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Boundary Regularity for Some Nonlinear Elliptic Degenerate Equations*

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Abstract. We consider the nonlinear elliptic degenerate equation

$$-x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+2u=f(u) \quad \text{in } \Omega_a, \tag{1}$$

where

$\Omega_a=\{(x,y)\in\mathbb{R}^2, 0<x<a, |y|<a\}$

for some constant $a>0$ and f is a C^∞ functions on $\mathbb{R}$ such that $f(0)=f'(0)=0$. Our main result asserts that: if $u\in C(\bar{\Omega}_a)$ satisfies

$$u(0,y)=0 \quad \text{for} \quad |y|<a, \tag{2}$$

then $x^{-2}u(x,y)\in C^\infty(\bar{\Omega}_{a/2})$ and in particular $u\in C^\infty(\bar{\Omega}_{a/2})$.

1. Introduction

This paper deals with the question of boundary regularity of solutions of a nonlinear elliptic degenerate equation of the form

$$-x^2\Delta u+2u=f(u) \quad \text{in } \Omega_a, \tag{1}$$

where

$$\Delta=D_x^2+D_y^2$$

$$\Omega_a=\{(x,y)\in\mathbb{R}^2; 0<x<a, |y|<a\}$$

for some constant $a>0$, and f is a C^∞ function on $\mathbb{R}$ such that

$$f(0)=f'(0)=0. \tag{2}$$

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Our main result is the following:

Theorem 1. *Assume* $u \in C^\infty(\Omega_a) \cap C(\bar{\Omega}_a)$ *satisfies* (1) *and*

$$u(0, y) = 0 \quad \textit{for} \quad |y| < a. \tag{3}$$

Then $x^{-2}u(x, y) \in C^\infty(\bar{\Omega}_{a/2})$ *and in particular* $u \in C^\infty(\bar{\Omega}_{a/2})$.

Equation (1) occurs in the theory of multimeron solutions to Yang-Mills field equations [2]. More precisely the equation in [2] is:

$$-x^2 \Delta \psi + \psi^3 - \psi = 0 \quad \text{in } \Omega_a$$

together with the boundary conditions:

$$\psi(0, y) = \pm 1.$$

If we set $u = \psi \mp 1$ we find

$$-x^2 \Delta u + (u \pm 1)^3 - (u \pm 1) = 0$$

that is (1) with $f(u) = -u^3 \mp 3u^2$. In [3] it is only proved that ψ is continuous up to the boundary (except at the points where ψ changes sign). Theorem 1 shows that ψ is C^∞ up to the boundary (except at the points where ψ changes sign).

2. Some Lemmas

The proof relies on some lemmas

Lemma 2. *Assume* $u \in C^2(\Omega_a) \cap C(\bar{\Omega}_a)$ *satisfies:*

$$|-x^2 \Delta u + 2u| \leqq \alpha(u^2 + x^4) \quad \textit{on } \Omega_a \tag{4}$$

for some constant α, *and*

$$u(0, y) = 0 \quad \textit{for} \quad |y| < a. \tag{5}$$

Then, there is a constant β such that

$$|u(x, y)| \leqq \beta x^2 \quad \text{on } \Omega_{a/2}.$$

Proof of Lemma 2. For $b < a$ set

$$M_b = \operatorname*{Sup}_{\Omega_b} |u|.$$

Since by (5) $M_b \to 0$ as $b \to 0$, we may fix b so small that

$$\alpha b^2 < 1/2 \tag{6}$$

$$\alpha M_b < 1/400. \tag{7}$$

We shall establish that

$$|u(x, 0)| \leqq A x^2 \quad \text{for} \quad 0 < x < b, \tag{8}$$

where

$$A = \operatorname{Max}\left\{\alpha b^2, \frac{100 M_b}{b^2}\right\}. \tag{9}$$

The conclusion of Lemma 2 follows easily. In order to prove (8) we introduce the function

$$v(x,y)=Ax^2-Bx^4+Cy^4, \tag{10}$$

where A is defined by (9),

$$B=\frac{A}{2b^2}, \tag{11}$$

$$C=\frac{M_b}{b^4}. \tag{12}$$

A direct computation shows that

$$-x^2\Delta v+2v\geqq\alpha(v^2+x^4) \quad \text{on } \Omega_b, \tag{13}$$

$$v(x,\pm b)\geqq M_b \quad \text{for} \quad 0<x<b, \tag{14}$$

$$v(b,y)\geqq M_b \quad \text{for} \quad 0<y<a, \tag{15}$$

$$\alpha \operatorname*{Sup}_{\Omega_b} v\leqq 1, \tag{16}$$

$$v\geqq 0 \quad \text{on } \Omega_b. \tag{17}$$

We now derive, using the maximum principle that

$$u\leqq v \quad \text{on } \Omega_b. \tag{18}$$

Indeed by (14) and (15), $u\leqq v$ on $\partial\Omega_b$.

Suppose, by contradiction, that $(u-v)$ achieves a positive maximum at $(x_0,y_0)\in\Omega_b$. We would have

$$\Delta(u-v)(x_0,y_0)\leqq 0.$$

On the other hand, we deduce from (4) and (13) that

$$-x^2\Delta(u-v)+2u-2v\leqq\alpha(u^2-v^2) \quad \text{on } \Omega_b.$$

Therefore

$$\begin{aligned}2&\leqq\alpha[u(x_0,y_0)+v(x_0,y_0)].\\ &\leqq\alpha M_b+1 \quad [\text{by (16)}]\end{aligned}$$

and thus $\alpha M_b\geqq 1$ – a contradiction with (7).

Lemma 3. *Under the assumptions of Theorem 1 there exist constant β_k such that*

$$|D_y^k u(x,y)|\leqq\beta_k x^2 \quad \text{on } \Omega_{a/2},$$

for all $k=0,1,2,\ldots$.

Proof of Lemma 3. Since $f(0)=0$ we have

$$|f(u)|\leqq C|u| \quad \text{on } \Omega_a$$

and by (1)

$$|\Delta u| \leqq (C+2)\frac{|u|}{x^2} \quad \text{on } \Omega_a .$$

It follows from Lemma 2 that $\Delta u \in L^\infty(\Omega_{a/2})$. We deduce from the L^p regularity theory (see e.g. [1]) that $u \in C^1(\bar{\Omega}_{a/4})$. In particular $D_y u \in C(\bar{\Omega}_{a/4})$ and

$$D_y u(0, y) = 0 \quad \text{for} \quad |y| < a/4$$

[since $u(0, y) = 0$ for $|y| < a$]. Also, differentiating (1) with respect to y we find

$$-x^2 \Delta(D_y u) + 2(D_y u) = f'(u) D_y u \quad \text{on } \Omega_a .$$

By (2) we have

$$|f'(u)| \leqq C|u|$$

and from Lemma 2 we see that

$$|f'(u)| \leqq C\beta x^2 , \quad \text{on } \Omega_{a/2} .$$

Consequently

$$|f'(u) D_y u|^2 \leqq C\beta(|D_y u|^2 + x^4) ,$$

and Lemma 2 applied to $D_y u$ shows that

$$|D_y u| \leqq \beta_1 x^2 \quad \text{on } \Omega_{a/8} .$$

The conclusion of Lemma 3 for $k=1$ follows directly. When $k \geqq 2$ we proceed in a similar way, by induction, differentiating (1) k times with respect to y.

Lemma 4. *Assume* $\varphi \in C^2(]0, a[) \cap C([0, a])$ *satisfies*

$$-x^2 D_x^2 \varphi(x) + 2\varphi(x) = h(x) , \qquad 0 < x < a ,$$

where $h \in L^\infty(0, a)$.

Set $\psi(x) = x^{-2}\varphi(x)$, *then*

$$D_x \psi(x) = -x^{-4} \int_0^x h(t)dt , \qquad 0 < x < a .$$

Proof. Indeed we necessarily have

$$\varphi(x) = \frac{C_1}{x} + C_2 x^2 + x^2 \int_x^a \frac{ds}{s^4} \int_0^s h(t)dt$$

for some constants C_1 and C_2. Since the last term remains bounded as $x \to 0$ we must take $C_1 = 0$, and the conclusion follows.

3. Proof of Theorem 1

We have by (1)

$$-x^2 D_x^2 u + 2u = x^2 D_y^2 u + f(u) .$$

Let $v(x,y)=x^{-2}u(x,y)$. We deduce from Lemma 4 that

$$D_x v(x,y) = -x^{-4} \int_0^x [t^2 D_y^2 u(t,y) + f(u(t,y))] dt . \tag{19}$$

Set $g(u)=u^{-2}f(u)$ so that by (2), g is a C^∞ function on $\mathbb{R}$. Changing the variable t in (19) into $s=\frac{t}{x}$ we find

$$D_x v(x,y) = -x \int_0^1 [D_y^2 v(sx,y) + v^2(sx,y) g(s^2x^2 v(sx,y))] s^4 ds . \tag{20}$$

It follows from Lemma 3 (applied with $k=0$ and $k=2$) that

$$|D_x v(x,y)| \leqq C|x| \quad \text{on } \Omega_{a/2} . \tag{21}$$

Next, if we differentiate (2) k times with respect to y we obtain, using Lemma 3, that

$$|D_x D_y^k v(x,y)| \leqq C_k \quad \text{on } \Omega_{a/2} , \tag{22}$$

for all k.

We may now differentiate (20) once with respect to x and k times with respect to y and we find that

$$|D_{xx} D_y^k v(x,y)| \leqq C_k \quad \text{on } \Omega_{a/2}$$

for all k. Proceeding by induction we obtain estimates for $D_x^\ell D_y^k v$ and the conclusion of Theorem 1 follows [note that we have even an estimate of the form $|D_x^\ell D_y^k v(x,y)| \leqq Cx$ when ℓ is odd].

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