Commun. Math. Phys. 69, 179–193 (1979)

Communications in
Mathematical
Physics

O(2) Symmetric Connections in an SU(2) Yang-Mills Theory**

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Abstract. The general O(2) symmetric Yang-Mills equations are derived. An ansatz for O(2) symmetric merons is presented and it is shown that any connection in this ansatz will have SU(2) topological charge density which is a sum of delta functions at points in a plane with weights $\pm\frac{1}{2}$. It is shown that any connection in this ansatz will be C^∞ away from these points.

1. Introduction

If the four-dimensional, Euclidean space, Yang-Mills equations are required to be O(3) symmetric, it was shown by Witten [1] that the equations reduced to the equations describing an interacting system on the Poincaré half-plane consisting of a $U(1)$ gauge field and a charged scalar field with self-couplings. DeAlfaro, Fubini, and Furlan [2] discovered an explicit O(3) symmetric solution to the Yang-Mills equations with the property that the topological charge density has values $\pm\frac{1}{2}$ concentrated at points; the 'two meron solution.' For arbitrary positive integer N, Glimm and Jaffe [3, 4] reduced the problem of finding N-meron, O(3) symmetric, solutions to the question of whether certain (infinite action) solutions to the scalar elliptic equation.

$$r^2(\partial_r^2 + \partial_r^2)\psi = \psi(\psi^2 - 1) \tag{1.1}$$

existed. These N-meron solutions of Glim and Jaffe are also characterized by a topological charge density which is equal to a sum of delta functions on a line with weights $\pm\frac{1}{2}$ at the merons. Jonsson, McBryan, Zirilli, and Hubbard [5] proved that these (infinite action) solutions to Eq. (1.1) do indeed exist. In this paper, an O(2) symmetric SU(2) Yang-Mills connection is defined and the form of the O(2) symmetric Yang-Mills equations are derived. It is then shown that certain classes of O(2) symmetric connections have topological charge densities which are the sum of delta functions in a plane with weights $\pm\frac{1}{2}$. The O(3) symmetric solutions of Glimm and Jaffe are included in these classes. It remains to be proven whether there are any other solutions to the Yang-Mills equations contained in these classes.

* National Science Foundation Pre-doctoral Fellow
** Supported in part by the National Science Foundation under Grant PHY 77-18762

0010-3616/79/0069/0179/$03.00

For convenience, this paper is divided into seven sections. In Sect. 2, the form of the Yang-Mills equations are derived when the fields are required to have O(2) symmetry. Section 3 relates the O(3) symmetric Yang-Mills equations derived by Witten to the O(2) symmetric equations. In Sect. 4, the O(3) symmetric meron solutions of Glim and Jaffe are discussed in the O(2) format. In Sect. 5, we define our classes of O(2) symmetric connections. It is shown in Sect. 6 and 7 that these classes are disjoint and that any connection in a given class defines a Yang-Mills connection on $\mathbb{R}^4$ which has topological charge density concentrated at points in a plane with weights $\pm\frac{1}{2}$.

2. The O(2) Symmetric Yang-Mills Equations

The following index notations will be used: Greek indices $\alpha, \beta, \gamma, \delta$ run from 0 to 3, where $x^0 = t, x^1 = x, x^2 = y, x^3 = z$. Greek indices $\mu, \nu, \sigma, \lambda$ run from 1 to 3 where $x^\nu(\nu = 1) = t, x^\nu(\nu = 2) \equiv \rho = (x^2 + y^2)^{1/2}, x^\nu(\nu = 3) = z$. Latin lower case indices i, j, k run from 1 to 3, Latin capitals A, B, C run from 1 to 2 where $x^A(A = 1) = x$ and $x^A(A = 2) = y$. Repeated indices are to be summed over. It is convenient to define the following space-time dependent representation of the SU(2) Lie algebra,

$$\begin{aligned} Q^1 &= \varepsilon^{3BA}\frac{x^B}{\rho}\sigma^A \\ Q^2 &= \sigma^3 \\ Q^3 &= \frac{x^A}{\sigma}\sigma^A \end{aligned} \tag{2.1}$$

where $\sigma^i(i = 1, 2, 3)$ are the Pauly matrices and $(\varepsilon)^{ijk}$ is the completely antisymmetric Levi-Civita symbol. Further define

$$\begin{aligned} Q_A^k &= \varepsilon^{3BA}\frac{x^B}{\rho}Q^k \\ R_{AB} &= \delta_{AB} - x_A x_B/\rho^2 \end{aligned} \tag{2.2}$$

then the following commutation algebra is satisfied by $\{Q^k\}_{k=1}^3$

$$\begin{aligned} [Q^i, Q^j] &= 2i\varepsilon^{ijk}Q^k \\ [Q^i, Q_A^i] &= 2i\varepsilon^{ijk}Q_A^k \\ [Q_A^i, Q_B^j] &= 2iR_{AB}\varepsilon^{ijk}Q^k. \end{aligned} \tag{2.3}$$

One also has the trace relations:

$$\begin{aligned} Tr(Q^iQ^k) &= 2\delta^{ik} \\ Tr(Q_A^iQ_A^k) &= 2\delta^{ik} \\ X_AQ_A^i &= 0. \end{aligned} \tag{2.4}$$

An arbitrary Yang-Mills connection may be written in the following form:

$$A_0 = \frac{i}{2}A_t^kQ^k$$

$$A_3 = \frac{i}{2} A_z^k Q^k$$

$$A_B = \frac{i}{2}\left\{ A_\rho^k \frac{x_B}{\rho} Q^k + \frac{1}{\rho}(\phi^1 Q_B^1 + (\phi^2 + 1)Q_B^2 + \phi^3 Q_B^3)\right\}. \tag{2.5}$$

We define an O(2) symmetric connection by the requirement [6]:

$A_\mu^k = A_\mu^k(t, \rho, z)$ and $\phi^k = \phi^k(t, \rho, z)$.

Here the vector A_μ^k for each $k = 1, 2, 3$ is a three component vector and

$A_\mu^k(\mu = 1) = A_t^k$; $A_\mu^k(\mu = 2) = A_\rho^k$ and $A_\mu^k(\mu = 3) = A_z^k$.

The Yang-Mills curvature tensor is given by the expression:

$$\mathscr{F}_{\alpha\beta} = \partial_\alpha \mathscr{A}_\beta - \partial_\beta \mathscr{A}_\alpha + [\mathscr{A}_\alpha \mathscr{A}_\beta] \tag{2.6}$$

If one defines the electric and magnetic parts of $\mathscr{F}_{\alpha\beta}$ in the usual way, $E_\alpha = \mathscr{F}_{0\alpha}$, $B_\alpha = \frac{1}{2}\varepsilon^{0\alpha\beta\gamma}\mathscr{F}_{\beta\gamma}$, then in terms of the fields A_μ^k and ϕ^k:

$$E_3 = i/2 F_{03}^k Q^k$$

$$E_A = i/2\left\{ F_{0\rho}^k \frac{x_A}{\sigma} Q_k + 1/\rho (D_0\phi)^k Q_A^k \right\}$$

$$B_3 = i/2(1/\rho (D_\rho \phi)^k Q^k)$$

$$B_A = i/2\left\{ F_{3\rho}^k Q_A^k - \frac{x_A}{\rho}(D_3\phi)^k Q^k \right\}. \tag{2.7}$$

In Eq. (2.7) the three-dimensional field tensor $F_{\mu\nu}^k(\mu, \nu = 1, 2, 3)$ and $(D_\mu \phi)^k$ are defined by:

$$\begin{aligned} F_{\mu\nu}^k &= \partial_\mu A_\nu^k - \partial_\nu A_\mu^k - \varepsilon^{ijk} A_\mu^i A_\mu^j \\ (D_\mu \phi)^k &= \partial_\mu \phi^k - \varepsilon^{ijk} A_\mu^i \phi^j \end{aligned} \tag{2.8}$$

where

$$\partial_\mu(\mu = 1) = \frac{\partial}{\partial t}; \quad \partial_\mu(\mu = 2) = \frac{\partial}{\partial \rho}; \quad \partial_\mu(\mu = 3) = \frac{\partial}{\partial z}.$$

In terms of the three-dimensional fields of Eq. (2.5), the Yang-Mills action is

$$a = 2\pi \int_{\rho \geqq 0} \iint dt d\rho dz\, \mathscr{L}(\rho, t, z) \quad \text{and}$$

$$\mathscr{L} = \rho/4 F_{\mu\nu}^k F_{\mu\nu}^k + \frac{1}{2\rho}(D_\mu \phi)^k (D_\mu \phi)^k. \tag{2.9}$$

The SU(2) topological charge is

$$Q^{\mathrm{SU}(2)} = 2\pi \int_{\rho \geqq 0} \iint dt d\rho dz\, q(\rho, z, t) \quad \text{and}$$

$$q(\rho, z, t) = \frac{-1}{16\pi^2} \varepsilon_{\mu\nu\sigma} F_{\mu\nu}^k D_\sigma \phi^k. \tag{2.10}$$

The SU(2) charge density may be written as the divergence of a non-gauge invariant current

$$I_\alpha = \frac{-1}{8\pi^2}\varepsilon_{\delta\alpha\beta\gamma}Tr(\mathscr{A}_\delta\partial_\beta\mathscr{A}_\gamma + \tfrac{2}{3}\mathscr{A}_\delta\mathscr{A}_\beta\mathscr{A}_\gamma)$$

whose components are:

$$\begin{aligned}
I_A &= \frac{-x_A}{\rho^2}\frac{1}{16\pi^2}(D_3\phi^k A_0^k - D_0\phi^k A_3^k + (\partial_0 A_3^k - \partial_3 A_0^k)(\phi^k + \delta^{k2}))\\
I_3 &= \frac{1}{16\pi^2}\frac{1}{\rho}(D_\rho\phi^k A_0^k - D_0\phi^k A_\rho^k + (\partial_0 A_\rho^k - \partial_\rho A_0^k)(\phi^k + \delta^{k2}))\\
I_0 &= \frac{-1}{16\pi^2}\frac{1}{\rho}(D_\rho\phi^k A_3^k - D_3\phi^k A_\rho^k + (\partial_\rho A_3^k - \partial_3 A_0^k)(\phi^k + \delta^{k2})).
\end{aligned}\tag{2.11}$$

The Yang-Mills equations in terms of the fields A^k and ϕ^k are

$$\begin{aligned}
&\frac{1}{\rho}\partial_\mu\rho F_{\mu\nu}^k - \varepsilon^{ijk}A_\mu^i F_{\mu\nu}^j + \frac{1}{\rho^2}\varepsilon^{ijk}\phi^i D_\nu\phi^j = 0\\
&D_\mu\frac{1}{\rho}D_\mu\phi^k = 0.
\end{aligned}\tag{2.12}$$

Define a metric tensor $g^{\mu\nu} = \rho^2\delta^{\mu\nu}$ with determinant $\sqrt{g} = \frac{1}{\rho^3}$. The lagrangian of Eq. (2.9) can be rewritten as

$$\mathscr{L}(\rho, z, t) = \tfrac{1}{4}\sqrt{g}g^{\mu\sigma}g^{\nu\pi}F_{\mu\nu}^k F_{\sigma\pi}^k + \tfrac{1}{2}\sqrt{g}g^{\mu\nu}D_\mu\phi^k D_\nu\phi^k \tag{2.13}$$

This is the metric covariant lagrangian for an SU(2) Yang-Mills field coupled to a triplet of Higg's mesons on a three dimensional space (with boundary) of constant negative curvature. The boundary is the plane $\rho = 0$. The Yang-Mills equations (Eq. 2.12) are the variational equations of this lagrangian when written in terms of the metrix $g^{\mu\nu}$. To avoid confusion in the remainder of this paper, all raising and lowering of the three dimensional indices ($[\mu, \nu, \sigma] = [t, \rho, z]$) will be done by the Kronecker delta $\delta_{\mu\nu}$. Factors of ρ will be written explicitly.

The self dual and anti-self dual instanton solutions to the Yang-Mills equations will be solutions to the first order equations $\mathscr{F}_{\alpha\beta} = \pm\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\mathscr{F}_{\gamma\delta}$. In terms of the three-dimensional formalism, this is just

$$D_\mu\phi^k = \pm\tfrac{1}{2}\rho\varepsilon_{\mu\sigma\gamma}F_{\sigma\lambda}. \tag{2.14}$$

Here $\varepsilon_{\mu\sigma\lambda}$ is the completely antisymmetric three-dimensional Levi-Civita tensor: $\varepsilon_{t\rho z} \equiv +1$.

The O(2) symmetric equations have this form for the following reasons. In coordinates (t, ρ, z, μ) on $\mathbb{R}^4$ with $\mu = \operatorname{Arg}(x + iy)$, the restriction to O(2) symmetry means that all fields and gauge transformations must be independent of μ. Under a guage transformation U, the four-dimensional Yang-Mills connection $\mathscr{A}_\alpha$ transforms as $\mathscr{A}_\alpha \to U\mathscr{A}_\alpha U^{-1} + U\partial_\alpha U^{-1}$. If U is independent of the angle μ then the component $\mathscr{A}_\mu$ of the connection transforms as a Higgs scalar. The form of the metric $g^{\mu\nu}$ follows from the fact that the Yang-Mills lagrangian in its original form on $\mathbb{R}^4$ is scale invariant.

From this point on, we shall work solely in terms of the three-dimensional variables (t, ρ, z), and fields A_μ^k and ϕ^k. Define the manifold, M, with boundary by

$$M = \{(t, \rho, z) \in \mathbb{R}^3 \,|\, \rho \geqq 0\} \tag{2.15}$$

Let Λ_M denote the exterior algebra of differential forms on M. The forms $dx^1 = dt$, $dx^2 = d\rho, dx^3 = dz$ form a basis of 1-forms for Λ_M; d will denote the exterior derivative in the usual way on Λ_M; $\wedge$ will denote the exterior product on Λ_M.

The symbol A will refer to an SU(2) connection on M, a Lie algebra valued 1-form; A^k will refer to its components in some specified basis for the SU(2) algebra. The symbol ϕ will refer to the Higg's field; ϕ^k to the components with respect to some specified basis. Notice that on $M, dQ^k = 0$ for $k = 1, 2, 3$.

The manifold M has a boundary, ∂M; the boundary is the plane $\rho = 0$. To discuss certain topological notions it will be necessary to compactify M. The conformal compactification of M will be denoted $\hat{M}$ and it is the three ball; the sphere at infinity is mapped to the South Pole. The boundary is the two sphere and will be denoted $\partial\hat{M}$.

As it is written in Eq. (2.9), the Yang-Mills lagrangian is a functional of A and ϕ and may be interpreted as a scalar function on M. Similarly the SU(2) charge density $q(x)$ in Eq. (2.10). The current I_α of Eq. (2.11), when interpreted as the components of a 1-form in Λ_M has the form

$$I = \frac{1}{8\pi^2}\varepsilon_{\mu\nu\sigma}\operatorname{Tr}\left(\partial_\nu A_\sigma\left(\phi + \frac{i}{2}Q^2\right) + A_\nu D_\sigma \phi\right)dx^\mu. \tag{2.16}$$

If one defines the natural duality operation in Λ_M by

$$*dx_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\sigma}dx^\nu \wedge dx^\sigma \tag{2.17}$$

then $q(x)$ is related to I by

$$q(x) = d^*I \equiv *d*I$$

and

$$*I = -\frac{1}{8\pi^2}\operatorname{Tr}((\phi + i/2Q^2)dA + A \wedge D\phi) \tag{2.18}$$

The Yang-Mills equations on M (Eq. (2.12) are not complete without specification of the boundary conditions for the connection A and the Higg's field ϕ on ∂M. Because of the factor $1/\rho$ in the lagrangian of Eq. (2.9), we take as our boundary condition on the Higg's field ϕ the requirement that $D\phi = 0$ on ∂M ($D\phi$ is defined by $D\phi = d\phi + [A, \phi]$). The field ϕ may be considered as a cross section of a three-dimensional vector bundle over M. We can use the basis $\{Q^i\}_{i=1}^3$ as a basis for the cross sections of this vector bundle. Locally we can pick coordinates in the bundle so that

$$\phi = \frac{i}{2}\hat{\phi}U^{-1}Q^2U \tag{2.19}$$

This is just the polar decomposition where $\hat{\phi}$ is a scalar function on M and U is a cross section of the principal SU(2) bundle over M. The condition that $D\phi = 0$ on ∂M implies that

$$d\hat{\phi} = 0$$
$$A = \hat{A}\frac{i}{2}U^{-1}Q^2U + U^{-1}dU \tag{2.20}$$

at $\rho = 0$. $\hat{A}$ is a 1-form in Λ_M.

The lagrangian of Eq. (2.9) has formally six independent degrees of freedom. Equations (2.20) constrain three of them on the boundary of M.

3. The O(3) Symmetric Ansatz of E. Witten in O(2) Form

Witten's O(3) symmetric ansatz [1] takes the SU(2) connection to have the following form:

$$\mathscr{A}_0 = \frac{i}{2}\bar{A}_0\frac{x_k}{r}\sigma^k$$
$$\mathscr{A}_k = \frac{i}{2}\bar{A}_1\frac{x_k}{r}x_j\sigma^j + \frac{i}{2}\frac{\bar{\phi}_0}{r}\left(\delta_{kj} - \frac{x_kx_j}{r^2}\right)\sigma^j + \frac{i}{2}(\bar{\phi}_1 + 1)\frac{1}{r}\varepsilon_{kji}x_i\sigma^j \tag{3.1}$$

where $r = (\rho^2 + z^2)^{1/2}$ and $\bar{A}_0, \bar{A}_1, \bar{\phi}_0$ and $\bar{\phi}_1$ are functions of r and t only. The O(3) symmetric fields $\bar{A}_i, \bar{\phi}_i i = 0, 1$ determine a set of O(2) symmetric fields (A^k_μ, ϕ^k) defined in Sect. 2. The correspondence is given by the following table:

$$A_0^1 = 0 \qquad A_0^2 = \bar{A}_0\frac{z}{r} \qquad A_0^3 = \bar{A}_0\frac{\rho}{r}$$
$$A_\rho^1 = (\bar{\phi}_1 + 1)\frac{z}{r^2} \qquad A_\rho^2 = \frac{z\rho}{r^2}\left(\bar{A}_1 - \frac{1}{r}\bar{\phi}_0\right) \qquad A_\rho^3 = \frac{1}{r^2}\left(\rho^2\bar{A}_1 + \frac{z^2}{r}\bar{\phi}_0\right)$$
$$A_3^1 = -(\bar{\phi}_1 + 1)\frac{\rho}{r^2} \qquad A_3^2 = \frac{1}{r^2}\left(z^2\bar{A}_1 + \frac{\rho^2}{r}\bar{\phi}_0\right) \qquad A_3^3 = \frac{\rho z}{r^2}\left(\bar{A}_1 - \frac{\bar{\phi}_0}{r}\right)$$
$$\phi^1 = \frac{\rho}{r}\bar{\phi}_0 \qquad \phi^2 = \frac{\rho^2}{r^2}(\bar{\phi}_1 + 1) - 1 \qquad \phi^3 = -\frac{\rho z}{r^2}(\bar{\phi}_1 + 1) \tag{3.2}$$

As an example, the solution to the Yang-Mills equations corresponding to a single instanton [7] at the origin of $\mathbb{R}^4$ has the following form in terms of (A^k_μ, ϕ^k);

$$A = \frac{2}{1 + x^2}\left\{(zd\rho - \rho dz)\frac{i}{2}Q^1 + (zdt - tdz)\frac{i}{2}Q^2 + (\rho dt - td\rho)\frac{i}{2}Q^3\right\}$$
$$\phi = \frac{2\rho}{1 + x^2}\left\{-t\frac{i}{2}Q^1 + \frac{i}{\rho_2}Q^2 - z\frac{i}{2}Q^3\right\} - \frac{i}{2}Q^2. \tag{3.3}$$

The single meron solution of DeAlfaro, Fubini and Furlan [2] has the following form in terms of (A^k_μ, ϕ^k):

$$A = \frac{1}{x^2}\left\{(zd\rho - \rho dz)\frac{i}{2}Q^1 + (zdt - tdz)\frac{i}{2}Q^2 + (\rho dt - td\rho)\frac{i}{2}Q^3\right\}$$

$$\phi = \frac{\rho}{x^2}\left\{-t\frac{i}{2}Q^1 + \rho\frac{i}{2}Q^2 - z\frac{i}{2}Q^3\right\} - \frac{i}{2}Q^2. \tag{3.4}$$

4. The O(3) Symmetric Merons

By a gauge transformation, the two meron solution of DeAlfaro, Fubini and Furlan [2] corresponding to a meron on the t-axis at the point (t, ρ, z) given by $S_+ = (\varepsilon, 0, 0)$ and an antimeron at $S_- = (-\varepsilon, 0, 0)$ on the t-axis may be put in the following form:

$$A = \frac{2\varepsilon}{x_+x_-}\frac{1}{x_+x_- + 2\rho\varepsilon}\left\{-2\rho z d\rho - 2ztdt + (\rho^2 - z^2 + t^2 - \varepsilon^2)dz\right\}\frac{i}{2}Q^1$$

$$\phi = \frac{i}{2}\left(-\frac{x^2 - \varepsilon^2}{x_+x_-}Q^2 - \frac{2z\varepsilon}{x_+x_-}Q^3\right)$$

$$\text{where } x_\pm = (\rho^2 + z^2 + (t \mp \varepsilon)^2)^{1/2}. \tag{4.1}$$

This gauge is manifestly neutral [4] as can be seen by examining the expression for the SU(2) current, equation (2.16). Both the connection, A, and the Higgs field, ϕ, are C^∞ in $M - \{S_+\} - \{S_-\}$. There is no singular string [3] in this gauge.

On the boundary $\partial M - \{S_+\} - \{S_-\}$, the two meron solution takes the following form:

$$A = \frac{2\varepsilon}{x_+^2 x_-^2}\{-2ztdt + (t^2 - z^2 - \varepsilon^2)dz\}\frac{i}{2}Q^1$$

$$\phi = \frac{i}{2}\left\{-\frac{(z^2 + t^2 - \varepsilon^2)}{x_+x_-}Q^2 - \frac{2z\varepsilon}{x_+x_-}Q^3\right\}. \tag{4.2}$$

If one defines $\tan\omega = \dfrac{2z\varepsilon}{t^2 + z^2 - \varepsilon^2}$, then the boundary values of Eq. (4.2) may be written as

$$A = d\omega\frac{i}{2}Q^1$$

$$\phi = \frac{i}{2}\left(\cos\omega Q^2 + \sin\omega Q^3\right). \tag{4.3}$$

Since the ρ component of A is zero on $\partial M - \{S_+\} - \{S_-\}$ and ϕ is a function only of ρ^2, it is clear that the conditions of equation (2.20) are identically satisfied on $\partial M - \{S_+\} - \{S_-\}$. Furthermore, the connection, A, and Higgs field ϕ are constant as $\rho \to \infty$ so that A and ϕ are a C^∞ connection and Higgs field on the compactified space, $\hat{M}$, except at the points $\{S_+\}$ and $\{S_-\}$. Equation (2.20) is satisfied on $\partial\hat{M} - \{S_+\} - \{S_-\}$.

The function ω is homogeneous of degree zero and is such that if Γ is any closed curve in $\partial M - \{S_+\} - \{S_-\}$,

$$\int_\Gamma d\omega = 2\pi(n_+^\Gamma - n_-^\Gamma) \tag{4.4}$$

where $n_\pm^\Gamma$ is the winding number of Γ about $S_\pm$.

The more general O(3) symmetric meron solutions of Glimm and Jaffe corresponding to N merons on the t-axis at points $S_{+i} = (t_{+i}0,0)$ and N antimerons on the t-axis at points $S_{-i} = (t_{-i},0,0)$ with $t_{-i} < t_i < t_{-(i+1)}i = 1$ to $N(t_{-(N+1)} \equiv \infty)$ can be put into a form analogous to the form of the two meron solutions given by Eq. (4.1). Let ψ_N be a solution to Eq. (1.1) corresponding to N merons at the points S_{+i} and N antimerons at the points S_{-i} for $i = 1$ to N. There exists a gauge in which the connections and Higgs field for the solution ψ_N has the form:

$$A = \frac{-1}{(\rho(1-\psi_N^2)^{1/2} + r)}\frac{r}{(1-\psi_N^2)^{1/2}}\left\{\psi_N(1-\psi_N^2)\frac{\rho}{r}\frac{1}{r^2}(\rho dz - z d\rho) - \frac{z}{r}d\psi_N\right\}\frac{i}{2}Q^1$$

$$\phi = \frac{i}{2}\left(\psi_N Q^2 - \frac{z}{r}(1-\psi_N^2)^{1/2}Q^3\right) \tag{4.5}$$

if one defines the angle ω by

$$\cos\omega = \psi_N\,; \quad \sin\omega = -\frac{z}{|z|}(1-\psi_N^2)^{1/2}. \tag{4.6}$$

Then on $\partial M - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ the boundary values of the connection and Higgs field of Eq. (4.5) are given precisely by Eq. (4.3) also. Equation (2.20) is satisfied on $\partial M - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ by the N-meron solution.

It was shown in reference [5] that $|\psi_N| \to 1$ pointwise as $\rho^2 + z^2 \to 0$ and t different from $t_{\pm i}$ for $i = 1$ to N. This implies that the angle ω defined in Eq. (4.6) is well defined in $\partial M - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$. Taking into account that ψ_N is equal to $+1$ on the t-axis between meron and antimeron, i.e., for $t_{-i} < t < t_{+i}$ (for $i = 1$ to N) and equal to -1 everywhere else on the t-axis one can show that for any closed curve Γ in $\partial M - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$,

$$\int_\Gamma d\omega = 2\pi\sum_{i=1}^{N}(n_{+i}^\Gamma - n_{-i}^\Gamma) \tag{4.7}$$

where n_{+i}^Γ is the winding number of Γ about the point $S_{\pm i}$.

The curvature form, F, for the connection in Eq. (4.5) is given by

$$F = -d\psi_N \wedge \frac{1}{r^2}(\rho dz - z d\rho)\frac{i}{2}Q^1. \tag{4.8}$$

As a distribution in the plane $\rho = 0$, the (t,z) component of F is formally

$$F_{tz}dt \wedge dz\frac{i}{2}Q^1\Big|_{\rho=0} = \left[2\pi\sum_{i=1}^{N}(\delta(z)\delta(t-t_{+i}) - \delta(z)\delta(t-t_{-i}))\right]dz \wedge dt\frac{i}{2}Q^1. \tag{4.9}$$

This is another way of expressing the loop integral of Eq. (4.7).

Both A and ϕ of Eq. (4.5) have been shown in reference [5] to be real analytic where $(\rho^2 + z^2) \neq 0$. The differentiability of ψ_N at $\rho^2 + z^2 = 0$ is not known but it seems a reasonable conjecture to postulate C^∞ behavior in $M - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$.

5. An Ansatz for O(2) Symmetric Merons

The O(3) symmetric N-meron solution of Glimm and Jaffe in the neutral gauge of reference [3] correspond to the following connection on $\mathbb{R}^4$:

$$\begin{aligned} \mathscr{A}_0 &= 0 \\ \mathscr{A}_k &= \frac{i}{2}(\psi_N + 1)\frac{1}{r}\varepsilon_{kji}x_i\sigma^j. \end{aligned} \tag{5.1}$$

In Eq. (5.1), ψ_N is a solution to Eq. (1.1). Equation (1.1) for ψ_N was derived by taking the Witten ansatz of Eq. (3.1) and looking for solutions with delta functions for the SU(2) topological charge density. This reduced the number of degrees of freedom in the ansatz to just one. Glimm and Jaffe showed that by a gauge transformation, $\bar{A}_0, \bar{A}_1$ and $\bar{\phi}_0$ could be set equal to zero yielding the connection in Eq. (5.1). The SU(2) charge density and current (Eq. (2.10) and (2.16)) are manifestly zero for the connection in Eq. (5.1). The delta functions of weight $\pm\frac{1}{2}$ in the SU(2) charge density are regained when a gauge transformation which is singular at the merons is made. The SU(2) charge density becomes formally $\frac{1}{2}\sum_{i=\pm 1}^{\pm N} \pm\, \delta(\vec{r})\delta(t - t_{\pm i})$ corresponding to N merons at the points $(t = t_{+i}, \vec{r} = 0)$ and N antimerons at $(t = t_{-i}, \vec{r} = 0)$ for $i = 1$ to N. To find N merons in the plane $\rho = 0$ at points $(t = t_{+i}, \rho = 0, z = z_{+i})$ and N anti-merons at $(t = t_{-i}, \rho = 0, z = z_{-i})$ a similar reduction of the number of degrees of freedom of the connection defined by Eq. (2.5) will be proposed in order to simplify the Yang-Mills equations (Eq. 2.12). For this reduced connection, the SU(2) charge density and current will be manifestly zero as for the connection in Eq. (5.1), but it will be shown that by a gauge transformation which is singular at the merons, the SU(2) charge density will formally be a sum of delta functions,

$$q = \frac{1}{2}\sum_{i=1}^{N} \pm\, \delta(x)\delta(y)\delta(t - t_{\pm i})\delta(z - z_{\pm i}).$$

Further, the connection on $\mathbb{R}^4$ will be C^∞ except at the merons in this gauge. If one restricts A and ϕ to be of the following form:

$$\begin{aligned} A &= \frac{i}{2}A^1 Q^1 \\ \phi &= \frac{i}{2}\phi^2 Q^2 + \frac{i}{2}\phi^3 Q^3 \end{aligned} \tag{5.2}$$

with A^1 in Λ_M, then the lagrangian of Eq. (2.9) reduces to

$$\mathscr{L} = \frac{\rho}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\rho}(D_\mu \phi^2 D_\mu \phi^2 + D_\mu \phi^3 D_\mu \phi^3)$$

with $F = dA^1$

$$D\phi^k = d\phi^k - \varepsilon^{k1j} A^1 \phi^j \tag{5.3}$$

with the restriction of the connection to that of Eq. (5.2), $F^2_{\mu\nu} = F^3_{\mu\nu} = 0$. The Yang-Mills equations become:

$$\frac{1}{\rho}\partial_\mu \rho F_{\mu\nu} + \frac{1}{\rho^2}(\phi^2 D_\nu \phi^3 - \phi^3 D_\nu \phi^2) = 0$$

$$D_\nu \frac{1}{\rho} D_\nu \phi^k = 0 \qquad k = 2,3 \tag{5.4}$$

These equations describe the interaction of a charged, scalar field with a $U(1)$ gauge potential on M. The ansatz of Eq. (5.2) transforms covariantly under gauge transformations generated by $\frac{i}{2}Q^1$. Given the field configuration of Eq. (5.2), the SU(2) charge density and current (Eq. (2.10) and (2.16)) are manifestly zero. The Yang-Mills action for the ansatz of Eq. (5.2) is

$$a = 2\pi \int_M \mathscr{L} d\rho dt dz.$$

A given pair (A, ϕ) of the form given by Eq. (5.2) defines an equivalence class $[A, \phi]$ of cross sections ϕ of the complex line bundle over M and connections A on this bundle. Two pairs (A_1, ϕ_1) and (A_2, ϕ_2) represent the same equivalence class if there exists a $U(1)$ gauge transformation, g, generated by $\frac{i}{2}Q^1$ such that g transforms (A_1, ϕ_1) into (A_2, ϕ_2).

Fix N pairs of points, $S_{\pm i} = (t_{\pm i}, 0, z_{\pm i})_{i=1}^N$. Define F_M^N to be the set of $U(1)$ gauge equivalent classes of connections and cross sections (of the form given by Eq. (5.2)) of the $\mathbb{C}^1$ bundle over M which are C^∞ in $M - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$ and satisfy the following three conditions:

(i) $D\phi = 0$ on $M - \bigcup_{i=\pm 1}^{\pm N}$

$$|\phi| = 1 \tag{5.5}$$

(ii) For any closed curve Γ in $\partial M - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$,

$$\mathrm{Tr}\left(- iQ^1 \int_\Gamma A \right) = 2\pi \sum_{i=1}^{N} (n^\Gamma_{+i} - n^\Gamma_{-i}) \tag{5.6}$$

where $n^\Gamma_{\pm i}$ is the winding number of Γ about the point of $S_{\pm i}$.

(iii) If $[A,\phi]\in F_M^N$, then for all $(A',\phi')\in[A,\phi]$, (A',ϕ') is C^∞ on $\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$.

Notice that each F_M^N is indexed by a set of distinct points.

In Sect. 4 it was shown that the two meron solution and the N-meron generalizations of Glimm and Jaffe define equivalence classes in some F_M^N. The remainder of this paper is devoted to studying the properties of the F_M^N and their members.

6. Properties of the F_M^N

In this section we prove two facts about the F_M^N. The first fact is that if one gauge transforms the boundary values on ∂M of a cross section and connection in a given F_M^N, then that gauge transformation can be extended in a C^∞ manner to all of M to define a gauge transformed connection and cross section. This will follow from Proposition 1. From Proposition 2 it will follow that the sets F_M^N and $F_M^{N'}$ are disjoint unless the set of points defining F_M^N and $F_M^{N'}$ are identical. Thus for $N \neq 0$, F_M^N does not contain equivalence classes which are pure gauges.

Define $F_{\partial M}^N$ indexed by the set of points $\{S_{\pm i}\}_{i=1}^N$ analogously to the definition in Sect. 5 of F_M^N. The set $F_{\partial M}^N$ is the set of $U(1)$ gauge equivalent classes of connections and cross sections of the $\mathbb{C}^1$ bundle over ∂M which are C^∞ in $\partial M - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$ and satisfy conditions (i), (ii) and (iii) of Sect. 5 on ∂M. Denote the principal $U(1)$ bundle over M by $P(M, U(1))$ and the principal $U(1)$ bundle over ∂M by $P(\partial M, U(1))$. A C^∞ gauge transformation on $M(\partial M)$ defines a C^∞ cross-section of $P(M, U(1))(P(\partial M, U(1)))$. For a connection and cross section (A,ϕ), let $(A,\phi)_{\partial M}$ denote the restriction of (A,ϕ) to ∂M. $(A,\phi)_{\partial M}$ is just the pull back under the inclusion $i\colon \partial M \to M$ of the pair (A,ϕ). The action of a cross section g of $P(M, U(1))$ on (A,ϕ) will be denoted by $g(A,\phi)$. Similarly the action of a cross section $\tilde{g}$ of $P(\partial M, U(1))$ will be denoted $\tilde{g}(A,\phi)_M$. Given $(A,\phi)\in[A,\phi]$ and $[A,\phi]$ in some F_M^N, a cross section $\tilde{g}$ of $P(\partial M, U(1))$ defines a gauge equivalent boundary value $\tilde{g}(A,\phi)_{\partial M}$.

Proposition 1. *Let $(A,\phi)\in[A,\phi]$ in some F_M^N. Let g be any cross section of $P(\partial M, U(1))$. Then g extends to a C^∞ cross section of $P(M, U(1))$. Further, there exists $(A',\phi')\in[A,\phi]$ such that $(A',\phi')_{\partial M} = g(A,\phi)_{\partial M}$.*

Proof. Note, $\hat{M}$ is the three-dimensional closed ball and its boundary, $\partial\hat{M}$, is the two sphere. Standard topological arguments (see for example reference [8] pp. 150–151) imply that $\tilde{g}$ can be extended to a C^∞ cross section, g, of $P(\hat{M}, U(1))$. From condition (iii) we see that because all members of $[A,\phi]$ may be defined on $\hat{M}$, there exists a pair $(A',\phi')\in[A,\phi]$ with boundary values equal to $\tilde{g}(A,\phi)_{\partial M}$, namely $g(A,\phi)$ where g is the extension of $\tilde{g}$.

Proposition 2. *Let $[A,\phi]\in F_M^N$ for $N > 0$. Then $[A,\phi]$ is not a pure gauge.*

Proof. The conditions (5.5) and (5.6) are invariant under the action of $U(1)$ gauge transformations which are C^∞ on $M - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$. If $[A,\phi]$ were a pure gauge then one could find, for each $(A,\phi)\in[A,\phi]$ a C^∞ gauge transformation, g, on $M-\bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ such that $g(A,\phi)=(0,1)$. Hence, $(0,1)\in[A,\phi]$ which contradicts the condition of Eq. (5.6). A similar argument can be used to show that $F_M^N\cap F_M^{N'}=\varnothing$ unless the points defining F_M^N and $F_M^{N'}$ coincide.

7. The Local Topological Charge

All field configurations of the form defined in Eq. (5.2) have zero SU(2) topological charge and zero SU(2) current as defined in Eq. (2.18). The total topological charge is gauge invariant but the current is not, the gauge transformations which are generated by $\frac{i}{2}Q^2$ and $\frac{i}{2}Q^3$ will produce nonzero SU(2) current. One can define the local topological charge of the merons by integrating the dual form to the current, Eq. (2.18), over the surface of a small ball surrounding each meron [2]. The concept of local topological charge is, of course, not gauge invariant [3].

From this point on, all work will be done on the conformal compactification, $\hat{M}$, of M. For each $i=1$ to N and for $\lambda>0$ define the set

$$B_{\pm i}^\lambda=\{(t,\rho,z)\in\hat{M}\,|\,(t-t_{\pm i})^2+\rho^2+(z-z_{\pm i})^2\leqq\lambda^2\} \tag{7.1}$$

and int $B_{\pm i}^\lambda$ to be the interior of $B_{\pm i}^\lambda$. Define $\hat{M}^\lambda$ to be the compact manifold with boundary resulting from the excision of the interiors of the $B_{\pm i}^\lambda$ from $\hat{M}$,

$$\hat{M}^\lambda=\hat{M}-\bigcup_{i=\pm 1}^{\pm N}\operatorname{int}B_i^\lambda. \tag{7.2}$$

The boundary of $\hat{M}^\lambda$ is the set

$$\partial\hat{M}^\lambda=\partial\hat{M}\cup\left(\bigcup_{i=\pm 1}^{\pm N}\partial B_i^\lambda\right)-\bigcup_{i=\pm 1}^{\pm N}\operatorname{int}D_i^\lambda$$

where

$$\begin{aligned}\partial B_{\pm i}^\lambda&=\{(t,\rho,z)\in\hat{M}\,|\,(t-t_{\pm i})^2+\rho^2+(z-z_{\pm i})^2=\lambda^2\}\\ D_{\pm i}^\lambda&=\{(t,0,z)\in\partial\hat{M}\,|\,(t-t_{\pm i})^2+(z-z_{\pm i})^2\leqq\lambda^2\}.\end{aligned} \tag{7.3}$$

Of course λ is taken to be small enough that all $B_{\pm i}^\lambda$ are disjoint.

The dual form to the current 1-form in Eq. (2.16) is

$$*I((A,\phi))=-\frac{1}{8\pi^2}\operatorname{Tr}\left(\left(\phi+\frac{i}{2}Q^2\right)dA+A\wedge D\phi\right) \tag{7.4}$$

The bundle $P(\hat{M}^\lambda,U(1))$ of Sect. 4 may be considered as a sub-bundle of the principal SU(2) bundle over $\hat{M}$, $P(\hat{M}^\lambda,\mathrm{SU}(2))$ by the obvious inclusion. If U is C^∞ cross section of $P(\hat{M}^\lambda,\mathrm{SU}(2))$ then U acts on $*I((A,\phi))$ in the following way: For fixed, but general (A,ϕ)

$$\begin{aligned}U(*I)=-\frac{1}{8\pi^2}&[\operatorname{Tr}\{\phi dA+A\wedge D\phi\}\\ &+d(\operatorname{Tr}\{(U^{-1}AU+U^{-1}dU)\tfrac{i}{2}Q^2-U^{-1}dU(U^{-1}\phi U)\})].\end{aligned} \tag{7.5}$$

For (A, ϕ) of the form defined by Eq. (5.2), Eq. (7.5) becomes

$$U(*I) = -\frac{1}{8\pi^2} d(\mathrm{Tr}\{(U^{-1}AU + U^{-1}dU)\tfrac{1}{2}Q^2 - U^{-1}dU(U^{-1}\phi U)\}). \tag{7.6}$$

Define the local topological charge of $(A, \phi) \in F_M^N$ in the gauge U by:

$$Q_{\mathrm{Loc}}^{\pm i}[U, (A, \phi)] = 2\pi \int_{\partial B^\lambda_{\pm 1} - \mathrm{int} D^\lambda_{\pm 1}} U(*I). \tag{7.7}$$

A calculation shows that $Q_{\mathrm{Loc}}^{\pm i}$ depends only on $[A, \phi] \in F_M^N$

Proposition 3. *For each $[A, \phi] \in F_M^N$, and $(A, \phi) \in [A, \phi]$ there exist a C^∞ cross section U of $P(\hat{M}^\lambda, \mathrm{SU}(2))$ such that*

a) $U(*I((A, \phi))) = 0$ *on* $\partial\hat{M}^\lambda - \bigcup_{i=\pm 1}^{\pm N}$

b) $Q_{\mathrm{Loc}}^{\pm i}[U, [A, \phi]] = \pm\frac{1}{2}$ *for* $i = 1$ *to* N.

Proof. On $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$ the condition of Eq. (5.5) and (5.6) require that $(A, \phi)_{\partial M}$ have the form:

$$\begin{aligned} A &= d\omega \tfrac{1}{2} Q^1 \\ \phi &= \tfrac{1}{2}(\cos\omega Q^2 + \sin\omega Q^3) \end{aligned} \tag{7.8}$$

where ω is C^∞ in $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$ and has the property that if Γ is any closed curve in $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \{S_i\}$

then

$$\int_\Gamma d\omega = 2\pi \sum_i \{n^\Gamma_{+i} - n^\Gamma_{-1}\} \tag{7.9}$$

where $n^\Gamma_{\pm i}$ is the winding number of Γ about $S_{\pm i}$.

Define U on $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \mathrm{int}\, D_i^\lambda$ to be

$$U = \exp\left(-(\omega + \pi)\frac{i}{2}Q^1\right)\exp\left(\omega\frac{i}{2}Q^2\right). \tag{7.10}$$

U need not be known explicitly over the rest of $\hat{M}^\lambda$. All one need show is that given U on $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} \mathrm{int}\, D_i^\lambda$, it can be extended in a C^∞ fashion to a cross section of $P(\hat{M}^\lambda, \mathrm{SU}(2))$. This is done in two steps. To extend U over $\partial B^\lambda_{\pm i} - \mathrm{int}\, D^\lambda_{\pm i}$ note that $\partial B^\lambda_{\pm i} - \mathrm{int}\, D^\lambda_{\pm i}$ is topologically a disc. Its boundary (as a subspace),

$$\partial(\partial B^\lambda_{\pm i} - \mathrm{int}\, D^\lambda_{\pm i}) = \{(t, \rho, z) \in \partial\hat{M} \,|\, (t - t_{\pm i})^2 + (z - z_{\pm i})^2 = \lambda^2\} \tag{7.11}$$

is a circle. U is defined on $\partial(\partial B^\lambda_{\pm i} - \mathrm{int}\, D^\lambda_{\pm i})$ for $i = 1$ to N, and the arguments previously cited [8] with the fact that the fundamental group of SU(2), $\pi_1(\mathrm{SU}(2))$, $=0$ imply that a C^∞ extension to $\partial B^\lambda_{\pm i} - \operatorname{int} D^\lambda_{\pm i}$ exists. These same arguments imply that a C^∞ extension of U from $\partial\hat{M}^\lambda$ to $\hat{M}^\lambda$ exists since $\pi_2(\mathrm{SU}(2))=0$ also. Thus if U is defined on $\partial\hat{M}^\lambda - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ by Eq. (7.9), then it can be extended to a C^∞ cross section of $P(\hat{M}^\lambda, \mathrm{SU}(2))$.

The extended cross section, U, acts on the boundary values of (A,ϕ) on $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ given by Eq. (5.8) to give the gauge transformed boundary values of (A,ϕ) which are

$$U^{-1}AU + U^{-1}dU = d\omega\frac{i}{2}Q^2$$

$$U^{-1}\phi U = -\frac{i}{2}Q^2 \tag{7.12}$$

The current 2-form, $U(*I)$ from Eq. (7.7), can be evaluated on $\partial\hat{M} - \bigcup_{i=\pm 1}^{\pm N} D^\lambda_i$ explicitly and is

$$U(*I) = \frac{1}{8\pi^2}d(d\omega). \tag{7.13}$$

Because ω is C^∞ on $\partial\hat{M}^\lambda - \bigcup_{i=\pm 1}^{\pm N} D^\lambda_i$, this is zero which proves part *a*) of Proposition 3. Both (A,ϕ) and U are C^∞ in $\hat{M}^\lambda$ so it follows that $U(*I)$ is also C^∞ in $\hat{M}^\lambda$.

$U(*I)$ is nonzero on each $\partial B^\lambda_{\pm i} - \operatorname{int} D^\lambda_{\pm i}$ for $i=1$ to N and here its integral can be evaluated also. In full, Eq. (7.7) is

$$Q^{\pm i}_{\mathrm{Loc}}(U,[A,\phi]) = -\frac{1}{4\pi}\int_{\partial B^\lambda_{\pm i} - \operatorname{int} D^\lambda_{\pm i}} d(\mathrm{Tr}\{U^{-1}AU + U^{-1}dU)\frac{i}{2}Q^2 - U^{-1}dU(U^{-1}\phi U)\}. \tag{7.14}$$

Using Stoke's theorem, this can be transformed to an integral over $\partial(\partial B^\lambda_{\pm i} - \operatorname{int} D^\lambda_{\pm i})$ giving

$$Q^{\pm i}_{\mathrm{Loc}}(U,[A,\phi]) = \frac{1}{4\pi}\int_{\partial(\partial B^\lambda_{\pm i} - \operatorname{int} D^\lambda_{\pm i})} d\omega = \pm\tfrac{1}{2}. \tag{7.15}$$

This completes the proof that U gives (A,ϕ) local topological charge $\pm\frac{1}{2}$ at $S_{\pm i}$ (for $i=1$ to N) for any $[A,\phi]\in F^N_M$.

Finally, since (A,ϕ) and U are C^∞ in $\hat{M}^\lambda - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$ as one takes $\lambda\to 0$ and $U^{-1}AU + U^{-1}dU$ and $U^{-1}\phi U$ satisfy Eq. (7.12), it follows that the connection defined on $\mathbb{R}^4$ by (A,ϕ) in the gauge given by U (Eq. (2.5)) is in fact C^∞ in $\mathbb{R}^4 - \bigcup_{i=\pm 1}^{\pm N}\{S_i\}$. For $\rho\neq 0$ this fact follows from the C^∞ behavior of (A,ϕ) and U.

At $\rho = 0$, the connection on $\mathbb{R}^4$ from Eq. (2.5) has the boundary values:

$$\mathscr{A}_0 = \frac{i}{2}\partial_0 \omega Q^2$$

$$\mathscr{A}_3 = \frac{i}{2}\partial_3 \omega Q^2$$

$$\mathscr{A}_B = 0.$$

Since $Q^2 \equiv \sigma^3$, the connection is C^∞ everywhere but at the points $\{S_{i=1}\}^N$.

Acknowledgements. The author would like to express gratitude to Thordur Jonsson for his many suggestions and criticisms, and to Arthur Jaffe for suggesting this problem and serving as a constant source of advice.

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Communicated by A. Jaffe

Received January 11, 1979