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Some Results for the Exponential Interaction in Two or More Dimensions

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Abstract. We show that for the regularized exponential interaction $\lambda : e^{\alpha\varphi} :$ in d space-time dimensions the Schwinger functions converge to the Schwinger functions for the free field if $d>2$ for all α or if $d=2$ for all α such that $|\alpha|>\alpha_0$.

1. Notations and Results

In this paper we study the space-time cut-off exponential interaction in d space-time dimensions $V_\Lambda = \lambda \int_\Lambda :e^{\alpha\xi(x)}:dx$, where Λ is a bounded subset of $\mathbb{R}^d$, $\lambda > 0$ and the corresponding Euclidean measure $d\mu_\Lambda(\xi) = Z_\Lambda^{-1} e^{-V_\Lambda(\xi)} d\mu_0(\xi)$, μ_0 being the free Euclidean field of mass 1 on $\mathbb{R}^d$ [1], $\alpha \in \mathbb{R}$ and : : being the Wick ordering (see below for details on notation). Such models of quantum fields were introduced in [2], and in [2, 3] it was shown that if $d=2, |\alpha| < \sqrt{4\pi}$ then $V_\Lambda \in L_2(d\mu_0)$ and μ_Λ is a (non Gaussian) probability measure. The existence of a measure μ_Λ of the above form was shown for all $d \geqq 2$ and arbitrary α in [4], see also [5] and, for a different proof, [6][1]. In [5] it was shown that in the case $d \geqq 4$ the regularized (ultraviolet cut off) version of the measure μ_Λ converges as the regularization is removed to μ_0. In the present paper we tackle, using a modification of the basic idea of [5] together with methods of [7], the case $d \geqq 3$ and also the case $d=2$ for $|\alpha|$ large. The results of the present paper were announced in [10]. Let us now give the notations and state the results. We define the free field on $\mathbb{R}^d$ with ultraviolet cut off at distance γ^{-N}, $\gamma > 1$, N a positive number, as the Gaussian field ξ_N

$$\xi_N(x) = \int A_N(x-y)\xi(y)dy, \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where A_N is the kernel of the operator

$$A_N = \left(\frac{\gamma^{2N}}{\gamma^{2N} - \Delta}\right)^{\frac{1+k_d}{2}} \tag{1.2}$$

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1 Other references for the exponential interaction are e.g. [8, 9]

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and $\xi(x)$ is the free Euclidean field of mass 1 on $\mathbb{R}^d$ [1] i.e. the Gaussian field with mean zero and covariance operator $(1-\Delta)^{-1}$, Δ being the Laplacian on $\mathbb{R}^d$. k_d is the smallest integer larger or equal to $\frac{d-1}{2}$. Thus ξ_N has mean zero and covariance operator

$$C_N = \frac{1}{1-\Delta}\left(\frac{\gamma^{2N}}{\gamma^{2N}-\Delta}\right)^{1+k_d}. \tag{1.3}$$

The choice of k_d is such that $C_N(x)$ is a Lipschitz continuous function on $\mathbb{R}^d$ vanishing at infinity together with its local Lipschitz constant.

Let I be a cube. The exponential interaction with infrared (i.e. volume) cut-off I and ultraviolet (i.e. momentum) cut-off γ^{-N} is defined as

$$V_{N,I}(\xi) = \lambda \int_I :e^{\alpha\xi_N(x)}: dx = \lambda \int_I e^{-\frac{\alpha^2}{2}C_N(0)} e^{\alpha\xi_N(x)} dx. \tag{1.4}$$

Since C_N is Lipschitz-continuous with bounded Lipschitz constant, the distributions ξ_N are almost surely continuous with respect to the probability measure μ_0 of the free Euclidean field ξ.

The Euclidean probability measure associated with (1.4) is defined by

$$d\mu_{N,I}(\xi) = Z(N,I)^{-1} e^{-V_{N,I}(\xi)} d\mu_0(\xi), \tag{1.5}$$

$$Z(N,I) = \int e^{-V_{N,I}(\xi)} d\mu_0(\xi). \tag{1.6}$$

The Schwinger functions of $\mu_{N,I}$ are defined as

$$S_{N,I}(f_1, \ldots, f_n) = \int \prod_{j=1}^{n} \langle f_i, \xi \rangle d_{\mu_{N,I}}(\xi) \tag{1.7}$$

for $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^d)$, where $\langle\, , \rangle$ denotes the duality between $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d)$. The ground state energy $E_{N,I}(\lambda, \alpha)$ is defined as

$$E_{N,I}(\lambda, \alpha) = \frac{1}{|I|} \log Z(N,I). \tag{1.8}$$

Our result is the following:

Proposition. 1) *If $d=2$ there exists $\alpha_0 > \sqrt{4\pi}$ such that for all $\alpha > \alpha_0$ and all $\lambda \geq 0$:*

(i) $\lim_{N\to\infty} Z(N,I) = 1$;

(ii) $\lim_{N\to\infty} S_{N,I}(f_1, \ldots, f_n) = \int \prod_{j=1}^{n} \langle f_i, \xi \rangle d\mu_0(\xi)$.

for all $f_i \in \mathscr{S}(\mathbb{R}^d)$, $i=1, \ldots, n$.

2) *If $d>2$ the above relations hold for all $\alpha \geq 0$, all $\lambda \geq 0$, and, furthermore, the limit $\lim_{N\to\infty} E_{N,I} = 0$ is attained uniformly in I.*

Remark. 1) It is known [3] that for $d=2$ and $\alpha^2 < 4\pi$ the $\lim_{N\to\infty} V_{N,I}(\xi) = V_I(\xi)$ exists in $L^2(d\mu_0)$ and is non zero. Hence (ii) does not hold in this case. In this case

$\lim_{N\to\infty} S_{N,I}(f_1, \dots, f_n)$ are the Schwinger functions of a non Gaussian measure and in fact the limit $I \to \mathbb{R}^d$ gives the non trivial Schwinger functions of the exponential interaction studied in [3, 8].

2) The existence of $\lim_{N\to\infty} S_{N,I}(f_1, \dots, f_n)$ was shown for all $d \geqq 2$, $\forall \alpha \in \mathbb{R}$, $\forall \lambda \geqq 0$ in [4] (see also [5] and for a different proof [6]). In [5] it was shown that for $d \geqq 4$, $\forall \alpha \in \mathbb{R}$, $\forall \lambda \geqq 0$ $\lim_{N\to\infty} V_{N,I}$ exists μ_0 – a.s. and is zero μ_0 – a.s., which a fortiori yields the statements in the proposition, in this case. This is related to the irreducibility of the energy representation of Sobolev-Lie groups (Ismagilov and Vershik, Gelfand, Graev [11]).

2. Proof of the Proposition

Since $V_{N,I} \geqq 0$ we see that (ii) follows from (i), by dominated convergence. In fact (i) is equivalent to $\lim_{N\to\infty} e^{-V_{N,I}} = 1$ in μ_0-measure.

Let us now prove (i). We introduce an auxiliary Gaussian measure $\widetilde{\mu_N}$ describing a Gaussian field $\tilde{\xi}_N$ with covariance operator:

$$\begin{aligned} \tilde{C}_N &= \gamma^{2N(1+k_d)}(\gamma^{2N} - \Delta)^{-2-k_d} \quad \text{if} \quad d>2 \\ \tilde{C}_N &= C_N \quad \text{if} \quad d=2. \end{aligned} \tag{2.1}$$

It is easy to check that $\tilde{C}_N \leqq C_N$. We shall realize the field $\tilde{\xi}_N$ in the form

$$\tilde{\xi}_N(x) = \tilde{C}_N(0)^{1/2} \eta(\gamma^N x), \tag{2.2}$$

where η is a Gaussian random field with covariance $\overline{C_N}$ satisfying the inequality

$$0 \leqq \overline{C_N}(0) - \overline{C_N}(x-y) \leqq a|x-y|, \qquad \overline{C_N}(0) = 1, \tag{2.3}$$

for some N-independent constant $a>0$. Notice that for $d>2$, $\overline{C_N}$ is actually N-independent. Also $\tilde{C}_N(0)$ is proportional to $\gamma^{(d-2)N}$ if $d>2$ and to N if $d=2$. Since $\tilde{C}_N \leqq C_N$ we can use the "conditioning inequality" (e.g., [9a]) to obtain (for a derivation in our case see the Appendix)

$$\begin{aligned} 1 \geqq Z(N,I) &= \int e^{-V_{N,I}} d\mu_0 \\ &\geqq \int e^{-\tilde{V}_{N,I}} d\tilde{\mu}_N, \end{aligned}$$

with

$$\widetilde{V_{N,I}}(\tilde{\xi}_N) = \lambda \int_I e^{-\frac{\alpha^2}{2}\widetilde{C_N}(0)} e^{\alpha \tilde{\xi}_N(x)} dx. \tag{2.4}$$

Hence, if A is an arbitrary $\widetilde{\mu_N}$-measurable set,

$$1 \geqq Z(N,I) \geqq \int_A e^{-\widetilde{V_{N,I}}(\tilde{\xi}_N)} d\widetilde{\mu_N}(\tilde{\xi}_N). \tag{2.5}$$

We choose

$$\begin{aligned} A &= \{\widetilde{\xi_N} \,|\, |\widetilde{\xi_N}(x)| \leqq B\sqrt{N}(\widetilde{C_N}(0))^{1/2} \forall x \in I\} \\ &= \{\widetilde{\xi_N} \,|\, |\eta(\gamma^N x)| \leqq B\sqrt{N}, \forall x \in I\} \\ &= \bigcap_{q \subset \gamma^N I} A_q, \end{aligned} \tag{2.6}$$

where γ^{N_I} is the homothetic image of I by the scale factor γ^N, q denotes an element of a pavement of γ^{N_I} by unit cubes (assuming the sides of γ^{N_I} integer, for simplicity) and

$$A_q \equiv \{\tilde{\xi}_N \,|\, |\eta(x)| \leqq B\sqrt{N}, \forall x \in q\}.$$

Denote by χ_{A_q} the characteristic function of the event A_q. The choice (2.6) yields then from (2.5)

$$1 \geqq Z(N,I) \geqq \exp\{-\lambda|I| e^{-\frac{\alpha^2}{2}\tilde{C}_N(0)} e^{\alpha B\sqrt{N}\tilde{C}_N(0)^{1/2}}\} \cdot \int \prod_{q \subset \gamma^{N_I}} \chi_{A_q}(\tilde{\xi}_N) d\tilde{\mu}_N(\tilde{\xi}_N). \tag{2.7}$$

We now see that to prove (i) it is enough to prove

$$\int \prod_{q \subset \gamma^{N_I}} \chi_{A_q} d\tilde{\mu}_N \to 1 \quad \text{as} \quad N \to \infty$$

[using the observations following (2.3)]. This can however be obtained from

$$\int \prod_{q \subset \gamma^{N_I}} \chi_{A_q} d\tilde{\mu}_N \geqq 1 - \sum_{q \subset \gamma^{N_I}} \int (1 - \chi_{A_q}) d\tilde{\mu}_N. \tag{2.8}$$

In fact it is well known that (2.3) implies

$$\int (1 - \chi_{A_q}) d\tilde{\mu}_N \leqq c_1 e^{-c_2 B^2 N}, \qquad \forall B \geqq B_0, \tag{2.9}$$

for suitable chosen constants $c_1, c_2, B_0 > 0$ (see, e.g., [7], where some of the original proofs are quoted: the earlier proof goes, essentially, back to Wiener). Hence from (2.8), (2.9):

$$\int \prod_{q \subset \gamma^{N_I}} \chi_{A_q} d\tilde{\mu}_N \geqq 1 - |I|\gamma^{dN} c_1 e^{-c_2 B^2 N} \to 1, \quad N \to \infty, \tag{2.10}$$

for all B large enough ($B^2 > c_2^{-1} \log \gamma^d$). This then proves (i), hence by what we remarked at the beginning of the proof, all the statements in the proposition, except for the uniformity statement. The latter follows however from the better bound in [7]

$$\int \prod_q \chi_{A_q} d\tilde{\mu}_N \geqq \exp\{-|I|\gamma^{dN} \bar{c}_1 e^{-\bar{c}_2 B^2 N}\} \tag{2.11}$$

valid for $B > \overline{B_0}$ and for suitable chosen constants $\bar{c}_1, \bar{c}_2, \bar{B}_0 > 0$, (N, I)-independent. This ends the proof.

Remark. The methods of [7] should, in principle, allow to deduce (2.11) even in the case $d = 2$: the proof however cannot be trivially extracted from [7]. Originally we were hoping to obtain such a bound with some extra work: it was then pointed out to us by Fröhlich that, in any event, the elementary bound (2.9) could be used for the same purpose, losing only the uniformity property.

3. Comments

The above proof is clearly based on the following heuristic argument: the covariance of the field ξ_N is smooth and $C_N(0) \sim \gamma^{(d-2)N}$ if $d > 2$ and $C_N(0) \sim N$ if

$d=2$. Hence the field $|\xi_N(x)|$ should never exceed by too much $B_N\sqrt{C_N(0)}$ if B_N is very large. Therefore

$$V_{N,I} \leqq \lambda |I| e^{-\frac{\alpha^2}{2}\tilde{C}_N(0)} e^{B_N \alpha \sqrt{C_N(0)}}. \tag{3.1}$$

If B_N is so chosen that the r.h.s. of (3.1) tends to zero the result will follow provided one can show that the phase space restriction $|\xi_N(x)| \leqq B_N \sqrt{\tilde{C}_N(0)}$ does not cause a substantial loss of phase space: this is indeed the content of the estimates (2.10), (2.11). Actually if $d>2$ any choice $B_N = BN^\delta$, $\delta \geqq \frac{1}{2}$, would be sufficient: for $d=2$ the situation is quite delicate and one has to choose $B_N = B\sqrt{N}$ which, however, is good only for α large enough.

Appendix

We shall derive the "conditioning inequality" (2.4). Since $\tilde{C}_N \leqq C_N$ we have that $C'_N = C_N - \tilde{C}_N$ is a positive definite operator, hence there exists a Gaussian random field ξ'_N with covariance C'_N. If we consider $\tilde{\xi}_N$ and ξ'_N as two independent random fields then ξ_N has the same distribution as $\tilde{\xi}_N + \xi'_N$. Hence

$$Z(N,I) = \int \left[\int \exp\left[-\lambda \int_I e^{-\frac{\alpha^2}{2}\tilde{C}_N(0)} e^{\alpha \tilde{\xi}_N(x)} e^{-\frac{\alpha^2}{2}C'_N(0)} e^{\alpha \xi'_N(x)} dx \right] \cdot d\mu'_N(\xi'_N) \right] d\tilde{\mu}_N(\tilde{\xi}_N),$$

where μ'_N is the probability measure corresponding to ξ'_N, thus by Jensen's inequality

$$Z(N,I) \geqq \int \exp\left[-\lambda \int_I \left[e^{-\frac{\alpha^2}{2}\tilde{C}_N(0)} e^{\alpha \tilde{\xi}_N(x)} \cdot \int e^{-\frac{\alpha^2}{2}C'_N(0)} e^{\alpha \xi'_N(x)} d\mu'_N(\xi'_N) dx \right] d\tilde{\mu}_N(\tilde{\xi}_N) \right].$$

But since

$$\int e^{-\frac{\alpha^2}{2}C'_N(0)} e^{\alpha \xi'_N(x)} d\mu'_N(\xi'_N) = 1,$$

(2.4) follows.

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Note added in proof. E. P. Osipov has recently shown $\alpha_0 = \sqrt{8\pi}$ (Novosibirsk preprint).