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Borel Summability of the Mass and the S-Matrix in φ^4 Models

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Abstract. We show that, in the φ_2^4 theory, the physical mass and the two-body S-matrix are Borel summable in the coupling constant λ at $\lambda=0$.

1. Introduction

In this paper we show that in the φ_2^4 theory the following objects are Borel summable in the coupling constant at zero: (i) the momentum space analytic functions for every complex momentum in an open set containing the Euclidean points; (ii) the physical mass and the field strength renormalization constant; (iii) the two-body S-matrix in the elastic region. The proofs of (i) and (ii) have been written so that they extend straightforwardly to the case of φ_3^4 with the help of the cluster expansion as given by Magnen and Sénéor [14] and Burnap's work [3]. By contrast, the proof of (iii) uses the analyticity in the coupling constant of the irreducible kernels, known for the even φ_2^4 theory from Spencer's analysis [16]. It could be extended to non-even φ_2^4 theories by using the work of Koch [13]. The method extends to the massive Sine-Gordon model [9], where it yields analyticity in the coupling constant around 0. The principle of the method is clearly present in [9].

The Schwinger functions of the φ_2^4 theory are given by

$$S_n(x_1, \ldots, x_n, \lambda, \zeta)$$
$$= \lim_{\Lambda \uparrow \mathbb{R}^2} N^{-1}(\Lambda, \lambda, \zeta) \int d\mu_\zeta(\varphi)\varphi(x_1)\ldots\varphi(x_n) \exp\left| -\lambda \int_\Lambda :\varphi^4(x): d^2x \right|, \tag{1}$$

where $d\mu_\zeta$ is the Gaussian measure with (bare) mass $\zeta^{1/2}$ and : : denotes Wick ordering with the same mass. N is the obvious normalization factor. For $\lambda \geqq 0$ sufficiently small and $\zeta > 0$ sufficiently large the theory is known to exist [11]. Its physical mass will be denoted $m(\lambda, \zeta)$, and the first threshold above it, $2m'(\lambda, \zeta)$. The natural scaling law

$$S_n(\varrho x_1, \ldots, \varrho x_n, \varrho^{-2}\lambda, \varrho^{-2}\zeta) = S_n(x_1, \ldots, x_n, \lambda, \zeta) \tag{2}$$

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holds in the sense of tempered distributions for all n and all $\varrho>0$ and is also satisfied by the truncated Schwinger functions $S_n^T(x_1, \ldots, x_n, \lambda, \zeta)$.

Our starting point is the following known theorem:

Theorem 1. 1. [8]. *Let the unit of length be fixed. There exist constants* $\varepsilon>0$, $M>0$, $\tau_0>0$ *and, for each integer* $n \geqq 2$, *constants* C_n, B_n, *and some* L^p *norm* $\| \;\|_{(n)}$ *such that, for each* $f \in \mathscr{S}(\mathbb{R}^{2n})$, *all* $a \in \mathbb{R}^{2(n-1)}$,

$$\left(\frac{\partial}{\partial \lambda}\right)^r \int S_n^T(x_1, \ldots, x_n, \lambda, \zeta)\, f(x_1+a_1, \ldots, +a_{n-1}, x_n)dx_1 \ldots dx_n \tag{3}$$

can be extended to a function of λ *and* ζ, *holomorphic in the domain*

$$\{\lambda \,|\, |\lambda|<\varepsilon, \operatorname{Re}\lambda>0\} \times \{\zeta \,|\, |\operatorname{Arg}(\zeta-M^2)|<\tau_0\}, \tag{4}$$

continuous on the boundary, and bounded there in modulus by

$$(r!)^2 C_n^r \|f\|_{(n)} \exp[-(\operatorname{Re}\sqrt{\zeta}-B_n)\min(|a_1|, \ldots, |a_{n-1}|)]. \tag{5}$$

2. [11, 16]. *Furthermore, for real* $\zeta \geqq M^2$ *and any* $\delta_0>0$, *there is an* $\eta>0$ *such that* $0 \leqq \lambda \leqq \eta$ *implies*

$$|m(\lambda, \zeta)-\sqrt{\zeta}|<\delta_0\sqrt{\zeta}, \quad m'(\lambda, \zeta)-\sqrt{\zeta}+\delta_0\sqrt{\zeta}>0. \tag{6}$$

The Fourier transform of S_n^T will be denoted

$$\tilde{S}_n^T(p_1, \ldots, p_n, \lambda, \zeta)\delta\left(\sum_{j=1}^n p_j\right) = (2\pi)^{-2n}\int \exp\left(i\sum_{j=1}^n p_j x_j\right) S_n^T(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

It is easy to derive from the first part of Theorem 1 that there exist, for each n, constants $C_n'>0$ and $\nu_n \geqq 0$ such that, for every (λ, ζ) in (4), $\tilde{S}_n^T(p_1, \ldots, p_n)$ can be extended to a holomorphic function of $p_1, \ldots, p_n$, (with $p_1+\ldots+p_n=0$), in the domain

$$\left\{p \,\middle|\, \sum_{j=1}^{n-1} |\operatorname{Im} p_j| < \operatorname{Re}\sqrt{\zeta}-B_n\right\} \tag{7}$$

and satisfying there

$$\left|\left(\frac{\partial}{\partial \lambda}\right)^r \tilde{S}_n^T(p, \lambda, \zeta)\right| < C_n'^{\,r+1}(r!)^2\left[\operatorname{Re}\sqrt{\zeta}-B_n-\sum_{j=1}^{n-1}|\operatorname{Im} p_j|\right]^{-\nu_n} \tag{8}$$

(we assume, of course, $\operatorname{Re}\sqrt{\zeta}>B_n$).

In a theory such as $P(\varphi)_2$, φ_3^4, (for *real* coupling constants) the Schwinger functions are naturally related to a set of time-ordered functions [6, 7]. The relation is:

$$\tilde{S}_n^T(p_1, \ldots, p_n)=H_n(-ip_1^0, p_1^1, \ldots, -ip_n^0, p_n^1),$$

where $H_n(k_1, \ldots, k_n)$, the "momentum-space analytic function", is holomorphic in the "axiomatic domain". This domain contains the Euclidean points. In theories of the type of $P(\varphi)_2$, φ_3^4, the same holds for all derivatives in λ, ζ. The scaling property

$$H_n(k, \lambda, \zeta) = \varrho^{2(n-1)} H_n(\varrho k, \varrho^2 \lambda, \varrho^2 \zeta), \tag{9}$$

combined with (7) shows that for $\varrho > 1$, H_n is holomorphic in

$$\left\{k, \lambda, \zeta \,\middle|\, \sum_{j=1}^{n-1} (|\mathrm{Re}\, k_j^0|^2 + |\mathrm{Im}\, k_j^1|^2)^{1/2} < \mathrm{Re}\sqrt{\zeta} - 2B_n \varrho^{-1}, \right.$$
$$\left. |\lambda| < \varepsilon \varrho^{-2}, \mathrm{Re}\,\lambda > 0, |\mathrm{Arg}(\zeta - M^2 \varrho^{-2})| < \tau_0 \right\}, \tag{10}$$

where it satisfies, by (8),

$$\left|\left(\frac{\partial}{\partial \lambda}\right)^r H_n(k, \lambda, \zeta)\right| < (r!)^2 C_n''^{(r+1)} \varrho^{2(r+n-1)+\nu_n}, \qquad r = 0, 1, 2, \ldots. \tag{11}$$

We want to extend the analyticity in λ by using complex scaling. For this purpose, we first observe that the domain (10) contains

$$\left\{k, \lambda, \zeta \,\middle|\, \sum_{j=1}^{n-1} |k_j| < |\zeta|^{1/2} \cos(\tau_1/2) - 2B_n \varrho^{-1}, \right.$$
$$\left. |\lambda| < \varepsilon \varrho^{-2}, \mathrm{Re}\,\lambda > 0, |\zeta| > M^2, |\mathrm{Arg}\,\zeta| < \tau_1 \right\}$$

for any τ_1 satisfying $0 < \tau_1 < \tau_1'$, where τ_1' is defined by $\sin(\tau_0 - \tau_1') = \varrho^{-2} \sin \tau_0$ ($< \sin \varrho^{-2} \tau_0$). For $\varrho \geqq 2$, we find $\tau_1' > 3\tau_0/4$. As a next step, we fix $\tau_1 \in (0, 3\tau_0/4)$ and we define $\varrho(\tau_1) = \max\{2, B_n(M \sin(3\tau_1/2) \sin(\tau_1/2))^{-1}\}$ and $u(\tau_1) = \varepsilon \varrho(\tau_1)^{-2}$. Then one checks easily that with $\varrho = \varrho(\tau_1)$ the domain (10) contains

$$\left\{k, \lambda, \zeta \,\middle|\, \sum_{j=1}^{n-1} |k_j| < |\zeta|^{1/2} \cos \tau_1, \right.$$
$$\left. |\lambda| < u(\tau_1), \mathrm{Re}\,\lambda > 0, |\zeta| > M^2, |\mathrm{Arg}\,\zeta| < \tau_1 \right\}, \tag{12}$$

and there

$$\left|\left(\frac{\partial}{\partial \lambda}\right)^r H_n(k, \lambda, \zeta)\right| < (r!)^2 C_n(\tau_1)^{r+1}, \qquad r = 0, 1, 2, \ldots. \tag{13}$$

A complex scaling with parameter $\varrho^2 = \exp(\pm i\tau_1/2)$ (using (9)) shows that H_n is analytic in the domain

$$\mathscr{L}_n(\tau_1) = \left\{k, \lambda, \zeta \,\middle|\, \sum_{j=1}^{n-1} |k_j| < |\zeta|^{1/2} \cos \tau_1, \right.$$
$$|\lambda| < u(\tau_1), |\mathrm{Arg}\,\lambda| < (\pi + \tau_1)/2,$$
$$\left. |\zeta| > M^2, |\mathrm{Arg}\,\zeta| < \tau_1/2 \right\} \tag{14}$$

and the bounds (13) continue to hold.

2. The Two-Point Function

Let $F(k^2, \lambda, \zeta) = H_2(k, -k, \lambda, \zeta)$ denote the propagator of the φ_2^4 theory, i.e., for real Euclidean p, with $p^2 = (p^0)^2 + (p^1)^2$,

$$F(-p^2, \lambda, \zeta) = \tilde{S}_2^T(p, -p, \lambda, \zeta).$$

This is a holomorphic function of k^2, λ, ζ in $\mathscr{L}_2(\tau_1)$. Additional information is provided by the spectral analysis of [3, 11, 16] in the $P(\varphi)_2$ and φ_3^4 theories:

For every $\varrho > 1$, for real $\zeta > M^2/\varrho^2$ and $0 \leqq \lambda < \varepsilon \varrho^{-2}$, $\left(\frac{\partial}{\partial \lambda}\right)^r F$ is holomorphic in k^2 in the cut plane

$$\hat{\mathscr{D}} = \{k^2 | k^2 \neq m^2(\lambda, \zeta) \quad \text{and} \quad k^2 \notin 4m'^2(\lambda, \zeta) + \mathbb{R}^+\}.$$

Since, by Theorem 1, the mass is a continuous function of λ, ζ, we obtain a more useful description as follows: Let ζ_0 be fixed real and $> M^2$. For fixed τ_1 and $0 < \sigma < 1$, it is possible to find $K_0(\tau_1, \sigma)$ and $u(\tau_1, \sigma) < u(\tau_1)$ such that, for $0 < \lambda < u(\tau_1, \sigma)$, $\zeta_0 - K_0(\tau_1, \sigma) < \zeta < \zeta_0 + K_0(\tau_1, \sigma)$ the domain $\hat{\mathscr{D}}$ contains

$$\hat{\mathscr{D}}_{\zeta_0, \sigma, \tau_1} = \mathbb{C} \setminus \{\gamma(\zeta_0, \sigma) \cup \Gamma(\zeta_0, \tau_1)\} \tag{15}$$

with

$$\begin{aligned} &\gamma(\zeta_0, \sigma) = \{z \in \mathbb{R} | \zeta_0(1-\sigma) \leqq z \leqq \zeta_0(1+\sigma)\}, \\ &\Gamma(\zeta_0, \tau_1) = \{z \in \mathbb{R} | z \geqq 4\zeta_0(1 - \tau_1^2)\}. \end{aligned} \tag{16}$$

If we also choose $K_0(\tau_1, \sigma)$ such that $|\zeta - \zeta_0| < K_0(\tau_1, \sigma)$ implies $|\zeta| > M^2$, $|\mathrm{Arg}\,\zeta| < \tau_1/2$ and $|\zeta| \cos^2 \tau_1 > R(\tau_1) \equiv \zeta_0(1 - 2\tau_1^2)$, then $\mathscr{L}_2(\tau_1)$ contains

$$\begin{aligned} &\{k^2, \lambda, \zeta | \, |k^2| < R(\tau_1), |\lambda| < u(\tau_1, \sigma), |\mathrm{Arg}\,\lambda| < (\pi + \tau_1)/2 \\ &\quad |\zeta - \zeta_0| < K_0(\tau_1, \sigma)\}. \end{aligned} \tag{17}$$

We now perform an analytic interpolation between this information and (15), i.e., an analytic completion of the union of the two corresponding sets. The analytic interpolation between (17) and (15) is reduced to a semi-tube problem by the conformal maps:

$$\lambda'(\lambda) = -(2/\pi) \log(\Lambda^{-1} - \Lambda), \; \Lambda = [\lambda/u(\tau_1, \sigma)]^{\pi/(\tau_1 + \pi)},$$

which maps $\{\lambda | \, |\lambda| < u(\tau_1, \sigma), |\mathrm{Arg}\,\lambda| < (\pi + \tau_1)/2\}$ onto the strip $\{\lambda' | \, |\mathrm{Im}\,\lambda'| < 1\}$, and

$$\zeta'(\zeta) = (2/\pi) \log[(K_0 + \zeta - \zeta_0)/(K_0 - \zeta + \zeta_0)]$$

which maps the disk $\{\zeta | \, |\zeta - \zeta_0| < K_0(\tau_1, \sigma)\}$ onto the strip $\{\zeta' | \, |\mathrm{Im}\,\zeta'| < 1\}$.

Let h be the function, defined and continuous on $\mathbb{C}$, with $0 \leqq h \leqq 1$, taking the value 1 on

$$\{k^2 | \, |k^2| \leqq R(\tau_1)\}, \tag{18}$$

the value 0 on $\gamma(\zeta_0, \sigma) \cup \Gamma(\zeta_0, \tau_1)$, and harmonic elsewhere. The solution of the semi-tube problem is the domain:

$$\{k^2, \lambda, \zeta | \, |\mathrm{Im}\,\lambda'(\lambda)| + |\mathrm{Im}\,\zeta'(\zeta)| < h(k^2)\}. \tag{19}$$

For sufficiently small σ, the length of the "little cut" $\gamma(\zeta_0,\sigma)$ becomes small and its influence on h can thus be made arbitrarily weak: h can be thought of as the electrostatic potential produced by conductors (18) and $\gamma(\zeta_0,\sigma)$, $\Gamma(\zeta_0,\tau_1)$ respectively at potentials 1, 0, 0; as the size of conductor $\gamma(\zeta_0,\sigma)$ shrinks, so does its capacity. More precisely, let $\hat{h}$ be the function continuous on $\mathbb{C}$, $0 \leqq \hat{h} \leqq 1$, taking the value 1 on (18), 0 on $\Gamma(\zeta_0,\tau_1)$ and harmonic elsewhere. Clarly $\hat{h}-h \geqq 0$. For any fixed τ_1, for any fixed compact K, not containing ζ_0, for any $\mu>0$, there is an $\eta>0$ such that, for all $\sigma<\eta$ and all $z\in K$, $\hat{h}(z)-h(z)<\mu$. (This can be explicitly checked with the help of a conformal mapping.) In particular, for any fixed τ_1, it is possible to find a contour $\mathscr{C}(\tau_1)$, surrounding ζ_0 and intersecting the real segment $[-R(\tau_1), R(\tau_1)]$, and an open neighborhood $K(\tau_1)$ of $\mathscr{C}(\tau_1)$, such that $\zeta_0 \notin K(\tau_1)$ and, for all $z\in K(\tau_1)$, $\hat{h}(z)>\hat{h}(\zeta_0)-\tau_1^2$. Then it is possible to find $\eta(\tau_1)>0$ such that, for all $\sigma<\eta(\tau_1)$, $K(\tau_1)\cap\gamma(\zeta_0,\sigma)=\emptyset$, and, for all z in $K(\tau_1)$, $h(z)>\hat{h}(z)-\tau_1^2>\hat{h}(\zeta_0)-2\tau_1^2$. Since $\hat{h}$ is differentiable (uniformly in τ_1) on $[R(\tau_1), 2\zeta_0]$, it satisfies $\hat{h}(\zeta_0)>1-\text{const.}\,\tau_1^2$. Thus there exist a constant $C>0$ and, for each τ_1, an $\eta(\tau_1)>0$ such that $0<\sigma<\eta(\tau_1)$ implies $h(z)>1-C\tau_1^2$ for all $z\in K(\tau_1)$. As a consequence, for every $z\in K(\tau_1)$, $F(z,\lambda,\zeta)$ is analytic in λ and ζ in

$$\{\lambda \,|\, |\lambda|<s(\tau_1)y(\delta), |\mathrm{Arg}\,\lambda|<(1-\varphi)(1-\delta)(1-C\tau_1^2)(\pi+\tau_1)/2,$$
$$|\zeta-\zeta_0|<K_0(\tau_1,\eta(\tau_1))\,\mathrm{tg}(\varphi\pi/4)\},$$

where $0<\delta<1$ and $0<\varphi<1$ are arbitrary, $s(\tau_1)$ and $y(\delta)>0$. Thus fixing first τ_1 sufficiently small, then $\sigma<\eta(\tau_1)$, and finally δ and φ, we obtain the analyticity of F in

$$\{k^2,\lambda,\zeta \,|\, k^2\in K(\tau_1), |\lambda|<v, |\mathrm{Arg}\,\lambda|<(\pi/2)+\tau_2, |\zeta-\zeta_0|<\alpha\} \tag{20}$$

with $v>0$, $\tau_2>0$, $\alpha>0$.

The following information is now available:

1. $$\left(\frac{\partial}{\partial\lambda}\right)^n F(k^2,\lambda,\zeta)\bigg|_{\lambda=0} \equiv n!\,a_n(k^2,\zeta),$$

where $(\zeta-k^2)^{n+1}a_n(k^2,\zeta)$ is analytic in k^2, for real $\zeta>M^2$, in

$$D=\{k^2 \,|\, k^2\notin 4M^2+\mathbb{R}^+\} \tag{21}$$

and, in this domain

$$|(\zeta-k^2)^{n+1}a_n(k^2,\zeta)|<C(k^2,\zeta)^n n!, \tag{22}$$

where $C(k^2,\zeta)$ is bounded in any compact in (21). (This property of perturbation theory can be proved by using Sect. 2.2 of [10].) Furthermore for $k^2<0$, (i.e., k Euclidean), $a_n(k^2,\zeta)$ is analytic in ζ for $\mathrm{Re}\,\zeta>0$ and again satisfies (22) (since each graph is bounded in modulus by the same graph with ζ replaced by $\mathrm{Re}\,\zeta$). Hence $(\zeta-k^2)^{n+1}a_n(k^2,\zeta)$ is analytic in

$$\begin{gathered}\{k^2,\zeta \,|\, k^2\notin 4M^2+\mathbb{R}^+, |\mathrm{Arg}(\zeta-M^2)|<\varphi(k^2)\},\\ 2\varphi(k^2)=\mathrm{Re}\,\mathrm{Arccos}\left(\frac{k^2}{2M^2}-1\right).\end{gathered} \tag{23}$$

2. $F(k^2, \lambda, \zeta)$ is analytic in the domain (20), continuous at its boundary and bounded there by a constant (depending on $\zeta_0, \tau_1, \tau_2, \alpha$) and the same is true for $\frac{\partial}{\partial \lambda} F(k^2, \lambda, \zeta)$.

3. In the domain $\mathscr{L}_2(\tau_1)$, [see (14)],

$$\left|\left(\frac{\partial}{\partial \lambda}\right)^n F(k^2, \lambda, \zeta)\right| < C'''^{n+1}(n!)^2 . \tag{24}$$

The first information implies the existence of the function

$$B_1(k^2, \mu, \zeta) = \sum_{n=0}^{\infty} (n!)^{-1} a_n(k^2, \zeta)\mu^n , \tag{25}$$

which is holomorphic in

$$\{k^2, \mu, \zeta | k^2 \neq \zeta, k^2 \notin 4M^2 + \mathbb{R}^+, |\mathrm{Arg}(\zeta - M^2)| < \varphi(k^2), |\mu| < C(k^2, \zeta)^{-1}|\zeta - k^2|\} . \tag{26}$$

Denoting $\Delta = \{\lambda | |\lambda| < v, |\mathrm{Arg}\lambda| < (\pi/2) + \tau_2\}$ and $\omega = \{z | |z| < R(\tau_1)\} \cup K(\tau_1)$, we can also define

$$B_2(k^2, \mu, \zeta) = \frac{1}{2\pi i} \oint_{\partial \Delta} \lambda^{-1} F(k^2, \lambda, \zeta) \exp(\mu/\lambda) d\lambda = F(k^2, 0, \zeta)$$
$$+ \frac{1}{2\pi i} \oint_{\partial \Delta} \lambda^{-1} (F(k^2, \lambda, \zeta) - F(k^2, 0, \zeta)) \exp(\mu/\lambda) d\lambda \tag{27}$$

which is analytic in

$$\{k^2, \mu, \zeta | k^2 \in \omega, |\mathrm{Arg}\mu| < \tau_2, |\zeta - \zeta_0| < \alpha\} \tag{28}$$

and bounded there in modulus by

$$|B_2(k^2, \mu, \zeta)| < G \exp|\mu/v| . \tag{29}$$

The bound (24) and Watson's theorem [12, p. 192, Theorem 136] show that, for $|k^2| < R(\tau_1)$, B_1 and B_2 coincide in the intersection of their domain of definition. By analytic continuation, they define a unique function $B(k^2, \mu, \zeta)$ analytic in the union of the domains (26) and (28). This fact and the bound (29) show that

$$F'(k^2, \lambda, \zeta) = \int_0^{\infty} dt\, e^{-t} B(k^2, \lambda t, \zeta) \tag{30}$$

is well defined and analytic in k^2 and ζ for $0 < \lambda < v$, $k^2 \in \omega$, $|\zeta - \zeta_0| < \alpha$. For $|k^2| < R(\tau_1)$, it coincides with $F(k^2, \lambda, \zeta)$ by Watson's theorem, hence it continues to do so for $k^2 \in \omega$. Thus F is Borel summable as a function of λ.

For real λ and ζ, ($|\zeta - \zeta_0| < \alpha$) the physical mass and wave-function renormalization constants are given by

$$Z(\lambda, \zeta) = \frac{1}{2\pi i} \oint_{\mathscr{C}(\tau_1)} F(z, \lambda, \zeta) dz , \tag{31}$$

$$m^2(\lambda, \zeta) Z(\lambda, \zeta) = \frac{1}{2\pi i} \oint_{\mathscr{C}(\tau_1)} z F(z, \lambda, \zeta) d\zeta . \tag{32}$$

Let $g(\lambda, \zeta)$ denote either of the functions (31) or (32). These functions are analytic in λ and ζ in

$$\{\lambda, \zeta \mid |\lambda| < v, |\mathrm{Arg}\,\lambda| < (\pi/2) + \tau_2, |\zeta - \zeta_0| < \alpha\} \tag{33}$$

[see (20)], continuous and bounded on its boundary. The same is true for their respective first derivatives in λ. Therefore, reducing v if necessary, they do not vanish in the domain (33). The properties just derived for B_1, B_2, and F' show that $g(\lambda, \zeta)$ is Borel summable in λ at $\lambda = 0$. To reach the same conclusion for $m^2(\lambda, \zeta)$, it is necessary to use some elementary and well-known properties [15] which we briefly recall.

Let U be a bounded domain in $\mathbb{C}$, and denote $\mathscr{F}^r(U)$, for $r \geqq 0$, the class of functions f holomorphic in U, such that, for some constants $A \geqq 0$, $B \geqq 0$ (depending on f), for all $n \geqq 0$, and all $\lambda \in U$,

$$\left|\left(\frac{d}{d\lambda}\right)^n f(\lambda)\right| \leqq AB^n (n!)^{r+1}\,. \tag{34}$$

$\mathscr{F}^{(r)}(U)$ is a complex vector space and (by Leibniz's formula) is closed under multiplication.

Lemma 2. *If $f \in \mathscr{F}^{(r)}(U)$ satisfies* (34) *and, for all λ in U, $|f(\lambda)^{-1}| < C$ then, for all $n \geqq 1$ and all $\lambda \in U$,*

$$\left|\left(\frac{d}{d\lambda}\right)^n (f(\lambda))^{-1}\right| \leqq \frac{AC^2}{AC+1}[B(AC+1)]^n (n!)^{r+1}\,.$$

We conclude that $m^2(\lambda, \zeta)$ and $Z(\lambda, \zeta)$ are both Borel summable in λ and analytic in ζ. In order to be able to fix the physical mass, we shall use the following known facts.

Lemma 3. *Let F be a function of two complex variables, λ and z, holomorphic in $W = U \times \{z \mid |z| < R\}$, and satisfying there, for all n,*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^n F(\lambda, z)\right| < AB^n (n!)^{r+1}\,.$$

Let $g \in \mathscr{F}^{(r)}(U)$ and assume that $g(U) \subset \{z \mid |z| < \varrho\}$, $\varrho < R$. Denote $\Phi(\lambda) = F(\lambda, g(\lambda))$. Then $\Phi \in \mathscr{F}^{(r)}(U)$.

Lemma 4. *Let F have the same properties as in Lemma 3, and assume that, for some $C > 0$, and all $(\lambda, z) \in W$,*

$$\left|\frac{\partial}{\partial z} F(\lambda, z)\right| > C\,.$$

i) *Let $V(\lambda, \varrho) = F(\lambda, \{z \mid |z| < \varrho\})$. There exists a $\varrho > 0$, $(\varrho < R)$, such that, for all $\lambda \in U$, $z \to F(\lambda, z)$ maps $\{z \mid |z| < \varrho\}$ one-to-one onto $V(\lambda, \varrho)$ and has an inverse $m^2 \to g(\lambda, m^2)$. This defines a holomorphic function on $\{(\lambda, m^2) \mid \lambda \in U, m^2 \in V(\lambda, \varrho)\}$.*

ii) *There are constants A', B' such that for all n, $\lambda \in U$, $m^2 \in V(\lambda, \varrho/2)$,*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^n g(\lambda, m^2)\right| < A'B'^n (n!)^{r+1}\,.$$

Proof. (i) is an immediate application of the inverse function theorem [4, VIII,7 p. 250] together with the uniform bounds postulated in the lemma, and of the implicit function theorem. (ii) follows from the identity, valid for $m^2 \in V(\lambda, \varrho/2)$,

$$g(\lambda, m^2) = \frac{1}{2\pi i} \int_{|z|=\varrho} \frac{z \dfrac{\partial}{\partial z} F(\lambda, z)}{F(\lambda, z) - m^2} dz$$

and from Lemma 2. Note that, for fixed q, $|z| < R - \varepsilon$, $\left(\dfrac{\partial}{\partial z}\right)^q F(\lambda, z) \in \mathscr{F}^{(r)}(U)$ by the Cauchy inequalities.

We apply Lemma 4 to the function

$$F(\lambda, z) = m^2(\lambda, \zeta_0 + z)$$

with $U = \{\lambda \mid |\lambda| < v_1, |\mathrm{Arg}\,\lambda| < (\pi/2) + \tau\}$ and $0 < v_1 \leqq v$. The uniform bounds on the derivatives of $m^2(\lambda, \zeta)$ and the proof of the inverse function theorem show that, for sufficiently small $v_1 > 0$ and $\varrho > 0$, there is a $\kappa > 0$ such that for all $\lambda \in U$, $V(\lambda, \varrho/2)$ contains the disk $\{m^2 \mid |m^2 - \zeta_0| < \kappa\}$. Hence, the inverse function $g(\lambda, m^2)$ defined in Lemma 4 is well-defined for $\lambda \in U$ and $|m^2 - \zeta_0| < \kappa$, and, for fixed m^2, determines a function of λ belonging to $\mathscr{F}^{(r)}(U)$. This proves:

Lemma 5. *For fixed* $\zeta_0 > M^2$, *there are constants* $v_1 > 0$, $\tau > 0$, *and there is a unique function* $g(\lambda, w)$, *holomorphic in*

$$\{\lambda, w \mid |\lambda| < v_1, |\mathrm{Arg}\,\lambda| < \pi/2 + \tau, |w - \zeta_0| < \kappa\} \tag{35}$$

such that

$$m^2(\lambda, g(\lambda, w)) = w\,.$$

There is a constant K *such that, in the domain* (35), *for all* n,

$$\left|\left(\frac{\partial}{\partial \lambda}\right)^n g(\lambda, w)\right| < K^{n+1}(n!)^2\,.$$

Hence this function is Borel summable in λ.

3. The Two-Point Vertex Function

This section and the next use methods specific to the even φ_2^4 theory. The preceding results can be extended by using further known information [16, 17] about the one-particle-irreducible, or vertex, two-point function

$$\Gamma_2(k^2, \lambda, \zeta) = F(k^2, \lambda, \zeta)^{-1}\,.$$

The estimates of Spencer [16] can be straightforwardly extended to show that, for all $\zeta > M^2$, this function can be analytically continued in both k^2 and λ in the domain

$$\{k^2, \lambda \mid |\mathrm{Re}\,\sqrt{k^2}| < 3(\sqrt{\zeta} - B'), |\lambda| < \varepsilon(\zeta), \mathrm{Re}\,\lambda > 0\}$$

where it is bounded in modulus by Const.$(|k^2|+\zeta)$. (Here all constants depend on ζ, with the exception of B'). By real scaling, Γ_2 is analytic in

$$\{k^2,\lambda|\,|\mathrm{Re}\sqrt{k^2}|<3(\sqrt{\zeta}-B'\varrho^{-1}),|\lambda|<\varrho^{-2}\varepsilon(\varrho^2\zeta),\mathrm{Re}\lambda>0\}\,,\quad(\varrho>1)\,,\tag{36}$$

where it satisfies a similar bound. Furthermore, it is possible to show, by the method of partial integration that, for $\zeta_0>M^2$, any $\ell>0$, for some $v>0$, $\alpha>0$, $C>0$, in the domain

$$\{k^2,\lambda,\zeta|\,|\mathrm{Re}\sqrt{k^2}|<\sqrt{\zeta_0}-\ell,|\lambda|<v,\mathrm{Re}\lambda>0,|\zeta-\zeta_0|<\alpha\}\,,\tag{37}$$

the bounds

$$|(\zeta-k^2)F(k^2,\lambda,\zeta)|<C\,,\qquad \left|(\zeta-k^2)\frac{\partial}{\partial\lambda}F(k^2,\lambda,\zeta)\right|<C$$

hold. Thus, after reducing v if necessary, we have in (37),

$$|(\zeta-k^2)F(k^2,\lambda,\zeta)|>|(\zeta-k^2)F(k^2,0,\zeta)|-vC=1-vC\geqq 1/2\,,$$

hence

$$|\Gamma_2(k^2,\lambda,\zeta)|<2|\zeta-k^2|\text{ in (37)}.\tag{38}$$

In particular, interpolating between the domains (36) and (37) we find that Γ_2 is analytic in

$$\{k^2,\lambda,\zeta|\,|\mathrm{Re}\sqrt{k^2}|<3\sqrt{\zeta_0}-\ell,|\lambda|<v,\mathrm{Re}\lambda>0,|\zeta-\zeta_0|<\beta\}\tag{39}$$

for some $\beta>0$, and arbitrarily small $\ell>0$. In this domain it is again bounded by Const.$(\zeta_0+|k^2|)$. The same is true for $\frac{\partial}{\partial\lambda}\Gamma_2(k^2,\lambda,\zeta)$.

Since $\Gamma_2(k^2,0,\zeta)=\zeta-k^2$, there is a constant B such that, in (39),

$$|\zeta-k^2|-vB(\zeta_0+|k^2|)<|\Gamma_2(k^2,\lambda,\zeta)|<|\zeta-k^2|+vB(\zeta_0+|k^2|)\,.$$

As a consequence v and β may be chosen so small that in

$$\{k^2,\lambda,\zeta|\,|\mathrm{Re}\sqrt{k^2}|<3\sqrt{\zeta_0}-\ell,|k^2-\zeta_0|>\zeta_0/2,|\lambda|<v,\mathrm{Re}\lambda>0,|\zeta-\zeta_0|<\beta\}\,,$$
$$|\Gamma_2(k^2,\lambda,\zeta)|>|\zeta_0-k^2/4\,.\tag{40}$$

Applying Rouché's theorem in the variable k^2 for $|k^2-\zeta_0|<\zeta_0/2$ shows that inside this disk, for sufficiently small v and β, $|\lambda|<v$, $\mathrm{Re}\lambda>0$, $|\zeta-\zeta_0|<\beta$, Γ_2 has no other zero than $k^2=m^2(\lambda,\zeta)$. For v sufficiently small, $|m^2(\lambda,\zeta)-\zeta_0|<\zeta_0/4$ so that, on the circle $\{k^2|\,|k^2-\zeta_0|=\zeta_0/2\}$, and consequently inside,

$$|(k^2-m^2(\lambda,\zeta))^{-1}\Gamma_2(k^2,\lambda,\zeta)|>1/8\,.$$

Finally we see that, with suitable choice of $v>0$ and $\beta>0$, there is a constant $K>0$ such that, in (39),

$$|(k^2-m^2(\lambda,\zeta))^{-1}\Gamma_2(k^2,\lambda,\zeta)|>K\tag{41}$$

and consequently

$$(k^2 - m^2(\lambda, \zeta))F(k^2, \lambda, \zeta)$$

is also analytic in (39) and its modulus is bounded there by K^{-1}.

A first consequence of these facts is the possibility of substituting $\zeta = g(\lambda, \zeta_0)$ in $F(k^2, \lambda, \zeta)$ and $\Gamma_2(k^2, \lambda, \zeta)$ so as to fix the physical mass to ζ_0. More precisely it has been seen that, for sufficiently small $\alpha > 0$, one can choose $v > 0$ and $\tau > 0$ such that $|\lambda| < v$ and $|\mathrm{Arg}\,\lambda| < (\pi/2) + \tau$ imply $|g(\lambda, \zeta_0) - \zeta_0| < \alpha$ so that

$$(\zeta_0 - k^2)^{-1} G(k^2, \lambda, \zeta_0) = \Gamma_2(k^2, \lambda, g(\lambda, \zeta_0))(\zeta_0 - k^2)^{-1} \tag{42}$$

and

$$(\zeta_0 - k^2)F(k^2, \lambda, g(\lambda, \zeta_0)) = (\zeta_0 - k^2)G(k^2, \lambda, \zeta_0)^{-1} \tag{43}$$

are both analytic and bounded in

$$\{k^2, \lambda \,|\, |\mathrm{Re}\sqrt{k^2}| < 3\sqrt{\zeta_0} - \ell, |\lambda| < v, \mathrm{Re}\,\lambda > 0\}.$$

4. The Four-Point Function

The Bethe-Salpeter kernel $\tilde{K}(x_1, x_2; x_3, x_4; \lambda, \zeta)$ has been studied by Spencer and Zirilli in the φ_2^4 model and, in the general framework proposed by Symanzik [18], by Bros and Lassalle [1, 2, 16, 17]. (See also [5].) This is a Euclidean invariant tempered distribution over $(\mathbb{R}^2)^4$ which, considered as a kernel, satisfies the equation

$$R_0 S_4^T R_0 = \tilde{K} + \tilde{K} S_4^T R_0 . \tag{44}$$

Here S_4^T is considered as also defining a kernel and

$$\begin{aligned} R_0(x_1, x_2; x_3, x_4; \lambda, \zeta) = {} & \tfrac{1}{2}[S_2^T(x_1, x_3, \lambda, \zeta)]^{-1}[S_2^T(x_2, x_4, \lambda, \zeta)]^{-1} \\ & + \tfrac{1}{2}[S_2^T(x_1, x_4, \lambda, \zeta)]^{-1}[S_2^T(x_2, x_3, \lambda, \zeta)]^{-1} . \end{aligned}$$

We denote

$$\begin{aligned} & K(k, p, q, \lambda, \zeta) = (2\pi)^{-6} \int \tilde{K}(x_1, x_2; x_3, x_4; \lambda, \zeta) \\ & \exp\left[-\frac{k^0}{2}(x_1^0 + x_2^0 - x_3^0 - x_4^0) + i\frac{k^1}{2}(x_1^1 + x_2^1 - x_3^1 - x_4^1) \right. \\ & \left. - p^0(x_1^0 - x_2^0) + ip^1(x_1^1 - x_2^1) - q^0(x_3^0 - x_4^0) + iq^1(x_3^1 - x_4^1) \right] \\ & d^2(x_1 - x_2) d^2(x_3 - x_4)(1/4) d^2(x_1 + x_2 - x_3 - x_4) . \end{aligned}$$

The methods of Spencer [16] apply without modification for sufficiently small complex λ with $\mathrm{Re}\,\lambda > 0$. In particular they provide the following theorem:

Theorem 6 [16]. *Let $\zeta_0 > M^2$ be fixed. Then for any sufficiently small $\ell > 0$, there are constants $\alpha > 0$, $C > 0$, $\varepsilon > 0$, such that, for all real ζ with $|\zeta - \zeta_0| < \alpha$, K is well defined as an analytic function of k, p, q, and λ in the domain*

$$\begin{aligned}\{k, p, q, \lambda \,|\, &|\mathrm{Re}\,k^0| < (1-\theta)(4\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,k^1| < \theta(4\sqrt{\zeta_0} - \ell),\\ &|\mathrm{Re}\,p^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,p^1| < \theta(\sqrt{\zeta_0} - \ell),\\ &|\mathrm{Re}\,q^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,q^1| < \theta(\sqrt{\zeta_0} - \ell),\\ &\quad |\lambda| < \varepsilon, \mathrm{Re}\,\lambda > 0\}\end{aligned} \tag{45}$$

for any $\theta \in [0, 1]$, and in this domain,

$$|K| < |\lambda| C .$$

In the intersection of the domain (45) with

$$\{|\mathrm{Re}\,k^0| < (1-\theta)(2\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,k^1| < \theta(2\sqrt{\zeta_0} - \ell)\}, \tag{46}$$

K may be obtained by solving the integral equation (44) in momentum space, with a purely Euclidean integration variable, as explained in [1]; in fact, for sufficiently small $|\lambda|$, this can be done by using a Neumann series. Since, in the relevant domain H_4 can be analytically continued in ζ in the disk $\{\zeta \,|\, |\zeta - \zeta_0| < \alpha\}$, with appropriate redefinition of α, the same is true for K (provided k is in (46)). Analytic completion then shows that, (for suitable redefinition of ℓ and $\beta > 0$) K is holomorphic, and bounded by $C'|\lambda|$, in the topological product of (45) and $\{|\zeta - \zeta_0| < \beta\}$. It is then possible to substitute $\zeta = g(\lambda, \zeta_0)$ in K, i.e., to define

$$K'(k, p, q, \lambda, \zeta_0) = K(k, p, q, \lambda, g(\lambda, \zeta_0))$$

which, for fixed ζ_0, is again analytic and bounded by $C''|\lambda|$ in the domain

$$\begin{aligned}\{k, p, q, \lambda \,|\, &|\mathrm{Re}\,k^0| < (1-\theta)(4\sqrt{\zeta_0} - 2\ell), |\mathrm{Im}\,k^1| < \theta(4\sqrt{\zeta_0} - 2\ell),\\ &|\mathrm{Re}\,p^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,p^1| < \theta(\sqrt{\zeta_0} - \ell),\\ &|\mathrm{Re}\,q^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\,q^1| < \theta(\sqrt{\zeta_0} - \ell),\\ &\quad |\lambda| < \varepsilon, \mathrm{Re}\,\lambda > 0\},\end{aligned} \tag{47}$$

with possible redefinition of constants.

The original four-point function can now be reconstructed, and its analyticity extended, with the help of K (or K') [1, 2, 17]. We briefly recall the principle of this method. Let

$$R'_0(k, p, \lambda, \zeta_0) = G\left(\left(\frac{k}{2} + p\right)^2, \lambda, \zeta_0\right) G\left(\left(\frac{k}{2} - p\right)^2, \lambda, \zeta_0\right),$$

$$H'_4(k, p, q, \lambda, \zeta_0) = H_4\left(\left(\frac{k}{2} + p\right), \left(\frac{k}{2} - p\right), \left(-\frac{k}{2} + q\right), \left(-\frac{k}{2} - q\right), \lambda, g(\lambda, \zeta_0)\right)$$

$$L'(k, p, q, \lambda, \zeta_0) = R'_0(k, p, \lambda, \zeta_0) H'_4(k, p, q, \lambda, \zeta_0) R'_0(-k, q, \lambda, \zeta_0)$$

Then, by (44) and a straightforward analytic continuation, the equation

$$\begin{aligned} L'(k,p,q,\lambda,\zeta_0) &= K'(k,p,q,\lambda,\zeta_0) \\ &+ \int_{\mathscr{C}} K'(k,p,Z,\lambda,\zeta_0) R'_0(-k,Z,\lambda,\zeta_0)^{-1} L'(k,-Z,q,\lambda,\zeta_0) d^2 Z \end{aligned} \tag{48}$$

holds in the intersection of the domains (46) and (47), with the "contour $\mathscr{C}$" taken as the Euclidean space in the variable Z, i.e., $\mathscr{C} = \{Z \mid \mathrm{Re}\, Z^0 = 0 = \mathrm{Im}\, Z^1\}$.

We now fix $k = (\sigma, 0)$, and interpret (48) as an equation for L', given K'. The analyticity of K' in the domain (47), the bound $|K'| < C''|\lambda|$, and (43), allow the solution of (48) to be obtained as a Neumann series, the integration contour being deformed as σ varies, as explained in [1, 2]. (Alternatively, the methods of [5, 17] can be used.) The result is that L', as a function of σ, p, q, λ, is holomorphic and bounded in

$$\begin{aligned} &\mathscr{A} \times \{\lambda \mid |\lambda| < v, \mathrm{Re}\,\lambda > 0\}, \\ &\mathscr{A} = \{\sigma, p, q \mid \sigma \in \mathscr{D}(\gamma), |\mathrm{Re}\, p^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\, p^1| < \theta(\sqrt{\zeta_0} - \ell), \\ &\qquad |\mathrm{Re}\, q^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\, q^1| < \theta(\sqrt{\zeta_0} - \ell)\}, \end{aligned} \tag{49}$$

where the constants $v > 0$, $\gamma > 0$ have to be chosen sufficiently small (ℓ can be fixed arbitrarily small).

The domain $\mathscr{D}(\gamma)$ is a *two-sheeted domain* which can be described with the help of the variable $w = (\sigma^2 - 4\zeta_0)^{1/2} : \mathscr{D}(\gamma)$ is the (ramified) image of

$$\{w \mid 0 < \mathrm{Re}(w^2 + 4\zeta_0)^{1/2} < 4\sqrt{\zeta_0} - 2\ell, \mathrm{Im}\, w > -\gamma\}.$$

Note that $\{w \mid \mathrm{Im}\, w > 0\}$ corresponds, in the variable σ to the domain $\{\sigma \mid \sigma^2 \notin 4\zeta_0 + \mathbb{R}^+\}$. Of course L' is also analytic in the domain obtained from (49) by changing k to $-k$. The fact that the Riemann domain $\mathscr{A}$ goes beyond the "physical sheet" is essential for our purposes. Indeed we know from (14) and from Lemmas 3 and 5 that, in the domain

$$\{k, p, q, \lambda \mid |\lambda| < v, |\mathrm{Arg}\,\lambda| < (\pi + \tau_1)/2, |\mathrm{k}| + |\mathrm{p}| + |\mathrm{q}| < \tfrac{1}{2}\sqrt{\zeta_0}\}, \tag{50}$$

$\left(\dfrac{\partial}{\partial\lambda}\right)^r L'(k, p, q, \lambda, \zeta_0)$ is holomorphic and bounded in modulus by $(\mathrm{Const.})^{r+1}(r!)^2$. Fixing $k = (\sigma, 0)$ we may analytically interpolate between the domains (49) and (50) and obtain the existence of $\tau(\sigma, p, q) > 0$, $v' > 0$ such that L' is holomorphic in

$$\begin{aligned} \mathscr{U} = \{&\sigma, p, q, \lambda \mid \sigma \in \mathscr{D}(\gamma), |\mathrm{Re}\, p^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\, p^1| < \theta(\sqrt{\zeta_0} - \ell), \\ &|\mathrm{Re}\, q^0| < (1-\theta)(\sqrt{\zeta_0} - \ell), |\mathrm{Im}\, q^1| < \theta(\sqrt{\zeta_0} - \ell), \\ &|\lambda| < v', |\mathrm{Arg}\,\lambda| < \pi/2 + \tau(\sigma, p, q)\}. \end{aligned} \tag{51}$$

$\tau(\sigma, p, q)$ tends to 0 as (σ, p, q) tend to the boundary of their allowed domain $\mathscr{A}$, but is strictly positive inside. Furthermore the domain (51) extends in an obvious manner through the Lorentz invariance of L' to a domain $\mathscr{U}'$. The mechanism described in Sect. 1 ensures that, at all $(k, p, q) \in \mathscr{A}' = \bigcup_{\Lambda \in L^+(\mathbb{C})} \Lambda\mathscr{A}$, L' is Borel summable as a function of λ. There remains to prove that this implies the Borel summability of the two-particle S-matrix in the purely elastic region. Let (k, p, q) be a physical point on the mass-shell, i.e.,

$$\left(\frac{k}{2}+p\right)^2=\left(\frac{k}{2}-p\right)^2=\left(\frac{k}{2}+q\right)^2=\left(\frac{k}{2}-q\right)^2=\zeta_0$$

$$\left(\frac{k}{2}+p\right)\in V^+,\ \left(\frac{k}{2}-p\right)\in V^+,\ 4\zeta_0<k^2<(4\sqrt{\zeta_0}-2\ell)^2.$$

We may, for instance, choose $k=(\sigma,0)$. Then the domain $\mathscr{A}'$ contains a full complex neighborhood of this point (the part of this neighborhood with $\operatorname{Im}k^2>0$ (resp. $\operatorname{Im}k^2<0$) being taken in the physical (resp. second) sheet). This neighborhood intersects the mass-shell. Thus the S-matrix is analytic in a neighborhood of the given point on the mass-shell and, by the same arguments as in Sect. 2, is Borel summable in λ at $\lambda=0$.

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