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On Systems of Particles with Finite-Range and/or Repulsive Interactions

L. N. Vaserstein

Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France

Abstract. In an arbitrary system of particles with central repulsive interactions, right and left velocities exist at each moment of time, including infinity. An arbitrary system of particles with finite-range interactions splits into independent bounded clusters. The number of collisions in Sinai's billiard is finite.

Professor Sinai has asked if a finite system of hard balls (spheres) in infinite space has only a finite number of collisions over the infinite time interval; one assumes that the spheres are homogeneous, and that momentum and kinetic energy are conserved. There is a similar question for a finite time interval and Sinai's billiard, i.e., a system in the space with convex obstacles (walls).

Some results obtained by Sinai and other authors [1–4] led to the hope that the above questions have a positive answer. This hope is confirmed in the present paper.

The theorem asserting the finiteness of the number of collisions (including reflections by the walls) does not extend to hard bodies with arbitrary shapes: even between two convex bodies in the plane there can be infinitely many collisions in a finite time interval. Also, the shape of the walls is essential: one ball in a convex domain in the plane can hit the boundary an infinite number of times in a finite amount of time [8]; it is clear that in convex vessels (billiard tables) a ball can also follow the boundary around at unit speed (in our vessel with convex walls this is possible only along a straight line interval contained in the boundary). If we drop the condition of conservation of energy it becomes possible to get an infinite number of collisions between three balls on the line.

We obtain however, in the context of this paper, a number of results on general systems of particles with finite-range or repulsive interactions. Apart from continuity no smoothness condition is imposed a priori on the trajectories.

In Sect. 1 the existence of right and left velocities is established for all values of the time, including infinity, for an arbitrary system of particles with central repulsive interactions. Nothing is assumed about the energy in Sects. 1–4.

In Sect. 2 it is shown that an arbitrary system of particles with finite-range interactions splits into independent bounded clusters in a neighborhood of every value of time, including infinity (cf. [6]).

Sect. 3 establishes, among other things, that, for an arbitrary system of particles with repulsive interactions, which is bounded at time infinity, the integral over all time of the inner kinetic energy converges.

In Sect. 4 we state results for the smooth case.

Finally, in Sect. 5 the finiteness of the number of collisions is proved for a system of hard spheres in a vessel (container) with convex walls (obstacles). Conservation of energy is assumed. The walls are not required to be smooth; unique continuation of trajectories is not implied by our axioms (when, for example, a multiple collision occurs or a ball goes into a corner).

Notation

Let $N \geqq 1$ be an integer, $\mathbf{R}^N$ be N-dimensional Euclidean space. The length (modulus) of a vector u in $\mathbf{R}^N$ is denoted by $|u|$, and the scalar product of two vectors by $\langle u, v \rangle$.

Let T be a positive number or $+\infty$. Numbers t in the interval $[0, T)$ will be called time points or moments. We consider a finite system, indexed by I, of particles in $\mathbf{R}^N$, with positive masses $m_i(t)$, where $i \in I$, $t \in [0, T)$. The trajectories $x_i(t)$ of all particles are assumed to be continuous functions of the time t.

For any subsystem of particles, indexed by $J \subset I$, we let

$$m_J(t) = \sum_{j \in J} m_j(t), x_J(t) = \sum_{j \in J} x_j(t) m_j(t)/m_J(t)$$

be the total mass and the center of mass of the subsystem.

1. System of Centrally Repulsing Particles in a "Directed" External Field of Forces: The Existence of Velocities

In this section it is supposed that some convex closed cone V in $\mathbf{R}^N$ without whole straight lines is given and that

for every vector u from the conjugate cone

$$V^* = \{u : \langle u, v \rangle \geqq 0 \quad \text{for all} \quad v \in V\},$$

for every number C, and every time $t \in (0, T)$, there exists $\delta > 0$ such that for the subsystem

$$J = \{j \in I : \langle u, x_j(t) \rangle > C\}$$

on the time interval $(t - \delta, t + \delta)$ the mass $m_J(t)$ is constant, and the projection $\langle u, x_J(t) \rangle$ of the mass center is a convex (downwards) function of time t.

The physical sense of the axiom is transparent: the particles repel each other centrally and, besides, some exterior forces with directions in the cone V act on them (Fig. 1). In the case $V = 0$, there are no exterior forces and x_I is linear.

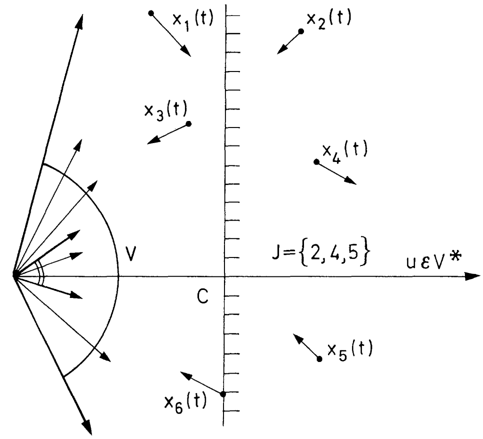

Fig. 1

Theorem (1.1). *For such a system, for each* $i \in I$ *and each* $t \in (0, T)$ *such that* $\liminf_{\tau \to t} m_i(\tau) \neq 0$, *the following double limits exist and are finite:*

$$\lim_{t \geqq t'' > t' \to t} \lim \frac{x_i(t'') - x_i(t')}{t'' - t'} =: Lx_i(t);$$

$$\lim_{t \leqq t' < t'' \to t} \lim \frac{x_i(t'') - x_i(t')}{t'' - t'} =: Rx_i(t).$$

In particular, one has the following limits and equalities:

$$Lx_i(t) = \lim_{t > \tau \to t} \frac{x_i(t) - x_i(\tau)}{t - \tau} = \lim_{t > \tau \to t} Lx_i(\tau) = \lim_{t > \tau \to t} Rx_i(\tau),$$

$$Rx_i(t) = \lim_{t < \tau \to t} \frac{x_i(t) - x_i(\tau)}{t - \tau} = \lim_{t < \tau \to t} Rx_i(\tau) = \lim_{t < \tau \to t} Lx_i(\tau).$$

Moreover, if $\liminf_{t \to T} m_i(t) \neq 0$ *and* $\max_{j \in I} |Rx_j(t)| \not\to +\infty$ *when* $t \to T$, *then the following finite limits exist and are equal:*

$$\lim_{T > t'' > t' \to T} \lim \frac{x_i(t'') - x_i(t')}{t'' - t'}$$

$$= \lim_{t \to T} Rx_i(t) = \lim_{t \to T} Lx_i(t) =: Lx_i(T).$$

To prove the theorem we will need the two following lemmas.

Lemma (1.2). (*Convexity criterion for a function.*) *If* Φ *is continuous function on an interval* $[0, T)$, *and for each* $t \in (0, T)$ *there is* $\delta_0 > 0$ *such that* $0 \leqq t - \delta_0 < t + \delta_0 < T$ *and* $\Phi(t + \delta) + \Phi(t - \delta) \geqq 2\Phi(t)$ *for all* $\delta \in [0, \delta_0]$, *then* Φ *is convex on* $[0, T)$.

Proof. Take any $t' < t''$ in $[0, T)$. The inequality

$$\alpha\Phi(t') + (1-\alpha)\,\Phi(t'') \geqq \Phi(\alpha t' + (1-\alpha)\,t''),$$

which is to be proved for all $\alpha \in [0, 1]$, can be rewritten in terms of the function

$$\Psi(t) = \Phi(t) - \Phi(t') - (t - t')\,(\Phi(t'') - \Phi(t'))/(t'' - t')$$

as the equality $\max\limits_{[t', t'']} \Psi(t) = 0$ [we have $\Psi(t') = \Psi(t'') = 0$]. Among $t \in [t', t'']$, on which the maximum is obtained, we choose the minimal number t. This t cannot lie inside $[t', t'']$, because of the condition on δ_0 [the inequality $\Phi(t+\delta) + \Phi(t-\delta) \geqq 2\Phi(t)$ is equivalent to inequality $\Psi(t+\delta) + \Psi(t-\delta) \geqq 2\Psi(t)$]. Thus, $t = t'$, that is $\Psi(t) \leqq 0$ for all $t \in [t', t'']$, as required.

Lemma (1.3). (*An inequality.*) *Given an integer* $n \geqq 1$, *and real numbers* $m_i \geqq 0$ *and* x_i $(1 \leqq i \leqq n)$. *Let*

$$m = \sum_{i=1}^{n} m_i,\ x = \sum_{i=1}^{n} m_i x_i/m,\ S = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j |x_i - x_j|.$$

Then $S \geqq \sum\limits_{i=1}^{n} m_i^2 |x - x_i|$.

Proof. Let us proceed by induction on n. If $n = 1$, both sides of the inequality vanish, so let $n \geqq 2$. Without loss of generality one can assume that $x_n \geqq x_i$ for all i. Let $m' = m - m_n$, $x' = (mx - m_n x_n)/m'$. By induction

$$S' := \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} m_i m_j |x_i - x_j| \geqq \sum_{i=1}^{n-1} m_i^2 |x' - x_i|.$$

Furthermore

$$\begin{aligned} S - S' &= m_n(m' x_n - m' x') = m'(mx - m'x') - m_n(mx - m_n x_n) \\ &= m_n^2 x_n - m_n m x - m'^2 x' + m' m x \\ &= m_n^2 (x_n - x) + m'^2 (x - x'). \end{aligned}$$

From this, using the inequality

$$|x' - x_i| \geqq |x - x_i| - (x - x'),$$

we get:

$$\begin{aligned} S &= S' + m_n^2 (x_n - x) + m'^2 (x - x') \\ &\geqq \sum_{i=1}^{n-1} m_i^2 |x' - x_i| + m_n^2 |x_n - x| + m'^2 (x - x') \\ &\geqq \sum_{i=1}^{n} m_i^2 |x - x_i| - \sum_{i=1}^{n-1} m_i^2 (x - x') + m'^2 (x - x') \\ &\geqq \sum_{i=1}^{n} m_i^2 |x - x_i|, \end{aligned}$$

as $m'^2 = (m_i + \ldots + m_{n-1})^2 \geqq m_i^2 + \ldots + m_{n-1}^2$.

Having these two general lemmas, let us prove the theorem. The condition on the cone V means that the conjugate cone V^* contains a basis of $\mathbf{R}^N$ as vector space. So, projecting our system on rays from V^*, we see that it is enough to check the theorem in one-dimensional case $N=1$. Choosing a basis on the line, we will consider $x_i(t)$ as real functions. These continuous functions have the following properties:

for each time moment $t\in(0,T)$ and each number C there is $\delta>0$ such that, for the subsystem $J=\{j\in I: x_j(t)>C\}$ on time interval $(t-\delta, t+\delta)$, the mass m_J is constant and the function x_J is convex.

Physically, "acceleration" of every "right" subsystem is directed to the right, as it is caused by repulsion from particles on the left and by external forces directed from the left.

Lemma (1.4). *Let μ_i be real numbers such that $|\mu_i|\leqq m_i(t)^2$ for all i and t. Then the function*

$$\Phi(t)=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_i(t)m_j(t)|x_i(t)-x_j(t)| + m_I^2 x_I(t)+\sum_{i=1}^{n}\mu_i x_i(t)$$

is convex on $[0,T)$.

Proof. Because of Lemma (1.2) it is enough to show that for any $t\in(0,T)$ there is $\delta_0>0$ such that $0\leqq t-\delta_0<t+\delta_0<T$ and $\Phi(t+\delta)+\Phi(t-\delta)\geqq 2\Phi(t)$ for all $\delta\in[0,\delta_0]$.

Fix t and choose $\delta_0>0$ such that $\delta_0\leqq t$, $\delta_0<T-t$, and such that $x_i(\tau)\neq x_j(\tau)$ for all $\tau\in[t-\delta_0, t+\delta_0]$ whenever $x_i(t)\neq x_j(t)$ [we used the continuity of $x_i(t)$ and the finiteness of I]. Take a positive $\delta\leqq\delta_0$. We want to check that

$$\Delta^2\Phi(t):=\Phi(t+\delta)+\Phi(t-\delta)-2\Phi(t)\geqq 0$$

[the same notation Δ^2 will be used later for other functions on $[0,T)$].

Case 1. $x_i(t)=x_j(t)$ for all i,j [then $\Phi(t)=m_I^2x_I(t)+\Sigma\mu_ix_i(t)$]. Using twice the inequality of Lemma (1.3) we get:

$$\begin{aligned}\Delta^2\Phi &\geqq \sum_{i\in I}(m_i(t-\delta)^2|x_I(t-\delta)-x_i(t-\delta)|+\Delta^2x_Im_I^2\\ &\quad+\sum_{i\in I}(m_i(t+\delta)^2|x_i(t+\delta)-x_I(t+\delta)|+\sum_{i\in I}\mu_i\Delta^2x_i\\ &\geqq \Delta^2x_Im_I^2+\sum_{i\in I}\mu_i\Delta^2x_I=\left(m_I^2+\sum_{i\in I}\mu_i\right)\Delta^2x_I\end{aligned}$$

[the inequalities $m_i(t\pm\delta)^2\geqq|\mu_i|$ have been used]. Now it is enough to note that $m_I^2+\sum_{i\in I}\mu_i\geqq 0$.

General Case. Let J be the set of particles with minimal coordinate at t, and let K be the complement of J in I: $J=\left\{j\in I: x_j(t)=\min_{i\in I}x_i(t)\right\}$, $K=I-J$. Since Case 1 is already considered, one can suppose that K is non-empty. Let Φ_J and Φ_K be the functions for subsystems J and K, analogous to the function $\Phi=\Phi_I$ for whole system $I=J\sqcup K$. Then

$$\begin{aligned}\Phi(\tau)&=\Phi_J(\tau)+\Phi_K(\tau)+m_J(\tau)m_K(\tau)(x_K(\tau)-x_J(\tau))+m_J(\tau)m_K(\tau)(x_J(\tau)+x_K(\tau))\\&=\Phi_J(\tau)+\Phi_K(\tau)+2m_J(\tau)m_K(\tau)x_K(\tau).\end{aligned}$$

By induction, the theorem is valid for system K (with the number of particles smaller than in I), so $\Delta^2\Phi_K\geqq 0$. It was shown in the above consideration of Case 1 that (in fact without assumption on convexity of Φ)

$$\Delta^2\Phi_J\geqq(\mu_J+m_J(t)^2)\Delta^2 x_J,$$

where

$$\mu_J:=\sum_{j\in J}\mu_j. \quad \text{Thus} \quad \Delta^2\Phi\geqq 2m_J m_K\Delta^2 x_K+(\mu_J+m_J^2)\Delta^2 x_J.$$

If $\Delta^2 x_J\geqq 0$, then $\Delta^2\Phi\geqq 2m_J m_K\Delta^2 x_K\geqq 0$, because the function x_K is convex on the segment $[t-\delta_0, t+\delta_0]$.

If, on the contrary, $\Delta^2 x_J<0$, then $\Delta^2\Phi\geqq 2m_J^2\Delta^2 x_J+2m_J m_K\Delta^2 x_K=2m_K\Delta^2 x_I\geqq 0$, because x_I is convex on $[0, T)$.

Hence, Lemma (1.4) is proved. This implies that, for any pair (i, t) such that $\liminf_{\tau\to t} m_i(\tau)\neq 0$, in a neighborhood of time t the function x_i can be represented as a difference of two convex functions, so that it has right and left derivatives at t.

Thus, the first part of Theorem (1.1) is proved. If $\liminf_{t\to T} m_i(t)\neq 0$ and $\liminf_{t\to T}\max_{j\in I}|Rx_j(t)|\neq +\infty$, then, in some neighborhood of T, the function x_i can be represented as the difference of the two convex functions with bounded derivative and consequently has a finite left derivative at T, and the left derivative is continuous from the left at T. Hence, Theorem (1.1) is proved.

Besides Theorem (1.1) one can obtain from Lemma (1.4):

Corollary (1.5). *If all* $|x_i(t)|$ *are bounded when* $t\to T$, *then for any* i *with* $\liminf_{t\to T} m_i(t)\neq 0$ *there is a finite* $\lim x_i(t)=:x_i(T)$.

Proof. Indeed, by Lemma (1.4) $x_i(t)$ can be represented as the difference of two convex functions bounded at T, in the case $N=1$; this implies the assertion for any N.

In the next section the following lemma (with $N=1$) will be useful:

Lemma (1.6). *The function* $f(t):=\max_{i\in I} x_i(t)$ *is convex on all intervall* $[0, T)$.

Proof. We take arbitrary $t\in(0, T)$ and will show that there exists $\delta_0>0$ such that $2f(t)\leqq f(t-\delta)+f(t+\delta)$ for all $\delta\in[0, \delta_0]$ [see Lemma (1.2)]. Consider the subset $J=\{j\in I: x_j(t)=f(t)\}$. By the above condition of repulsion, there exists $\delta_0>0$ such that function x_J is convex on $(t-\delta, t+\delta)$. Since $f(\tau)\geqq x_i(\tau)$ for all i and all τ,

$f(\tau) \geqq x_J(\tau)$ for all τ. So, for $\delta \in [0, \delta_0]$, we have:

$$2f(t) = 2x_J(t) \leqq x_J(t-\delta) + x_J(t+\delta) \leqq f(t-\delta) + f_J(t+\delta),$$

as required.

In the remaining part of this section we consider some examples. In all these examples our system consists of three particles ($I = \{1, 2, 3\}$) interacting only at moments of collisions (coincidences) and V is empty, that is the function x_J in the axiom of repulsion is not only convex but linear.

Let $t_0 < t_1 < \ldots$ be the following sequence: $t_k = k$ in the case $T = +\infty$; $t_k = T(1 - 2^{-k})$ in the case $T \neq +\infty$. We have: $t_k \to T$. In the examples below, we assume that on every segment $[t_k, t_{k+1}]$ all three particles move with constant velocities. To give such system it is therefore enough to indicate numbers $x_i(t_k)$ where $i = 1, 2, 3$; $k = 1, 2, \ldots$. We leave to the reader the checking of the axiom of repulsion.

Example (1.7). Let $T = 1$, $q = (3 - \sqrt{7})/2$;

$$x_1(t_k) = -(1 + (3 + (-1)^k) q^k/2)/2^k,$$
$$x_2(t_k) = (1 + q^k)/(-2)^k,$$
$$x_3(t_k) = (1 + (3 - (-1)^k) q^k/2)/2^k;$$
$$m_1(t) = (1 + q^{2k})/(2 + 3q^{2k}) \quad \text{for} \quad t \in [t_{2k-1}, t_{2k+1}),$$
$$m_3(t) = (1 + q^{2k-1})/(2 + 3q^{2k-1}) \quad \text{for} \quad t \in [t_{2k-2}, t_{2k}),$$
$$m_2(t) = 1 - m_1(t) - m_3(t) \quad \text{for all} \quad t.$$

Then $x_i \to 0$ for all i when $t \to 1$; m_1 and m_3 tend to $\frac{1}{2}$; $m_2 \to 0$; $Rx_1 \to 1$, $Rx_3 \to -1$, and Rx_2 oscillates between -1 and 1 when $t \to 1$. This example shows that the condition $\liminf m_i(t) \neq 0$ is essential in Theorem (1.1); note that in this example we can continue the trajectories for all time $[0, +\infty)$ with preservation of the axiom of repulsivity, puting, for example, $x_i(t) = 0$ and $m_i(t) = 1/3$ for all i and $t \geqq 1$.

Example (1.8). Let $q = 2 - \sqrt{3}$ in the case $T = +\infty$, $q = (3 - \sqrt{7})/2$ in the case $T < +\infty$;

$$m_1(t) = (1 - q^{2k})/(2 - 3q^{2k}) \quad \text{for} \quad t \in [t_{2k-2}, t_{2k}),$$
$$m_3(t) = (1 - q^{2k+1})/(2 - 3q^{2k+1}) \quad \text{for} \quad t \in [t_{2k-1}, t_{2k+1}),$$
$$m_2(t) = 1 - m_1(t) - m_3(t) \quad \text{for all} \quad t;$$
$$x_1(t_k) = \tfrac{1}{2}(1 - (-1)^k) q^{k+1} - 1, \qquad x_2(t_k) = (-1)^k (q^{k+1} - 1),$$
$$x_3(t_k) = 1 - \tfrac{1}{2}(1 + (-1)^k) q^{k+1}.$$

Then $x_1(t) \to -1$, $x_3(t) \to 1$, x_2 oscillates between x_1 and x_3; Rx_1 and Rx_3 tend to 0, the velocity Rx_2 oscillates between $-\infty$ and $+\infty$ in the case $T < +\infty$ and between -2 and 2 in the case $T = +\infty$ (when $t \to T$); m_1 and m_3 tend to $\frac{1}{2}$, $m_2 \to 0$. This example shows that the condition $\liminf m_i(t) \neq 0$ is essential in Corollary (1.5).

Example (1.9). Let $m_i(t)=1$ for all i, t; $q=2+\sqrt{3}$ in the case $T=+\infty$, $q=(3+\sqrt{7})/2$ in the case $T<+\infty$;

$$x_1(t_k)=-\tfrac{3}{4}q^k+\tfrac{1}{4}(-q)^k, \quad x_2(t_k)=-\tfrac{1}{2}(-q)^k, \quad x_3(t_k)=\tfrac{3}{4}q^k+\tfrac{1}{4}(-q)^k.$$

Then $Rx_1(t)\to-\infty$, $Rx_3\to+\infty$, Rx_2 oscillates between $-\infty$ and $+\infty$ when $t\to T$. This example shows that the condition $\limsup|Rx_i|<+\infty$ is essential for the existence of $\lim Rx_i$.

Example (1.10). Let $m_i(t)=1$ for all i, t; q is the same as in Example (1.8);

$$x_1(t_k)=(-q)^k-3q^k, \quad x_2(t_k)=-2(-q)^k, \quad x_3(t_k)=3q^k+(-q)^k.$$

Then x_i and Rx_i tend to 0 for $i=1, 2, 3$ when $t\to T$. This example shows that, without conservation of energy, three equal particles on the line can have an infinite number of collisions in finite of infinite interval of time $[0, T)$, momentum being conservated.

Remark. If the cone V above contained a whole line, then, evidently, the assertion of Theorem (1.1) would be false.

2. System of Particles with Finite-Range Interactions: Splitting into Bounded Independent Clusters

Let $r_{i,j}(t)$ be non-negative functions on $[0, T)$, where $i, j\in I$. Two particles i, j will be called *remote enough at t* if $|x_i(t)-x_j(t)|>r_{i,j}(t)$. A subsystem of particles will be called *independent at t* if every one of its particles is remote enough at t from every particle outside the subsystem.

Our axiom (hypothesis) of finite-range interaction is as follows:

for each time moment $t\in(0, T)$ and each subsystem $J\subset I$ independent at t there must exist $\delta>0$ such that on the time interval $(t-\delta, t+\delta)$ the total mass m_J of the subsystem is constant and the function x_J is linear.

Thus, in this section we assume that the mass of a subsystem is conserved and its center of mass moves with constant velocity vector when this subsystem is independent, i.e., its particles are remote enough from other particles.

Theorem (2.1). *Suppose* $\min_{i\in I}\liminf_{t\to T} m_i(t)\neq 0$ *and*

$$\max_{i\neq j}\limsup_{t\to T} r_{i,j}(t)/(m_i(t)+m_j(t))=:\varrho<+\infty.$$

Then there is $t_0\in(0, T)$ *and there is a splitting of the particles into subsystems (clusters) such that every cluster J is independent at any* $t\geq t_0$ *and it has one of the following two properties:*

a) *for any* $C>0$ *there exists* $t_C\in(t_0, T)$ *such that, for any* $t\in(t_C, T)$, *a subsystem* $K\subset J$, *independent at t, can be found with the modulus of the velocity of* x_K *at t greater than C;*

b) *the distance* $|x_j(t)-x_J(t)|$ *from the cluster center of mass to any particle* $j\in J$ *is bounded when* $t\to T$ *by* $(m_J(t_0)-\liminf_{t\to T} m_j(t))\varrho$.

Thus, speaking informally, if the diameter of an independent cluster is not bounded, then its energy tends to $+\infty$, whatever is meant by energy [we will not give any formal definition of energy because in this section particles are not bound to have velocities, see Example (2.2) below].

An example of a system with finite-range interactions in the sense of this section (with bounded $r_{i,j}$) is a system of mass centers of bounded bodies moving in infinite space with conservation of momentum and interacting at moments of collisions only.

If all $r_{i,j}(t)$ above are identically equal to 0 then the hypothesis of finite-range interaction with such $r_{i,j}$ implies the hypothesis of Sect. 1 with cone $V=0$. The examples of Sect. 1 show that the case a), or b), or both can occur in a cluster.

Proof of Theorem (2.1). Two particles i, j will be called remote enough at T if there is $t_0 \in (0, T)$ such that they are remote enough at all $t > t_0$. We place i, j in the same cluster if and only if there is an integer $k \geqq 1$ and there are particles $p_0, \ldots, p_k$ such that $p_0 = i$, $p_k = j$, and p_{s-1}, p_s are not remote enough at T for $s = 1, \ldots, k$.

Since the number of all particles is finite there exists a moment of time $t_0 \in (0, T)$ such that every cluster is independent at all $t \geqq t_0$, and therefore there will be no interactions between the clusters for $t \geqq t_0$. Hence each cluster can be considered separately.

We take a cluster J and forget the other clusters. We want to prove that if the property a) does not hold, then the property b) of the theorem holds. This property can be reformulated as follows: for every $\varepsilon > 0$ and every line in $\mathbf{R}^N$ there is $T_\varepsilon \in (t_0, T)$ such that the distance between the projections of $x_J(t)$ and $x_j(t)$ on this (straight) line does not surpass $(m_J(t_0) - m_j(t))(\varrho + 2\varepsilon)$ for all $j \in J$ and $t \geqq T_\varepsilon$.

Fixing the line we are reduced to the case $N=1$. Choosing a basis (an orientation and the origin) on the line, connected with the center of mass of J, we will assume that $x_j(t)$ are real functions and that $x_J(t) = 0$ for all $t \geqq t_0$.

We surround now every point $x_i(t)$ on the line by the segment

$$[x_i(t) - m_i(t)(\varrho + \varepsilon), x_i(t) + m_i(t)(\varrho + \varepsilon)]$$

of the length $2m_i(t)(\varrho + \varepsilon)$. A subsystem of particles and the corresponding subsystem of segments are called *tight* if the diameter of the union of the segments is not greater than the sum of the lengths of the segments. It is clear that if two tight subsystems of segments have a common point then their union will be a tight subsystem too. In particular, if a particle belongs to two tight subsystems, then the union of these subsystems is also a tight subsystem.

So, in every instant, the whole system (the cluster under consideration) of particles on the line splits *uniquely* into maximal tight subsystems which will be called *accumulations.*

Taking greater t_0 if necessary, we can assume that

$$r_{i,j}(t) \leqq (m_i(t) + m_j(t))(\varrho + \varepsilon) \quad \text{for all} \quad t \geqq t_0$$

and all $i, j \in J$. Then particles are remote enough at $t \geqq t_0$ whenever the corresponding segments have no intersection. In particular, in each instant $t \geqq t_0$, every accumulation is an independent subsystem at t.

In each moment of time, we surround the mass center of every accumulation by the segment whose length is the sum of the lengths of all segments in the accumulation. This segment will be called a *spindle*. The obtained spindles cover all our segments, every spindle covers an independent subsystem, its length is proportional to the mass of the subsystem. The sum of the lengths of all spindles is equal to the sum of the lengths of all segments, and is equal to $2(\varrho+\varepsilon)m_J$, where

$$m_J = m_J(t_0) = m_J(t) \quad \text{for} \quad t \geqq t_0 .$$

Now we contract the spindles into points as follows. For every $t \in (t_0, T)$ and $i \in J$, let $y_i(t)$ denote the middle of the corresponding spindle (i.e. the mass center of the corresponding accumulation). We set

$$z_i(t) = y_i(t) + (\varrho+\varepsilon)\Bigg(\sum_{y_j(t) > y_i(t)} m_j(t) - \sum_{y_j(t) < y_i(t)} m_j(t)\Bigg).$$

According to this formula, the distance between neighbouring points $z_i(t)$ and $z_j(t)$ is equal to the distance between the corresponding spindles. Particles i, j from the same accumulation and only such particles correspond to the same $z_i(t) = z_j(t)$. We have

$$\sum_{i \in J} m_i(t) z_i(t) = \sum_{i \in J} m_i(t) y_i(t) = \sum_{i \in J} m_i(t) x_i(t) = 0 \quad \text{for all} \quad t .$$

Although the functions $y_i(t)$ are discontinuous, the functions $z_i(t)$ are continuous on (t_0, T). We can consider the system J of the particles with the masses $m_i(t)$ and the trajectories $z_i(t)$. For this system, the hypothesis of finite-range interaction holds with $r_{i,j} = 0$. Hence, the hypothesis of repulsivity (see Sect. 1) holds with empty V.

Therefore, we can apply the results of Sect. 1 for $z_i(t)$. Since the maximum of the moduli of the velocities of the mass centers of the independent subsystems does not tend to $+\infty$ (as it was assumed above) and since $\liminf m_i(t) \neq 0$ for all i (see the condition of the theorem) there exist right $Rz_i(t)$ and left $Lz_i(t)$ velocities in some neighborhood of T, and there are $\lim_{t \to T} Rz_i(t) = \lim_{t \to T} Lz_i(t) =: Lz_i(T) \neq +\infty$.

In the case $T \neq +\infty$ it follows that there are the finite limits $\lim_{t \to T} z_i(t) =: z_i(T)$. If particles i, j are not remote enough at T, then, obviously, $z_i(T) = z_j(T)$. By our definition of the splitting into clusters, we get $z_i(T) = 0$ for all $i \in J$, so

$$\limsup_{t \to T} x_j(t) \leqq (\varrho+\varepsilon)\Big(m_J - \liminf_{t \to T} m_j(t)\Big) \quad \text{for any} \quad j \in J ,$$

as required.

In the case $T = +\infty$ we have $Lz_i(T) = 0$ for all $i \in J$ by our definition of cluster. It follows, by Lemma (1.6), that the left and right derivatives of the function $\max_{j \in J} z_j(t)$ tend to 0 when $t \to T$ and hence this function does not increase for $t > t_0$. Analogously, $\min_{j \in J} z_j(t)$ does not decrease for $t > t_0$. In view of Corollary (1.5) there exist finite limits $z_i(T) := \lim_{t \to T} z_i(t)$. It follows, as in the case $T = +\infty$, that $z_i(T) = 0$ for all i, that implies the assertion b) of Theorem (2.1), as required (Fig. 2).

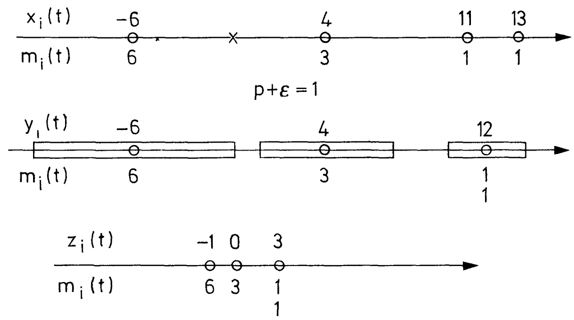

Fig. 2

Example (2.2). Let $m_1(t)$ be a positive constant function on $[0, T)$, $x_1(t)$ be a continuous function on $[0, T)$, $N=1$, $I=\{1,2\}$;

$$m_2(t):=m_1(t), \qquad x_2(t):=-x_1(t), \qquad r_{1,2}(t):=r_{2,1}(t):=2|x_1(t)|$$

on $[0, T)$. Then the hypothesis of finite-range interaction holds. The trajectory x_1 may have velocity nowhere.

Example (2.3). Change in Example (1.10) $x_1(t)$ to $x_1(t)+d$ and $x_3(t)$ to $x_3(t)-d$ for all t, where $d \geqq 0$ is a real number. Then the hypothesis of finite-range interaction holds with $r_{i,j}(t)=d$ for all i, j, t. This example shows that, without conservation of energy, three equal hard spheres (of any diameter d) on the line can have an infinite number of collisions in finite or infinite interval of time $[0, T)$, momentum being conserved (the particles are the centers of the spheres).

Theorem (2.1) and Corollary (1.5) imply

Theorem (2.4). *Under the condition of Theorem* (2.1), *suppose the axiom of repulsivity of Sect. 1, and suppose that* $\max_{i \in I} |Lx_i(t)| \not\to +\infty$ *when* $t \to T$. *Then, for each* $i \in I$, *there exists a finite* $\lim_{t \to T} (x_i(t) - Lx_i(T)t)$.

In some particular cases this was proved in [5, 7].

Remark. In the case when masses $m_i(t)=m_i$ are constant the hypothesis of this section is equivalent to the following axiom:

for any $t_1 < t_2 < t_3$ from $(0, T)$ there are vectors $P_{i,j}$ in $\mathbf{R}^N$ such that

$$P_{i,j}+P_{j,i}=0 \quad \text{for all} \quad i, j \in I\,;$$

$$\sum_{j \in I} P_{i,j} = \left(\frac{x_i(t_3)-x_i(t_2)}{t_3-t_2} - \frac{x_i(t_2)-x_i(t_1)}{t_2-t_1} \right) m_i \quad \text{for all} \quad i \in I\,;$$

if $|x_i(t)-x_j(t)| > r_{i,j}(t)$ for all $t \in [t_1, t_3]$, then $P_{i,j}=0$.

3. System of Non-Attracting Particles: Above Estimation of the Integral of Kinetic Energy

In this section we consider a system of particles where repulsivity prevails over attraction. An exact formulation of this hypothesis is rather cumbersome because the existence of velocities is not assumed.

Namely, in Theorem (3.1) below we suppose that all masses $m_i(t)=m_i$ are constant and

for every $t\in(0, T)$ and every $\varepsilon>0$ there is $\delta>0$ such that for any t_1, t_2, t_3 satisfying

$$\max(0, t-\delta)\leqq t_1<t_2<t_3<\min(T, t+\delta)$$

there exist vectors $P_{i,j}$ in R^N with the following properties:

a) $P_{i,j}+P_{j,i}=0$ for all $i, j\in I$;

b) $\sum_{j\in I} P_{i,j}=\left(\frac{x_i(t_3)-x_i(t_2)}{t_3-t_2}-\frac{x_i(t_2)-x_i(t_1)}{t_2-t_1}\right)m_i$ for all $i\in I$;

c) $\langle P_{i,j}, x_i(t_2)-x_j(t_2)\rangle\geqq-\varepsilon(t_3-t_1)$ for all $i, j\in I$.

These $P_{i,j}$ have the physical meaning of impulses received by the particle i from the particle j.

Note, that conditions a), b) imply that the mass center $x_I(t)$ of the whole system moves with constant velocity, i.e. the system is a closed one. Changing $x_i(t)$ to $x_i(t)-x_I(t)$, we can assume that this center remains at 0.

The hypothesis above holds naturally in the following example: a system of mass centers of hard bodies, which are star-shaped with respect to their centers, interacting without friction at moments of collisions.

Theorem (3.1). *The function $D(t)=\sum_{i\in I} x_i(t)^2 m_i/2$ is convex on the whole of* $[0, T)$. *In particular, there exist the left derivative $LD(t)$ on* $(0, T]$ *and the right derivative $RD(t)$ on* $[0, T)$. *For any strictly increasing sequence $t_0<t_1<\ldots<t_l$ on* $[0, T)$ *we have*

$$\sum_{k=1}^{l}\sum_{i\in I}\frac{|x_i(t_k)-x_i(t_{k-1})|^2 m_i}{t_k-t_{k-1}}\leqq LD(t_l)-RD(t_0).$$

Proof. Let $\delta(t,\varepsilon)$ denote the $\delta>0$ which exists by the hypothesis on interaction of this section. It is enough to prove the assertion of the theorem for $t_0\neq 0$.

Using the compactness of $[t_0, t_l]$ we can find $\delta>0$ such that every segment of the length $\leqq 2\delta$ in $[t_0, t_l]$ belongs to some segment of the form

$$[t-\delta(t,\varepsilon), t+\delta(t,\varepsilon)] \quad \text{where} \quad t\in[t_0, t_l].$$

Let $s_0<s_1<\ldots<s_q$ be a sequence, containing all t_k, such that $s_0=t_0$, $s_q=t_l$, and $s_p-s_{p-1}<\delta$ for all p.

Then there exist vectors $P_{i,j}(p)$ such that

$$\langle P_{i,j}(p), x_i(s_p)-x_j(s_p)\rangle\geqq-\varepsilon(s_{p+1}-s_{p-1})$$

and

$$P_{i,j}(p)+P_{j,i}(p)=0 \quad \text{for all} \quad i,j\in I,$$

$$\sum_{j\in I} P_{i,j}(p)=\left(\frac{x_i(s_{p+1})-x_i(s_p)}{s_{p+1}-s_p}-\frac{x_i(s_p)-x_i(s_{p-1})}{s_p-s_{p-1}}\right)m_i \quad \text{for all} \quad i\in I,$$

where $p=1,\dots,q-1$.

We consider the sum

$$S=\sum_{p=1}^{q-1}\sum_{i,j\in I}\langle P_{i,j}(p), x_i(s_p)\rangle .$$

Since $P_{i,j}(p)=-P_{j,i}(p)$, this sum can be rewritten as

$$S=\frac{1}{2}\sum_{p=1}^{q-1}\sum_{i,j\in I}\langle P_{i,j}(p), x_i(s_p)-x_j(s_p)\rangle ,$$

whence $S\geqq -\varepsilon(t_l-t_0)$.

On the other hand,

$$\begin{aligned} S&=\sum_{p=1}^{q-1}\sum_{i\in I}\left\langle\frac{x_i(s_{p+1})-x_i(s_p)}{s_{p+1}-s_p}-\frac{x_i(s_p)-x_i(s_{p-1})}{s_p-s_{p-1}}, m_i x_i(s_p)\right\rangle\\ &=\frac{D(s_q)-D(s_{q-1})}{s_q-s_{q-1}}-\frac{D(s_1)-D(s_0)}{s_1-s_0}\\ &\quad-\sum_{p=1}^{q-2}\sum_{i\in I}\frac{|x_i(s_{p+1})-x_i(s_p)|^2 m_i}{s_{p+1}-s_p}\\ &\quad-\sum_{i\in I}\frac{m_i}{2}\left(\frac{|x_i(s_q)-x_i(s_{q-1})|^2}{s_q-s_{q-1}}-\frac{|x_i(s_1)-x_i(s_0)|^2}{s_1-s_0}\right). \end{aligned}$$

Thus,

$$(3.2)\quad \left\{\begin{aligned} &\sum_{p=1}^{q-2}\sum_{i\in I}\frac{|x_i(s_{p+1})-x_i(s_p)|^2 m_i}{s_{p+1}-s_p}\\ &\leqq\frac{D(s_q)-D(s_{q-1})}{s_q-s_{q-1}}-\frac{D(s_1)-D(s_0)}{s_1-s_0}+\varepsilon(t_l-t_0). \end{aligned}\right.$$

First of all, from (3.2) we get

$$\frac{D(s_q)-D(s_{q-1})}{s_q-s_{q-1}}-\frac{D(s_1)-D(s_0)}{s_1-s_0}\geqq -\varepsilon(t_l-t_0).$$

Since here $s_q=t_l$, $s_0=t_0$, and, for an ε as small as we please, s_1 and s_{q-1} can be chose freely in some neighborhoods of t_0 and t_l accordingly, we have

$$\liminf_{t_l>s_{q+1}\to t_l}\frac{D(t_l)-D(s_{q-1})}{t_l-s_{q-1}}\geqq\limsup_{t_0>s_1\to t_0}\frac{D(s_1)-D(t_0)}{s_1-t_0}.$$

Since here t_l and t_0 can be arbitrary under the condition $0<t_0<t_l<T$, we have obtained the first assertion of the theorem (convexity of D).

Hence, the right part of inequality (3.2) is not greater than $LD(t_l)-RD(t_0)+\varepsilon(t_l-t_0)$. On the other hand, note that $|u|^2/\alpha+|v|^2/\beta \geqq |u+v|^2/(\alpha+\beta)$ for any vectors u, v and positive numbers α, β (it follows from the inequality $|\beta u-\alpha v|^2\geqq 0$).

Therefore the left part of (3.2) is not less than

$$\sum_{i\in I}\left(\frac{|x_i(t_1)-x_i(s_1)|^2}{t_1-s_1}+\frac{|x_i(s_{q-1})-x_i(t_{l-1})|^2}{s_{q-1}-t_{l-1}}\right)m_i$$
$$+\sum_{i\in I}\sum_{k=2}^{l-1}\frac{|x_i(t_k)-x_i(t_{k-1})|^2 m_i}{t_k-t_{k-1}}$$

(without loss of generality, we can assume that $s_1<t_1$ and $s_{q-1}>t_{l-1}$). Since in (3.2) we can take ε as small as we please, and s_1, s_{q-1} as close to t_0, t_l respectively as we please, we obtained the second assertion of Theorem (3.1).

Corollary (3.3). *If $T=+\infty$ and the diameter of the system does not tend to $+\infty$ when $t\to+\infty$, then, for any sequence $t_0<t_1<\dots$ we have*

$$\sum_{k=1}^{\infty}\sum_{i\in I}\frac{|x_i(t_k)-x_i(t_{k-1})|^2 m_i}{t_k-t_{k-1}}\leqq RD(t_0).$$

Proof. Indeed, then $RD(t)\leqq 0$ for all t.

Corollary (3.4). *Suppose that the right derivatives $Rx_i(t)$ of the trajectories exist and are integrable in the sense of Riemann. Then, for the energy $E_I(t)=\frac{1}{2}\sum_{i\in I}|Rx_i(t)-Rx_I(t)|^2 m_i$ and for any $t_1<t_2$ in the segment $[0,T]$, we have*

$$\int_{t_1}^{t_2}E_I(t)\,dt\leqq\tfrac{1}{2}(LD(t_2)-RD(t_1)).$$

Suppose now, in addition, the condition of Theorem (2.1) on the locality of interaction, and let $T=+\infty$. Then there is a $t_0\in(0,T)$ and there is a splitting of the system into independent at $t\geqq t_0$ clusters such that, for the inner energy $E_J(t)=\frac{1}{2}\sum_{j\in J}|Rx_j(t)-Rx_J(t)|^2 m_j$ of every cluster J, either $E_J(t)\to+\infty$ when $t\to+\infty$, or $\int_{t_0}^{\infty}E_J(t)dt<+\infty$.

Proof. Indeed, the first assertion follows directly from Theorem (3.1). To prove the second assertion we have Theorem (2.1) and Corollary (1.3) to apply.

Remarks. The hypothesis of this section does not imply the existence of velocities, if $N\geqq 2$.

If $N=1$, this hypothesis is equivalent to the hypothesis of Sect. 1 with $N=1$, $V=0$, and constant masses m_i.

4. Smooth Case

Here the results of Sects. 1–3 are formulated in the case of smooth trajectories. Namely, in this section we suppose that all masses $m_i(t)=m_i$ are constant, and that all functions $x_i(t)$ have continuous first and second derivatives (velocities and accelerations) $dx_i(t)/dt$, $d^2x_i(t)/dt^2$. Let some continuous vector functions (forces) $F_{i,j}(t)$ and $F_i(t)$ be given, satisfying the following Newton laws:

a) $F_{i,j}(t)+F_{j,i}(t)=0$ for all $i,j\in I, t\in[0,T)$;

b) $F_i(t)+\sum_{j\in I} F_{i,j}(t)=m_i d^2x_i(t)/dt^2$ for all $i\in I, t\in[0,T)$.

From Theorems (1.1) and (2.4) we get

Theorem (4.1). *Suppose that*

$$\langle F_{i,j}(t), x_i(t)-x_j(t)\rangle = |F_{i,j}(t)||x_i(t)-x_j(t)|$$

for all $i,j\in I$, $t\in(0,T)$ and that the closure of the convex cone generated by all $F_i(t)$, where $i\in I$, $t\in(0,T)$, does not contain straight lines. Then, either there exist finite limits $Lx_i(T):=\lim_{t\to T} dx_i(t)/dt$ for all $i\in I$, or the energy $\sum_{i\in I}|dx_i(t)/dt|^2 m_i/2$ tends to $+\infty$ when $t\to T$.

If the energy does not tend to $+\infty$, all $F_i=0$, and, for some $r\geqq 0$, we have $F_{i,j}(t)=0$ whenever $|x_i(t)-x_j(t)|>r$, then there exist finite limits $\lim_{t\to T}(x_i(t)-Lx_i(T)\,t)$ where $i\in I$.

In some particular cases this was proved in [5, 7].

From Theorem (2.1) we get

Theorem (4.2). *Suppose that, for some non-negative functions $r_{i,j}(t)$, forces $F_{i,j}(t)=0$ whenever $|x_i(t)-x_j(t)|>r_{i,j}(t)$, and external forces $F_i(t)=0$ for all i, t. Then, if $\max_{i\in I}\limsup_{t\to T} r_{i,j}(t)/(m_i+m_j)=:\varrho\neq+\infty$, there is $t_0\in(0,T)$ and there is a splitting of the system into clusters such that $|x_i(t)-x_j(t)|>r_{i,j}(t)$ for $t\geqq t_0$ and i,j from different clusters, and for each cluster J either its inner kinetic energy $E_J(t):=\sum_{j\in J}\left|\frac{d}{dt}(x_j(t)-x_J(t)\right|^2 m_j/2$ tends to $+\infty$ when $t\to T$, or $\limsup_{t\to T}|x_j(t)-x_J(t)|\leqq(m_J-m_j)\varrho$ for all $j\in J$.*

Compare this with a splitting into clusters in a system of attracting particles [6].

From Theorem (3.1) we get

Theorem (4.3). *Suppose that $\langle F_{i,j}(t), x_i(t)-x_j(t)\rangle\geqq 0$ for all $i, j\in I$, $t\in(0,T)$, and $F_i=0$ for all i. Then $d^2D(t)/dt^2\geqq 2E(t)$ for all t, where*

$$D(t):=\sum_{i\in I}|x_i(t)-x_I(t)|^2 m_i/2,$$

$$E(t):=\sum_{i\in I}\left|\frac{d}{dt}(x_i(t)-x_I(t))\right|^2 m_i/2.$$

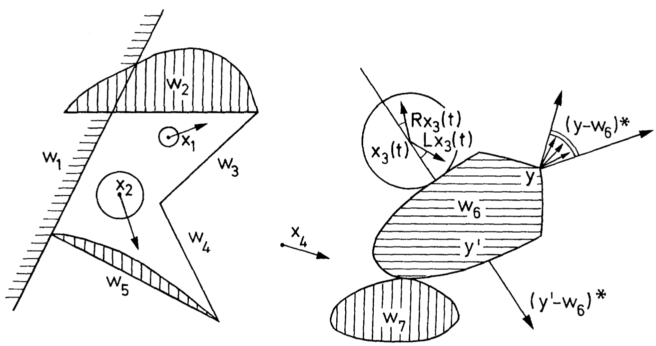

Fig. 3

5. Sinai's Billiard: The Finiteness of the Number of Collisions

Let a finite collection W of non-empty closed convex sets in $\mathbf{R}^N$ be given, and let Ω be the intersection of the closures in $\mathbf{R}^N$ of the complements in $\mathbf{R}^N$ of these sets.

We will call Ω a "vessel" with convex "walls" $w \in W$, although Ω may be unbounded and its complement in $\mathbf{R}^N$ may be unconnected. In particular, Ω can coincide with all $\mathbf{R}^N$ (for empty W) or with arbitrary, not necessarily convex, polyhedron (for suitable W). In the vessel Ω (i.e., in the space $\mathbf{R}^N$ outside the obstacles $w \in W$) we will consider a system of hard spheres (balls) moving with conservation of energy, centrally repulsing at moments of collisions between themselves, and normally repulsing from the walls at moments of collisions with them (Fig. 3).

Before giving exact axioms, we introduce some notation. For $x \in \mathbf{R}^N$, $w \in W$, let $(x-w)^* := \{u : \langle u, x-y\rangle \geqq 0$ for all $y \in w\}$. For any two subsets A, B in $\mathbf{R}^N$ let $d(A,B)$ be the infinum of the distances $|a-b|$ between points $a \in A$, $b \in B$.

In Theorem (5.3) below, besides constancy of the masses $m_i(t) = m_i$, we assume that, for some non-negative numbers r_i (radius of spheres), the following hypotheses hold:

(5.1) there is $E \geqq 0$ (energy) such that for any open subinterval of $(0, T)$ on which all x_i are linear we must have $\sum_{i \in I} |dx_i(t)/dt|^2 m_i/2 = E$;

(5.2) for any t_1, t_2, t_3 satisfying $0 \leqq t_1 < t_2 < t_3 < T$, there are vectors $P_{i,k}$ where $i \in I$, $k \in I \sqcup W$ such that

a) $P_{i,j} + P_{j,i} = 0$ for all $i, j \in I$;

b) $$\sum_{k \in I \sqcup W} P_{i,k} = \left(\frac{x_i(t_3) - x_i(t_2)}{t_3 - t_2} - \frac{x_i(t_2) - x_i(t_1)}{t_2 - t_1}\right) m_i$$

for all $i \in I$;

c) for any $i, j \in I$ either $r_i = r_j = 0$ and $x_i(t) = x_j(t)$ for some $t \in [t_1, t_3]$, or $P_{i,j}$ belongs to the convex cone generated by the vectors $x_i(t) - x_j(t)$ with $|x_i(t) - x_j(t)| = r_i + r_j$, $t \in [t_1, t_3]$;

d) for any $i \in I$, $w \in W$, the vector $P_{i,w}$ belongs to the convex cone generated: by the vectors $x_i(t)-y$ with $y \in w$, $|x_i(t)-y|=r_i=d(x_i(t),w)$, $t\in[t_1,t_3]$ in the case $r_i \neq 0$; and by the cones $(x_i(t)-w)^*$ with $x_i(t)\in w$, $t\in[t_1,t_3]$ in the case $r_i=0$.

The condition (5.1) means the conservation of kinetic energy; nothing is supposed about the existence or conservation of energy in moments of interaction.

The condition (5.2) means the absence of friction. The vector $P_{i,k}$ in this axiom have the sense of some mean impulse which the ball i receives from the ball or the wall k in the interval $[t_1,t_3]$. The condition (5.2a) means the equality of the action to the reaction. The condition (5.2b) means that balls interact only when they are in contact, and the centrally repulse each other (without friction) at such moments of time. The condition (5.2c) means that interaction of a ball with a wall occurs only in moments of contact and it is directed at each such moment along a normal to the wall going through the center of the ball; such normal is unique in the case $r_i \neq 0$.

We will not suppose that balls and walls are impenetrable, i.e., that $|x_i-x_j| \geqq r_i+r_j$, $x_i \in \Omega$, $d(x_i,w)\geqq r_i$, because this condition is not necessary to prove that the number of collisions is finite. So, balls can, for example, go through each other without interaction at all, if they please.

We will say that there is no interaction in the system at t if in some neighborhood of t all particles (i.e., the centers of balls) move with constant velocity vectors, that is, all x_i are linear. Other t, which will be called moments of interaction in the system, are, obviously, a closed subset in $[0,T)$. Such t is characterized as follows: there is no neighborhood of t, in which all x_i are linear.

Theorem (5.3). *Suppose that either $T \neq +\infty$, or there is $x\in\mathbf{R}^N$ such that $d(x,w)\leqq \min_{i\in I} r_i$ for all $w\in W$ (for example, W is empty). Then there exists only a finite number of moments of interaction in the system during the whole $[0,T)$.*

Proof. First of all, going to the configuration space, we reduce to the situation with one ball of mass 1 of radius 0. Namely, in the space $\mathbf{R}^{NI}$ we introduce an Euclidean scalar product of vectors $u=(u_i)_{i\in I}$, $v=(v_i)_{i\in I}$ according to the formula $\langle u,v\rangle = \sum_{i\in I}\langle u_i,v_i\rangle m_i$.

We will consider in R^{NI} the convex subsets

$$A_{i,w}=\{x=(x_k)_{k\in I}: d(x_i,w)\leqq r_i\}, \quad \text{where} \quad i\in I, w\in W,$$

and the convex subsets

$$A_{i,j}=\{x=(x_k)_{k\in I}: |x_i-x_j|\leqq r_i+r_j\}, \quad \text{where} \quad i,j\in I, i\neq j.$$

The set of these non-empty closed subsets (walls) we denote by W'.

We consider a particle (ball with the radius 0) with trajectory $x(t)=(x_i(t))_{i\in I}$, with mass 1. The condition (5.1) is equivalent to

(5.4) there is $E\geqq 0$ such that $|dx(t)/dt|=\sqrt{2E}$ at any t when there is no interaction at t, i.e., x is linear in some neighborhood of t.

The axiom (5.2) takes the following form:

(5.5) for every $t_1 < t_2 < t_3$ the vector $\frac{x(t_3)-x(t_2)}{t_3-t_2} - \frac{x(t_2)-x(t_1)}{t_2-t_1}$ belongs to the cone generated by the cones of the form

$$(x(t)-w')^*, \quad \text{where} \quad w' \in W', t \in [t_1, t_3], \quad x(t) \in w'.$$

The moments of interaction in $\mathbf{R}^N$ and $\mathbf{R}^{NI}$ are the same. The assumption $\min_{x \in \mathbf{R}^N} \max_{w \in W} d(x, w) \leqq \min_{i \in I} r_i$ of Theorem (5.3) is equivalent to the following: the intersection of all $w' \in W'$ is not empty [if W' is empty, i.e., W is empty and $\mathrm{Card}(I) = 1$, then the intersection is the whole $\mathbf{R}^{NT}$ by definition].

We want to prove that under the conditions of Theorem (5.3), which are now rewritten in terms of configuration space as the conditions (5.4), (5.5), and the assumption: either $T \neq +\infty$, or the intersection of all walls $w' \in W'$ is non-empty, – there is only a finite number of moments of interaction in the system, i.e., whole $[0, T)$ can be divided into a finite number of subintervals (semisegments) on each of which x being linear.

Case 1. $x(t) \in \bigcap_{w' \in W'} w' =: A$ for all $t \in [0, T)$. We will show that then there is no moments of interaction at all, i.e., x is linear on $(0, T)$. It is enough to show that for each time $t_2 \in (0, T)$ there is $\delta > 0$ such that $\delta \leqq t_2$, $\delta \leqq T - t_2$, and $\frac{x(t_3)-x(t_2)}{t_3-t_2} = \frac{x(t_2)-x(t_1)}{t_2-t_1}$ for all t_1, t_3 satisfying $t_2 - \delta \leqq t_1 < t_2 < t_3 \leqq t_2 + \delta$.

It is clear from the definition of $(y-A)^*$ that for any point $y \in A$ close enough to $x(t_2)$ and any vector $v \in (y-A)^*$ with $|v| = 1$, we have $d(v, (x(t_2)-A)^*) < \frac{1}{2}$. [Indeed, otherwise we could find a sequence $y_k \in A$, a sequence $v_k \in (y_k - A)^*$, and a vector v such that $y_k \to x(t_2)$, $v_k \to v$, and $|v_k| = 1$, $d(v_k, (x(t_2)-A)^*) \geqq \frac{1}{2}$ for all k, hence, $d(v, (x(t_2)-A)^*) \geqq \frac{1}{2}$. On the other hand, for each point $z \in A$ we have $\langle v, x(t_2) - z\rangle = \lim_{k \to \infty} \langle v_k, y_k - z\rangle \geqq 0$, i.e., $v \in (x(t_2) - A)^*$, i.e., $d(v, (x(t_2) - A^*) = 0$.]

We choose $\delta > 0$ such that $\delta \leqq t_2$, $\delta \leqq T - t_2$, and $d(u, (x(t_2)-A)^*) < \frac{1}{2}$ for all $t \in [t_2 - \delta, t + \delta]$ and all $v \in (x(t) - A)^*$ with $|v| = 1$. Then this inequality holds for all vectors v with $|v| = 1$ from the convex cone V generated by the cones $(x(t) - A)^*$ with $|t - t_2| \leqq \delta$, hence, for any $v \in V$, there is $u \in (x(t_2) - A)^*$ such that $\langle u, v\rangle \geqq |u|\,|v|\,\sqrt{3}/2$.

In view of the axiom (5.5) we can take here

$$v = \frac{x(t_3)-x(t_2)}{t_3-t_2} - \frac{x(t_2)-x(t_1)}{t_2-t_1}$$

(it is clear that $(x(t)-A)^* \supset (x(t)-w')^*$ for all $w' \in W'$) with t_1, t_3 satisfying $t_2 - \delta \leqq t_1 < t_2 < t_3 \leqq t_1 + \delta$. On the other hand, in view of the definition of $(x(t_2) - A)^*$, we have $\langle v, u\rangle \leqq 0$ for this v. Hence $v = 0$, as required.

To consider the further cases, the following lemma will be useful.

Lemma (5.6). *Let B be a convex closed non-empty subset in the intersection of all walls $w' \in W'$. Then the function $f(t) := d(x(t), B)^2$ is convex on the whole interval $[0, T)$.*

Proof of the Lemma. We take any $t\in(0, T)$ and set $\delta_0 := \min(t, T-t)>0$. We want to show that $f(t+\delta)+f(t-\delta)-2f(t)\geqq 0$ for any positive $\delta\leqq\delta_0$ [see Lemma (1.2)].

Let y_1, y_2, y_3 be the points in B nearest to $x(t-\delta)$, $x(t)$, $x(t+\delta)$ respectively. We set $y := (y_1+y_3)/2\in B$, $u_1 := x(t-\delta)-y_1$, $u_2 := x(t)-y$, $u_3 := x(t+\delta)-y_3$. Then $f(t-\delta)=|u_1|^2$, $f(t+\delta)=|u_3|^2$, $f(t)\leqq|u_2|^2$.

From the condition (5.5) we have

$$\left\langle \frac{x(t+\delta)-x(t)}{\delta}-\frac{x(t)-x(t-\delta)}{\delta}, x(t)-y\right\rangle \geqq 0,$$

i.e.

$$\langle x(t+\delta)+x(t-\delta)-2x(t), x(t)-y\rangle\geqq 0,$$

i.e.

$$\langle u_1+u_3-2u_2, u_2\rangle\geqq 0,$$

i.e.

$$|u_1|^2+|u_2|^2-2|u_3|^2-|u_1-u_2|^2/2-|u_3-u_2|^2/2\geqq 0,$$

so

$$\begin{aligned} f(t+\delta)+f(t-\delta)-2f(t) &\geqq |u_1|^2+|u_3|^2-2|u_2|^2\\ &\geqq (|u_1-u_2|^2+|u_3-u_2|^2)/2\geqq 0. \end{aligned}$$

We continue now the proof of Theorem (5.3).

Case 2. $T\neq+\infty$ and there is a sequence $0<t_1<t_2<\ldots<T$ such that $t_k\to T$ when $k\to\infty$ and the trajectory $x(t)$ is linear on each segment $[t_k, t_{k+1}]$. We show then that, for some $t_0<T$, there is no interaction in the system during (t_0, T).

Let $v_k := (x(t_{k+1})-x(t_k))/(t_{k+1}-t_k)$ for $k=1, 2, \ldots$. From the condition (5.4) $|v_k|=\sqrt{2E}$. Let $x(T) := \lim_{t\to T} x(t)$. We denote by W_T the set of walls w' such that $x(t_k)\in w'$ for infinitely many k, and choose l such that $x(t_k)\notin w'$ for $k\geqq l$, $w'\notin W_T$.

In view of Lemma (5.6) the function $d(x(t), x(T))$ is convex for $t\geqq t_l$, when the walls outside W_T have no influence on the trajectory. It follows, using the convexity near t_{k+1}, that $\langle u_{k+1}-u_k, x(t_{k+1})-x(T)\rangle\geqq 0$ for $k\geqq l$. Consequently, the distance d_k from $x(T)$ to the ray, going from the point x_k in the direction v_k (i.e., towards the point x_{k+1}) does not decrease (the trivial case $E=0$ is excluded). Since $x(t_k)\to x(T)$, $d_k\to 0$. Hence $d_k=0$ for $k\geqq l$. Therefore $v_k=v_{k+1}$ for $k\geqq l$, as required (Fig. 4).

Case 3. $T\neq+\infty$. If we had infinitely many moments of interaction, then we could find a time moment $t_0\in[0, T]$ such that in every left or in every right neighborhood of t_0 there are infinitely many such moments. Taking into account the possibility to reverse time, we can, without loss of generality, assume that in every neighborhood of $t_0=T$ there are infinitely many moments of interaction.

Using induction on the number of the walls in W', we can assume that, for each time t such that $x(t)$ does not belong to the intersection A of all walls from W',

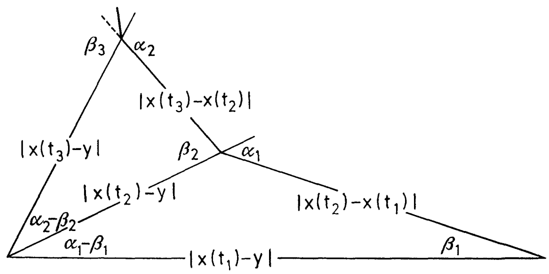

Fig. 4. For Case 2 of Theorem (5.3) and for Lemma (5.7)

there is some neighborhood of t without moments of interaction besides, perhaps, t itself. According to the Case 2, we get that as close to T as we please we can find t such that $x(t)\in A$. Applying Lemma (5.6) with $B=A$, we get: $x(t)\in A$ in some neighborhood of T, which contradicts Case 1.

Lemma (5.7). *Let a point y belong to all $w'\in W'$, and let $x(t)\neq y$ for all t. Then* $\operatorname*{var}_{t=0}^{T}\dfrac{x(t)-y}{|x(t)-y|}\leqq\pi.$

Proof of the Lemma. Since the set of moments of interaction is discrete (see Case 3), we can find a sequence $t_1<t_2\ldots$ such that $t_1=0$, $t_k\to T$, and $x(t)$ is linear on every segment $[t_k,t_{k+1}]$. Let $v_k:=(x(t_{k+1})-x(t_k))/(t_{k+1}-t_k)$. By the axiom (5.4), $|v_k|=\sqrt{2E}$ (the case $E=0$ is trivial, so that let $E>0$). From the convexity of $|x(t)-y|^2$ [see Lemma (5.6)] in some neighborhood of t_k, where $k\geqq 2$, it follows that $\langle v_{k-1},x(t_k)-y\rangle\leqq\langle v_k,x(t_k)-y\rangle$, i.e. $\alpha_{k-1}\leqq\beta_k$, where

$$\alpha_{k-1}:=\arccos\langle v_{k-1},x(t_k)-y\rangle/\sqrt{2E}\,|x(t_k)-y|,$$

$$\beta_k:=\arccos\langle v_k,x(t_k)-y\rangle/\sqrt{2E}\,|x(t_k)-y|.$$

Thus,

$$\begin{aligned}\operatorname*{var}_{0}^{T}\frac{x(t)-y}{|x(t)-y|}&=\sum_{k=1}^{\infty}\operatorname*{var}_{t_k}^{t_{k+1}}\frac{x(t)-y}{|x(t)-y|}\\&=\sum_{k=1}^{\infty}\arccos\frac{\langle x(t_{k+1})-y,x(t_k)\rangle}{|x(t_{k+1})-y||x(t_k)|}\\&=\sum_{k=1}^{\infty}(\alpha_k-\beta_k)\leqq\sum_{k=1}^{\infty}(\beta_{k+1}-\beta_k)\\&=\lim_{k\to\infty}\beta_k-\beta_1\leqq\pi-\beta_1\leqq\pi\quad\text{(Fig. 4)}.\end{aligned}$$

Lemma (5.8). *Under condition* (5.7), *$d(x(t),B)$ is convex.*

Proof of the Lemma. At first we will show that $d(x(t),B)$ is convex on every interval, in which $x(t)$ is linear. Let $t-\delta$, t, $t+\delta$ belong to such an interval, and let

$y_1, y_2, y_3, y=(y_1+y_3)/2$ be as in the proof of Lemma (5.6). Then

$$d(x(t+\delta), B)+d(x(t-\delta), B)-2d(x(t), B)$$
$$\geqq |x(t-\delta)-y_1|+|x(t+\delta)-y_3|-2|x(t)-y|\geqq 0$$

since now

$$x(t-\delta)-y_1+x(t+\delta)-y_3-2(x(t)-y)=0.$$

In view of the discretness of the moments of interaction, it remains now to prove that $Ld(x(t), B)\leqq Rd(x(t), B)$ for each isolated moment of interaction t. This is obvious in the case $d(x(t), B)=0$ because the function $d(x(t), B)$ is non-negative. When $d(x(t), B)\neq 0$ it remains to use the inequality

$$Lf(t)=2d(x(t), B)\,Ld(x(t), B)$$
$$\leqq Rf(t)=2d(x(t), B)\,Rd(x(t), B)$$

which follows from Lemma (5.6).

We continue the proof of Theorem (5.3).

Case 4. $T=+\infty$, the intersection A of all $w'\in W'$ is non-empty, and there is a sequence $0\leqq t_1<t_2<\ldots$, such that $t_k\to+\infty$ and x is linear on each segment $[t_k, t_{k+1}]$. We want to show that $x(t)$ is linear for t large enough.

Now, the function $d(x(t), A)$ is convex by Lemma (5.8). If it is constant on some interval (t_0, T), then, as it can be seen from the proof of Lemma (5.6), x is linear for $t\geqq t_0$.

Otherwise there are $C>0$, $\varepsilon>0$ such that

$$d(x(t), A)>\varepsilon t-C \quad \text{for all} \quad t.$$

Using induction on the number of the walls in W', we see that it is enough to consider the case when there are infinitely many k with $x(t_k)\in w'$, for any given $w'\in W'$.

We choose now a point y in A. Since $d(x(t), A)>\varepsilon t-C$, $x(t)\neq y$ for t large enough. By Lemma (5.7) there exists

$$\text{(5.9)} \quad \lim_{t\to+\infty} \frac{x(t)-y}{|x(t)-y|}=:e.$$

For any $w'\in W'$, we have $x(t)\in w'$ for infinitely many k. So, taking in account (5.9) and the fact that $|x(t)-y|\to+\infty$, we get that w' contains the ray V, going from y in direction e. Consequently, this ray belongs to A.

It is clear from (5.9) that $d(V, x(t))/|x(t)-y|\to 0$ when $t\to+\infty$. But, on the other hand,

$$d(V, x(t))\geqq d(A, x(t))\geqq \varepsilon t-C, |x(t)-y|$$
$$|x(t)-y|\leqq|x(0)-y|+\sqrt{2E}\,t,$$

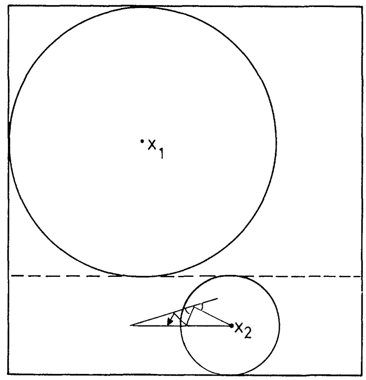

Fig. 5

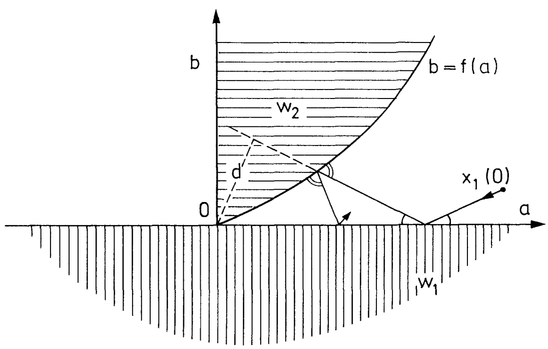

Fig. 6

hence

$$d(V, x(t))/|x(t)-y| \geqq (\varepsilon t - C)/(|x(0)-y| + \sqrt{2E}\, t) \to \varepsilon/\sqrt{2E} > 0 \quad \text{when} \quad t \to +\infty .$$

Thus, in the case of a non-constant $d(x(t), A)$ we get a contradiction with the assumed infiniteness, for every w', of the number of k such that $x(t_k) \in w'$.

Theorem (5.3) is proved. It follows

Corollary (5.10). *In the case $T \neq +\infty$, the trajectories x_i can be continued on all time interval $[0, +\infty)$ with conservation of the axioms of this section.*

Such continuation is not necessary unique. It can be shown that the assertion (5.10) holds also under the additional axiom of impenetrability. In our context the statement about unique continuation for almost all trajectories makes sense.

In [1–4] the authors did not formulate exact axioms for the systems under consideration; in fact, excluding multiple collisions, and prohibiting to balls to go into corners, they assumed a priori the finiteness of the number of the collisions in a finite time (more exactly, in every closed subsegment of $[0, T)$). Modulo this, Theorem (5.3) is proved: in [2] – in the case $N=1$, W empty; in [1] – in the case,

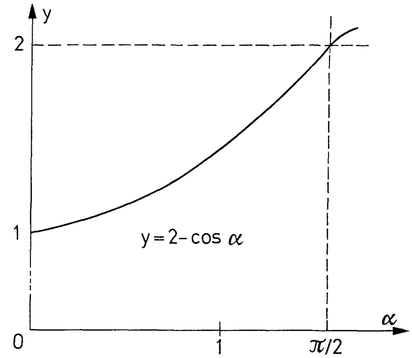

Fig. 7

when every $w \in W$ is an impenetrable hyperplane going through 0 and $\mathrm{Card}(I)=1$ (in fact, [1] covers [2]); in [3] – in the case $N=2=\mathrm{Card}(W)$, $\mathrm{Card}(I)=1$. In [4], the maximal number of collisions between 3 equal balls is computed in the case $N=3$. A result of [3] leads to the conjecture, that in the condition $\min_x \max_w d(x, w) \leqq \min_i r_i$ of Theorem (5.3), the first min could be replaced by inf.

Sinai asked if it is possible to get some estimation on the number of collisions in his billiards. To get it we have to introduce some new conditions on the system, and we intend to do this in a future publication. For some illustrations of the difficulties see Examples (5.11), (5.12) below.

Example (*5.11*). *Let* $I=\{1,2\}$, $m_1=m_2=1$, $r_1+r_2=1$, $E=1$, $N=2$, $\Omega=\{(a,b)\in\mathbf{R}^2 : |a|\leqq 1, |b|\leqq 1\}$ [such Ω can be realized with $\mathrm{Card}(W)=4$]. Then, for any $T>0$ and any real C, there exist trajectories $x_i(t)$ with a number of moments of interaction during $[0, T)$ greater than C (Fig. 5).

Example (*5.12*). Let f be a non-negative convex function on $[0, +\infty)$, $f \neq 0$, $f(0)=0$. Let $I=\{1\}$, $m_1=1$, $r_1=0$, $E=1$, $N=2$, $W=\{w_1, w_2\}$, where $w_1=\{(a,b)\in\mathbf{R}^2 : b\leqq 0\}$, $w_2=\{(a,b)\in\mathbf{R}^2 : a\geqq 0, b\geqq f(a)\}$ (Fig. 6).

One can show that, when $Rf(0)\neq 0$, the number M of the reflections of the particle by the walls (i.e. the number of moments of interaction in the system) is less than $\pi/\mathrm{arctg}(Rf(0))+1$ for any trajectory and any T. If $Rf(0)=0$, then for any $T>0$ and any real C there exists a trajectory with $M>C$ reflections.

However, this number M can be estimated via the distance d from $0=(0,0)\in\mathbf{R}^2$ to the ray $x_1(t_1)+Rx_1(t_1)\mathbf{R}^+$, where t_1 is the instant of the first reflection, as follows: $M<\pi/\mathrm{arctg}(f(a)/a)+1$, where $a^2+f(a)^2=d^2$, and T can be $+\infty$; if $d=0$, then $M=1$.

Example (*5.13*). 2 heterogeneous balls on the plane rolling round each other.

Let a point $(\alpha(t), y(t))$ moves in the plane (α, y) along the curve $y=2-\cos\alpha$ *at unit speed* from the point $(\alpha(0)=0, y(0)=1)$ to the point $(\alpha(T)=\pi/2, y(T)=1)$ (Fig. 7).

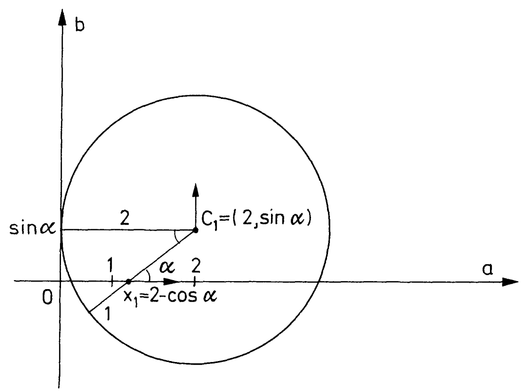

Fig. 8

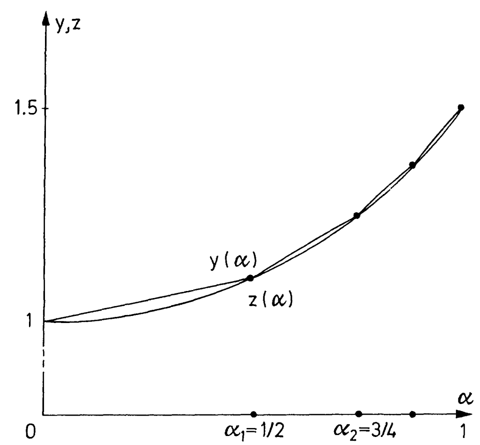

Fig. 9

We take a 2-dimensional ball of radius 2, of mass 1, of moment of gyration 1, with the distance 1 between the center of mass x_1 and the geometric center c_1 (Fig. 8).

We put the ball on the plane (a, b) so that $x_1(t)=(y(t), 0)$, $c_1(t)=(y(t)+\cos(\alpha(t)), \sin(\alpha(t)))$, and put another equal ball on the same plane so that $x_2(t)=-x_1(t)$, $c_2(t)=(-y(t)-\cos(\alpha(t)), \sin(\alpha(t)))$.

Then these hard balls interact at each moment $t\in[0, T)$; the forces $d^2x_1(t)/dt^2=-d^2x_2(t)/dt^2\neq 0$ are directed from the point of contact $(0, \sin(\alpha(t)))$ along the radiuses of the circles; the kinetic energy $|dx_i(t)/dt|^2/2+|d\alpha(t)/dt|^2/2$ of each ball is constant.

Example (5.14). Infinitely many collisions between 2 convex hard bodies in the plane.

Let $\alpha_k=1-\frac{1}{2}^k$ for $k=0, 1, \ldots$; y be the function on $[0, 1)$ which is linear on every $[\alpha_k, \alpha_{k+1}]$ and $y(\alpha_k)=2-\cos\alpha_k$ for all $k\geqq 0$; z be a convex function on $[0, 2\pi]$ such that $z(\alpha_k)=y(\alpha_k)=2-\cos\alpha_k$ for all $k\geqq 0$, $Lz(\alpha_k)\leqq(Ly(\alpha_k)+Ry(\alpha_k))/2\leqq Rz(\alpha_k)$

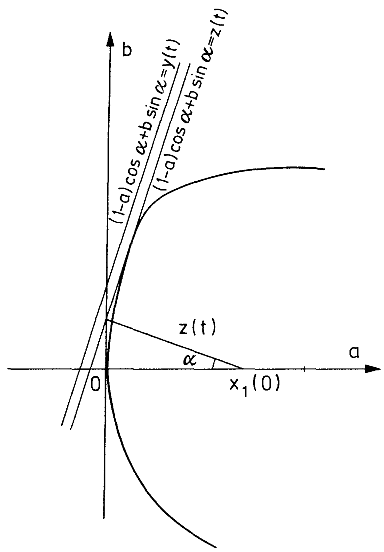

Fig. 10

for all $k \geqq 1$, and $z(\alpha)=2-\cos\alpha$ for $\alpha\in[1, 2\pi]$. (We can take $z=y$ on $[0, 1)$, or choose an infinitely smooth z under this condition.)

Let a point $(\alpha(t), y(\alpha(t)))$ move along the curve $y=y(\alpha)$ at unit speed from the point $(\alpha(0)=0, y(0)=1)$ to the point $(\alpha(T)=1, y(1)=2-\cos 1)$ (Fig. 9).

We consider a hard convex body on the plane, of mass 1, of moment of gyration 1, with the center of mass $x_1(t)=(y(\alpha(t)), 0)$, with the initial position (at $t=0$, see Fig. 10)

$$\{(a, b)\in \mathbf{R}^2 : (1-a)\cos\alpha + b\sin\alpha \leqq z(\alpha) \quad \text{for all} \quad \alpha\in[0, 2\pi)\},$$

and the angle coordinate $\alpha(t)$. We take another copy of the body symmetric with the first body relative to the line $a=0$.

Then the kinetic energy of each body is constant; the moments of interaction in the system are t_k such that $\alpha(t_k)=\alpha_k$; the moments of contact between the bodies are t such that $y(\alpha(t))=z(\alpha(t))$.

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