X}_\Lambda}\left[\prod_{\beta\in X}\psi^1(\beta) - \prod_{\beta\in X}\psi^{12}(\beta)\right]G_T(X),$$

$$\psi^1(\beta) = \int_{\mathscr{G}_{[\beta]}}\prod_{x\in[\beta]} d\tilde{v}_x \prod_{B\in\beta}\left(e^{\frac{1}{kT}\phi_B(\boldsymbol{\theta}+\boldsymbol{\theta}^1_{\Lambda^c})}-1\right),$$

$$\psi^{12}(\beta) = \int_{\mathscr{G}_{[\beta]}}\prod_{x\in[\beta]} d\tilde{v}_x \prod_{B\in\beta}\left(e^{\frac{1}{kT}\phi_B(\boldsymbol{\theta}+\boldsymbol{\theta}^{12}_{\Lambda^c})}-1\right),$$

and we have as before:

Lemma 3

$$-\operatorname{Log}\frac{Z(\Lambda;\boldsymbol{\theta}^{12})}{Z(\Lambda;\boldsymbol{\theta}^1)} = \sum_{\substack{X\in\mathscr{X}_\Lambda\\ X\cap\mathscr{B}^1_\Lambda(s)\neq\emptyset\\ X\cap\partial\Lambda\neq\emptyset}}\left[\prod_{\beta\in X}\psi^1(\beta) - \prod_{\beta\in X}\psi^{12}(\beta)\right]G_T(X).$$

Theorem 4. *If the interactions are such that $|\psi^0(\beta)| < c(T)^{|\beta|}$ with $c(T)\to 0$ as $T\to\infty$, then the surface tension between the states defined by $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2=\tau\boldsymbol{\theta}^1$ where τ is any internal symmetry, is zero as soon as $T>T_0$.*

Corollary. *Let $\boldsymbol{\theta}^2=\tau\boldsymbol{\theta}^1$ with τ any internal symmetry. If the interactions are uniformly bounded then the surface tension $\tau^{(12)}$ is always zero at high temperature.*

Indeed in that case $c(T)=\left(e^{\frac{1}{kT}\sup_B\|\phi_B\|_{\mathscr{L}^\infty}}-1\right)$.

5. Surface Tension at Low Temperature for Discrete Groups

In this last section we restrict ourselves to the case where $\mathscr{G}_x$ *is discrete* for all x in $\mathscr{L}$ and therefore $\mathscr{G}^\wedge_b$ is also discrete for all b in $\mathscr{B}$. The boundary conditions we shall consider are $\boldsymbol{\theta}^1=0$ and $\boldsymbol{\theta}^2\in\mathscr{G}_{\mathscr{L}}$; we denote by "+" the boundary condition defined by $\boldsymbol{\theta}^1=0$.

To discuss the properties in the low temperature domain we shall follow the method introduced to construct duality transformations [1]. In fact, as we shall see on explicit examples (Sect. 6), the duality transformation provides a technique to evaluate the constant $C^\blacktriangledown_{12}$ Eq. (17), i.e., to obtain a lower bound on the critical temperature. We should insist at this point on the fact that $\mathscr{G}_x$ could be of infinite order (as far as low temperature properties are concerned the results for all examples of Sect. 6 remain in particular valid with $\mathscr{G}_x=\mathbb{Z}$).

Let Γ, $\Gamma^{(f)}$, $\Gamma_{(f)}$ be the groups defined by means of the homeomorphism γ as

$$\Gamma=\{\boldsymbol{\vartheta}=\gamma(\boldsymbol{\theta})\in\mathscr{G}^\wedge_{\mathscr{B}};\ \boldsymbol{\theta}\in\mathscr{G}_{\mathscr{L}}\}=\operatorname{Im}\gamma,$$

$$\Gamma^{(f)}=\{\boldsymbol{\vartheta}=\gamma(\boldsymbol{\theta})\in\mathscr{G}^\wedge_{\mathscr{B}};\ \boldsymbol{\theta}\in\mathscr{G}_{\mathscr{L},f}\},$$

$$\Gamma_{(f)}=\Gamma\cap\mathscr{G}^\wedge_{\mathscr{B},f},$$

where $\mathscr{G}_{\mathscr{L}}, \mathscr{G}^\wedge_{\mathscr{B}}$ denotes respectively the complete direct sum of $\{\mathscr{G}_x\}$, $\{\mathscr{G}^\wedge_b\}$, while $\mathscr{G}_{\mathscr{L},f}$, $\mathscr{G}^\wedge_{\mathscr{B},f}$ denotes the direct sum.

Furthermore let $\mathscr{K}$ and $\mathscr{K}_f$ denote the group of closed graphs and the group of closed graphs with finite length, i.e.

$$\mathscr{K} = \{\mathbf{k} \in \mathscr{G}_{\mathscr{B}};\ \langle \gamma(\boldsymbol{\theta});\mathbf{k}\rangle = 1 \qquad \forall \boldsymbol{\theta} \in \mathscr{G}_{\mathscr{L},f}\},$$
$$\mathscr{K}_{(f)} = \mathscr{K} \cap \mathscr{G}_{\mathscr{B},f} \equiv \Gamma^{\perp}.$$

To construct duality transformations one introduces first a family $\mathbf{k}^{(j)}$, $j \in J$ of generators for the group $\mathscr{K}_{(f)}$, i.e.

$$\mathbf{k}^{(j)} \in \mathscr{K}_{(f)} \qquad \forall j \in J$$

and

$$\boldsymbol{\vartheta} \in \Gamma \quad \text{iff} \quad \langle \boldsymbol{\vartheta};\mathbf{k}^{(j)}\rangle = 1 \qquad \forall j \in J.$$

Using this family of generators we introduce on the set $\mathscr{B}$ of bonds a "dual" graph structure;

b_1 and b_2 are *-connected if there exists some j in J such that $k^{(j)}_{b_1} \neq 0$ and $k^{(j)}_{b_2} \neq 0$.

We shall then denote by $\mathscr{B}^*$ the set $\mathscr{B}$ with this dual graph structure.

The study of the surface tension at low temperature start from the "*Low Temperature Expansion*" [1][7]

$$\mu_{(\Lambda;+)}(\boldsymbol{\varrho}_\Lambda) = \frac{\sum\limits_{\vartheta \in \Gamma_\Lambda} \prod\limits_{b \in \mathscr{B}_\Lambda} \omega(b;\vartheta_b + \varrho_b)}{\sum\limits_{\vartheta \in \Gamma_\Lambda} \prod\limits_{b \in \mathscr{B}_\Lambda} \omega(b;\vartheta_b)},$$

where $\hat{\mathscr{G}}_{\mathscr{B}_\Lambda} = \prod\limits_{b \in \mathscr{B}_\Lambda} \hat{\mathscr{G}}_b = \{\boldsymbol{\vartheta} = (\vartheta_b)\}$ is discrete

$$\Gamma_\Lambda = \{\vec{\boldsymbol{\vartheta}} = \gamma(\boldsymbol{\theta}_\Lambda) \in \hat{\mathscr{G}}_{\mathscr{B}_\Lambda};\ \boldsymbol{\theta}_\Lambda \in \mathscr{G}_\Lambda\} \subset \hat{\mathscr{G}}_{\mathscr{B}_\Lambda},$$
$$\omega(b;\vartheta_b) = \exp\{V_b[\vartheta_b] - V_b[0]\}; \qquad \varrho_b = \gamma_b(\boldsymbol{\theta}^2_{\mathscr{L}^d}).$$

Following the procedure of Sect. 4 any element $\boldsymbol{\vartheta} \in \Gamma_\Lambda$ can be uniquely decompose into *-connected component in $\Gamma \cap \hat{\mathscr{G}}_{\mathscr{B}_\Lambda}$, where $\boldsymbol{\vartheta}'$ and $\boldsymbol{\vartheta}''$ are said to be disconnected if the graphs $\beta_{\boldsymbol{\vartheta}'} \subset \mathscr{B}^*$ and $\beta_{\boldsymbol{\vartheta}''} \subset \mathscr{B}^*$ are *-disconnected. Indeed if $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}' + \boldsymbol{\vartheta}''$ with $\boldsymbol{\vartheta}'$ and $\boldsymbol{\vartheta}''$ *-disconnected, then

$$\prod_b \langle \vartheta_b; k^{(j)}_b\rangle = 1 = \prod_{b \in \beta_{\vartheta'}} \langle \gamma_b(\boldsymbol{\theta}_\Lambda); k^{(j)}_b\rangle \prod_{b \in \beta_{\vartheta''}} \langle \gamma_b(\boldsymbol{\theta}_\Lambda); k^{(j)}_b\rangle$$

and

$$\prod_{b \in \beta_{\vartheta'}} \langle \gamma_b(\boldsymbol{\theta}_\Lambda); k^{(j)}_b\rangle = 1 \quad \text{for all} \quad j \in J$$

therefore

$$\{\gamma_b(\boldsymbol{\theta}_\Lambda)\}_{b \in \beta_{\vartheta'}} \in \Gamma \cap \hat{\mathscr{G}}_{\mathscr{B}_\Lambda}.$$

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7 In the case of compact group such an expansion can also be obtained from the H.T. expansion by means of Poisson Formula [1]

In conclusion if the system satisfies the condition

$$\Gamma_\Lambda = \Gamma \cap \mathcal{G}_{\hat{\mathcal{B}}_\Lambda}$$

then any element of Γ_Λ can be uniquely decomposed into *-connected components which are also in Γ_Λ.

Let us then introduce:

$$\beta_0 = \{b; \varrho_b \neq 0\} \qquad \beta_\vartheta = \{b; \vartheta_b \neq 0\}.$$

Since any graph $\beta \subset \mathcal{B}_\Lambda^*$ can be uniquely decomposed into *-connected component we can write

$$\beta = \bigcup_i \beta_i \bigcup_j \tilde{\beta}_j,$$

where the β_i are connected to β_0 while the $\tilde{\beta}_j$ are not; therefore

$$\sum_{\vartheta \in \Gamma_\Lambda} = \sum_{\beta \subset \mathcal{B}_\Lambda^*} \sum_{\substack{\vartheta \in \Gamma_\Lambda \\ \beta_{\vartheta+\varrho} = \beta}}$$

$$= \sum_{\substack{\bar{\beta} = (\beta_1, \ldots, \beta_q) \\ \beta_i \text{ *-conn to } \beta_0}} \sum_{\substack{\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_r) \\ \tilde{\beta}_j \text{ not *-conn to } \beta_0}} \sum_{\substack{\vartheta \in \Gamma_\Lambda \\ \beta_{\vartheta+\varrho} = \bar{\beta}}} \sum_{\substack{\tilde{\vartheta} \in \Gamma_\Lambda \\ \beta_{\tilde{\vartheta}+\varrho} = \tilde{\beta}}}$$

It then follows that:

$$\mu_{(\Lambda;+)}(\boldsymbol{\varrho}_\Lambda) \leqq \sum_{\substack{\bar{\beta} = (\beta_1, \ldots, \beta_q) \\ \beta_i = \text{*-conn to } \beta_0}} \prod_{i=1}^{q} \left[\sum_{\substack{\vartheta \in \Gamma_\Lambda \\ \beta_{\vartheta+\varrho} = \beta_i}} \prod_{b \in \beta_i} \omega[b; \vartheta_b + \varrho_b] \right]$$

and thus

$$\mu_{(\Lambda;+)}(\boldsymbol{\varrho}_\Lambda) \leqq \sum_{\substack{\beta = (\beta_1, \ldots, \beta_q) \\ \beta_i = \text{*-conn to } \beta_0}} \prod_{i=1}^{q} t^{|\beta_i|},$$

where

$$t = \sup_b \left[\sum_{\vartheta_b \neq 0} e^{-\{V_b[0] - V_b[\vartheta_b]\}} \right].$$

Theorem 5. $(\boldsymbol{\theta}^1 = 0; \boldsymbol{\theta}^2 \in \mathcal{S})$
Let $\{\mathcal{L}, \mathcal{B}, \gamma, V\}$ be a discrete system in $\mathbb{R}^\nu$, with finite density of lattice sites and finite range interaction.

If the interactions are such that i) *$\mathcal{K}_f$ separates $\mathcal{B}$;*

ii) *$\Gamma_\Lambda = \Gamma \cap \mathcal{G}_{\hat{\mathcal{B}}_\Lambda}$ for a sequence of parallelepiped $\Lambda \to \mathbb{R}^\nu$;*

iii) *there exists a family of generators $\mathbf{k}^{(j)}$ for $\mathcal{K}_f$ which separates $\mathcal{B}$ and such that $\sup_{b \in \mathcal{B}} |\{j \in J; k_b^{(j)} \neq 0\}| = a^* < \infty$;*

iv) *$V_b[0] > V_b[\vartheta_b]$ for all $\vartheta_b \neq 0$ and $e^{V_b[]} \in \mathcal{L}^1$ then there exists a temperature $\mathbf{T}_0^1$ such that the surface tension $\tau^{(12)}$, if it exists, satisfies the inequality:*

$$\tau^{(12)} \geqq C_2 C_{12}^{\blacktriangledown} \quad \textit{for} \quad T < T_0^1$$

where C_2 is a positive constant and $C_{12}^{\blacktriangledown}$ is the constant introduced in Theorem 1.

Remarks. 1. The condition i) means that for all b_1, b_2 in $\mathscr{B}$, $\exists \mathbf{k} \in \mathscr{K}_f$ such that $k_{b_1} \neq 0$ and $k_{b_2} = 0$.

2. The condition ii) in Theorem 5 is equivalent to the conditions

a) $\Gamma_{(f)} = \Gamma^{(f)}$.

b) For any $\boldsymbol{\theta} \in \mathscr{G}_{\mathscr{L},f}$ the condition $\gamma_b(\boldsymbol{\theta}) = 0$ for all $b \notin \mathscr{B}_\Lambda$ implies that there exists $\boldsymbol{\theta}'_\Lambda \in \mathscr{G}_\Lambda$ such that $\boldsymbol{\theta} \cdot \boldsymbol{\theta}'_\Lambda \in \mathscr{S}$.

Condition a) is an essential condition which appears in all discussions on Peierls arguments. On the other hand condition b) is expected to be always satisfied for $\mathbb{Z}^\nu$-invariant systems.

3. The condition iii) appears because we have no assumptions on $\mathscr{L}$; in particular it is known that if $\mathscr{L} = \mathbb{Z}^\nu$ and if the system is $\mathbb{Z}^\nu$ invariant then this condition will always be satisfied.

4. Using the family $\mathbf{k}^{(j)}$ of generators for $\mathscr{K}_{(f)}$ we can construct the dual system; if $\mathscr{G}_x = \mathbb{Z}_q$, $q \leqq \infty$, $\forall x \in \mathscr{L}$ then

$$\mathscr{L}^* = J \qquad \mathscr{G}_{x^*} \cong \mathbb{Z}_{q_j} \text{ if } \mathbf{k}^{(j)} \text{ is of order } q_j < \infty\,,$$

$$\mathscr{G}_{x^*} \cong T^{(1)} \text{ otherwise}\,,$$

$$\mathscr{B}^* = \mathscr{B} \qquad \mathscr{G}_{b^*} = \hat{\mathscr{G}}_b\,,$$

$$\gamma_b^*(\boldsymbol{\theta}^*) = \sum_{j \in J} \theta_j^* k_b^{(j)}\,.$$

Since $\langle \gamma(\boldsymbol{\theta}_\Lambda); \mathbf{k}^{(j)} \rangle = 1$ for all j we have $\langle \gamma(\boldsymbol{\theta}_\Lambda); \gamma^*(\boldsymbol{\theta}^*_{\Lambda^*}) \rangle = 1$ for all $\boldsymbol{\theta}^*_{\Lambda^*} \in \mathscr{G}^*_{\Lambda^*}$ and using the condition i)–iii), we have:

$$\begin{cases} \Gamma_\Lambda \cong \mathscr{K}^*_{\Lambda^*} = \{\mathbf{k}^* \in \mathscr{K}^*_{\Lambda^*}; \mathbf{k}^* \in \mathscr{K}^*_f\} \\ \Lambda^* = \{j; \exists b \in \mathscr{B}_\Lambda \quad \text{s.t.} \quad k_b^{(j)} \neq 0\}\,. \end{cases}$$

We thus have:

$$C^{\blacktriangledown}_{12} = \limsup_{\Lambda \to \infty} \frac{1}{\prod L_i} c_\Lambda$$

$$c_\Lambda = \min_{\boldsymbol{\theta} \in \mathscr{G}_\Lambda} |\gamma(\boldsymbol{\theta}) + \gamma(\boldsymbol{\theta}^2_{\mathscr{L}^d}) \cap \mathscr{B}_\Lambda| = \min_{\mathbf{k}^* \in \mathscr{K}_{\Lambda^*}} |\mathbf{k}^* + \boldsymbol{\varrho}^*_\Lambda|$$

= length of the shortest graphs $\mathbf{l}^* \in \mathscr{G}_{\mathscr{B}^*}$ such that $\mathbf{l}^* - \varrho^*_\Lambda$ is a closed graph.

Proof of Theorem 5. To prove this theorem we need the following lemma whose proof is given in Appendix 2.

Lemma 5. *The number of graphs $\beta \subset \mathscr{B}_\Lambda$ with $|\beta| = q$ such that all its connected components are connected to a given β^0 is bounded by*

$$4^q \alpha^q \quad \text{if} \quad q \geqq |\beta^0|$$

$$4^{|\beta_0|} \alpha^q \quad \text{if} \quad q \leqq |\beta^0|\,.$$

To prove Theorem 5 we first remark that if $C^{\blacktriangledown}_{12} = 0$ then Theorem 1 implies that $\tau^{(12)} = 0$. Let us then assume that $C^{\blacktriangledown}_{12} > 0$; using Lemma 5 and the fact that $c_\Lambda \leqq |\boldsymbol{\varrho}_\Lambda|$ we have:

$$\mu_{(\Lambda;+)}(\boldsymbol{\varrho}_\Lambda) \leqq \sum_{q=c_\Lambda}^{|\boldsymbol{\varrho}_\Lambda|} 4^{|\boldsymbol{\varrho}_\Lambda|} t^q \alpha^q + \sum_{q > |\boldsymbol{\varrho}_\Lambda|} (4t\alpha)^q$$

$$\leqq 4^{|\boldsymbol{\varrho}_\Lambda|} \left(\frac{(\alpha t)^{c_\Lambda}}{1 - \alpha t} + \frac{(\alpha t)^{|\boldsymbol{\varrho}_\Lambda|}}{1 - 4\alpha t} \right) \leqq \frac{4^{|\boldsymbol{\varrho}_\Lambda|}}{1 - 4\alpha t} (\alpha t)^{c_\Lambda}$$

as soon as $4\alpha t<1$, which is satisfied as soon as $T<T_0^1$ because of the condition on the potential.

Therefore

$$-\frac{1}{\prod L_i}\log\mu_{(\Lambda;+)}(\varrho_\Lambda)\geqq\frac{1}{\prod L_i}\left(\log(1-4\alpha t)+c_\Lambda\log\frac{1}{\alpha t}-|\varrho_\Lambda|\log 4\right)$$

which concludes the proof because of the condition on the potential, and the fact that $\frac{1}{\prod L_i}|\varrho_\Lambda|$ is uniformly bounded (since $\boldsymbol{\theta}^2\in\mathcal{S}$ and $\mathcal{L}$ has finite density of lattice sites).

Conclusions

For *finite spin* systems with finite range interaction such that $V_b[0]>V_b[\vartheta_b]$ and which satisfy condition i) and ii) of Theorem 5 we have obtained the following *criterion* for any boundary condition $\boldsymbol{\theta}^1=0$, $\boldsymbol{\theta}^2\in\mathcal{S}$:

either $\limsup\limits_{\Lambda\to\mathcal{L}}\frac{c_\Lambda}{\prod L_i}=0$, in which case the surface tension $\tau^{(12)}$ is zero for all temperature

or $\limsup\limits_{\Lambda\to\mathcal{L}}\frac{c_\Lambda}{\prod L_i}>0$, in which case there exists a phase transition associated with surface tension, i.e.

$\tau^{(12)}=0$ at high temperature

$\tau^{(12)}\neq 0$, or does not exist, at low temperature.

6. Examples

In this section we shall illustrate our results on explicit examples; it should be remarked that all these examples have a phase transition in the usual sense.

1. Generalized Potts Models on Square Lattice

$$\mathcal{L}=\mathbb{Z}^2\qquad \mathcal{G}_x=\mathbb{Z}_q\qquad q\leqq\infty\qquad \forall x\in\mathcal{L}$$

$$\mathcal{G}_b=\mathbb{Z}_q\quad\text{if}\quad q<\infty\,;$$

$$\mathcal{G}_b=T^1\quad\text{if}\quad q=\infty\,,$$

$V_b(\boldsymbol{\theta})=V_{\langle xy\rangle}[\theta_x-\theta_y]$ where $\langle xy\rangle$ denotes oriented nearest neighbours on $\mathbb{Z}^2$.

Therefore, in this example

$$\mathcal{B}=\mathcal{B}^1=\{b=\langle xy\rangle\,;\,(x,y)\subset\mathbb{Z}^2\,|x-y|=1\}\,,$$

$$\hat{\mathcal{G}_b}=\mathcal{G}_x\,,$$

$$\gamma_b(\boldsymbol{\theta})=\theta_x-\theta_y\bmod q\,;\qquad \mathcal{S}=\{\boldsymbol{\theta}^{(s)}=(\theta_x^{(s)}=\theta^{(s)})\,;\,\theta^{(s)}\in\mathbb{Z}_q\}\,.$$

Let us then take the boundary conditions

$$\boldsymbol{\theta}^{(1)}=0\qquad \boldsymbol{\theta}^{(2)}=\boldsymbol{\theta}^{(s)}\in\mathcal{S}\,.$$

We then have:

$$\boldsymbol{\varrho}_\Lambda = \gamma(\boldsymbol{\theta}^{(s)}_{\mathscr{L}a}) \cap \mathscr{B}_\Lambda = \begin{cases} \varrho_b = \theta^{(s)} & \text{if} \quad b = \langle xy \rangle \quad \text{with} \quad x \in \Lambda^d ; y \in \Lambda^u \\ \varrho_b = 0 & \text{otherwise.} \end{cases}$$

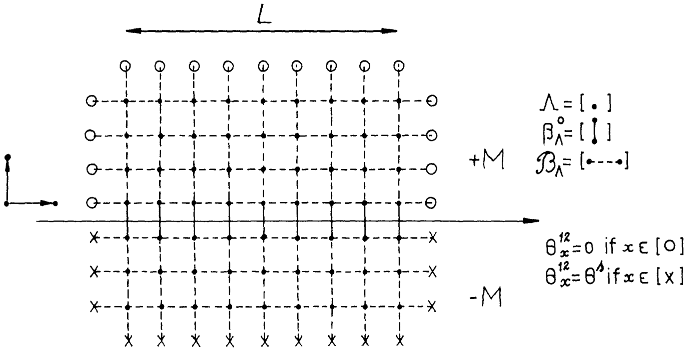


Fig. 2. Generalized Potts Model with boundary condition $\boldsymbol{\theta}^{12}$

Using the duality transformation (see Fig. 3), $\boldsymbol{\varrho}_\Lambda \mapsto \boldsymbol{\varrho}^*_\Lambda$ where

$$\varrho^*_{b^*} = \begin{cases} \theta^s & \text{if} \qquad \langle x^* y^* \rangle \subset \{x^*_v = 0\} \\ 0 & \text{otherwise} \end{cases}$$

the shortest graph $\mathbf{l}^*_{\Lambda^*}$ such that $\mathbf{l}^*_{\Lambda^*} - \varrho^*_\Lambda$ is a closed graph has length L.

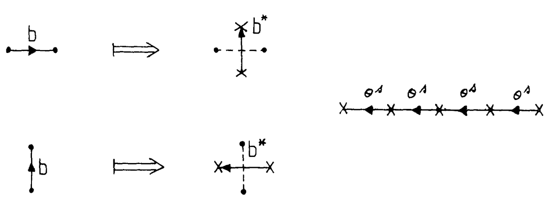


Fig. 3. a Duality transformation for Potts model. **b** $\boldsymbol{\varrho}^*_\Lambda$

Therefore if the potential is such that

$$V_{\langle xy \rangle}[0] > V_{\langle xy \rangle}[\theta] \; \forall \theta \neq 0 \quad \text{and} \quad e^{V_{\langle xy \rangle}[\cdot]} \in \mathscr{L}^1(\mathbb{Z}_q)$$

then $\tau^{(12)} \neq 0$ at low temperature.

Furthermore *for $q < \infty$ we conclude that there exists a phase transition associated with surface tension.*

2. Triangular Model with 3 Body forces

$\mathscr{L}$ = triangular lattice; for all $x \in \mathscr{L}$, $\mathscr{G}_x = \mathbb{Z}_q$ $q \leqq \infty$

$V_b(\boldsymbol{\theta}) = V_{\langle xyz \rangle}[\theta_x + \theta_y + \theta_z]$ where $\langle xyz \rangle$ denotes elementary triangles.

In this case:

$$\mathscr{B} = \mathscr{B}^1 = \{b = \langle xyz \rangle ; (x, y, z,) \subset \mathscr{L}\} = \text{elementary triangle},$$

$$\mathscr{G}_b^{\hat{}} = \mathbb{Z}_q \quad \begin{array}{ll} \mathscr{G}_b = \mathbb{Z}_q & \text{if} \quad q < \infty \\ \mathscr{G}_b = T^{(1)} & \text{if} \quad q = \infty , \end{array}$$

$$\gamma_b(\boldsymbol{\theta}) = \theta_x + \theta_y + \theta_z \bmod q ,$$

$$\begin{aligned} \mathscr{S} = \{\boldsymbol{\theta}^{(s)} ; \theta_x^{(s)} = \theta_a \in \mathbb{Z}_q \quad & \text{if} \quad x \in \mathscr{L}^A \\ \theta_y^{(s)} = \theta_b \in \mathbb{Z}_q \quad & \text{if} \quad y \in \mathscr{L}^B \\ \theta_z^{(s)} = -(\theta_a + \theta_b) = \theta_c \quad & \text{if} \quad z \in \mathscr{L}^C\} , \end{aligned}$$

where $\mathscr{L} = \mathscr{L}^A \cup \mathscr{L}^B \cup \mathscr{L}^C$ (see Fig. 4).

Let us then take the boundary conditions,

$$\boldsymbol{\theta}^1 = 0 \qquad \boldsymbol{\theta}^2 = \boldsymbol{\theta}^{(s)} \in \mathscr{S}$$

we then have:

$$\boldsymbol{\varrho} = \gamma(\theta^{(s)}_{\mathscr{L}^d}) = \begin{cases} \varrho_{\langle xyz \rangle} = \theta_x^{(s)} & \text{if} \quad x \in \mathscr{L}^d, (yz) \subset \mathscr{L}^u \\ \varrho_{\langle xyz \rangle} = \theta_x^{(s)} + \theta_y^{(s)} & \text{if} \quad (xy) \subset \mathscr{L}^d \, z \in \mathscr{L}^u \\ \varrho_{\langle xyz \rangle} = 0 & \text{otherwise}. \end{cases}$$

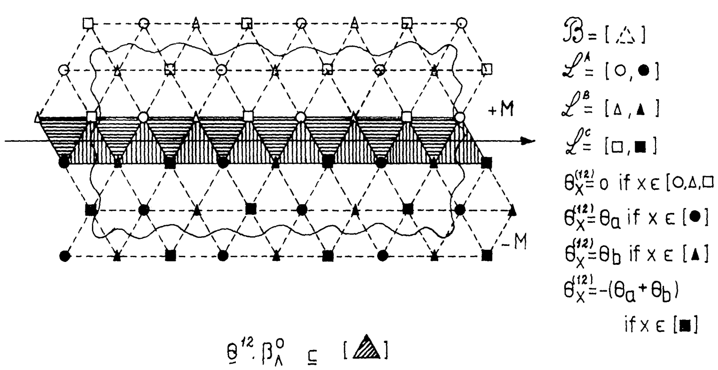


Fig. 4. Triangular model with 3-body forces and boundary condition

Using a duality transformation [1] $b \mapsto b^* = b$ and

$$\gamma^*_{b^*}(\boldsymbol{\theta}^*) = \theta^*_x + \theta^*_y + \theta^*_z \qquad \text{if} \quad \begin{array}{c} x \\ y \;\; z \end{array}$$

$$\gamma^*_{b^*}(\boldsymbol{\theta}^*) = -(\theta^*_x + \theta^*_y + \theta^*_z) \quad \text{if} \quad \begin{array}{c} y \;\; z \\ x \end{array}$$

it follows that the shortest graph $\mathbf{l}^*_\Lambda$ such that $\mathbf{l}^*_\Lambda - \varrho^*_\Lambda$ is a closed graph has length L.

Therefore if the potential is such that

$$V_b[0] > V_b[\theta] \quad \text{for all} \quad \theta \neq 0$$

then *there exists a phase transition with surface tension*, if $q<\infty$. Note that for $q=\infty$ we can conclude that the surface tension is non zero at low temperature if $V_b[0] > V_b[\theta]$ and $e^{V_b} \in \mathscr{L}^1$.

3. Ashkin-Teller's Model

$$\mathscr{L}=\mathbb{Z}_A^2 \cup \mathbb{Z}_B^2 \quad \text{for all} \quad x \in \mathscr{L}, \mathscr{G}_x=\mathbb{Z}_2,$$

$$\begin{aligned}\mathscr{B}=\{\langle xy\rangle, \langle x'y'\rangle, \langle xyx'y'\rangle; \quad & (xy)\subset \mathbb{Z}_A^2 \quad |x-y|=1\\ & (x'y')\subset \mathbb{Z}_B^2 \quad |x'-y'|=1\\ & |x-x'|<1 \quad |y-y'|<1\},\end{aligned}$$

$$\hat{\mathscr{G}_b}=\mathbb{Z}_2,$$

$$\gamma_b(\boldsymbol{\theta})=\theta_x+\theta_y \bmod 2 \qquad \text{if} \quad b=\langle xy\rangle \quad \text{or} \quad \langle x'y'\rangle,$$

$$\gamma_b(\boldsymbol{\theta})=\theta_x+\theta_y+\theta_{x'}+\theta_{y'} \bmod 2 \quad \text{if} \quad b=\langle xyx'y'\rangle,$$

$$\mathscr{S}=\left\{\boldsymbol{\theta}^{(\mathrm{I})}=(0); \quad \boldsymbol{\theta}^{\mathrm{II}}=(1); \quad \boldsymbol{\theta}^{\mathrm{III}}=\begin{cases}1 & \text{if} \quad x\in\mathbb{Z}_A^2\\ 0 & \text{if} \quad x\in\mathbb{Z}_B^2\end{cases}; \quad \boldsymbol{\theta}^{\mathrm{IV}}=\boldsymbol{\theta}^{\mathrm{II}}+\boldsymbol{\theta}^{\mathrm{III}}\right\}.$$

Let us then take the boundary condition

$$\boldsymbol{\theta}^1=(0) \qquad \boldsymbol{\theta}^2=\boldsymbol{\theta}^{\mathrm{II}}$$

we then have:

$$\boldsymbol{\varrho}_\Lambda=\gamma(\boldsymbol{\theta}^{\mathrm{II}}_{\mathscr{L}^d})\cap\mathscr{B}_\Lambda=\begin{cases}\varrho_b=1 & \text{if} \quad b=\langle xy\rangle \quad x\in\Lambda^u y\in\Lambda^d\\ \varrho_b=1 & \text{if} \quad b=\langle x'y'\rangle \quad x'\in\Lambda^u y'\in\Lambda^d\\ \varrho_b=0 & \text{otherwise.}\end{cases}$$

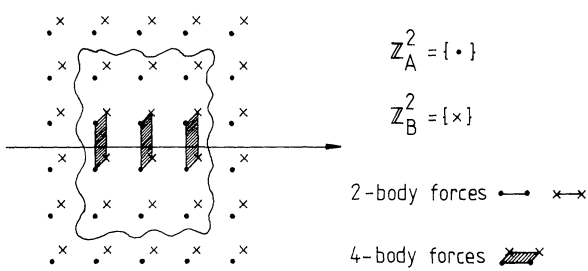


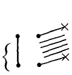
Fig. 5. Ashkin-Teller model

i) $\boldsymbol{\theta}^2=\boldsymbol{\theta}^{(\mathrm{II})}$

$$\boldsymbol{\varrho}_\Lambda=\begin{cases}\varrho_b=1 & \text{if} \qquad b\in\{\ \}\\ \varrho_b=0 & \text{otherwise}\end{cases}$$

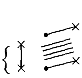
ii) $\boldsymbol{\theta}^2=\boldsymbol{\theta}^{(\mathrm{III})}$

$$\boldsymbol{\varrho}_\Lambda=\begin{cases}\varrho_b=1 & \text{if} \qquad b\in\{\ \}\\ \varrho_b=0 & \text{otherwise}\end{cases}$$



iii) $\boldsymbol{\theta}^2=\boldsymbol{\theta}^{(\mathrm{IV})}$

$$\varrho_\Lambda=\begin{cases}\varrho_b=1 & \text{if} \qquad b\in\{\ \}\\ \varrho_b=0 & \text{otherwise.}\end{cases}$$



Applying a duality transformation, [1]

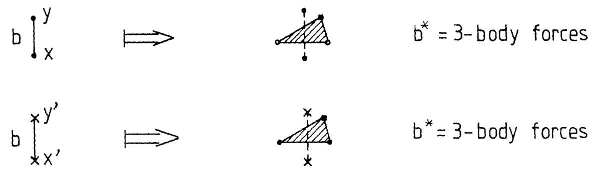


Fig. 6a

we obtain for ϱ_Λ^* the graph:

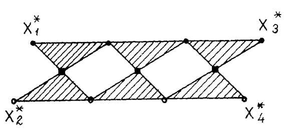


Fig. 6b

Therefore for each site $x^* \in \mathscr{L}^*$, except x_i^*, $i=1, \ldots, 4$, the number of bonds of ϱ_Λ^* incident on x^* is even: it thus follows that the shortest graph $\mathbf{l}_{\Lambda^*}^*$ such that $\mathbf{l}_{\Lambda^*}^* - \varrho_\Lambda^*$ is closed has length of the order of L. Again, we then conclude that there exists a phase transition with surface tension if the interactions are ferromagnetic.

4. Generalized Potts Model on Cubic Lattice

$\mathscr{L} = \mathbb{Z}^3$; for all $x \in \mathscr{L}$ $\quad \mathscr{G}_x = \mathbb{Z}_q \quad q \leqq \infty$,

$V_b(\boldsymbol{\theta}) = V_{\langle xy \rangle}[\theta_x - \theta_y]$ where $\langle xy \rangle$ denotes oriented nearest neighbours on $\mathbb{Z}^3$.

$\mathscr{G}_{\hat{b}} = \mathbb{Z}_q$

Let us take the boundary conditions $\boldsymbol{\theta}^1 = (0) \quad \boldsymbol{\theta}^2 = (\theta_x^{(s)} = \theta^{(s)})$.

Again introducing a duality transformation we conclude that there exists a phase transition associated with surface tension, if $q < \infty$.

If $q = \infty$ the conclusion is the same as in Examples 1 and 2.

5. "Plaquettes"

The model is defined on Fig. 7

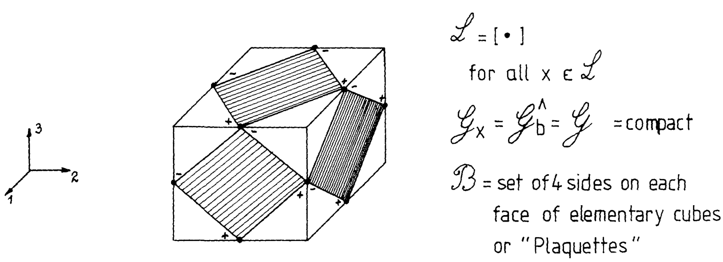


Fig. 7. The "Plaquette" model

To define the interactions V_b we associate with each site x of the plaquette b a given sign ε_x (see Fig. 7) and we have:

$$V_b(\boldsymbol{\theta}) = V\left[\sum_{x\in b} \varepsilon_x \theta_x\right]$$

i.e.

$$\gamma_b(\boldsymbol{\theta}) = \sum_{x\in b} \varepsilon_x \theta_x .$$

Taking the boundary conditions

$$\boldsymbol{\theta}^{(1)} = 0, \boldsymbol{\theta}^{(2)} \in \mathscr{S}, \text{i.e.} \sum_{x\in b} \varepsilon_x \theta_x^{(2)} = 0 \text{ we have:}$$

$$\varrho_b = \begin{cases} 0 & \text{if } b \text{ does not intersect } x_\nu = 0 \\ \theta_x^{(2)} & \text{if } b \text{ intersect } x_\nu = 0 \text{ and } x \in b \cap \mathscr{L}^u . \end{cases}$$

Furthermore we note that the condition $\boldsymbol{\theta}^{(2)} \in \mathscr{S}$ implies

$$\sum_{\substack{b\in \text{Elementary cubes} \\ \text{Intersecting } x_\nu = 0}} \varepsilon_b^* \varrho_b = 0 \quad \text{where} \quad \varepsilon_b^* = \varepsilon_{b\cap \mathscr{L}^u} .$$

The standard duality transformation yields the generalized Potts Model on $\mathbb{Z}^3$ with $b^* = \langle x^* y^* \rangle =$ nearest neighbour orthogonal to the plaquette b oriented along the axis; indeed

$$\gamma(\theta_x) = \begin{cases} \varepsilon_x \theta_x & \text{if } b \ni x \text{ otherwise} \\ 0 \end{cases} \quad \xleftrightarrow{\text{Duality}}$$

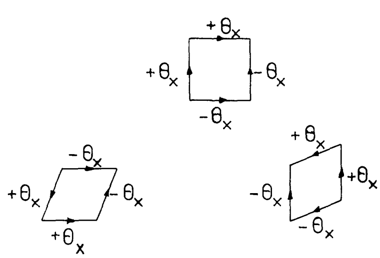


Fig. 8a

$$\text{Generators of } \Gamma_\Lambda \underset{\text{duality}}{\Longleftrightarrow} \text{Generators of } \mathscr{K}^*_{\Lambda^*} .$$

For this duality transformation β_0^* is a subset of the bonds in the $x_\nu = 0$ plane and we have for all $x^* \in \Lambda^* \backslash \partial \Lambda^*$

$$\sum_{b^* \ni x^*} \varepsilon_{b^*} \varrho_{b^*} = 0 \quad \text{where} \quad \varepsilon_{b^*} = +1 \quad \text{if} \quad \underset{x^*}{\bullet}\!\xrightarrow{b^*}$$

$$\varepsilon_{b^*} = -1 \quad \text{if} \quad \rightarrow\!\bullet\, x^*$$

$$\text{i.e. } \prod_{b^*} \langle \gamma^*_{b^*}(\theta^*_{x^*}); \varrho_{b^*} \rangle = 1 \qquad \forall x^* \in \Lambda^* \backslash \partial \Lambda^*$$

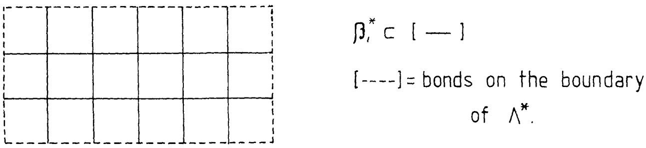


Fig. 8b. The $x_\mu = 0$ plane for the dual of the "Plaquette" model

By induction on the bonds on the boundary of Λ^* (see Fig. 8) we can always find $\mathbf{k}^*$ in $\mathscr{K}_{\Lambda^*}^*$ such that $|\mathbf{\varrho}_\Lambda^* + \mathbf{k}^*| \leqq 2(L_1 + L_2 + 2)$.

Therefore for any $\boldsymbol{\theta}^{(s)}$ *in* $\mathscr{S}$ *the surface tension is zero between the state* $\omega^{(+)}$ *and* $\omega^{(s)}$ *at all temperature.*

This result suggests that the surface tension could be non-zero only between distinct states since it is known that $\omega^{(+)} = \omega^{(s)}$.

Acknowledgements. One of us (C.G.) would like to thank Prof. J.-P. Antoine for his kind invitation at the University of Louvain-La-Neuve, where this work was started.

Appendix 1

Let $\mathscr{B}_\Lambda$ be the family of bonds intersecting Λ; b_1 and b_2 in $\mathscr{B}_\Lambda$ are connected by a line if $B_{b_1} \cap B_{b_2} \neq \emptyset$. We consider $\mathscr{X}_\Lambda$ the space of subsets with repetition of the family of *connected* graphs $\beta \subset \mathscr{B}_\Lambda$ with $|\beta| \geqq 2$. For all $X = (\beta^1, \ldots, \beta^q) \in \mathscr{X}_\Lambda$

$$G(X) = \prod_{(\beta^i, \beta^j) \subset X} g(\beta^i, \beta^j),$$

$$g(\beta^i, \beta^j) = \begin{cases} 1 & \text{if } \beta_i \text{ is not connected to } \beta_j \\ 0 & \text{if } \beta_i \text{ is connected to } \beta_j, \end{cases}$$

$$[X] = \bigcup_{\beta \in X} \left[\bigcup_{b \in \beta} B_b \right] = \text{sites of } \mathscr{L} \text{ covered by } X.$$

Lemma 2. *There exists* $z_0 \in]0, 1]$ *such that*

$$\sup_{x \in \mathscr{L}} \sum_{\substack{X \in \mathscr{X}_\Lambda \\ [X] \ni x \\ \operatorname{diam}[X] > d}} z_0^{|X|} |G_T(X)| \leqq (c_1 + c_2 d^\nu) e^{-c_3 d^{1/2}},$$

$$c_i \geqq 0,$$

$$|X| = \sum_{\beta \in X} |\beta| \qquad |\beta| = \text{cardinality of } \beta \subset \mathscr{B}_\Lambda.$$

Proof. This proof is the immediate generalization of the one given in [5] for the Ising model and we shall indicate only the changes to be introduced.

Let

$$\phi(X) = z^{|X|} G(X)$$

then

$$\phi_T(X) = z^{|X|} G_T(X)$$

and $\phi_T(X) = 0$ if X is not connected.

Since the first part of [5] are purely algebraic relations Eq. (4.27) of [5] becomes

$$I_{m+1} \leqq I_m z^{\frac{|\beta|}{2}} \exp\left[\sum_{\beta' \text{ connected } \beta} z^{\frac{|\beta'|}{2}}\right].$$

Let a be the maximum coordination number of the graph $\mathscr{B}_\Lambda$ (where the coordination number of b is the number of b' such that $B_b \cap B_{b'} \neq \emptyset$), then

$$\sum_{\beta' \text{ connected} \beta} z^{1/2|\beta'|} = \sum_{n \geqq 2} z^{1/2n}. \,(\#\text{ of } \beta' \text{ with } |\beta'| = n \text{ connected to } \beta.)$$

But the number of β' containing a given b, with $|\beta'|=n$, is bounded by $a^{(2n-2)}$; therefore

$$\sum_{\beta' \text{ connected } \beta} z^{1/2|\beta'|} \leqq (a+1)|\beta| \sum_{n \geqq 2} z^{1/2n} a^{(2n-2)} = (a+1)|\beta| \frac{a^2 z}{1-a^2 z^{1/2}}$$

and:

$$I_{m+1} \leqq I_m \exp|\beta| \left[-\frac{\mu}{2} + (a+1)\frac{a^2 e^{-\mu}}{1-a^2 e^{-\mu/2}}\right]$$

with

$$z = e^{-\mu}.$$

It follows that there exists $z_0 < a^{-4}$ such that

$$I_{m+1} \leqq I_m z^\kappa \quad \text{with} \quad 1/2 < \kappa < 1 \quad \text{for} \quad z < z_0.$$

Furthermore $I_1 = \sup_\beta |\phi(\beta)| z^{-1/2|\beta|} = \sup_{|\beta| \geqq 2} z^{1/2|\beta|} \leqq z$ which yields $I_m \leqq z^{\kappa m}$.

Introducing $\delta = \sup_b (\operatorname{diam} B_b)$ we conclude the proof as in [5].

Appendix 2

Lemma 5. *The number of graphs $\beta \subset \mathscr{B}_\Lambda$ with $|\beta| = q$, such that all its connected components are connected to a given β_0 is bounded by:*

$$4^q \alpha^q \quad \textit{if} \quad q \geqq |\beta_0|, \tag{1}$$

$$4^{|\beta_0|} \alpha^q \quad \textit{if} \quad q \leqq |\beta_0|. \tag{2}$$

Proof. Define $A^q = \left\{f^q;\ f^q : [1|\beta_0|] \to [0q] \text{ and } \sum_{p \in [1|\beta_0|]} f^q(p) = q\right\}$ with

$[1|\beta_0|] \subset \mathbb{N}, [0q] \subset \mathbb{N}$.

Each f^q defines for each $b \in \beta_0$ the length of the connected component of β which is connected to it. Since the number of connected graphs of length q containing a given b is bounded by α^q (Appendix 1) to each f^q there corresponds at most α^q graphs which satisfies the conditions of our lemma. It remains therefore to bound $\# A^q$.

$$\# A^q = \binom{|\beta_0| + q - 1}{q}$$

for

$$q \geqq |\beta_0| \binom{|\beta_0|+q-1}{q} \leqq \binom{2q-1}{q}, \tag{1}$$

for

$$q \leqq |\beta_0| \binom{|\beta_0|+q-1}{q} \leqq \binom{2|\beta_0|-1}{q}. \tag{2}$$

By application of the Stirling's formulae:

$$n^n \sqrt{2\pi n} \exp - n \leqq n! \leqq n^n \sqrt{2\pi n} \exp \left[\frac{1}{12n} - n\right]$$

One has:

$$\binom{2n-1}{n} \leqq 4^n$$

α^q. ($\# A^q$) gives for cases (1) and (2) the desired bounds.

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