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Scattering Theory and Dispersion Relations for a Class of Long-Range Oscillating Potentials*

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Abstract. If a spherically symmetric potential is such that $\int_r^{\to\infty} V(r')dr' = O(\exp - \mu r)$, and if an additional regularity condition is imposed [a sufficient one being that $rV(r)$ is L^1], the partial wave amplitudes are meromorphic in a strip of width μ in the complex momentum plane, and the full scattering amplitude is analytic inside an ellipse at fixed energy and satisfies fixed momentum transfer $(\sqrt{-t})$ dispersion relations for $|t| < \mu^2$.

Such a class of potentials includes not only exponentially decreasing potentials but also long-range oscillating potentials such as $(1+r^2)^{-2}\sin(\exp \mu r)$. In fact the results can partly be extended to a still broader class of potentials with increasing amplitude at infinity. It is argued that these results might lead to a revision of conventional ideas on what is the potential between physical hadrons.

Appendices may be of interest to special functions addicts.

I. Introduction

In this paper we propose to study non-relativistic scattering theory and dispersion relations for a class of spherically symmetric, long-range potentials which are very rapidly oscillating at large distances. As an example, consider the potential

$$V_1(r) = (1+r)^{-3}\cos(\exp(\mu r)) \tag{I.1}$$

This potential satisfies the condition

$$rV(r) \in L^1(0, \infty) \tag{I.2}$$

and it is well-known that for such potentials all the machinery of usual scattering theory, including the use of the Jost functions to define the S matrix and the bound

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states, in each partial wave, applies [1–3]. However, $V_1(r)$ having only a power-like decrease at infinity, it seems at first that there is no reason why the partial-wave amplitudes should be analytic in some domain in k[1], or that the full scattering amplitude be analytic in the $\cos\theta$ plane in some ellipse [2, 3], and satisfy dispersion relations in the energy variable, except in the forward direction [4].

The striking fact which we shall show in this paper is that because of its very rapid oscillations at large distances (due to the exponential factor), the above potential behaves as a short-range potential having the asymptotic tail $O(e^{-\mu r})$. Oscillations at infinity, if they are very rapid, introduce a damping factor in the tail of the potential, and could make it short-range even if the apparent tail is only an inverse power law. Once this fact is shown, it will become clear that there is no real distinction, as far as the analytic and asymptotic properties of the partial wave amplitudes in the k plane are concerned, between the potential (I.1) and

$$V_2(r) = \underset{r\to\infty}{O}(\exp - \mu r) \tag{I.3}$$

In fact, it will become clear that the factor $(1+r)^{-3}$ in V_1 can be replaced by $\exp(\alpha r^{1-\varepsilon}), \alpha, \varepsilon > 0$, without modifying anything.

In general, it is known [5, 6] that any potential for which the primitive

$$gW(r) = -g \lim_{R\to\infty} \int_r^R V(t)dt \tag{I.4}$$

satisfies the conditions

$$W \in L^1(0,\infty) \tag{I.5a}$$

$$rW(r)|_{0,\infty} = 0 \tag{I.5b}$$

is a regular potential for scattering theory, i.e., the Hamiltonian is self-adjoint and unique, is bounded from below with a finite number of (only) negative energy bound states, we have the completeness of the Möller wave operators (asymptotic completeness), and, moreover, the S matrix is an analytic function of the coupling constant g at $g=0$, within a circle of finite radius.

If, moreover, we assume that, for large values of r,

$$|e^{\mu r} W(r)| \leqq C \tag{I.6}$$

then we shall see that we obtain analyticity of the partial wave amplitudes $A_\ell(k)$ in the strip

$$|\mathrm{Im}\, k| < \mu/2 \tag{I.7}$$

We also establish, using, however, the extra assumption (I.2), the analyticity of the full amplitude in the variable $z=\cos\theta$ in the small Lehmann ellipse with

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1 We shall use throughout this paper the by-now standard units $\hbar = 2M = 1$, so that the energy becomes $E = k^2, k$ being the wave number.

foci ± 1 and semi-major axis

$$x = \sqrt{1 + \frac{2\mu^2}{k^2}}. \tag{I.8}$$

Likewise, it will be shown that the imaginary part of the full amplitude is analytic in the large Lehmann ellipse, with a semi-major axis

$$X = 1 + \frac{4\mu^2}{k^2} \tag{I.9}$$

Concerning the analyticity of the full amplitude F in the energy variable E for fixed momentum transfer t, we shall find the usual cut-plane. In fact, it turns out that F is analytic in both variables E and t in the domain

$$\{E, t \,|\, E \in \mathbb{C}, E \notin \mathbb{R}^+ ; |t| < \mu^2\} \tag{I.10}$$

Using this, together with the asymptotic behaviour of the amplitude for large E, which turns out to be as usual, and analyticity in t, we finally show that the scattering amplitude satisfies the Khuri dispersion relations [4] with exactly the same restriction on t:

$$|t| < \mu^2. \tag{I.11}$$

We believe that while condition (I.6) is essential to derive these results, condition (I.2) is probably purely technical.

As an example, we may consider the potential

$$V_3(r) = e^{\mu r} \cos e^{2\mu r} \tag{I.12}$$

It is easily seen, by integration by parts, that we have

$$W_3(r) = O(e^{-\mu r}), \tag{I.13}$$

and so all the properties of the amplitudes we have just described are true for the above potential. There seems therefore to be no difference between (I.12), or in general (I.6), and (I.3). One may, of course, give many more striking examples. In resumé, no matter how fast the increase of the potential at infinity is, it becomes effectively a short-range potential provided it is multiplied by an oscillating factor whose oscillations at infinity are sufficiently fast.

Section II is devoted to the study of the radial Schrödinger equation, and the proof of the properties of partial-wave amplitudes described above. In Sect. III, we study the large ℓ behaviour of partial wave amplitudes A_ℓ, and show that (I.6) implies an exponential decrease in ℓ for large ℓ, as expected from a real short-range potential. This in turn implies analyticity in the Lehmann ellipses. Section IV is devoted to the proof of the analyticity domain (I.10) of the total amplitude, and the proof of Khuri dispersion relations. There are three appendices dealing with bounds and properties of Bessel functions and related kernels.

II. The Radial Schrödinger Equation

To begin with, and in accordance with what was said in the Introduction, we assume that the potential satisfies (I.2). We are therefore on the safe side as far as axiomatic time-dependent scattering theory is concerned, and may consider each partial wave separately [1, 3]. We assume also (I.6) for the integral of the potential. The above conditions are clearly satisfied by potentials similar to (I.1). Indeed

$$\begin{aligned} W(r) &= -\mu^{-1}\int_r^\infty (1+t)^{-3}e^{-\mu t}d(\sin(e^{\mu t})) \\ &= \mu^{-1}(1+r)^{-3}e^{-\mu r}\sin(e^{\mu r}) \\ &\quad + \mu^{-1}\int_r^\infty \sin(e^{\mu t})d[e^{-\mu t}(1+t)^{-3}] \\ &= \underset{r\to\infty}{O}(e^{-\mu r}), \end{aligned} \tag{II.1}$$

The reduced radial Schrödinger equation reads

$$u_\ell''(k,r) + \left(k^2 - \frac{\ell(\ell+1)}{2}\right)u_\ell(k,r) = V(r)u_\ell(k,r). \tag{II.2}$$

Because of (I.1), we know that this equation has a unique solution—up to a multiplicative factor—which vanishes at the origin (for complete detail, see Ref. [2], Chapter 3, and Ref. [3], Chapter 12). We shall call it φ_ℓ, and normalize it according to

$$\lim_{r\to 0}[(2\ell+1)!!]r^{-l-1}\varphi_\ell(k,r) = 1 \tag{II.3}$$

As shown in the above references, it satisfies the Volterra integral equation

$$\varphi_\ell(k,r) = k^{-\ell-1}\tilde{j}_\ell(kr) + \int_0^r G_\ell(k,r,r')V(r')\varphi_\ell(k,r')dr' \tag{II.4}$$

where [7]

$$\tilde{j}_\ell(z) = \sqrt{\frac{\pi z}{2}}J_{\ell+1/2}(z) \tag{II.5}$$

$$\tilde{n}_\ell(z) = \sqrt{\frac{\pi z}{2}}N_{\ell+1/2}(z) \tag{II.6}$$

$$G_\ell(k;r,r') = k^{-1}[\tilde{j}_\ell(kr')\tilde{n}_\ell(kr) - \tilde{j}_\ell(kr)\tilde{n}_\ell(kr')], \tag{II.7}$$

Notice that G_ℓ is an entire function of k for $r, r' \neq 0$.

By iterating the above integral equation, and using the bounds (Appendix A)

$$|\tilde{j}_\ell(kr)| < C_\ell\left(\frac{|k|r}{1+|k|r}\right)^{\ell+1}\exp[|\mathrm{Im}\,k|r] \tag{II.8}$$

$$|G_\ell(k;r,r')| < C_\ell\left(\frac{r}{1+|k|r}\right)^{\ell+1}\left(\frac{r'}{1+|k|r'}\right)^{-\ell}e^{|\mathrm{Im}\,k|(r-r')} \tag{II.9}$$

$$\left|\frac{\partial}{\partial r}\tilde{j}_\ell(kr)\right| < C_\ell|k|\left(\frac{|k|r}{1+|k|r}\right)^{\ell}e^{|\mathrm{Im}\,k|r} \tag{II.10}$$

$$\left|\frac{\partial}{\partial r}G_\ell\right| < C_\ell\left(\frac{r}{1+|k|r}\right)^\ell\left(\frac{r'}{1+|k|r'}\right)^{-\ell} e^{|\mathrm{Im}\,k|(r-r')} \tag{II.11}$$

valid for $r' < r$, one can easily show that φ_ℓ is the unique regular solution of the Schrödinger equation, and one obtains the bounds [2, 3] (see also Ref. [5] in which only W is used)

$$|\varphi_\ell(k,r)| < A_\ell\left(\frac{r}{1+|k|r}\right)^{\ell+1} e^{|\mathrm{Im}\,k|r} \tag{II.12}$$

$$|\varphi'_\ell(k,r)| < A_\ell\left(\frac{r}{1+|k|r}\right)^{\ell} e^{|\mathrm{Im}\,k|r} \tag{II.13}$$

where A_ℓ is an appropriate constant. From these, one obtains that for fixed ℓ and r, φ_ℓ and φ'_ℓ are even entire functions of k of exponential type. Moreover, they satisfy the following asymptotic estimates [2, 3]

$$\begin{pmatrix}1\\ \partial/\partial r\end{pmatrix}\varphi_\ell \underset{\substack{r\to\infty\\ k\,\mathrm{real}}}{=} C_\ell(k)\begin{pmatrix}\sin\\ k\cos\end{pmatrix}\left[kr-\frac{\ell\pi}{2}+\delta_\ell(k)\right]+O(1) \tag{II.14}$$

$$k^{\ell+1}\begin{pmatrix}1\\ \partial/\partial r\end{pmatrix}\varphi_\ell \underset{|k|\to\infty}{=} \begin{pmatrix}\sin\\ k\cos\end{pmatrix}\left(kr-\frac{\ell\pi}{2}\right)+e^{|\mathrm{Im}\,k|r}o(1) \tag{II.15}$$

where δ_ℓ is the phase shift, and $C_\ell(k)$ an appropriate constant which turns out to be related to the modulus of the Jost function (see below). Note that, so far, only (I.2) has been used.

We come now to the Jost solution f_ℓ, defined by its asymptotic behaviour

$$\lim_{r\to\infty} e^{-ikr}e^{i\ell\pi/2}f_\ell(k,r)=1 \tag{II.16}$$

It satisfies the Volterra equation (Ref. [2], Chapt. 4; [3], Chapt. 12)

$$f_\ell(k,r)=\tilde{h}_\ell^{(1)}(kr)-\int_r^\infty G_\ell(k;r,r')V(r')f_\ell(k,r')dr' \tag{II.17}$$

where G_ℓ is the same as before, with the modification that now $r < r'$, and [5]

$$h_\ell^{(1)}(kr)=i\left(\frac{\pi kr}{2}\right)^{1/2}H_{\ell+1/2}^{(1)}(kr) \tag{II.18}$$

Again, assuming only (I.2), it is shown in the above references that the Jost solution exists, and that it is unique. f'_ℓ also exists and is continuous in r away from the origin. Moreover, for fixed $r\neq 0, f_\ell$ and f'_ℓ are holomorphic in k in $\mathrm{Im}\,k>0$, continuous in $\mathrm{Im}\,k \geqq 0, k\neq 0$, and satisfy the estimates

$$\begin{bmatrix}1\\ \partial/\partial r\end{bmatrix}f_\ell(k,r) \underset{\substack{r\to\infty\\ \mathrm{Im}\,k\geqq 0,k\neq 0}}{=\!=\!=} \begin{pmatrix}1\\ ik\end{pmatrix}[e^{i(kr-l\pi/2)}+o(1)e^{-\mathrm{Im}\,kr}] \tag{II.19}$$

$$\begin{bmatrix}1\\ \partial/\partial r\end{bmatrix}f_\ell(k,r) \underset{\substack{k\to\infty\\ \mathrm{Im}\,k\geqq 0,r\neq 0}}{=\!=\!=} \begin{pmatrix}1\\ ik\end{pmatrix}[e^{i(kr-l\pi/2)}+o(1)e^{-\mathrm{Im}\,kr}] \tag{II.20}$$

We come now to the crucial point which is to show that if W is exponentially decreasing at infinity, condition (I.6), f_ℓ becomes holomorphic in $\mathrm{Im}\, k > -\mu/2$, except for a pole of order ℓ at $k=0$. This is well-known for potentials having the exponential decrease (I.3) at infinity (Ref. [2], Chapt. 4; [3], Chapt. 12). The method of proof is quite similar to the method used in a previous paper devoted to the study of the potentials which are singular and oscillating at the origin [5]. It consists in using $V = W'$ in the integral equation (II.17) and then integrating by parts to obtain formally the following two coupled integral equations for f_ℓ and f_ℓ':

$$f_\ell(k,r) = \tilde{h}_\ell(kr) + \int_r^\infty W(r') f_\ell(k,r') \frac{\partial}{\partial r'} G_\ell(k;r,r')dr' + \int_r^\infty W(r') G_\ell(k;r,r') \frac{\partial}{\partial r'} f_\ell(k,r')dr' \tag{II.21}$$

and

$$f_\ell'(k,r) = \frac{\partial}{\partial r}\tilde{h}_\ell(kr) + W(r)f_\ell(k,r) + \int_r^\infty W(r') f_\ell(k,r') \frac{\partial^2}{\partial r \partial r'} G_\ell(k;r,r')dr' + \int_r^\infty W(r') \frac{\partial}{\partial r} G_\ell(k;r,r') f_\ell'(k,r')dr' \tag{II.22}$$

Notice that here $r' \geqq r$. We have assumed that

$$\lim_{r'\to\infty} W(r') G_\ell(k;r,r') f_\ell(k,r') = 0 \tag{II.23}$$

and we shall justify it a posteriori. In order to show the existence and uniqueness of the solution of the above integral equations, we need the bounds (Appendix A)

$$|\tilde{h}_\ell(kr)| < C_\ell \left(\frac{|k|r}{1+|k|r}\right)^{-\ell} e^{-\mathrm{Im}\, kr} \tag{II.24}$$

$$|\tilde{h}_\ell'(kr)| < C_\ell \left(\frac{|k|r}{1+|k|r}\right)^{-\ell-1} |k| e^{-\mathrm{Im}\, kr} \tag{II.25}$$

$$|G_\ell| < C_\ell \left(\frac{r'}{1+|k|r'}\right)^{\ell+1} \left(\frac{r}{1+|k|r}\right)^{-\ell} e^{|\mathrm{Im}\, k|(r'-r)} \tag{II.26}$$

$$\left|\frac{\partial}{\partial r'} G_\ell\right| < C_\ell \left(\frac{r'}{1+|k|r'}\right)^{\ell} \left(\frac{r}{1+|k|r}\right)^{-\ell} e^{|\mathrm{Im}\, k|(r'-r)} \tag{II.27}$$

$$\left|\frac{\partial}{\partial r} G_\ell\right| < C_\ell \left(\frac{r'}{1+|k|r'}\right)^{\ell+1} \left(\frac{r}{1+|k|r}\right)^{-\ell-1} e^{|\mathrm{Im}\, k|(r'-r)} \tag{II.28}$$

$$\left|\frac{\partial^2}{\partial r \partial r'} G_\ell\right| < C_\ell \left(\frac{r'}{1+|k|r'}\right)^{\ell} \left(\frac{r}{1+|k|r}\right)^{-\ell-1} e^{|\mathrm{Im}\, k|(r'-r)} \tag{II.29}$$

Using all these bounds in the coupled integral equations (II.21)–(II.22) and putting

$$\bar{f}_\ell(r) = \left(\frac{|k|r}{1+|k|r}\right)^{\ell} e^{\mathrm{Im}\, kr} |f_\ell|$$

and

$$\bar{g}_\ell(r) = \left(\frac{|k|r}{1+|k|r}\right)^{\ell+1} e^{\operatorname{Im} kr} \frac{|f'_\ell|}{|k|}$$

we obtain the integral inequalities

$$\bar{f}_\ell(r) < C + C\int_r^\infty |W(r')|(\bar{f}_\ell(r') + \bar{g}_\ell(r'))e^{(|\operatorname{Im} k| - \operatorname{Im} k)(r'-r)}dr'$$

$$\bar{g}_\ell(r) < C + \frac{r|W(r)|}{1+|k|r}\bar{f}_\ell(r) + C\int_r^\infty |W(r')|(\bar{f}_\ell(r') + \bar{g}_\ell(r'))e^{(|\operatorname{Im} k| - \operatorname{Im} k)(r'-r)}dr'$$

Notice that the dimensionality of these inequalities is satisfactory since C and rW are dimensionless. Now

$$\frac{r|W(r)|}{1+|k|r} \leqq r|W(r)| \leqq r\int_r^\infty |V(t)|dt \leqq \int_0^\infty t|V(t)|dt = C_1$$

Using this and solving the above inequalities (one has to solve the first order differential inequality for $\bar{f} + \bar{g}$), we obtain

$$\bar{f}_\ell(r) \leqq \bar{f}_\ell(0) < A_\ell + B_\ell \int_0^\infty e^{(|\operatorname{Im} k| - \operatorname{Im} k)r}|W(r)|dr = \bar{C}_\ell$$

$$\bar{g}_\ell(r) \leqq \bar{g}_\ell(0) = A_\ell + B_\ell \int_0^\infty e^{(|\operatorname{Im} k| - \operatorname{Im} k)r}|W(r)|dr = \bar{C}_\ell$$

where A and B are appropriate constants. These bounds show clearly that whenever (I.6) is satisfied the iteration of the coupled integral equations (II.21)–(II.22) leads to absolutely and uniformly convergent series for f_ℓ and f'_ℓ as long as $\operatorname{Im} k > -\mu/2$, $k \neq 0$, and also to the bounds

$$|f_\ell| < \bar{C}_\ell\left(\frac{|k|r}{1+|k|r}\right)^{-\ell} e^{-\operatorname{Im} kr} \tag{II.30}$$

$$|f'_\ell| < \bar{C}_\ell\left(\frac{|k|r}{1+|k|r}\right)^{-\ell-1} |k|e^{-\operatorname{Im} kr} \tag{II.31}$$

where $\bar{C}_\ell$ is the constant defined previously. Since all the terms of the series are uniform in k (remember that G_ℓ is an entire function of k) it follows as usual that f_ℓ and f'_ℓ are analytic in k in $\operatorname{Im} k > -\mu/2$, except for a pole of order ℓ at $k = 0$, exactly as for the free solution $\tilde{h}_\ell$. Also, using the above bounds in the integral equations for f_ℓ and f'_ℓ, it is easily seen that (II.19) and (II.20) hold. One also checks easily that (II.23) holds in $\operatorname{Im} k > -\mu/2$.

Notice that the above analysis is quite similar to that of the case where the potential itself is exponentially decreasing at infinity and the bounds (II.30) and (II.31) are the same in both cases except for the constants $\bar{C}$, which are given in the usual case as integrals of V and now as integrals of W with the same exponential factor $\exp[(|\operatorname{Im} k| - \operatorname{Im} k)r]$. In short it makes no difference as far as the Jost solution is concerned if we replace (I.3) by (I.6). This means that the notion of short-range potentials with exponential decrease at infinity can be generalized to those for which only W has such a property.

In the case where V or W decrease at infinity faster than any exponential, for instance like $\exp(-\mu r^\alpha)$, $\alpha > 1$, one reaches the conclusion that f_ℓ is well-defined and holomorphic everywhere, and that all the bounds and asymptotic estimates previously given hold in $\operatorname{Im} k > -K$, no matter how large (finite) K may be.

We come now to the study of the Jost function, defined by [2, 3]:

$$F_\ell(k) = \lim_{r\to 0} \frac{(-kr)^\ell}{(2\ell-1)!!} f_\ell(k,r) \equiv k^\ell W[f_\ell, \varphi_\ell], \tag{II.32}$$

where W is the Wronskian $f'\varphi - f\varphi'$. It follows that since φ_ℓ and φ'_ℓ are entire functions of k, and $k^\ell f_\ell$ and its derivative are holomorphic in $\operatorname{Im} k > -\mu/2$, the Jost function is holomorphic in $\operatorname{Im} k > -\mu/2$.

Using also the asymptotic estimates given for $|k| \to \infty$ we find that

$$\lim_{\substack{k\to\infty \\ \operatorname{Im} k > -\mu/2}} F_\ell(k) = 1 \tag{II.33}$$

which is known for exponential potentials [2, 3].

So it turns out again that (I.3) can be replaced by the weaker condition (I.6) without modifying the holomorphy domain of F_ℓ or its asymptotic behaviour.

Having established the existence and the properties of the Jost solution and the Jost function, everything else (bound states and their properties, the phase shifts and the S matrix, ...) can now be studied in the standard manner. We refer the reader to the literature [2, 3] and quote only the results.

Concerning the bound states, we know that they are given by the (simple) zeros of F_ℓ on the positive imaginary axis of the k plane. That their number is finite follows from the Bargmann bound [2, 3] or the bound [8]

$$n_\ell \leqq \frac{4}{2\ell+1} \int_0^\infty r W^2 dr. \tag{II.34}$$

As for the phase shift $\delta_\ell(k)$ it is just minus the phase of the Jost function for real values of k, and is therefore real and continuous. Because of (II.33) we can choose it such that $\delta_\ell(\infty) = 0$. The Levinson theorem then reads

$$\delta_\ell(0) - \delta_\ell(\infty) = \delta_\ell(0) = n_\ell \pi \tag{II.35}$$

Notice that all the properties given in this paragraph follow from (I.2) only [2, 3]. Also the constant $C_\ell(k)$ in (II.14) turns out to be equal to $k^{-\ell-1}|F_\ell(k)|$. We consider now the S matrix. Its ℓ^{th} element is given, for k real, by

$$S_\ell(k) = \exp[2i\delta_\ell(k)] = \frac{F_\ell(-k)}{F_\ell(k)} \tag{II.36}$$

and we have, quite generally, assuming only (I.2),

$$\lim_{k=\infty} S_\ell(k) = 1 \tag{II.37}$$

If we assume also (I.3) or (I.6), we know from what we learned above that $S_\ell(k)$

can be continued analytically in the domain (II.38)

$$|\mathrm{Im}\, k| < \mu/2 \tag{II.38}$$

where it becomes a meromorphic function with poles at the bound states and resonances. Again it follows from (II.33) that (II.37) holds in the above strip.

The S matrix being analytic around the origin we obtain in the usual manner (Ref. [2], p.40; [3], p. 308; [9]) the effective range

$$k^{2\ell+1} \cot \delta_\ell(k) = -\frac{1}{a_\ell} + \frac{1}{2} r_0 k^2 + \ldots \tag{II.39}$$

where a_ℓ is the scattering length. All the coefficients of this expansion are always finite, except a_ℓ, which becomes infinite when there is a resonance ($\ell = 0$) or a bound state ($\ell \geqq 1$) at zero energy since then $F_\ell(0) = 0$.

Notice that, for long range potentials having a power-like decrease at infinity, (II.39) breaks down after a few terms depending on the tail of the potential and on ℓ [9]. It is only for short-range potentials that the effective range expansion is true for all ℓ. Therefore, we see again that we do not lose anything in assuming (I.6) instead of (I.3).

Finally, the physical solution of the radial equation, ψ_ℓ, which satisfies the boundary conditions

$$r\psi_\ell|_{r=0} = 0 \quad r\psi_\ell|_{r\to\infty} \simeq e^{i\delta_\ell} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) + o(1) \tag{II.40}$$

is given by (Ref. [2], Appendix C; [3], pp. 341 and 374)

$$u_\ell(k,r) = r\psi_\ell = k^{\ell+1} \frac{\varphi_\ell(k,r)}{F_\ell(k)} \tag{II.41}$$

and u_ℓ satisfies the Fredholm integral equation (see the references just given)

$$u_\ell(k,r) = \tilde{j}_\ell(kr) + \frac{1}{k} \int_0^\infty K_\ell(kr, kr') V(r') u_\ell(k,r') dr' \tag{II.42}$$

where

$$K_\ell(kr, kr') = \frac{i}{2} \pi k \sqrt{rr'}\, J_{\ell+1/2}(kr_<) H^{(1)}_{\ell+1/2}(kr_>) \tag{II.43}$$

From (II.41) the bounds (II.8), (II.10), (II.12) and (II.13), and the definition of the S matrix (Ref. [3], p. 303)

$$\begin{aligned} S_\ell(k) &= 1 - \frac{2i}{k} \int_0^\infty \tilde{j}_\ell(kr) V(r) u_\ell(k,r) dr \\ &= 1 + \frac{2i}{k} \int_0^\infty W(r) [\tilde{j}_\ell' u_\ell + \tilde{j}_\ell u_\ell'] dr \end{aligned} \tag{II.44}$$

we easily find, for k real,

$$|S_\ell(k)-1| \leqq C_\ell \frac{|k|^{2\ell+1}\int_0^\infty |W(r)|\left(\frac{r}{1+|k|r}\right)^{2\ell+1} dr}{|F_\ell(k)|} \tag{II.45}$$

This inequality shows that, for short range W's, we indeed have, for $k \to 0$, and in the absence of a bound state at zero energy,

$$S_\ell - 1 = 2ie^{i\delta_\ell}\sin\delta_\ell = 2i\sin\delta_\ell = O(k^{2\ell+1}) \tag{II.46}$$

This is in agreement with (II.39).

When $F_\ell(0)=0$, we find [2, 3]

$$\delta_0(k) = O(1) \tag{II.47a}$$

$$\delta_\ell(k) \underset{\ell \geqq 1}{=} O(k^{2\ell-1}). \tag{II.47b}$$

We consider now briefly the case of potentials for which the primitive

$$W(r) = -\lim_{R\to\infty}\int_r^R V(t)dt \tag{II.48}$$

satisfies the conditions previously enumerated, namely (I.5a) and (I.5b). As previously mentioned this class of potentials is perfectly regular for scattering theory [5, 6]. To see that imposing (I.6) leads again to the analyticity of the partial wave amplitudes, we introduce a cut-off at infinity into the potential so that it now satisfies (I.2) and then let the cut-off disappear at the end. For instance, we use

$$V_R(r) = V(r)\theta(R-r) \tag{II.49}$$

and let $R \to \infty$ at the end. Or, with the potential (I.12), we consider the limit $\varepsilon \downarrow 0$ of

$$V_\varepsilon(r) = e^{-\varepsilon r^2} e^{\mu r}\cos(e^{2\mu r}) \tag{II.50a}$$

for which

$$W_\varepsilon(r) \underset{r\to\infty}{=} O(e^{-\varepsilon r^2} e^{-\mu r}) \tag{II.50b}$$

etc.

In the presence of the cut-off, the partial wave amplitudes are holomorphic in the entire k plane. Now, since when the cut-off disappears we still have a regular potential whose primitive W satisfies (I.6) all the way, we end up with partial wave amplitudes which are analytic in the strip $|\mathrm{Im}\, k| < \mu/2$.

III. Exponential Decrease of Partial Wave Amplitudes with Angular Momentum and "Lehmann" Ellipse

It is well-known that if $V = O(e^{-\mu r})$ for $r \to \infty$, the partial wave amplitude decreases exponentially with ℓ, the angular momentum. We shall prove that in fact it is sufficient to have $W = O(e^{-\mu r})$ [condition (I.6)], together with the possibly technical condition (I.2), to get this exponential decrease. Once this is established, we shall reconstruct the full scattering amplitude by summing over angular momenta and show that it is analytic inside an ellipse in the $\cos\theta$ complex plane.

We use now the "Fredholm" form of the Schrödinger equation (II.42)

$$u_\ell(r) = \tilde{j}_\ell(kr) + \frac{1}{k}\int_0^\infty K_\ell(kr, kr')V(r')u_\ell(r')dr' \tag{III.1}$$

from which one can get the partial wave amplitude:

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{1}{k}\int_0^\infty \tilde{j}_\ell(kr)V(r)u_\ell(r)dr, \tag{III.2}$$

If we want to study the behaviour of this quantity for large ℓ for the class of potentials satisfying (I.6), we have to integrate (III.2) by parts:

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{1}{k}\int_0^\infty W(r)\frac{d}{dr}[\tilde{j}_\ell(kr)u_\ell(r)]dr \tag{III.3}$$

and we need bounds or estimates on $u_\ell(r)$ and $u'_\ell(r)$. Bounds on $u_\ell(r)$ can be easily obtained by a method which has been used previously [10] and will be summarized and extended here.

It is possible to find a bound on the kernel of the integral equation (III.1). An improved bound is derived in the Appendix, it is

$$|K_\ell(x, x')| = |\tilde{j}_\ell(x_<)\tilde{h}_\ell(x_>)| < C_1 x(\ell + \tfrac{1}{2})^{-2/3} \tag{III.4}$$

with $C_1 \leqq 1.08$.

It is also clear, by choosing $x' > x$ and using $|\tilde{h}_\ell^{(1)}(x')| > 1$ that

$$|\tilde{j}_\ell(x)| < C_1 \times (\ell + \tfrac{1}{2})^{-2/3} \tag{III.5}$$

(this can be improved!).

Inserting in (III.1) we get

$$|u_\ell(r)| < krC_1(\ell + \tfrac{1}{2})^{-2/3}\left[1 + \frac{I}{k}\right] \tag{III.6}$$

with $I = \int_0^\infty |V(r)|\,|u_\ell(r)|dr$. Notice that here k is always real.

At this point, we make use of property (I.2), which allows us to multiply (III.6) by $|V(r)|$ and integrate. This gives

$$I\left[1 - C_1(\ell + \tfrac{1}{2})^{-2/3}\int_0^\infty r|V(r)|dr\right] \leqq C_1 k(\ell + \tfrac{1}{2})^{-2/3}\int_0^\infty r|V(r)|dr \tag{III.7}$$

for $\ell \geqq \ell_0$, with

$$\ell_0 + \tfrac{1}{2} = \left[2C_1\int_0^\infty r|V(r)|dr\right]^{3/2}. \tag{III.8}$$

The bracket is larger than $\frac{1}{2}$, and we get an upper bound on I and on u_ℓ:

$$|u_\ell| < D_1(\ell + \tfrac{1}{2})^{-2/3} kr \tag{III.9}$$

Another bound can be obtained by using instead of (III.4) the bound on the kernel

$$K_\ell(x, x') < C_2(\ell + \tfrac{1}{2})^{1/3} \tag{III.10}$$

with $C_2 \leqq 1.2$.

Inserting (III.9) and (III.10) into (III.1) we get

$$|u_\ell - \tilde{j}_\ell(kr)| < D_1 C_2 (\ell + \tfrac{1}{2})^{-1/3} \int r|V(r)|\,dr < D_2(\ell + \tfrac{1}{2})^{-1/3}$$

$$\text{for } \ell > \ell_0. \tag{III.11}$$

The next step is to calculate a bound on $u_\ell'(r)$ by differentiating (III.1). We need bounds on the derivative of K_ℓ. In Appendix B we find:

$$\left|\frac{d}{dx} K_\ell(x, x')\right| \leqq 1 + \sup_x |\tilde{j}_\ell(x)| \leqq C_3(\ell + \tfrac{1}{2})^{1/6} \tag{III.12}$$

with $C_3 \leqq 2$, and

$$\left|\frac{d\tilde{j}_\ell(x)}{dx}\right| < 1. \tag{III.13}$$

Inserting (III.9), (III.12) and (III.13) into (III.1) after differentiation we get

$$|u_\ell'(r)| \leqq D_3 k \quad \text{for} \quad \ell \geqq \ell_0. \tag{III.14}$$

We can now evaluate (III.3). We get

$$\begin{aligned} |e^{i\delta_\ell} \sin\delta_\ell| \leqq & \int_0^\infty |\tilde{j}_\ell'(kr)|\,|\tilde{j}_\ell(kr)|\,|W(r)|\,dr \\ & + D_2(\ell + \tfrac{1}{2})^{-1/3} \int_0^\infty |\tilde{j}_\ell'(kr)|\,|W(r)|\,dr \\ & + D_3 \int_0^\infty |\tilde{j}_\ell(kr)|\,|W(r)|\,dr. \end{aligned} \tag{III.15}$$

Using now

$$\frac{d}{dx}\tilde{j}_\ell = \frac{\ell + 1}{2\ell + 1}\tilde{j}_{\ell-1} - \frac{\ell}{2\ell + 1}\tilde{j}_{\ell+1}$$

and (III.13) together with Schwarz inequality we get, for $\ell > \sup\{\ell_0, 1\}$

$$|e^{i\delta_\ell} \sin\delta_\ell| < \left[\int_0^\infty r|W(r)|^2 e^{2\mu' r}\,dr\right]^{1/2} \ldots$$

$$\cdot\left[\begin{array}{l}(1+D_3)\left[\int(\tilde{j}_\ell(kr))^2\frac{e^{-2\mu' r}}{r}dr\right]^{1/2}\\ +D_2(\ell+\frac{1}{2})^{-1/3}\left[\left(\int_0^\infty(\tilde{j}_{\ell-1}(kr))^2\frac{e^{-2\mu' r}}{r}dr\right)^{1/2}+\left(\int_0^\infty(\tilde{j}_{\ell+1}(kr))^2\frac{e^{-2\mu' r}}{r}dr\right)^{1/2}\right]\end{array}\right] \tag{III.16}$$

where μ' is any number less than μ, appearing in (I.6). Conditions (I.2) and (I.6) guarantee the convergence of the integral appearing in the right-hand side of (III.16). Indeed

$$\int_0^\infty r|W(r)|^2 e^{2\mu' r}dr < e^{2\mu' R}\int_0^R r|W(r)|^2 dr + \int_R^\infty r|W(r)|^2 e^{2\mu' r}dr$$

If $W = O(e^{-\mu r})$ the second integral is convergent. The first integral becomes, after partial integration

$$\int r^2 VW dr$$

and

$$\int_0^R r^2 VW dr < (\sup|rW|)\int_0^\infty r|V(r)|dr$$

and

$$|rW| \leqq r\left|\int_r^\infty V dr'\right| < \int_r^\infty r'|V(r')|dr' < \int_0^\infty r|V(r)|dr.$$

In the second bracket of (III.16) we recognize integrals defining Q_ℓ functions [11]. We use also the recursion relation

$$zQ_\ell = \frac{\ell+1}{2\ell+1}Q_{\ell-1} + \frac{\ell}{2\ell+1}Q_{\ell+1}$$

and get for $\ell > \sup\{\ell_0, 1\}$:

$$|e^{i\delta_\ell}\sin\delta_\ell| < \text{const}\left[1+\sqrt{1+\frac{4\mu'^2}{2k^2}}\right]\left[Q_\ell\left(1+\frac{4\mu'^2}{2k^2}\right)\right]^{1/2} \tag{III.17}$$

This inequality is probably the main result of this Section. It shows that the full scattering amplitude, defined through its partial wave expansion

$$F(k^2,\cos\theta) = \frac{1}{k}\sum_0^\infty(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) \tag{III.18}$$

is analytic inside an ellipse in the $\cos\theta$ complex plane, with foci at $\cos\theta = \pm 1$ and right extremity $\cos\theta = \cosh\phi$, with

$$\cosh 2\phi = 1 + \frac{4\mu'^2}{2k^2}.$$

In fact, since μ' is arbitrarily close to μ

$$\cosh 2\phi = 1 + \frac{4\mu^2}{2k^2}. \tag{III.19}$$

The absorptive part

$$\operatorname{Abs} F = \frac{1}{k}\sum_0^\infty (2\ell + 1)\sin^2\delta_\ell P_\ell(\cos\theta) \tag{III.20}$$

is analytic inside an ellipse whose extremity is precisely $\cosh 2\phi$.

If $\ell_0 = 0$, i.e., if $\int r|V(r)|dr$ is sufficiently *small*, a simple bound can be obtained on the full absorptive part, from the formula

$$\frac{1}{z-x} = \sum_0^\infty (2\ell+1)P_\ell(x)Q_\ell(z)$$

valid for

$$|x + \sqrt{x^2-1}| < |z + \sqrt{z^2-1}|.$$

It is, including the S wave contribution:

$$|\operatorname{Abs} F| < \text{const}\left[\left(\frac{1}{k} + \frac{C}{k^3}\right)\frac{1}{\left|\left(1 + \frac{4\mu'^2}{2k^2} - \cos\theta\right)\right|} + \frac{1}{k}\right] \tag{III.21}$$

and hence, changing variables to

$$t = -2k^2(1-\cos\theta) \tag{III.22}$$

the square of the momentum transfer,

$$|\operatorname{Abs} F| < \text{const}\left[\frac{k + \frac{1}{k}}{|4\mu'^2 - t|} + \frac{1}{k}\right] \tag{III.23}$$

for $|t| < 4\mu^2$. In (III. 21), (III.23), the constants are functions of μ' and can become infinite as $\mu' \to \mu$.

At this point, we have, at fixed energy, an analyticity domain for the amplitude and the absorptive part which is *almost* the same as the one obtained for a potential decreasing like $e^{-\mu r}$. The restriction "almost" is due to the fact that in the low energy limit the analyticity domain of the amplitude shrinks to zero, in the t variable. However, this is only provisional and due to the fact that in estimating (III.3) very poor bounds on u and u' have been used.

One can ask what happens if $\ell_0 > 0$. At any given energy the analyticity domains are the same, but the bounds (III.21) and (III.23) must be modified. If $0 < \ell_0 < 1$ there is no real change, because the S wave has been replaced in (III.21) by its upper bound anyway. If $\ell_0 > 1$, one cannot be contented by replacing the first $[\ell_0]$ partial wave amplitudes by their unitarity bounds because then, for any fixed t, the bound on the absorptive part would be more singular than $1/k^2$ as $k^2 \to 0$ and would lead to a divergence in the dispersion integrals that will be considered in the following section. However, we have seen in Sect. II that any given partial wave has the normal threshold behaviour, $\delta_\ell \sim k^{2\ell+1}$; if the corresponding Jost function does not vanish at $k = 0$. Then the product

$$\frac{1}{k} P_\ell\left(1 + \frac{t}{2k^2}\right) e^{i\delta_\ell} \sin \delta_\ell$$

is not singular at $k = 0$ for fixed t. The only assumption we have to make is that there are no zero energy bound states. Incidentally this assumption, never explicitly stated, is implicit in all previous work on dispersion relations for exponentially decreasing potentials.

IV. Fixed Momentum Transfer Dispersion Relations

One of the classical results for potentials of the order of $\exp(-\mu r)$ for $r \to \infty$ is the validity of dispersion relations for fixed momentum transfer squared, t such that $|t| < \mu^2$. If we use as variables $E = k^2$ and t the dispersion relation can be written as [4]

$$F(E,t) = F^B(t) + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Abs} F(E',t) dE'}{E' - E} \tag{IV.1}$$

where $F^B(t)$ represents the Born approximation:

$$F^B(t) = \int_0^\infty \frac{\sin qr}{q} V(r) r dr \tag{IV.2}$$

with $t = -q^2$.

An obvious question is to know whether these dispersion relations also hold for $|t| < \mu^2$ for potentials satisfying only (I.2) and (I.6). We see already easily that the Born term, which can be written as

$$\frac{1}{q} \int_0^\infty \frac{d}{dr} [r \sin qr] W(r) dr \tag{IV.3}$$

is analytic, under conditions (I.2) and (I.6) in

$$|\operatorname{Im} q| < \mu \tag{IV.4}$$

the image of which, in the t variable, is a parabola with focus $t = 0$ and summit $t = \mu^2$ containing precisely the circle $|t| < \mu^2$.

We also see, from the results of Sect. III, that the integrand, in (IV.1), can be continued for $|t| < \mu^2$. However, there are two problems left. First of all, the bound (III.23) is not good enough to guarantee the convergence of the dispersion integral without subtraction. There is no evidence, however, that this bound is optimal. Second, even if there was no problem of convergence, it would still be necessary to prove that the right-hand side of (IV.1) coincides with the scattering amplitude.

To prove the validity of (IV.1) we shall use a regularisation procedure replacing the potential V by

$$V_\varepsilon(r) = V(r) \exp(-\varepsilon r) \tag{IV.5}$$

For V_ε the conditions of Refs. [4] are satisfied at least for $|t| < \varepsilon^2$, but we also have

$$W_\varepsilon = O(\exp - (\mu + \varepsilon) r),$$

and therefore the *scattering amplitude* at the energy $E = k^2$ is analytic inside an ellipse in the $\cos\theta$ plane with right extremity $\sqrt{1 + (2(\mu+\varepsilon)^2)/2k^2}$, containing the circle, defined, using the t variable as

$$|t| < \frac{(\mu+\varepsilon)^2}{\frac{1}{2}\left[1 + \sqrt{1 + \frac{\mu+\varepsilon)^2}{k^2}}\right]} \tag{IV.6}$$

For k^2 large enough, this circle is as close as one wishes to $|t| < (\mu+\varepsilon)^2$.

The procedure we shall use now is inspired by the method by which dispersion relations in elementary particle scattering for complex t were derived [12] and more precisely the version by Sommer [13] in which the size of the complex t domain is explicitly found.

In (IV.1) we subtract the low energy part of the dispersion integral and the Born term:

$$\phi_\varepsilon(E, E_0, t) = F_\varepsilon(E,t) - F_\varepsilon^B(t) - \frac{1}{\pi}\int_0^{E_0} \frac{\operatorname{Abs} F_\varepsilon(E',t)\,dE'}{E'-E} \tag{IV.7}$$

By construction ϕ_ε is analytic in $|t| < \varepsilon^2$ with a cut in the energy plane starting at $E = E_0$. ϕ_ε has no discontinuity on $0 < E < E_0$ and we can take the limit of $E_{\text{real}} < E_0$ in (IV.7) without problem. However, then we have a second piece of information: $\phi_\varepsilon(E, E_0, t)$ for $E_{\text{real}} < E_0$ is analytic in t in the intersection of

$$|t| < 4(\mu+\varepsilon)^2,\ |t| < (\mu+\varepsilon)^2,\ |t| < \frac{(\mu+\varepsilon)^2}{\frac{1}{2}\left[1 + \sqrt{1 + \frac{\mu^2}{E}}\right]}, \tag{IV.8}$$

and

$$\phi_\varepsilon(E, E_0, t) = \frac{1}{\pi}\int_{E_0}^{\infty} \frac{\operatorname{Abs} F_\varepsilon(E',t)\,dE'}{E'-E}. \tag{IV.9}$$

Following the same lines as in [12], and noticing the positivity properties

$$\left|\left(\frac{d}{dt}\right)^n \operatorname{Abs} F_\varepsilon(E',t)\right| < \left(\frac{d}{dt}\right)^n \operatorname{Abs} F_\varepsilon(E',0), \quad -2E' \leqq t \leqq 0, \tag{IV.10}$$

it is possible to prove that the successive derivatives of $\phi_\varepsilon(E, E_0, t)$ for $-2E_0 < t \leqq 0$, for E real, can be obtained by differentiation under the integral sign in (IV.9) [the derivatives exist because of the existence of the analyticity domain (IV.8)]:

$$\left(\frac{d}{dt}\right)^n \phi_\varepsilon(E, E_0, t)\Big|_{t=0} = \frac{1}{\pi}\int_{E_0}^{\infty} \frac{\left(\frac{d}{dt}\right)^n \operatorname{Abs} F_\varepsilon(E',t)\Big|_{t=0} dE'}{E'-E} \tag{IV.11}$$

Clearly (IV.11) can be continued for complex E. The positivity of the integrand for real $E < E_0$ guarantees the convergence of the integral for complex E.

We can then try to resum the series defining the scattering amplitude

$$\sum_n \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n \phi_\varepsilon(E, E_0, t=0) \tag{IV.12}$$

and noticing that

$$\frac{\left(\frac{d}{dt}\right)^n \phi_\varepsilon(E, E_0, 0)}{\left(\frac{d}{dt}\right)^n \phi_\varepsilon(E_1, E_0, 0)} < \sup_{E'}\left|\frac{E'-E_1}{E'-E}\right|,$$

for $0 < E_{1\,\mathrm{real}} < E_0$, we see that the analyticity domain in t of the series (IV.12) is a circle with the same size as the circle defined by (IV.8) for $E = E_1$. We conclude that $\phi(E, E_0, t)$ is analytic in the topological product

$$|t| < |\mu - \eta(E_0, E_1)|^2 \otimes E - E_0 \notin \mathbb{R}^+, \tag{IV.13}$$

where $\eta(E_0, E_1)$ goes to zero as E_0 and E_1 go to infinity.

Therefore from (IV.7) $F_\varepsilon(E, t)$ is analytic in the topological product

$$|t| < |\mu - \eta(E_0, E_1)|^2 \otimes E \notin \mathbb{R}^+,$$

and in fact, since, by choosing E_0 and E_1 large enough, η can be made arbitrarily small

$$|t| < \mu^2 \otimes E \notin \mathbb{R}^+ \tag{IV.14}$$

At the same time, it is possible to show that ϕ_ε satisfies an unsubtracted dispersion relation.

We notice that $(d/dt)^n \operatorname{Abs} F_\varepsilon(E, 0)$ is a *positive* distribution in E, i.e., a measure. If we consider a *positive* value of t, and choose $E_{\mathrm{real}} < E_0$ we have

$$\phi_\varepsilon(E, E_0, t) = \sum \frac{t^n}{n!}\frac{1}{\pi}\int_{E_0}^{\infty} \frac{\left(\frac{d}{dt}\right)^n \operatorname{Abs} F_\varepsilon(E', 0) dE'}{E'-E},$$

and because of positivity the summation and the integration can be *interchanged*, thus defining a positive distribution in E:

$$\operatorname{Abs} F_\varepsilon(E', t) = \sum \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n \operatorname{Abs} F_\varepsilon(E', 0)$$

with

$$\phi_\varepsilon(E, E_0, t) = \frac{1}{\pi}\int_{E_0}^{\infty} \frac{\operatorname{Abs} F_\varepsilon(E', t) dE'}{E'-E} \tag{IV.15}$$

However, by construction

$$|\operatorname{Abs} F_\varepsilon(E', t)| < \operatorname{Abs} F_\varepsilon(E', |t|)$$

and therefore the representation holds in fact for any $|t| < \mu^2$, provided E_0 is large enough. We conclude that $F_\varepsilon(E, t)$, for $|t| < \mu^2$, can be written as

$$F_\varepsilon(E, t) = F_\varepsilon^B(t) + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Abs} F_\varepsilon(E', t)}{E' - E} dE'. \tag{IV.16}$$

Now comes the final step, which is to let ε go to zero. Because of the condition (I.2) the partial wave amplitudes for any finite energy will converge to the $\varepsilon = 0$ value as $\varepsilon \to 0$. We also know that the partial wave expansions are uniformly convergent in ε inside the Lehmann ellipses for physical energies because W_ε converges to W. Also, clearly $F_\varepsilon^B(t) \to F^B(t)$. From this it is possible to deduce that $\phi_\varepsilon(E, E', t)$ converges to $\phi(E, E', t)$ using the definition (IV.7). However, a precaution is necessary because ϕ_ε contains in its definition an apparent principal value integral. So what one has to do is to consider $\int w(E')\phi_\varepsilon(E', E_0, t)dE'$, w is positive, C_∞ with compact support in $]0, E_0[$. Such a quantity, for *arbitrary* w converges toward $\int w(E')\phi(E', E_0', t)dE'$ if $|t| < (\mu - \eta)^2$, and we conclude that $\phi_\varepsilon(E, E_0, t) \to \phi(E, E_0, t)$ at least in a distribution sense. However, for *positive* t, $E < E_0$, $\phi_\varepsilon(E, E_0, t)$ is a monotonous function of E from (IV.15) and its limit is necessarily an ordinary monotonous function. We conclude that the representation (IV.15) also holds in the limit $\varepsilon = 0$, because the integral is uniformly bounded and because the integrand, for any finite E' approaches the value corresponding to $\varepsilon = 0$. Of course, $\phi(E, E_0, t)$ can be uniquely continued to complex E. Therefore $F_\varepsilon(E, t)$, for E outside the cut approaches $F(E, t)$ which satisfies the dispersion relation (IV.16) with $\varepsilon = 0$. This dispersion relation implies that not only the absorptive part but also the amplitude is analytic in $|t| < \mu^2$ at all energies. This means that by using unitarity the analyticity domain of the absorptive part could be enlarged and that the analyticity domain of the amplitude could be further improved by reinjecting the absorptive part into the dispersion relation. For the most general potential satisfying (I.2) and (I.6), the best one can hope is to prove dispersion relations for t inside the parabola with focus $t = 0$ and extremity $t = \mu^2$. Even this is not at all certain and, in any case, difficult to prove. It is very likely necessary to use other variables in the dispersion relations.

V. Concluding Remarks

The results we have obtained give new support to the idea that what is really important in scattering theory is the primitive of the potential rather than the potential itself. We must admit that this fact seems to us difficult to understand from a classical point of view. We are led to a revision of our ideas on short-range potentials. Potentials with exponentially decreasing primitive produce scattering amplitudes which possess essentially the same analytic properties as those due to ordinary exponentially decreasing potentials. The next step would be to find the full analyticity domain of the scattering amplitudes. One question is, for instance, whether one can find in our new set of potentials a nontrivial subset for which the Mandelstam representation is valid; for ordinary short-range potentials we know that this is the case with superpositions of real exponentials. This investigation might be difficult. For instance, a potential of the form $V(r) = P(r)\cos(\exp \mu r)$, where P is a rational function, produces a Born approximation which is an entire function of t, and therefore, it is likely that we cannot have a Mandelstam representation with a finite number of subtractions with this potential.

We are also led to re-examine the inverse problem, in particular the inverse problem at fixed energy. We know that if a scattering amplitude is analytic in a cut plane in $\cos\theta$ there is a unique superposition of Yukawa potentials which reproduce the scattering amplitude as was shown first by G. Targonski and one of us (A.M.) [14]. The problem is to know what happens now if we allow this new class of potentials. In practice we only know from field theory that the scattering amplitude of two elementary particles stable with respect to strong interactions is analytic in some ellipse in the $\cos\theta$ plane. It was believed that the only potentials which would reproduce such an amplitude would be exponentially decreasing, but this is not any more necessarily the case. As a matter of fact this new freedom might be welcomed to avoid some conflicts between the axiomatic field theory description in which the only fields one considers are associated with stable particles and the description in terms of quarks, i.e., fields without asymptotic fields. In the latter case, it is not at all obvious that the potential between two nucleons will decrease exponentially. Of course, it is not at all obvious either that the potential will belong to this new class.

Finally, let us notice that the assumptions we have made might not be yet minimal. When obtaining the analyticity properties of the Jost function, for given angular momentum, the condition $\int r|V(r)|\,dr < \infty$ has never been used. However, we need the convergence of this integral in the proof of the existence of a Lehmann ellipse. Admittedly our proof is rather clumsy. In this proof we have just altered in a minimal way the standard proof of convergence to the partial wave series for ordinary potentials.

Appendix A. Bounds associated to the volterra form of the Schrödinger equation

In this Appendix, we establish the bounds (II.8) to (II.11) and (II.24) to (II.29) on spherical Bessel and Hankel functions which were used in Section II for the study of partial wave amplitudes. Equations (II.8), (II.9), (II.24) and (II.26) are well known (Ref. [2], Appendix A; [3] p. 372). To establish the others we start with (II.10). It can easily be seen that the recursion formulae for the Bessel functions lead to

$$z_\ell'(x) = \frac{\ell}{x} z_\ell(x) + z_{\ell-1}(x) \tag{A.1}$$

$$z_\ell'(x) = \frac{\ell(\ell+1)}{x} z_\ell(x) - z_{\ell+1}(x) \tag{A.2}$$

where z is any of the functions $\tilde{j}_\ell, \tilde{n}_\ell$ or $\tilde{h}_\ell$. Now

$$\begin{aligned} |\tilde{j}_\ell'(kr)| &= |k\tilde{j}_\ell'(x)|_{x=kr} = k\left|\frac{\ell}{kr}\tilde{j}_\ell(kr) + \tilde{j}_{\ell-1}(kr)\right| \\ &< |k|\left[\frac{C_\ell \ell}{|k|r}\left(\frac{|k|r}{1+|k|r}\right)^{\ell+1} + C_{\ell-1}\left(\frac{|k|r}{1+|k|r}\right)^{\ell}\right]\exp|\operatorname{Im} k|r \\ &< \bar{C}_\ell|k|\left(\frac{|k|r}{1+|k|r}\right)^{\ell}\exp|\operatorname{Im} k|r \end{aligned} \tag{A.3}$$

where

$$\bar{C}_\ell = \ell C_\ell + C_{\ell-1}.$$

Likewise, for proving (II.25), we start from (II.24), use the recursion formula (A.2) and proceed as above:

$$e^{\mathrm{Im}\,kr}|\tilde{h}'_\ell(kr)| < |k|\left[\frac{\ell(\ell+1)}{|k|r}\left(\frac{|k|r}{1+|k|r}\right)^{-\ell} C_\ell + C_{\ell+1}\left(\frac{|k|r}{1+|k|r}\right)^{-\ell-1}\right]$$
$$\leqq \bar{\bar{C}}_\ell |k| \left(\frac{|k|r}{1+|k|r}\right)^{-\ell-1} \tag{A.4}$$

where

$$\bar{\bar{C}}_\ell = C_{\ell+1} + \ell(\ell+1)C_\ell.$$

Now, in order to prove (II.11), we use the relation [2, 3]

$$G_\ell = \frac{1}{k}[\tilde{j}_\ell(kr)\tilde{h}_\ell(kr') - \tilde{j}_\ell(kr')\tilde{h}_\ell(kr)]. \tag{A.5}$$

Differentiating now this relation with respect to r, using (II.8), (II.9), (II.24) and (II.25), and proceeding as above, we obtain, respectively, (II.11) for $r' < r$, and (II.27) for $r' > r$.

The remaining bounds are obtained in a similar manner by differentiating (A.5), and using the bounds which have been established.

Notice that, for ℓ integer, all the spherical Bessel functions are uniform with respect to their arguments, and that G_ℓ is an entire function of k for $r, r' \neq 0$.

Appendix B. Bounds on $K_\ell(x, x')$ and its derivative

1. We recall the definition

$$K_\ell(x,x') = \begin{cases} \tilde{j}_\ell(x)\tilde{h}_\ell^{(1)}(x') & \text{for } x < x' \\ \tilde{j}_\ell(x')\tilde{h}_\ell^{(1)}(x) & \text{for } x > x' \end{cases} \tag{B.1}$$

with

$$\tilde{j}_\ell(x) = \sqrt{\frac{\pi x}{2}} J_{\ell+1/2}(x)$$
$$\tilde{h}_\ell^{(1)}(x) = \sqrt{\frac{\pi x}{2}} H^{(1)}_{\ell+1/2}(x) \tag{B.2}$$

The function $|\tilde{h}_\ell^{(1)}(x)|$ is a decreasing function of x for $\ell > 0$ [15]. Therefore

$$|K_\ell(x,x')| < |K_\ell(x,x)| < x \sup_y \left|\frac{K_\ell(y,y)}{y}\right|, \text{ if } x < x'$$
$$|K_\ell(x,x')| < |K_\ell(x',x')| < x' \sup_y \left|\frac{K_\ell(y,y)}{y}\right| < x \sup_y \left|\frac{K_\ell(y,y)}{y}\right| \text{ if } x > x' \tag{B.3}$$

The problem of finding a bound on $K_\ell(x, x')$ is therefore reduced to finding a bound on

$$\sup\left|\frac{K_\ell(y,y)}{y}\right| = \frac{\pi}{2}\sup\left|J_{\ell+1/2}(y)H^{(1)}_{\ell+1/2}(y)\right|. \tag{B.4}$$

Naturally, (B.3) also implies

$$|K_\ell(x,x')| < \frac{\pi}{2}\sup\left|xJ_{\ell+1/2}(x)H^{(1)}_{\ell+1/2}(x)\right| \tag{B.5}$$

2. We now discuss the bounds on $(d/dx)K_\ell(x,x')$. Notice that this function is discontinuous at $x = x'$.

For $x < x'$ we have

$$\left|\frac{d}{dx}K_\ell(x,x')\right| < |\tilde{j}'_\ell(x)\tilde{h}^{(1)}_\ell(x')| < |\tilde{j}'_\ell(x)\tilde{h}^{(1)}_\ell(x)| \tag{B.6}$$

from the previous remark on h_ℓ.

For $x > x'$ the situation is different. We have

$$\left|\frac{d}{dx}K_\ell(x,x')\right| = |\tilde{j}_\ell(x)\tilde{h}^{(1)\prime}_\ell(x)|.$$

From the differential equation

$$\tilde{h}''_\ell + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)\tilde{h}_\ell = 0,$$

we get

$$\tilde{h}'^*_\ell\tilde{h}''_\ell + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)\tilde{h}_\ell\tilde{h}'^*_\ell = 0,$$

and hence

$$\frac{d}{dx}|\tilde{h}'_\ell|^2 + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)\frac{d}{dx}|\tilde{h}_\ell|^2 = 0$$

Therefore

$|\tilde{h}'_\ell|$ decreases for $0 < x < \sqrt{\ell(\ell+1)}$,

$|\tilde{h}'_\ell|$ increases for $\sqrt{\ell(\ell+1)} < x < \infty$.

We conclude

$$\left|\tilde{j}_\ell(x')\frac{d}{dx}\tilde{h}^{(1)}_\ell(x)\right| < \left|\tilde{j}_\ell(x')\frac{d}{dx'}\tilde{h}^{(1)}_\ell(x')\right| \quad \text{if } x' < x < \sqrt{\ell(\ell+1)} \tag{B.7}$$

$$\left|\tilde{j}_\ell(x')\frac{d}{dx}\tilde{h}_\ell(x)\right| < \left|\tilde{j}_\ell(x')\frac{d}{dy}\tilde{h}^{(1)}_\ell(y)\right|_{y\to\infty} = |\tilde{j}_\ell(x')| \quad \text{if } x > \sqrt{\ell(\ell+1)} \tag{B.8}$$

We notice also the Wronskian relation

$$\tilde{j}_\ell(x)\frac{d}{dx}\tilde{h}^{(1)}_\ell(x) - \frac{d}{dx}\tilde{j}_\ell(x)\tilde{h}^{(1)}_\ell(x) = i \tag{B.9}$$

Combining (B.6), (B.7), (B.8) and (B.9), we get

$$\left|\frac{d}{dx}K_\ell(x,x')\right| < \sup\begin{cases} \sup\limits_{x<\sqrt{\ell(\ell+1)}} |\tilde{j}'_\ell(x)\tilde{h}^{(1)}_\ell(x)| \\ \sup\limits_{x<\sqrt{\ell(\ell+1)}} |\tilde{j}_\ell(x)\tilde{h}^{(1)\prime}_\ell(x)|. \\ 1+\sup|\tilde{j}_\ell| \end{cases}$$

We now calculate

$$\sup_{x<\sqrt{\ell(\ell+1)}} \left|\tilde{j}_\ell(x)\frac{d}{dx}\tilde{h}_\ell(x)\right| = \sup\left|\tilde{j}_\ell(x)\left(\frac{d\tilde{j}_\ell}{dx}+i\frac{d\tilde{n}_\ell}{dx}\right)\right|$$

We have the following properties in $0<x<\sqrt{\ell(\ell+1)}, \tilde{j}_\ell>0$ increasing, $\tilde{n}_\ell<0$ algebraically increasing, and hence, from (B.9),

$$|\tilde{j}_\ell\tilde{n}'_\ell| + |\tilde{j}'_\ell n_\ell| = 1,$$

but since $|\tilde{h}_\ell|$ decreases

$$|\tilde{j}_\ell\tilde{j}'_\ell| < |\tilde{n}_\ell\tilde{n}'_\ell|.$$

Hence

$$|\tilde{j}_\ell\tilde{j}'_\ell| < |\tilde{j}_\ell\tilde{n}_\ell\tilde{j}'_\ell\tilde{n}'_\ell|^{1/2}$$

and, with

$$x = |\tilde{j}_\ell\tilde{n}'_\ell|,$$

$$|\tilde{j}_\ell\tilde{h}^{(1)\prime}_\ell| < \sqrt{x^2+x(1-x)} < 1$$

Similarly, with $Y = |\tilde{j}'_\ell\tilde{n}_\ell|$

$$|\tilde{j}'_\ell\tilde{h}^{(1)}_\ell| < \sqrt{y^2+y(1-y)} < 1.$$

Finally, we get

$$\left|\frac{d}{dx}K_\ell(x,x')\right| < 1+\sup|\tilde{j}_\ell|. \tag{B.10}$$

3. After this we are left with three problems: find

$$\sup|J_{\ell+1/2}(x)H^{(1)}_{\ell+1/2}(x)|,\ \sup|x\,J_{\ell+1/2}(x)H^{(1)}_{\ell+1/2}(x)|,\ \sup|\sqrt{x}\,J_{\ell+1/2}(x)|.$$

We now proceed to the next step which is to show that

$$|J_{\ell+1/2}(x)H^{(1)}_{\ell+1/2}(x)|$$

reaches its maximum for $x>\ell+\frac{1}{2}$, and has no relative maximum for $x<\ell+\frac{1}{2}$. If this is true, it is also true for the other quantities considered.

Consider first the product $P = -J_\nu N_\nu$, where $\nu=\ell+\frac{1}{2}$. P satisfies the differential equation (17)

$$\theta^3P - 4(\nu^2-x^2)\theta P + 4x^2P = 0 \tag{B.11}$$

with $\theta = x(d/dx)$, which admits as first integral

$$P\theta^2 P - \tfrac{1}{2}(\theta P)^2 + 2(x^2 - v^2)P^2 = -2v^2 P^2(0) \tag{B.12}$$

From (B.11) one deduces that P has at most one maximum in $0 < x < v$. Indeed if $\theta P < 0, \theta^3 P < 0$ and hence $\theta^2 P$ remains negative to the right of the maximum. From (B.12) one gets that, at the maximum x_M of P, if it exists,

$$(v^2 - x_M^2) < \frac{v^2 P^2(0)}{P^2_{\max}}. \tag{B.13}$$

With

$$v^2 P^2(0) = \left(\frac{1}{\pi}\right)^2$$

and

$$P^2_{\max} > P^2(v)$$

and, using the properties which can be found in the book of Watson [16]:

$$\left.\begin{array}{l} v^{1/3} J_v(v) \text{ increases with } v \\ |N_v(v)/J_v(v)| \text{ decreases with } v \end{array}\right\} \tag{B.14}$$

we get, taking $v > 9$

$$P^2(x = v) > \frac{[9^{1/3} J_9(9)]^4 (\sqrt{3})^2}{v^{4/3}} \simeq 0.11973 v^{-4/3}$$

(for $v < 9$ we can use tables).

Hence

$$x_M > v - \tfrac{1}{2} v^{1/3} \quad \text{for } v > 9. \tag{B.15}$$

Now the maximum of $J_v^2 |H_v|^2$, at x'_M is necessary such that $x'_M > x_M$ since $J_v^4(x)$ is increasing for $0 < x < v$. This maximum is given by

$$2 + \left(\frac{N}{J}\right)^2 + \frac{N}{J}\frac{N'}{J'} = 0. \tag{B.16}$$

(B.14) gives us some control on N/J, but we need also to control N'/J'. To do so we consider a new quantity

$$Q = -J_v N_v + \alpha J_v^2$$

which satisfies also (B.12) and has therefore at most one maximum in $0 < x < v$. We adjust α in such a way that $Q'(X = v) > 0$, by using properties (B.14), together with

$$JN' - NJ' = 2/(\pi z)$$

and

$$v^{2/3} J'_v(v) \text{ increasing} \tag{B.17}$$

We have

$$Q'(x=v) = -2J'N - \frac{2}{\pi v} + 2\alpha JJ'$$

$$> 2JJ'[\sqrt{3}+\alpha] - \frac{2}{\pi v} \text{ using } \left(\left|\frac{N}{J}\right| > \sqrt{3}\right)$$

$$> \frac{2}{v}\left[9J_9(9)J'_9(9)[\sqrt{3}+\alpha) - \frac{1}{\pi}\right], \text{ for } v > 9,$$

which is positive for $\alpha > 0.016$. Therefore we have

$$-J'N - N'J + 0.032JJ' > 0$$

for $0 < x < v$ and hence

$$\frac{N'}{J'} < 0.032 - \frac{N}{J} \tag{B.18}$$

The condition for the maximum of $J_v H_v$ implies therefore

$$2 + 0.032\frac{N}{J} < 0 \tag{B.19}$$

However, in $0 < x < v, |N/J|$ is a decreasing function of x. We shall now find a bound on $|N/J|$ at $x = x_M < x'_M$: similarly, to

$$\frac{d}{dv}\left|\frac{N_v(v)}{J_v(v)}\right| < 0$$

one can prove (Appendix C)

$$\frac{d}{dv}\left|\frac{N_v(v - tv^{1/3})}{J_v(v - t\gamma^{1/3})}\right| < 0 \tag{B.20}$$

This allows us to get, for $v > 9$

$$4.6 > \left|\frac{N_v}{J_v}\left(v - \frac{v^{1/3}}{2}\right)\right| > \left|\frac{N_v}{J_v}(x_M)\right| > \left|\frac{N_v}{J_v}(x'_M)\right| \tag{B.21}$$

which is incompatible with (B.19) and proves that for $v > 9\,|J_v H_v|$ is *monotonously increasing* in $0 < x < v$.

4. Now we shall estimate a bound on $|J_v H_v|$ by using the asymptotic formulae near the turning point and the error bounds given by Olver [17]. These are, after some majorizations, for $v > 9, x \geqq v$

$$\left(\frac{v}{2}\right)^{1/3} J_v < 1.025[A_i(v^{2/3}\zeta) + 0.025M(v^{2/3}\zeta)]$$

$$-\left(\frac{v}{2}\right)^{1/3} N_v < -B_i(v^{2/3}\zeta) + 0.032M(v^{2/3}\zeta) \tag{B.22}$$

where

$$M = \sqrt{(A_i)^2 + (B_i)^2}$$

Ai and Bi are the Airy functions, and

$$\zeta = -\left\{\frac{3}{2}\int_1^t \frac{(t^2-1)^{1/2}}{t}\,dt\right\}^{2/3} \quad \text{with } t = \frac{x}{\nu}$$

Maximizing with respect to ζ, we get

$$\nu^{2/3}\,|J_\nu H_\nu^{(1)}| < 0.5066 \quad \text{for } \nu > 9 \tag{B.23}$$

It can be checked numerically that this is also true for $\nu < 9$.

From this we deduce

$$|K_\ell(x, x')| < \frac{0.796x}{(\ell+\frac{1}{2})^{2/3}}. \tag{B.24}$$

We also need an independent bound on K_ℓ and on $\tilde{j}_\ell$. We notice that the absolute maximum of $|\tilde{j}_\ell|$ occurs between $x = \ell + \frac{1}{2}$ and the first zero of $\tilde{j}_\ell$. Indeed

$$\tilde{j}_\ell^2 + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)^{-1} (\tilde{j}_\ell')^2$$

is a decreasing function of x for $x > \sqrt{\ell(\ell+1)}$. Also the absolute maximum of $K_\ell(x, x)$ is for $x < j_{\ell+1/2},1$.

From the property

$$\frac{d}{d\nu} J_\nu(\nu + t\nu^{1/3}) N_\nu - \frac{dN_\nu}{d\nu} J_\nu < 0$$

established in Appendix C, one deduces that the first zero of $J_\nu, j_{\nu,1}$ is such that

$$\frac{j_{\nu,1} - \nu}{\nu^{1/3}} \text{ decreases} \tag{B.25}$$

Therefore we deduce that

$$|K_\ell(x, x')| < 0.796\,(\ell + \tfrac{1}{2})^{1/3}\,\frac{j_{9,1}}{9} \quad \text{for } \ell \geqq 9$$

i.e.,

$$|K_\ell(x, x')| < 1.2(\ell + \tfrac{1}{2})^{1/3} \tag{B.26}$$

Naturally one finds that the restriction $\ell \geqq 9$ can be removed. Similarly

$$|\tilde{j}_\ell(x)| < \sqrt{\frac{\pi}{2}}\max(J_{\ell+1/2})\sqrt{\frac{j_{9,1}}{9}}\,\nu^{1/2}$$

i.e.,

$$|\tilde{j}_\ell(x)| < 1.04(\ell + \tfrac{1}{2})^{1/6} \quad \text{for } \ell \geqq 9$$

from which (III.12) follows.

Appendix C. A property of monotonicity on bessel functions

In the book of Watson [16] the following property is established:

$$N_\nu(\nu)\frac{d}{d\nu}J_\nu(\nu) - J_\nu(\nu)\frac{d}{d\nu}J_\nu(\nu) < 0 \tag{C.1}$$

We want to prove, by using a generalization of the method, that

$$N_\nu(\nu + t\nu^{1/3})\frac{d}{d\nu}J_\nu(\nu + t\nu^{1/3}) - J_\nu(\nu + t\nu^{1/3})\frac{d}{d\nu}N_\nu(\nu + t\nu^{1/3}) < 0 \tag{C.2}$$

We shall specify the argument of the Bessel functions later and use

$$J_\nu\frac{d}{d\nu}N_\nu(f(\nu)) - N_\nu\frac{d}{d\nu}J_\nu(f(\nu))$$
$$= \frac{2}{\pi}\left\{\left(\frac{df}{d\nu}\Big/ f(\nu)\right) - \frac{1}{\nu}\int_0^\infty K_0\left[2f(\nu)\sinh\left(\frac{t}{2\nu}\right)\right]e^{-t}dt\right\}$$

Now we use the fact that $K_0'(z) < 0$ and that $\sinh x > x$

$$J_\nu\frac{d}{d\nu}N_\nu(f(\nu)) - N_\nu\frac{d}{d\nu}J_\nu(f(\nu))$$

$$> \frac{2}{\pi}\left[\frac{f'}{f} - \frac{1}{\nu}\int_0^\infty K_0\left(\frac{f(\nu)}{\nu}t\right)\exp(-t)dt\right]$$

$$= \frac{2}{\pi}\left[\frac{f'}{f} - \frac{\log\left(\frac{f(\nu)}{\nu} + \sqrt{\left(\frac{f(\nu)}{\nu}\right)^2 - 1}\right)}{\sqrt{(f(\nu))^2 - \nu^2}}\right] \quad \text{for } f(\nu) > \nu$$

$$= \frac{2}{\pi}\left[\frac{f'}{f} - \frac{\operatorname{Arc\,cos}\left(\frac{\nu}{f(\nu)}\right)}{\sqrt{\nu^2 - (f(\nu))^2}}\right] \quad \text{for } f(\nu) < \nu.$$

we want to show that for the choice $f(\nu) = \nu + t\nu^{1/3}$ the brackets corresponding to $t > 0$ or $t < 0$ are positive.

If $f(\nu) = \nu + t\nu^{1/3}$

$$\frac{\nu f'(\nu)}{f(\nu)} = \frac{2}{3}\frac{\nu}{f(\nu)} + \frac{1}{3}.$$

In the case $f(\nu) < \nu$, the problem is reduced to study the sign of

$$\frac{2x}{3} + \frac{1}{3} - \frac{\sin^{-1}\sqrt{1 - \frac{1}{x^2}}}{\sqrt{1 - \frac{1}{x^2}}} = \frac{\phi(x)}{\sqrt{1 - \frac{1}{x^2}}}, \quad x > 1.$$

By successive differentiations of $\phi(x)$ it can be shown indeed that $\phi(x)$ is positive.

Similarly one can also prove

$$\frac{2}{3y}+\frac{1}{3}-\frac{\log(y+\sqrt{y^2-1})}{\sqrt{y^2-1}}>0,\ y>1.$$

Inequality (C.2) is therefore established both for $t<0$ and $t>0$. From (C.2), one gets, by integration,

$$-\frac{N_{v_0}(v_0-tv_0^{1/3})}{J_{v_0}(v_0-tv_0^{1/3})}>-\frac{N_v(v-tv^{1/3})}{J_v(v-tv^{1/3})}>-\frac{B_i(2^{1/3}t)}{A_i(2^{1/3}t)} \tag{C.3}$$

when the denominators are positive for $v_0 \leqq v < \infty$.

One also gets, by adjusting $v+tv^{1/3}$ to coincide with $j_{v,1}$

$$\frac{d}{dv}\left(\frac{j_{v,1}-v}{v^{1/3}}\right)<0$$

and hence $j_{v,1}>v+1.85575v^{1/3}$. The same property can be shown to hold for the higher zeros, by using the interlacing of the zeros of J_v and N_v, and also holds for the zeros of N_v.

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