Commun. Math. Phys. 68, 195—207 (1979)

Communications in
Mathematical
Physics


# Lorentz Covariance and Kinetic Charge

P. Basarab-Horwath, R. F. Streater, and J. Wright
Bedford College, Department of Mathematics, London NW1 4NS, England

**Abstract.** There is a one-to-one correspondence between inequivalent covariant displaced Fock representations of the free relativistic field and the 1-cohomology of the Poincaré group with coefficients in the 1-particle space.

Representations with positive energy are obtained from cocycles with finite energy which have particle-like properties and are interpreted as condensed states of matter without a sharply defined mass.

The 1-cohomology groups of $\mathscr{P}_+^\uparrow$ are calculated. These are trivial in 3- or 4-dimensional space-time, or if the mass is non-zero. Non-trivial cocycles for subgroups lead to representations in which $\mathscr{P}$-invariance is spontaneously broken. We recover $\mathscr{P}$-invariance in a direct integral representation possessing a gauge group, and a superselection structure labelled by the velocities of the condensed states of matter which are the cocycles determining each irreducible component of the representation. A model in 4-dimensional space-time is constructed.

## 1. Cohomological Classification of Displaced Fock Representations

Let $U$ be a representation of the Poincaré group $\mathscr{P}_+^\uparrow$ acting on a Hilbert space $\mathscr{H}$ and satisfying the spectral condition:

$$P^0 \geqq 0 \; (P^0)^2 \geqq \mathbf{P}^2 ,$$

where $P^\mu$ is the self-adjoint generator of space-time translations in the direction $x^\mu$, $\mu = 0, 1, \ldots, s$. If $m$ is the mass of $U$, then $\mathscr{H}$ may be realised as a class of solutions of a manifestly covariant family of wave equations and subsidiary conditions:

$$(\square + m^2)\varphi_\alpha(\mathbf{x}, t) = 0 \quad D^{\alpha\beta}\varphi_\beta(\mathbf{x}, t) = 0 ,$$
$$\alpha = 1, 2, \ldots ,$$

where $D$ is a tensor of differential operators chosen to remove unwanted spin components.

0010-3616/79/0068/0195/$02.60

Let $\mathscr{M}$ be a dense complex-linear subspace of $\mathscr{K}$ invariant under $\mathscr{P}_+^\uparrow$ i.e. $U(a, \Lambda)\mathscr{M} \subseteqq \mathscr{M}$ for all $(a, \Lambda) \in \mathscr{P}_+^\uparrow$. Let $\mathfrak{A}$ be the Weyl algebra over $\mathscr{M}$ [1] and let $W$ be the Weyl map $\mathscr{M} \to \mathfrak{A}$; thus for each $\Psi \in \mathscr{M}$, $W(\Psi)$ is a unitary operator obeying

$$W(\Phi)W(\Psi) = \exp[i/2 \operatorname{Im}\langle \Phi, \Psi \rangle] W(\Phi + \Psi).$$

In order to interpret $\mathfrak{A}$ as the algebra of observables of a theory, it is usual to require $\mathscr{M}$ to be localised, in the sense that if $\varphi \in \mathscr{M}$ then each component $\varphi_\alpha(\mathbf{x}, t)$ has compact support in $\mathbf{x}$ for every $t$.

It may be that the field $\varphi$ itself should not be regarded as observable; for instance, in electromagnetism, $\mathbf{F}_{\mu\nu}$ but not $A^\mu$ is observable. In such cases we take $\mathscr{M}$ to consist of observable fields only.

In Segal's work [1], where only the case $\mathscr{M} = \mathscr{K}$ is discussed, it is shown that the usual Fock representation is the only irreducible one with a Lorentz-invariant vacuum and positive energy, which obeys a mild continuity condition. On the other hand by considering $\mathscr{M}$ to be properly smaller than $\mathscr{K}$, it is possible for the phenomenon of spontaneously broken symmetry to occur [2]; in this case, infinitely many Lorentz invariant states on the Weyl algebra can be constructed, all giving rise, through the Gelfand-Naimark-Segal construction, to representations with positive energy. It can be argued that there should be no physical distinction between these vacua, as only one vacuum state occurs in Nature. Therefore any observable operator in the Weyl algebra must have the same expectation value in any of these vacua. In [3] this is achieved by restricting $\mathscr{M}$ even more, so that $\mathscr{M}$ lies in the kernel of the Lorentz-invariant linear functionals on the space of localized elements of $\mathscr{K}$. In this way, there is again a unique representation of the Weyl algebra over $\mathscr{M}$ having positive energy and a vacuum, and being of displaced Fock type.

However, other representations of $\mathfrak{A}$ may be physically interesting, in that they are covariant, and satisfy the spectral condition, although they do not possess a vacuum state. Such representations correspond to the presence of condensed states of the field whose behaviour is somewhat particle-like. It may be noted that although we are discussing representations of a free field, obeying a linear differential equation, the fiducial states giving rise to the non-vacuum representations have considerable similarity of behaviour and physical interpretation to the soliton states giving rise to superselection sectors in the theories discussed by Fröhlich [4]. These latter states correspond to soliton solutions of the non-linear differential equations satisfied by the fields in [4].

Roepstorff [5] has shown in a discussion of the free electromagnetic field that the cohomology of the space-time translation group can be used to classify all inequivalent displaced Fock representations in which that group is implemented. We shall review this method of classification for the general case of a free field outlined above, and using arbitrary subgroups of the Poincaré group. In Sect. 2 we shall discuss the cohomology of the full Poincaré group in more detail.

Subsequent sections will be concerned with the induced representation construction and its use to provide models carrying a representation of the Poincaré group, obtained as a direct integral over inequivalent displaced Fock representations in which the Euclidean group and time-displacements are implemented. The resulting covariant theory has an interesting interpretation.

Let $\pi_F$ denote the Fock representation of $\mathfrak{A}$. This representation is determined using the GNS construction from the vacuum state $\omega_F$, which is defined uniquely by the characteristic function

$$\omega_F(W(\Psi))=\exp(-\tfrac{1}{2}\|\Psi\|^2_{\mathscr{K}}).$$

For $L\in\mathscr{P}^\uparrow_+$ let $\tau_L$ be the automorphism of $\mathfrak{A}$ induced by the action $U(L)$ of $\mathscr{P}^\uparrow_+$ on $\mathscr{M}$:

$$\tau_L(W(\Phi))=W(U(L)\Phi).$$

*Definition.* A representation $\pi$ of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\pi$ is said to be covariant if

1. For each $L\in\mathscr{P}^\uparrow_+$, $\tau_L$ is spatial in $\pi$ i.e. $\tau_L$ is implemented by a unitary operator $V_\pi(L)$;
2. $V_\pi(L)$ can be chosen to be continuous in $L$.

It is well known that the Fock representation is covariant. Denote by $\mathscr{M}^\times$ the algebraic dual of $\mathscr{M}$; to each $\Psi^\times\in\mathscr{M}^\times$ there is a displaced Fock representation $\pi_{\Psi^\times}$; it is determined by its characteristic function

$$\omega_{\Psi^\times}(W(\Phi))=\exp(i\,\mathrm{Im}\langle\Psi^\times,\Phi\rangle)\omega_F(W(\Phi)),\qquad \Phi\in\mathscr{M}.$$

Two such representations $\pi_{\Psi^\times}$ and $\pi_{\Xi^\times}$ are equivalent if and only if the map $\Phi\to\mathrm{Im}\langle\Psi^\times-\Xi^\times,\Phi\rangle$ is a continuous functional on $\mathscr{M}$ with respect to the topology induced by the norm on $\mathscr{M}$ ([6, 7]). This holds if and only if $\Psi^\times-\Xi^\times\in\mathscr{M}^*$ the topological dual of $\mathscr{M}$. If $\Psi^\times\in\mathscr{M}^*$, $\omega_{\Psi^\times}$ is a vector state in Fock space, called a coherent state. We shall denote $\pi_F(W(\Phi))$ by $W_F(\Phi)$ and $\pi_{\Psi^\times}(W(\Phi))$ by $W_{\Psi^\times}(\Phi)$. Clearly

$$W_{\Psi^\times}(\Phi)=\exp[i\,\mathrm{Im}\langle\Psi^\times,\Phi\rangle]\,W_F(\Phi).$$

The action $U^\times(L)$ of $\mathscr{P}^\uparrow_+$ on $\mathscr{M}^\times$ is defined by duality:

$$\langle U^\times(L)\Psi^\times,\Phi\rangle=\langle\Psi^\times,U(L)\Phi\rangle\quad\text{for all}\quad\Phi\in\mathscr{M},\ \Psi^\times\in\mathscr{M}^\times;$$

this action induces an action $\tau^\times_L$ on the set of displaced vacuum states: $\tau^\times_L\omega_{\Psi^\times}=\omega_{U^\times(L)\Psi^\times}$ and this coincides with the dual $\tau^\times_L$ of the automorphism group $\{\tau_L\}$ on $\mathfrak{A}$.

Let $\Psi^\times\in\mathscr{M}^\times$ and define $\psi_L=\Psi^\times-U^\times(L)\Psi^\times$. The action $\tau_L$ is implemented in $\pi_{\Psi^\times}$ if and only if the representations $\pi_{U^\times(L)\Psi^\times}$ and $\pi_{\Psi^\times}$ are unitarily equivalent. This holds if and only if the map

$$\Phi\to\mathrm{Im}\langle\Psi^\times-U^\times(L)\Psi^\times,\Phi\rangle=\mathrm{Im}\langle\psi_L,\Phi\rangle$$

is continuous, i.e. if and only if $\psi_L\in\mathscr{M}^*=\mathscr{K}$.

Suppose $\Psi^\times$ is such that $\psi_L\in\mathscr{M}^*$. Then the map $L\to\psi_L$ obeys the *cocycle condition*

$$U^*(M)\psi_L-\psi_{LM}+\psi_M=0.$$

A cocycle of the form $\psi_L=\Psi^\times-U^\times(L)\Psi^\times$ may be called a topological cocycle ([8], Theorem 7.3). Araki shows that every cocycle is a sum of a topological cocycle and an "algebraic" cocycle. The latter might arise if the representation space of $U$ contains non-zero vectors invariant under space-time translations. Such vectors do not arise for representations of mass $m \geqq 0$ and spin $s \geqq 0$.

Now suppose $\pi_{\Xi^\times}$ and $\pi_{\Phi^\times}$ are unitarily equivalent representations. Then for some $\Lambda \in \mathscr{M}^*$, $\langle \Psi^\times - \Xi^\times - \Lambda, \Phi \rangle = 0$ for all $\Phi \in \mathscr{M}$. Let $\psi_L = \Psi^\times - U^\times(L)\Psi^\times$ and $\chi_L = \Xi^\times - U^\times(L)\Xi^\times$. Since $\mathscr{M}$ is $\mathscr{P}_+^\uparrow$-invariant, we obtain $\langle \chi_L - \psi_L - (\Lambda - U^*(L)\Lambda), \Phi \rangle = 0$ for all $\Phi \in \mathscr{M}$. So as functionals on $\mathscr{M}$, the cocycles $\chi_L$ and $\psi_L$ differ by a coboundary, and they are therefore cohomologous cocycles in the group $Z^1(\mathscr{P}_+^\uparrow, \mathscr{M}^*)$, i.e. belong to the same element of $H^1(\mathscr{P}_+^\uparrow, \mathscr{M}^*)$. Let us find conditions under which the converse result would hold. Suppose $\chi_L$ and $\psi_L$ are cohomologous cocycles. Then they differ by a coboundary i.e.

$$\psi_L - \chi_L = \Lambda - U^*(L)\Lambda \quad \text{for some} \quad \Lambda \in \mathscr{M}^*.$$

Suppose $\Psi^\times$ and $\Xi^\times$ are such that $\psi_L = \Psi^\times - U^\times(L)\Psi^\times$ and $\chi_L = \Xi^\times - U^\times(L)\Xi^\times$; then the fact that they are cohomologous implies $\Psi^\times - \Xi^\times - \Lambda$ is $\mathscr{P}_+^\uparrow$-invariant for some $\Lambda \in \mathscr{M}^*$. If $\mathscr{M}$ has been restricted to lie in the kernel of $\mathscr{P}_+^\uparrow$-invariant functionals, then

$$\langle \Psi^\times - \Xi^\times - \Lambda, \Phi \rangle = 0$$

for all $\Phi \in \mathscr{M}$, and by Shale's criterion we find that $\pi_{\Xi^\times}$ and $\pi_{\Psi^\times}$ are unitarily equivalent. We summarize these remarks as:

**Theorem 1.** *If all $\mathscr{P}_+^\uparrow$-invariant functionals vanish on $\mathscr{M}$, then there is a one-to-one correspondence between the equivalence classes of displaced Fock representations in which $\tau$ is implemented and the first cohomology group $H^1(\mathscr{P}_+^\uparrow, \mathscr{M}^*)$ with coefficients in $\mathscr{M}^*$.*

Roepstorff [5] proves a special case, where $\mathscr{M} = \mathscr{K}$, the one-particle space of the free electromagnetic field; in this case there are no invariant functionals. We now establish the form of the unitary operators $V_{\Psi^\times}(L)$ which implement $\mathscr{P}_+^\uparrow$ in $\pi_{\Psi^\times}$.

**Theorem 2.** *Let $\psi_L = \Psi^\times - U^\times(L)\Psi^\times$ be a cocycle for the action $U$ of $\mathscr{P}_+^\uparrow$ on $\mathscr{M}$, and let $\pi_{\Psi^\times}$ be the corresponding displaced Fock representation of the Weyl algebra over $\mathscr{M}$. Define $V_{\Psi^\times}(L) = V_F(L) W_F(-\psi_L)$, where $V_F(L)$ implements $\tau_L$ in the Fock representation. Then*

a) *$V_{\Psi^\times}(L)$ implements $\tau_L$ in $\pi_{\Psi^\times}$ i.e. for all $\Phi \in \mathscr{M}$ and $L \in \mathscr{P}_+^\uparrow$*

$$V_{\Psi^\times}(L)\pi_{\Psi^\times}(W(\Phi))V_{\Psi^\times}(L)^{-1} = \pi_{\Psi^\times}(W(U(L)\Phi)).$$

b) *$V_{\Psi^\times}$ is a multiplier representation of $\mathscr{P}_+^\uparrow$ with multiplier*

$$\omega(L, M) = \exp\left(\frac{-i}{2} \operatorname{Im} \langle \psi_L, \psi_{M^{-1}} \rangle\right).$$

c) *$V_{\Psi^\times}$ is continuous on $\mathscr{P}_+^\uparrow$ if and only if $\psi_L$ is a continuous cocycle.*

d) *$V_{\Psi^\times}$ is infinitely divisible.*

*Proof.*

a) $V_{\Psi^\times}(L)W_{\Psi^\times}(\Phi)V_{\Psi^\times}(L)^{-1}$

$$=V_F(L)W_F(-\psi_L)\exp[i\,\mathrm{Im}\langle\Psi^\times,\Phi\rangle]W_F(\Phi)W_F^{-1}(-\psi_L)V_F(L)^{-1}$$

$$=V_F(L)W_F(\Phi-\psi_L)W_F(\psi_L)V_F(L)^{-1}\exp\left[\frac{i}{2}\mathrm{Im}\langle-\psi_L,\Phi\rangle+i\,\mathrm{Im}\langle\Psi^\times,\Phi\rangle\right]$$

$$=V_F(L)W_F(\Phi)V_F(L)^{-1}\exp\left[\frac{i}{2}\mathrm{Im}\langle-\psi_L,\Phi\rangle+i\,\mathrm{Im}\langle\Psi^\times,\Phi\rangle\right.$$
$$\left.+\frac{i}{2}\mathrm{Im}\langle\Phi-\psi_L,\psi_L\rangle\right]$$

$$=W_F(U(L)\Phi)\exp i[\mathrm{Im}\langle\Psi^\times-\psi_L,\Phi\rangle]=W_F(U(L)\Phi)\exp i[\mathrm{Im}\langle U^\times(L)\Psi^\times,\Phi\rangle]$$
$$=W_{\Psi^\times}(U(L)\Phi),$$

b) $V_{\Psi^\times}(L)V_{\Psi^\times}(M)=V_F(L)W_F(-\psi_L)V_F(M)W_F(-\psi_M)$

$$=V_F(L)V_F(M)W_F(-U^*(M)\psi_L)W_F(-\psi_M)$$

$$=V_F(LM)W_F(-U^*(M)\psi_L-\psi_M)\exp\left[\frac{i}{2}\mathrm{Im}\langle U^*(M)\psi_L,\psi_M\rangle\right]$$

$$=V_F(LM)W_F(-\psi_{LM})\exp\left[\frac{i}{2}\mathrm{Im}\langle\psi_L,U(M)\psi_M\rangle\right].$$

Now, $U(M)\psi_M=U^*(M^{-1})\psi_M=-\psi_{M^{-1}}$ since $\psi_e=0$, ($e$ is the identity of $\mathscr{P}_+^\uparrow$). Hence

$$V_{\Psi^\times}(L)V_{\Psi^\times}(M)=\exp\left[\frac{-i}{2}\mathrm{Im}\langle\psi_L,\psi_{M^{-1}}\rangle\right]V_{\Psi^\times}(L,M).$$

c) If $L\to\psi_L$ is continuous, then $L\to V_\Psi(L)=V_F(L)W_F(-\psi_L)$ is continuous, since $L\to V_F(L)$ is continuous. Conversely, if $V_{\Psi^\times}(L)$ is continuous, then $\mathrm{Re}\log\langle\Omega_F,V_{\Psi^\times}(L)\Omega_F\rangle=-\frac{1}{4}\|\psi_{L^{-1}}\|^2$ is continuous in $L$ (here, $\Omega_F$ denotes the Fock vacuum). Suppose $L,M\in\mathscr{P}_+^\uparrow$. Then, from the cocycle condition,

$$\|\Psi_L-\Psi_M\|=\|\Psi_{LM^{-1}M}-\Psi_M\|=\|U^*(M)\Psi_{LM^{-1}}\|=\|\Psi_{LM^{-1}}\|\to0$$

as $L\to M$. Thus, $\Psi_L$ is a continuous cocycle.

d) The representation $V_{\Psi^\times}$ is already in the canonical form for an infinitely divisible projective representation [9, 10].

We may interpret $\Psi^\times$ as a generalised function; we parametrise $\mathscr{M}$ by the Cauchy data of the independent field components $\phi_\alpha$ $\alpha=1,2,\ldots$ and its canonical conjugate field denoted $\pi_\alpha$. Let us define $\Phi_\alpha^\times$, $\Pi_\alpha^\times\in\mathscr{S}^*(\mathbb{R}^3)$ as generalised functions by the formula

$$\mathrm{Im}\langle\Phi^\times,\Phi\rangle=\langle\Phi_\alpha^\times(\cdot,t),\pi_\alpha(\cdot,t)\rangle_{L^2}-\langle\Pi_\alpha^\times(\cdot,t),\phi_\alpha(\cdot,t)\rangle_{L^2}.$$

We now define $\Phi_\alpha^\times(\mathbf{x},t)$ to be the solution to the wave equation with these Cauchy data; they formally obey the same Hamiltonian equations as $\phi_\alpha$ itself (and

$\mathrm{Im}\langle\Phi^\times,\Phi\rangle$ is the Poisson bracket), and transform covariantly under the dual action $U^\times$ of $\mathscr{P}_+^\uparrow$ on $\mathscr{M}^\times$. Thus, each cocycle may be identified with a solution $\Phi^\times$ of the field equations and the (dual) subsidiary conditions. In Sect. 2, it is shown that the cohomology class of such a solution is determined by its infrared behaviour, i.e. by the behaviour of $\Phi^\times(\mathbf{x},t)$ at spatial infinity.

Roepstorff [5] has remarked that not all representations of the form $\pi_{\Phi^\times}$ (for the electromagnetic field) have positive energy. We can remark, as in [3, 5], that if the wave $\Phi^\times(\mathbf{x},t)$ has finite classical energy, then the energy in $\pi_{\Phi^\times}$ is bounded below. To see this, note that, in the representation $\pi_{\Phi^\times}$, the field may be taken to be $\hat{\phi}_\alpha^F+\Phi_\alpha^\times$ where $\hat{\phi}_\alpha^F$ is the relativistic Fock field. The energy operator is then the formally positive quadratic Hamiltonian, acting on Fock space:

$$H(\hat{\phi}^F+\Phi^\times,\hat{\pi}^F+\Pi^\times)=H^F+\int\hat{\phi}_\alpha^F\left.\frac{\partial H}{\partial\hat{\phi}_\alpha^F}\right|_0 d^3\mathbf{x}+\int\hat{\pi}_\alpha^F\left.\frac{\partial H}{\partial\hat{\pi}_\alpha^F}\right|_0 d^3\mathbf{x}+H(\Phi^\times,\Pi^\times)\,.$$

$H^F$ is self-adjoint; suppose $H(\Phi^\times,\Pi^\times)$, the classical energy of the wave $\Phi^\times$ is finite; then the total operator is self-adjoint on $D(H^F)$ provided the cross-term is self-adjoint and Kato – small relative to $H^F$. We now indicate the proof of this.

By Hamilton's equations:

$$\int\left(\hat{\phi}_\alpha^F\frac{\partial H}{\partial\hat{\phi}_\alpha^F}+\hat{\pi}_\alpha^F\frac{\partial H}{\partial\hat{\pi}_\alpha^F}\right)d^3\mathbf{x}=\int\left(\hat{\phi}_\alpha^F\left(-\frac{d}{dt}\Pi_\alpha^\times\right)+\hat{\pi}_\alpha^F\left(\frac{d}{dt}\Phi_\alpha^\times\right)\right)d^3\mathbf{x}$$

and this is self-adjoint, and Kato small relative to $H^F$, if $\frac{d}{dt}(\Phi_\alpha^\times(\mathbf{x},t))\in\mathscr{K}$, which is abstractly written $X_1\Phi^\times\in\mathscr{K}$, where $X_1$ generates time-translations in $\mathscr{K}$. We shall see in the next section that this, the infinitesimal form of the cocycle condition $(U(t)-1)\Phi^\times\in\mathscr{K}$, can always be arranged if $\Phi^\times$ is a cocycle. Conversely, as in [5], one shows that if the energy in $\pi_{\Phi^\times}$ is bounded below, then $\Phi^\times$ is cohomologous to a cocycle of finite energy, that is, a soliton-like bose condensate.

## 2. On the Cohomology of the Poincaré Group

Araki [8] has analysed the cohomology group $H^1(G,\mathscr{K})$, where $G$ is a Lie group and $\mathscr{K}$ is the carrier space of a unitary representation $U$ of $G$. A 1-coboundary is a map $\psi:G\to\mathscr{K}$ of the form

$$\psi_g=\Phi-U^{-1}(g)\Phi\quad\text{for some}\quad\Phi\in\mathscr{K}^*\,.$$

The topological cocycles are maps $\psi:G\to\mathscr{K}$ of the form $\psi_g=\Phi^\times-U^\times(g)\Phi^\times$ where $\Phi^\times$ lies in the space $\bar{D}^+$ constructed as follows.

Let $\{X_1,\dots,X_k\}$ be a basis in the Lie Algebra of $G$ and define

$$K_j=1-\int U(\exp tX_j)h(t)dt\qquad j=1,\dots,k\,,$$

where $h\in\mathscr{D}(\mathbb{R})$ is such that $0\leqq\tilde{h}(\lambda)<1$ for $\lambda\neq0$ and $\tilde{h}(0)=1$ and $\tilde{h}''(0)\neq0$.

Define $K=\sum_{1}^{k} K_j$. Each $K_j$ and hence $K$ is $\geqq 0$; $K^{-1/2}$ is possibly unbounded; $D^+$ is the image of $\mathscr{K}$ under $K^{-1/2}$. We define the norm $\|\phi\|_+ = \|K^{1/2}\phi\|_{\mathscr{K}}$ and $\bar{D}^+$ is the completion of $D^+$ in $\|\ \|_+$. Clearly $\bar{D}^+ \subseteqq \mathscr{K}^\times$ and $U^\times$ acts on $\bar{D}^+$.

**Theorem 3.** *Any representation of $\mathscr{P}_+^\uparrow$ with positive mass, in space-time of dimension $\geqq 1$, has trivial cohomology.*

Note: This result has also been obtained by Pinczon [11].

*Proof.* Let $X_1=(\mathbf{p}^2+m^2)^{1/2} \geqq m$ be the energy of the representation. Hence $K \geqq \delta > 0$ so $K^{-1/2}$ exists and is bounded (above and below). Hence

$$\|\phi\|_+ = \|K^{1/2}\phi\|_{\mathscr{K}} \leqq C\|\phi\|_{\mathscr{K}} = C\|K^{-1/2}K^{1/2}\phi\| \leqq CC'\|K^{1/2}\phi\|_{\mathscr{K}} = C''\|\phi\|_+$$

i.e. all norms are equivalent. Hence $\bar{D}^+ = \mathscr{K}^*$ and any topological cocycle is a coboundary. Since 0 is the only vector invariant under translations, the only algebraic cocycles are trivial. □

*Scholium* [12]. a) Let $U$ be a continuous representation of a connected Lie group $G$ on a Banach space $\mathscr{K}$. Then $H^1(G,\mathscr{K}) = H^1_\omega(G,\mathscr{K})$ where $H^1_\omega$ consists of 1-cocycles which are analytic in the group parameters about the identity.

By the cocycle law, $U^\times(g)\psi_h = \psi_{gh} - \psi_g$; thus $U^\times(g)\psi_h$ is analytic in $g$ for fixed $h$, i.e. the cocycles in $H^1_\omega$ are valued in the set $\mathscr{K}_\omega$ of analytic vectors for the group representation.

b) If $\mathbf{G}$ is the Lie algebra of $G$, the map $\Delta$, defined by

$$\frac{d}{dt}\psi(\exp tX)|_{t=0} = (\Delta\psi)(X)$$

maps $H^1_\omega(G,\mathscr{K})$ into $H^1(\mathbf{G},\mathscr{K}_\omega)$. If $G$ is simply connected, then $H^1_\omega(G,\mathscr{K}) = H^1(\mathbf{G},\mathscr{K}_\omega)$.

The Scholium implies that each cocycle has a representative in the set of $C^\infty$-vectors for the Lie algebra $\mathbf{G}$ of $G$; the cocycles of $\mathbf{G}$ are elements $\Phi^\times \in \mathscr{K}_\omega^\times$ such that $K_j^\times \Phi^\times \in \mathscr{K}^* = \mathscr{K}$ since in our case, $\mathscr{K}$ is a Hilbert space.

**Theorem 4.** *In space-time of dimension $\geqq 3$, $H^1(\mathscr{P}_+^\uparrow, \mathscr{K})=0$ for any representation of $\mathscr{P}_+^\uparrow$ of mass $m \geqq 0$ and* spin $s \geqq 0$.

*Proof.* A cocycle for $\mathscr{P}_+^\uparrow$ is also a cocycle for the subgroups SO(2) and SO(3), ... respectively. These are compact groups, and so their cohomology is trivial [8]. This means that any cocycle for $\mathscr{P}_+^\uparrow$ must be cohomologous to one derived from a $\Phi^\times \in \bar{D}^+$ that is rotation invariant. This excludes representations of spin $>0$ (see Appendix). For the spin 0, mass 0 case, the one-particle space is

$$\mathscr{K} = L^2(\mathbb{R}^s, p^{-1}d^sp), \qquad s = \text{space-dimension}.$$

We may take $\Phi^\times$ to be a generalized function, invariant under rotations:

$$\Phi^\times(\mathbf{p}) = u(|\mathbf{p}|).$$

We describe Lorentz transformations on $\mathscr{K}$ by $\Phi^\times(p_1,\ldots,p_s)\to\Phi^\times(\mathrm{ch}\,\eta\cdot p_1+\mathrm{sh}\,\eta|\mathbf{p}|, p_2,\ldots,p_s)$ where $\eta$ is the rapidity in the (0, 1)-plane. The infinitesimal generator $J_{01}$ is thus

$$\left.\frac{\partial}{\partial\eta}\right|_{\eta=0}=|\mathbf{p}|\,\frac{\partial}{\partial p_1}=p_1\frac{\partial}{\partial(|\mathbf{p}|)}$$

on rotation-invariant vectors. A non-trivial cocycle is therefore a function $\Phi^\times\cdot(\mathbf{p})=u(|\mathbf{p}|)$ such that $\int\limits_0^\infty |u|^2p^{s-2}dp=\infty$ but $\int\limits_0^\infty |pu|^2p^{s-2}dp<\infty$ and $\int\limits_0^\infty (p\cos\theta)^2\left|\frac{du}{dp}\right|^2 p^{s-2}dp\,d\sigma(\theta)<\infty$ where $d\sigma(\theta)$ is the surface element, and the angular coordinates are chosen so that $p_1=p\cos\theta$. The first condition expresses $\Phi^\times\notin\mathscr{K}$ and the second and third say that

$$P^0\Phi^\times\in\mathscr{K}\qquad J_{01}\Phi^\times\in\mathscr{K}$$

whence $P^j\Phi^\times\in\mathscr{K}$.

By the Scholium, $\Phi^\times$ may be chosen so that it is in the domain of all powers of $J_{01}$ so that

$$\left(p\frac{\partial}{\partial p}\right)^n\Phi^\times\in\mathscr{K},\qquad n=1,2,\ldots.$$

Thus, we may choose $u\in c^\infty(0,\infty)$ and the condition $J_{01}\Phi^\times\in\mathscr{K}$ is equivalent to $\int\limits_0^\infty (u')^2p^s dp<\infty$. The inequality [13]

$$\left(\frac{s-1}{2}\right)^2\int\limits_0^\infty |u|^2p^{s-2}dp\leqq\int\limits_0^\infty\left|\frac{du}{dp}\right|^2 p^s dp$$

now implies that $\Phi^\times\in\mathscr{K}$. This proves the theorem.

Redheffer's inequality fails if $s=1$. The function (where $-1<\alpha<1$) $\Phi^\times(p)=[\log(1/p)]^{-1/2\alpha}$ provides a counterexample and also a cocycle for $\mathscr{P}_+^\uparrow$. Other non-trivial (and inequivalent) cocycles are provided by functions of the form $\Phi^\times(p)+0(p)$, $p\to0$

$$\Phi^\times(p)=\log\log(1/p)\equiv\log_2(1/p),\ldots,\log_n(1/p)\ \equiv\log(\log_{n-1}(1/p))\qquad n=2,3,\ldots.$$

These cocycles provide new representations of the CCR with positive energy, not corresponding to any previously known model.

Further cocycles in 2-dimensional space-time can be parametrized by two real numbers $q$, $q_5$, where $\Phi^\times(p)\to q+iq_5$ as $|\mathbf{p}|\to0+\cdot$. These cocycles were discovered in [3] to give covariant strongly locally Fock representations of the $C^*$-algebra generated by the field gradient $\partial_\mu\phi$. It can be shown that the Thirring model takes place in the direct sum of representations labelled by $q$, $q_5$ [14]. It can be shown that a cocycle $\Phi^\times$ gives rise to a "local" cocycle in the sense of [15] if and only if $\Phi\to q+iq_5$ as $|\mathbf{p}|\to0$, for some real $q, q_5$. This may provide grounds for rejecting the strange examples above.

## 3. Kinetic Charge

The absence of cocycles for $\mathscr{P}_+^\uparrow$ in space-time dimensions of 3 and 4 suggests that a more general construction should be tried. In fact it is easy to construct cocycles for smaller groups than $\mathscr{P}_+^\uparrow$ such as $\mathbb{R}\times\mathbb{E}_3$, $\mathbb{R}$ being time-evolution. In this case we can proceed to construct a covariant representation of the field, albeit a reducible one. We now indicate the general procedure.

Let $\mathfrak{A}$ be a $C^*$-algebra and $\pi_0$ an irreducible representation of $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}_0$. Let $G$ be a Lie group of automorphisms of $\mathfrak{A}$ and let $G_0$ be the subgroup of automorphisms implemented in $\pi_0$. We assume $G_0$ to be closed. Then $G/G_0=V$ is a $G$-space, with a natural quasi-invariant measure, $\mu$ say. For simplicity, we assume $\mu$ is invariant. Let $U_0$ be the multiplier representation on $\mathscr{H}_0$ implementing $G_0$ in $\pi_0$. Let $\omega$ be the multiplier of $U_0$. For the present, $\omega$ does not need to be the restriction to $G_0$ of a multiplier for $G$.

Let $v_0\in\mathbb{V}=G/G_0$ be chosen, and for each $v\in\mathbb{V}$ choose a boost $b(v)\in G$ such that $b(v)v=v_0$. Let $\pi_v$ be the representation (on $\mathscr{H}_v\cong\mathscr{H}_0$) given by $\pi_v(A)=\pi_0(b(v)A)$, $A\in\mathfrak{A}$. Let

$$\pi=\int_{\oplus\mathbb{V}} d\mu(v)\pi_v$$

acting on

$$\mathscr{H}=\int_{\oplus\mathbb{V}} d\mu(v)\mathscr{H}_v\,.$$

It is known [16] that if $\omega$ is the restriction of a multiplier for $G$, then the representation of $G$ induced by the representation $U_0$ of $G_0$, is equivalent[1] to $U(g)$ on $\mathscr{H}$ defined by

$$(U(g)\psi)(v)=U_0(b(v)gb^{-1}(g^{-1}v))\psi(g^{-1}v)\,.$$

More generally, for any multiplier $\omega$ of $G_0$, we have:

**Theorem 5.** *$U(g)U(h)=\omega_1(g,h)U(gh)$ for all $g,h\in G$ where $\omega_1\in\pi(\mathfrak{A})'$ is unitary.*

*Proof.* Let $\psi\in\mathscr{H}$. Then

$$\begin{aligned}(U(g)U(h)\psi)(v)&=U_0(b(v)gb^{-1}(g^{-1}v))[U(h)\psi](g^{-1}v)\\&=U_0(b(v)gb^{-1}(g^{-1}v))\,U_0(b(g^{-1}v)hb^{-1}(h^{-1}g^{-1}v))\psi(h^{-1}g^{-1}v)\\&=\lambda(g,h,v)\,U_0(b(v)gb^{-1}(g^{-1}v)\cdot b(g^{-1}v)hb^{-1}((gh)^{-1}v)\psi((gh)^{-1}v)\,,\end{aligned}$$

where $\lambda(g,h,v)=\omega(b(v)gb^{-1}(g^{-1}v),\ b(g^{-1}v)hb^{-1}(h^{-1}g^{-1}v))$. Thus, $(U(g)U(h)\psi)(v)=[\lambda(g,h,v)U(gh)\psi](v)$, where $|\lambda|=1$. Clearly, $\lambda$ defines a unitary in $\pi(\mathfrak{A})'$.

**Lemma.** *Let $V$ denote the operator on $\mathscr{H}$ defined by*

$$(V\psi)(g^{-1}v)=\lambda^{-1}(g,g^{-1},v)[U(g^{-1})\psi](g^{-1}v)\,.$$

*The $V=U^{-1}(g)$.*

*Proof.* A straightforward calculation.

---

1 Up to a gauge transformation in $\pi(A)^1$

**Theorem 6.** *For each $g$, $U(g)$ implements $g:\mathfrak{A}\to\mathfrak{A}$ in $\pi$.*

*Proof.* Let $\psi\in\mathscr{H}$. Then, for any $A\in\mathfrak{A}$,

$$\begin{aligned}[U(g)\pi(A)U^{-1}(g)\psi](v) &= U_0(b(v)gb^{-1}(g^{-1}v))[\pi(A)U^{-1}(g)\psi](g^{-1}v)\\ &= U_0(b(v)gb^{-1}(g^{-1}v))\pi_0(b(g^{-1}v)A)[U^{-1}(g)\psi](g^{-1}v)\\ &= U_0(b(v)gb^{-1}(g^{-1}v))\pi_0(b(g^{-1}v)A)\lambda^{-1}(g,g^{-1},v)(U(g^{-1})\psi)(g^{-1}v)\\ &= \lambda^{-1}(g,g^{-1},v)\pi_0(b(v)gA)U_0(b(v)gb^{-1}(g^{-1}v))U_0(b(g^{-1}v)g^{-1}b^{-1}(v))\psi(v)\\ &= \pi_0(b(v)gA)\psi(v) = \pi(gA)\psi(v)\,. \quad \square\end{aligned}$$

*Remark.* The special case where $\omega=1$ and $\mathscr{H}_0$ contains a $G_0$-invariant state, is similar to Theorem III, 2.1 ii) of [17].

In this way we can construct a $\mathscr{P}_+^\uparrow$-covariant representation even in the absence of cocycles for $\mathscr{P}_+^\uparrow$.

If $\omega$ is not a multiplier for $\mathscr{P}_+^\uparrow$, we obtain a representation with multiplier in $\pi(\mathfrak{A})'$. This would appear to be satisfactory from a physical point of view, since it means that manifest covariance is achieved only if a Lorentz transformation is accompanied by a gauge transformation of the second kind, i.e. a unitary element in $\pi(\mathfrak{A})'$. This possibility is omitted from the usual analysis of Lorentz invariance in quantum mechanics [18].

## 4. A Model in Four Dimensions

Let $\mathscr{H}_0$ be the Fock space of the free field of mass zero (obeying $\Box\phi=0$), and $\hat{\phi}_F$ the Fock representation of the field. Let $\Phi^\times$ be the real solution of $\Box\Phi^\times(\mathbf{x},t)=0$ such that $\dot{\Phi}^\times(\mathbf{x},0)=0$ and $\Phi^\times(\mathbf{x},0)=\varrho(|\mathbf{x}|)$. We choose $\varrho(|\mathbf{x}|)$ to be a $C^\infty$-function, such that $\varrho(r)=q/r$, $r\geqq r_0$. Let $\pi_0$ be the displaced Fock representation: $\hat{\phi}_0(\mathbf{x},t)=\hat{\phi}_F(\mathbf{x},t)+\Phi^\times(\mathbf{x},t)$; we note that, from the wave-equation, $\dot{\Phi}^\times(\mathbf{x},0)=0$ and $\ddot{\Phi}^\times(\mathbf{x},0)=\nabla^2\Phi^\times(\mathbf{x},0)=0$ if $|\mathbf{x}|>r_0$. Thus, the wave is stationary outside $|\mathbf{x}|=r_0$ at $t=0$. The displaced Fock representation thus describes a localized state.

The function $\tilde{\varrho}(\mathbf{k})$, near $\mathbf{k}=0$, behaves as $q/k^2$. We can easily check that $\hat{\phi}_0$ is not the Fock representation. Indeed,

$$\begin{aligned}\langle\Psi,\Psi'\rangle_{\mathscr{H}} = \int\Psi(\mathbf{x},0)(-\nabla^2)^{1/2}\Psi'(\mathbf{x},0)d^3\mathbf{x} + \int\dot{\Psi}(\mathbf{x},0)(-\nabla^2)^{-1/2}\dot{\Psi}'(\mathbf{x},0)d^3\mathbf{x}\\ + i\int(\Psi\dot{\Psi}'-\dot{\Psi}\Psi')d^3\mathbf{x}\,.\end{aligned}$$

In our case

$$\Psi=\Psi'=\Phi^\times,\ \dot{\Phi}^\times=\dot{\Psi}=\dot{\Psi}'=0\,.$$

Hence

$$\langle\Phi^\times,\Phi^\times\rangle_{\mathscr{H}} = \int\Phi^\times(\mathbf{x})(-\nabla^2)^{1/2}\Phi^\times(\mathbf{x})d^3\mathbf{x} = \int\bar{\tilde{\varrho}}(\mathbf{k})|\mathbf{k}|\,\tilde{\varrho}(\mathbf{k})d^3\mathbf{k}$$

behaving like

$$q^2\int_0(1/k^2)\cdot k\cdot(1/k^2)k^2\,dk = q^2\int_0 dk/k = \infty\,.$$

Hence $\Phi^\times$ is not in $\mathscr{K}$, and $\hat{\phi}_0$ is not equivalent to $\hat{\phi}_F$. Clearly one (or two) more powers of $(-\nabla^2)^{1/2}$ would ensure convergence; therefore $\langle P^0\Phi^\times, P^0\Phi^\times\rangle < \infty$ and $\langle \Phi^\times, P^0\Phi^\times\rangle < \infty$. Thus, $\Phi^\times$ is a wave of finite energy. Similarly, one or two powers of $k_1$, $k_2$, or $k_3$ ensure convergence. Thus $\mathbf{P}\Phi^\times \in \mathscr{K}$. Hence $\Phi^\times$ is a cocycle of finite energy for $\mathbb{R}^4$. Since $\Phi^\times$ is rotation invariant, it defines a cocycle $\psi_g = \Phi^\times - U_g\Phi^\times$ for $\mathbb{R}\times\mathbb{E}^3$. One checks that $\Phi^\times - U(\lambda)\Phi^\times \notin \mathscr{K}$ for any pure Lorentz transformation, so we are in the situation of Sect. 5, with $G = \mathscr{P}^\uparrow_+$, $G_0 = \mathbb{R}\times\mathbb{E}^3$.

Let $\pi_0$ be the representation of the CCR defined by $\Phi^\times$ (denoted $\pi_{\Phi^\times}$ in Sect. 1). Scale automorphisms generated by $(\mathbf{x}, t) \to (\lambda\mathbf{x}, \lambda t)$, are not implemented in $\pi_0$, and so scale invariance is spontaneously broken in $\pi_0$, leading to the possibility that non-zero mass may appear. The scale of mass is fixed by the parameter $m = \langle \Phi^\times, P^0\Phi^\times\rangle$, a reasonable notation since $\langle \Phi^\times, \mathbf{P}\Phi^\times\rangle = 0$.

We can now proceed with the construction in Sect. 3. Here $G/G_0 = \mathbb{V}$ is isomorphic to the mass-shell parametrized by $\mathbf{p} = m\mathbf{v}/(1-v^2)^{1/2}$, with $(p^0)^2 - \mathbf{p}^2 = m^2$. Choose $v_0 = (1,0,0,0)$ and boosts $b$, pure Lorentz transformations. In this model, $\omega \equiv 1$.

The Hilbert space of the covariant representation is $\mathscr{H} = \int_{\mathbb{V}}^{\oplus} \mathscr{H}_v d\mu(v)$. The invariant measure is $(1-v^2)^{-2}d^3v = m^{-2}(\mathbf{p}^2+m^2)^{-1/2}d^3\mathbf{p}$. There is a natural isomorphism between $\mathscr{H}$ and $L^2(\mathbb{R}^3, d^3\mathbf{p}(\mathbf{p}^2+m^2)^{-1/2}) \otimes \mathscr{H}_F = \mathfrak{H}$ where $\mathscr{H}_F$ is the Fock space on which $\pi_0$ acts, and to which each $\mathscr{H}_v$ is isomorphic. The isomorphism maps $\psi(v)\otimes\Psi \in \mathfrak{H}$ to $\psi(v)\Psi \in \mathscr{H}$. We recognise $\mathfrak{H}$ as the natural space for describing a particle of mass $m$ in interaction with a massless Boson field.

We might hope that this theory gives a good description of asymptotically free particles of mass $m$ in interaction with massless Bosons.

Many other cocycles for $\mathbb{R}\times\mathbb{E}^3$ exist; the one we have chosen, behaving as $q/r$, $r\to\infty$, has the virtue that the field is stationary even if not zero, outside a compact set.

*Acknowledgements.* R.F.S. would like to thank the University of Bielefeld for assistance to attend the Z.i.f. symposium 1975, and Professors Kuiper and Ruelle of the IHES where part of this work was done. P.B.-H. thanks the Science Research Council of the U.K. for a reasearch studentship, and J.W. thanks the British Council for a Commonwealth Scholarship, which supported this research.

## Appendix

**Lemma.** *Let $G$ be a group with a compact subgroup $K$ and let its 1-cocycles be of the form $\Omega - U(g)\Omega$; then the function $\Omega$ may be assumed to invariant under the action of $K$.*

*Proof.* Suppose $\psi$ is a 1-cocycle of $G$. Then $\psi'$ defined by

$$\psi'(g) = \psi(g) + [U(g) - 1]\varphi,$$

where $\varphi = \int_K \psi(k)dk$, is a cohomologous 1-cocycle which vanishes on $K$. Since $\psi'$ is also of the form $\Omega - U(g)\Omega$ we may assume that $\Omega$ is invariant under the action of $K$.

**Proposition.** *For mass $m=0$ and spin $s \neq 0$* $H^1(\mathscr{P}_+^\uparrow, L^2(\mathbb{R}^3, d^3\mathbf{p}/|\mathbf{p}|))=0$.

*Proof.* Lomont and Moses [19] give the following realisation of the Lie algebra of the compact subgroup SO(3):

$$J_1 = s - i(\mathbf{p} \times \nabla)_1\,,$$

$$J_2 = \frac{p_2 s}{p+p_1} - i(\mathbf{p} \times \nabla)_2\,,$$

$$J_3 = \frac{p_3 s}{p+p_1} - i(\mathbf{p} \times \nabla)_3\,,$$

where $p=|\mathbf{p}|$.

The lemma allows us to assume that the cocycle function $\Omega$ is invariant under the action of SO(3). Therefore $J_i\Omega=0$ $i=1,2,3$. Then it follows that

$$\left(p_2\left[J_2 - \frac{p_2}{p+p_1}J_1\right] - p_3\left[J_3 - \frac{p_3}{p+p_1}J_1\right]\right)\Omega=0$$

from which we can deduce $p(p+p_1)\left[p_2\frac{\partial}{\partial p_3} - p_3\frac{\partial}{\partial p_2}\right]\Omega=0$. This implies that $s\Omega=0$ so that $\Omega=0$ as $s \neq 0$.

## References


1. Segal, I.E.: Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I. Mat. Fys. Medd. Danske Vid. Selsk **12**, 33 (1959)
   Segal treats only the case where $\mathscr{M}=\mathscr{K}$ but it is easy to extend his work to the case of $\mathscr{M} \subsetneqq \mathscr{K}$. For a modern alternative, see Slawny, J.: On factor representations and the $C^*$-algebra of canonical commutation relations. Commun. Math. Phys. **24**, 151–170 (1972)
   Segal, I.E.: Mathematical characterization of the physical vacuum for a linear Bose-Einstein field (foundations of the dynamics of infinite systems III). Ill. J. Math. **6**, 500–523 (1962)
2. Streater, R.F.: Spontaneous breakdown of symmetry in axiomatic theory. Proc. R. Soc. **287**, 510–518 (1965)
3. Streater, R.F., Wilde, I.F.: Fermion states of a Boson field. Nucl. Phys. B **24**, 561–575 (1970)
4. Fröhlich, J.: New superselection sectors in 2-dimensional Bose quantum field models. Commun. Math. Phys. **47**, 269–310 (1976)
5. Roepstorff, G.: Coherent photon states and spectral condition. Commun. Math. Phys. **19**, 301–314 (1970)
6. Shale, D.: Linear symmetries of free Boson fields. Trans. Am. Math. Soc. **103**, 149 (1962)
7. Manvuceau, J.: Ann. Inst. Henri Poincaré **8**, 139 (1968)
8. Araki, H.: Factorizable representations of current algebras. Publ. RIMS (Kyoto) **5**, 361–422 (1970)
9. Streater, R.F.: Current commutation relations, continuous tensor products and infinitely divisible group representations. In: Local quantum theory. Jost, R. (ed.). New York: Academic Press 1969
10. Parthasarathy, K.R., Schmidt, K.: Positive definite kernels, continuous tensor products, and central limit theorems of probability theory. In: Lecture notes in mathematics, Vol. 272. Berlin, Heidelberg, New York: Springer 1972
11. Pinczon, G.: Lett. Math. Phys. (to appear)
12. Pinczon, G., Simon, J.: The one-cohomology of Lie groups. Lett. Math. Phys. **1**, 83–91 (1975)
13. Redheffer, R.: Integral inequalities with boundary terms. In: Inequalities. II. Shisha, O. (ed.). New York: Academic Press 1970

14. Streater, R.F.: Currents and charges in the Thirring model. In: Physical reality and mathematical description. Enz, C.P., Mehra, J. (eds.). Dordrecht, Boston: Reidel 1974. See also: Gauge theories of superselection rules. In: Schladming Lectures, 1973. Urban, P. (ed.). Berlin, Heidelberg, New York: Springer 1974
15. Roberts, J.E.: Local cohomology and Superselection structure. Commun. Math. Phys. **51**, 107–120 (1976)
16. Mackey, G.W.: Theory of unitary group representations, p. 206. Chicago, London: University of Chicago Press 1976
17. Araki, H., Kastler, D., Takesaki, M., Haag, R.: Extension of KMS states and chemical potential. Commun. math. Phys. **53**, 97–134 (1977)
18. Wigner, E.P.: Unitary representations of the inhomogeneous Lorentz group. Ann. Math. **40**, 149 (1939)
19. Lomont, J.S., Moses, H.E.: Reduction of reducible representations of the infinitesimal generators of the proper, orthochronous, inhomogeneous Lorentz group. J. Math. Phys. **8**, 837 (1967)

Communicated by R. Haag

Received February 13, 1978; in revised form November 10, 1978