Commun. Math. Phys. 67, 241–250 (1979)

Communications in
Mathematical
Physics

Exact S-Matrix of the Adjoint SU(N) Representation

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Abstract. We have calculated the exact factorised S-matrices of the adjoint SU(N) representation in $1+1$ space-time dimensions. Besides the trivial solution the only realised solution exhibits an $O(N^2-1)$ symmetry.

1. Introduction

Recently a lot of work has been done [1–7] in calculating exact factorising S-matrices in two dimensions and investigating their relationship to quantum field theoretical models. In the present paper we calculate the factorising S-matrix for particles which transform under the adjoint representation of SU(N).

Our interest in the S-matrix of the adjoint SU(N) representation was stimulated by recent investigations [8–10] on CP^{N-1} models which were introduced by Eichenherr [8]. These models are in their construction similar to the nonlinear σ-model in two dimensions. In the nonlinear σ-model the interaction is introduced by restricting the (classical) field to an orbit of $O(N)$; in analogy the interaction of the CP^{N-1} models is introduced by the geometrical constraint of restricting the classical field to an idempotency orbit of the adjoint representation of SU(N) [8]. Much of the interest in the nonlinear σ-model in two dimensions is motivated by the analogies found with respect to the Yang-Mills theory in four dimensions. For the CP^{N-1} models this analogy goes even further. In particular the CP^{N-1} models possess instanton solutions for all N and the instanton effects can be investigated within the $1/N$ expansion [10]. The theory can be rewritten as an abelian gauge theory [9, 10] and the fundamental fields are then confined by a topological Coulomb force.

In complete analogy to the $O(N)$ σ-model the CP^{N-1} models exhibit higher order local and non-local conservation laws. If the conservation laws survive quantization and if the spectrum of outcoming particles has at the lowest level only the adjoint SU(N) representation, then by arguments analogous to those first worked out for the massive Thirring model [11] the S-matrix calculated in the present paper describes the scattering of the mesons of the CP^{N-1} models. Of course more precise information concerning the spectrum – e.g. within the

0010-3616/79/0067/0241/$02.00

semiclassical approximation – is necessary for a complete specification and this is currently under investigation.

We have obtained the (to us surprising) result that factorisation implies for the S-matrix of the SU(N) adjoint representation an $O(N^2-1)$ symmetry which goes far beyong the assumed initial symmetry. The S-matrix in question is therefore given by the result of Zamolodchikov and Zamolodchikov [2]. Our calculations assume $N \geqq 6$. Nevertheless we conjecture (from experience with prior calculations [4]) the result to remain valid for some smaller N. Especially we like to mention that for $N=2$ (but not for higher N) the classical CP^{N-1} model becomes equivalent with the $O(3)$ nonlinear σ-model and a confinement discussion can also be carried out on the S-matrix level [12].

For clarity our result is stated as a theorem in Sect. 2 where also the notation is introduced. Section 3 is concerned with the proof. Some technical details concerning unitarity and the factorisation equations are relegated to the Appendices A and B.

2. Notation and Result

For a reason outlined in the introduction we are interested in elastic scattering of the adjoint representation of SU(N). We introduce the matrix elements

$$\begin{aligned} {}^{\text{out}}\langle k(P'_1)\, l(P'_2) | i(P_1)\, j(P_2)\rangle^{\text{in}} &= {}_{ij}S_{kl}(\theta)\,\delta(\underline{p}'_1-\underline{p}_1)\,\delta(\underline{p}'_2-\underline{p}_2) \\ &\quad + {}_{ji}S_{kl}(\theta)\,\delta(\underline{p}'_1-\underline{p}_2)\,\delta(\underline{p}'_2-\underline{p}_1)\,, \end{aligned} \tag{1}$$

where

$$\operatorname{ch}\theta = \frac{P_1 P_2}{m^2}\,.$$

For convenience we define:

$$\begin{aligned} {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2} &= \tfrac{1}{16}\lambda^i_{\alpha_1\alpha_2}\lambda^j_{\beta_1\beta_2}\lambda^{k*}_{\gamma_1\gamma_2}\lambda^{l*}_{\delta_1\delta_2}\;{}_{ij}S_{kl} \\ &= \tfrac{1}{16}\lambda^i_{\alpha_1\alpha_2}\lambda^j_{\beta_1\beta_2}\lambda^k_{\gamma_2\gamma_1}\lambda^l_{\delta_2\delta_1}\;{}_{ij}S_{kl}\,. \end{aligned} \tag{2}$$

The λ's are the Hermitean traceless Gell-Mann λ-matrices and * denotes complex conjugation. ${}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2}$ fulfils the following properties:

a) Tracelessness:

$$\sum_{\alpha=1}^{N} {}_{\alpha\alpha\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2} = 0 \quad \text{etc.} \tag{3a}$$

b) Symmetry:

$${}_{ij}S_{kl} = {}_{ji}S_{lk} \leftrightarrow {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2} = {}_{\beta_1\beta_2\alpha_1\alpha_2}S_{\delta_1\delta_2\gamma_1\gamma_2}\,. \tag{3b}$$

c) Crossing:

$${}_{ij}S_{kl}(\theta) = {}_{il}S_{kj}(i\pi-\theta) \leftrightarrow {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2}(\theta) = {}_{\alpha_1\alpha_2\delta_2\delta_1}S_{\gamma_1\gamma_2\beta_2\beta_1}(i\pi-\theta)\,. \tag{3c}$$

d) PT invariance: (follows also from c)

$${}_{ij}S_{kl} = {}_{kl}S_{ij} \leftrightarrow {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2} = {}_{\gamma_2\gamma_1\delta_2\delta_1}S_{\alpha_2\alpha_1\beta_2\beta_1}\,. \tag{3d}$$

e) Hermitean analyticity:

$$ {}_{ij}S_{kl}(\theta)^* = {}_{ij}S_{kl}(-\theta^*) \curvearrowleft {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2}(\theta)^* = {}_{\alpha_2\alpha_1\beta_2\beta_1}S_{\gamma_2\gamma_1\delta_2\delta_1}(-\theta^*)\,. \tag{3e}$$

Here we have used the completeness relation of the λ-matrices.

$$\lambda^i_{\alpha\beta}\lambda^i_{\gamma\delta} = 2\left(\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right),$$

$$\mathrm{Tr}\,\lambda^i\lambda^j = 2\delta^{ij}\,.$$

f) Elasticity unitarity: Assuming absence of other particles degenerate with the adjoint representation under consideration

$$ {}_{ij}S_{kl}(\theta)\,{}_{kl}S_{mn}(-\theta) = \delta_{im}\delta_{jn}$$

$$\curvearrowleft {}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2}(\theta)\,{}_{\gamma_1\gamma_2\delta_1\delta_2}S_{\varepsilon_1\varepsilon_2\kappa_1\kappa_2}(-\theta)$$

$$= \frac{1}{16}\left(\delta_{\alpha_1\varepsilon_1}\delta_{\alpha_2\varepsilon_2} - \frac{1}{N}\delta_{\alpha_1\alpha_2}\delta_{\varepsilon_1\varepsilon_2}\right)\left(\delta_{\beta_1\kappa_1}\delta_{\beta_2\kappa_2} - \frac{1}{N}\delta_{\beta_1\beta_2}\delta_{\kappa_1\kappa_2}\right). \tag{3f}$$

There are (for $N \geqq 4$) 24 independent products of four δ-functions and the associated amplitudes are related by (3a)–(3f). They are assumed to be meromorphic functions of θ and (3e) ensures the usual Hermitean analyticity. We define the amplitudes by the following formulae; their graphical representation which is often convenient to use is given in Fig. 1

$$\begin{aligned} 4\,{}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2} = &+A\ \delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\delta_{\gamma_1\gamma_2}\delta_{\delta_1\delta_2} + B\ \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\gamma_1\delta_2}\delta_{\gamma_2\delta_1}\\ &+C_1\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\delta_{\gamma_1\delta_2}\delta_{\gamma_2\delta_1} + C_2\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\gamma_1\gamma_2}\delta_{\delta_1\delta_2}\\ &+D_1\delta_{\alpha_1\alpha_2}\delta_{\gamma_1\gamma_2}\delta_{\beta_1\delta_1}\delta_{\beta_2\delta_2} + D_2\delta_{\beta_1\beta_2}\delta_{\delta_1\delta_2}\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\gamma_2}\\ &+D_3\delta_{\alpha_1\alpha_2}\delta_{\delta_1\delta_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2} + D_4\delta_{\beta_1\beta_2}\delta_{\gamma_1\gamma_2}\delta_{\alpha_1\delta_1}\delta_{\alpha_2\delta_2}\\ &+E_1\delta_{\alpha_1\alpha_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\delta_2}\delta_{\gamma_2\delta_1} + E_2\delta_{\beta_1\beta_2}\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\delta_2}\delta_{\gamma_2\delta_1}\\ &+E_3\delta_{\alpha_1\alpha_2}\delta_{\beta_1\delta_1}\delta_{\beta_2\gamma_2}\delta_{\gamma_1\delta_2} + E_4\delta_{\beta_1\beta_2}\delta_{\alpha_1\delta_1}\delta_{\alpha_2\gamma_2}\delta_{\gamma_1\delta_2}\\ &+E_5\delta_{\gamma_1\gamma_2}\delta_{\alpha_1\delta_1}\delta_{\alpha_2\beta_1}\delta_{\beta_2\delta_2} + E_6\delta_{\delta_1\delta_2}\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_2}\\ &+E_7\delta_{\gamma_1\gamma_2}\delta_{\alpha_1\beta_2}\delta_{\alpha_2\delta_2}\delta_{\beta_1\delta_1} + E_8\delta_{\delta_1\delta_2}\delta_{\alpha_1\beta_2}\delta_{\alpha_2\gamma_2}\delta_{\beta_1\gamma_1}\\ &+F_1\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\beta_1}\delta_{\beta_2\delta_2}\delta_{\gamma_2\delta_1} + F_2\delta_{\alpha_1\beta_2}\delta_{\alpha_2\gamma_2}\delta_{\beta_1\delta_1}\delta_{\gamma_1\delta_2}\\ &+F_3\delta_{\alpha_1\delta_1}\delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_2}\delta_{\gamma_1\delta_2} + F_4\delta_{\alpha_1\beta_2}\delta_{\alpha_2\delta_2}\delta_{\beta_1\gamma_1}\delta_{\gamma_2\delta_1}\\ &+G_1\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\gamma_2}\delta_{\beta_1\delta_1}\delta_{\beta_2\delta_2} + G_2\delta_{\alpha_1\delta_1}\delta_{\alpha_2\delta_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2}\\ &+H_1\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\delta_2}\delta_{\beta_1\delta_1}\delta_{\beta_2\gamma_2} + H_2\delta_{\alpha_1\delta_1}\delta_{\alpha_2\gamma_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\delta_2}\,. \end{aligned} \tag{4}$$

In addition to (3a)–(3f) the requirement of factorisation gives the equation

$$ {}_{ij}S_{lm}(\theta)\,{}_{lk}S_{pn}(\theta+\theta')\,{}_{mn}S_{qr}(\theta') = {}_{nl}S_{pq}(\theta)\,{}_{im}S_{nr}(\theta+\theta')\,{}_{jk}S_{lm}(\theta')\,. \tag{5a}$$

Using completeness and tracelessness of the λ-matrices (5a) becomes equivalent to (5b):

$$\begin{aligned} &{}_{\alpha_1\alpha_2\beta_1\beta_2}S_{a_1a_2b_1b_2}(\theta)\,{}_{a_1a_2\gamma_1\gamma_2}S_{\delta_1\delta_2c_1c_2}(\theta+\theta')\,{}_{b_1b_2c_1c_2}S_{\mu_1\mu_2\nu_1\nu_2}(\theta')\\ &\quad = {}_{c_1c_2a_1a_2}S_{\delta_1\delta_2\mu_1\mu_2}(\theta)\,{}_{\alpha_1\alpha_2b_1b_2}S_{c_1c_2\nu_1\nu_2}(\theta+\theta')\,{}_{\beta_1\beta_2\gamma_1\gamma_2}S_{a_1a_2b_1b_2}(\theta')\,. \end{aligned} \tag{5b}$$

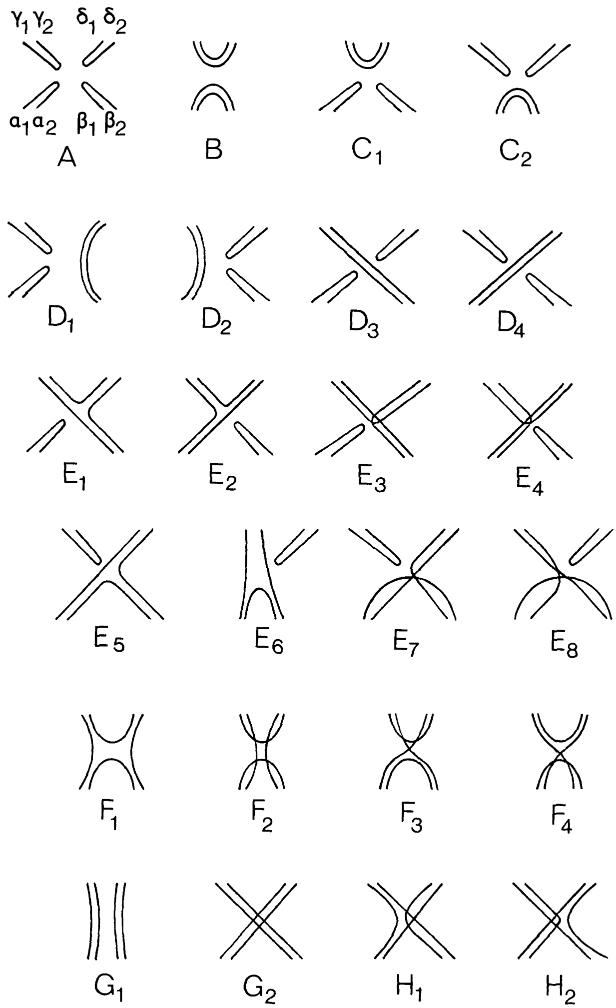

Fig. 1

The symmetry (3b), trace (3a) and crossing (3c) conditions imply (cf. Sect. 3)

$$\begin{aligned}
&E \equiv E_i \quad \forall i=1,\dots,8 \qquad (E=\hat{E})\\
&C \equiv C_1 = C_2, \qquad \hat{C} = D_3 = D_4\\
&F \equiv F_1 = F_2, \qquad \hat{F} = H_1 = H_2\\
&D \equiv D_1 = D_2 \qquad (D=\hat{D})\\
&A = -\frac{1}{N}(C+\hat{C}+D), \qquad B = -(NC+2E)\\
&F_3 = F_4 = -(NE+F+\hat{F})\\
&G_1 = -(ND+2E), \qquad G_2 = -(N\hat{C}+2E),
\end{aligned} \tag{6}$$

where we have introduced the notation: $\hat{f}(\theta)=f(i\pi-\theta)$.

We are now ready to state our Result.

Theorem. *For $N\geqq 6$ the only solutions of Eq.* (3a)–(3f) *and the factorisation Eq.* (5) *are the trivial solution,*

$$C=E=F=0 \tag{7}$$

$$_{ij}S_{kl}=-ND\delta_{ik}\delta_{jl}$$

with

$$D(\theta)D(-\theta)=\frac{1}{N^2}$$

the Zamolodchikov [2] $O(N^2-1)$ *solution,*

$$E=F=0$$

$$_{ij}S_{kl}=-N(D\delta_{ik}\delta_{jl}+C\delta_{ij}\delta_{kl}+\hat{C}\delta_{il}\delta_{jk}) \tag{8a}$$

with

$$\frac{\hat{C}}{D}=-\frac{2\pi i}{N^2-3}\,\frac{1}{\theta},\qquad N^2D(\theta)D(-\theta)=\frac{\theta^2}{\theta^2+\dfrac{4\pi^2}{(N^2-3)^2}},$$

and the Hortaçsu et al. [13] $O(N^2-1)$ *solution,*

$$E=F=D=0$$

$$_{ij}S_{kl}=-N(C\delta_{ij}\delta_{kl}+\hat{C}\delta_{il}\delta_{jk}) \tag{8b}$$

with

$$\frac{C}{\hat{C}}=\frac{\operatorname{sh}v\dfrac{\theta}{i\pi}}{\operatorname{sh}v(1-\theta/i\pi)},\qquad \operatorname{ch}v=\frac{N^2-1}{2},\qquad N^2\hat{C}(\theta)\hat{C}(-\theta)=1\,.$$

3. The Proof

It is easily checked that (7) and (8) are solutions. We now prove that these are the only solutions.

Symmetry (3b) yields for $N\geqq 4$

$$\begin{aligned}&E_1=E_4,\quad E_2=E_3,\quad E_5=E_8,\quad E_6=E_7\\&D_1=D_2,\quad D_3=D_4,\quad F_1=F_2,\quad F_3=F_4.\end{aligned} \tag{9}$$

Crossing (3c) implies

$$\begin{aligned}&A=\hat{A},\ B=\hat{G}_2,\ C_1=\hat{D}_3,\ C_2=\hat{D}_4,\ D_1=\hat{D}_1,\ D_2=\hat{D}_2\\&E_1=\hat{E}_3,\ E_2=\hat{E}_6,\ E_4=\hat{E}_8,\ E_5=\hat{E}_7\\&F_1=\hat{H}_1,\ F_2=\hat{H}_2,\ F_3=\hat{F}_4,\ G_1=\hat{G}_1\end{aligned} \tag{10}$$

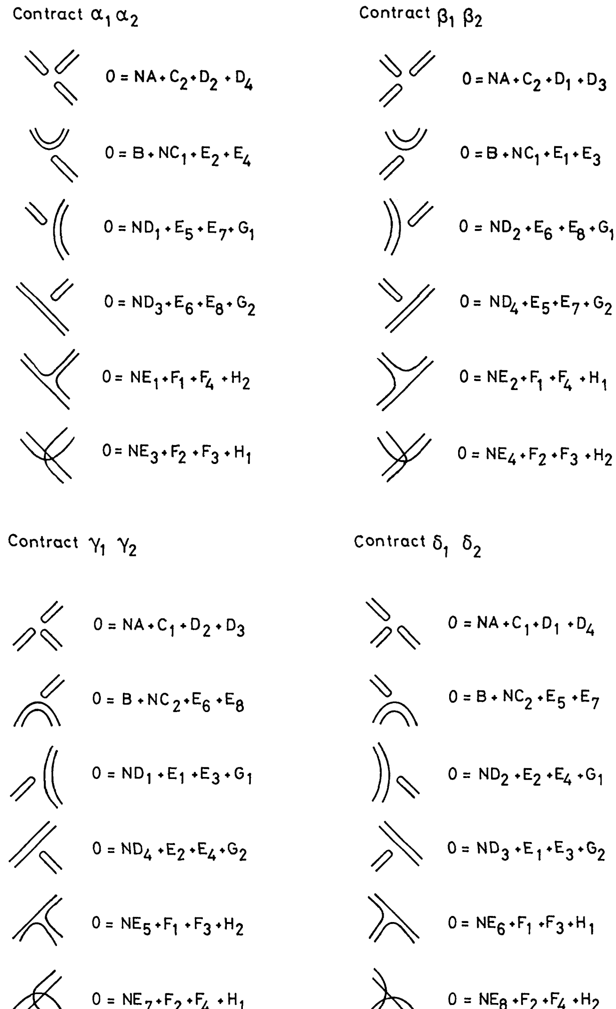

Fig. 2

The trace condition (3a) gives 24 (in part dependent) Eqs. (cf. Fig. 2). Putting symmetry, crossing and trace equations together we are left with 4 independent amplitudes

$$C, D, E, F \quad \text{with} \quad D = \hat{D}, E = \hat{E}$$

and the remaining amplitudes determined by (6).

From the unitarity Eq. (3f) we obtain six invariant amplitudes

$$U_i(\theta)\, U_i(-\theta) = 1 \qquad i = 1, \ldots, 6. \tag{11}$$

As shown in Appendix A they are ($N \geqq 4$ assumed):

$$
\begin{aligned}
U_1 &= N\hat{C} - ND \\
U_2 &= N\hat{C} - ND + N^2E + N\hat{F} + 2NF \\
U_3 &= N\hat{C} + ND + 4E + 2\hat{F} \\
U_4 &= N\hat{C} + ND + 4E - 2\hat{F} \\
U_5 &= N\hat{C} + ND + N^2E + N\hat{F} \\
U_6 &= N\hat{C} + ND + 4N^2E + 2N\hat{F} + N(N^2-1)C\,.
\end{aligned} \tag{12}
$$

Finally we have to make use of the factorization Eq. (5). This is the technically most involved part of the proof. For $N \geqq 6$ the products of six δ-functions remaining at the end of the calculation are all independent and we obtain $6! = 720$ equations with $2 \times 24^3 = 27{,}648$ terms involved. Using an algebraic computer program [14] we have calculated all these equations. Fortunately there are some very simple equations involved which give serious restrictions on the amplitudes leading immediately to the theorem. After sorting out the configurations of δ-function indices of these equations by the computer they can be checked by hand. Therefore we forget in the following presentation about the involved computer work.

First consider the coefficient of

$$\delta_{\alpha_1\mu_1}\delta_{\beta_1\nu_1}\delta_{\alpha_2\gamma_1}\delta_{\delta_1\nu_2}\delta_{\beta_2\mu_2}\delta_{\gamma_2\delta_2}$$

as shown in Appendix B this yields the simple equation,

$$H_2F'_3G''_1 + G_1F'_3H''_2 = 0\,, \tag{13}$$

where we have used the notation.

$$f' = f(\theta + \theta') \qquad f'' = f(\theta')\,.$$

It follows

$$H_2 = 0 \quad \text{or} \quad G_1 = 0 \quad \text{or} \quad F_3 = 0\,.$$

Case 1. $H_2 = 0$ i.e. $F = 0$.

By the method of Appendix B the coefficient of

$$\delta_{\alpha_1\beta_2}\delta_{\beta_1\delta_1}\delta_{\alpha_2\gamma_1}\delta_{\mu_1\mu_2}\delta_{\nu_1\delta_2}\delta_{\gamma_2\nu_2}$$

is calculated to give,

$$
\begin{aligned}
&NE_8F'_1D''_1 + (H_2 + F_2 + F_4)F'_1D''_1 + E_8F'_1(E''_5 + E''_7) + F_4G'_1E''_5 + F_2B'E''_7\,. \\
&\quad = E_8B'H''_1\,.
\end{aligned} \tag{14}
$$

Then:

$$
\begin{aligned}
F = 0 &\Rightarrow F_4G'_1E''_5 = 0 \Rightarrow EG'_1E'' = 0 \\
&\Rightarrow E = 0 \quad \text{or} \quad G_1 = 0
\end{aligned}
$$

$E=F=0$ implies the solutions (8) or trivial (7) solution. $F=G_1=0$ through unitarity $\Rightarrow E=0=D=F$ which implies the Hortaçsu et al. [13] solution.

Case 2. $G_1=0$.

The coefficient of $\delta_{\alpha_1\mu_1}\delta_{\beta_1\gamma_2}\delta_{\gamma_1\nu_1}\delta_{\delta_1\nu_2}\delta_{\alpha_2\mu_2}\delta_{\beta_2\delta_2}$ yields

$$G_2F'_2G''_1=H_1F'_2F''_2+G_2G'_1F''_2. \tag{15}$$

Then [via (6)] $G_1=0\Rightarrow F=0$, i.e. reduces to a subcase of 1.

Case 3. $F_3=0$.

The coefficient of $\delta_{\alpha_1\mu_1}\delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_1}\delta_{\delta_1\mu_2}\delta_{\nu_1\delta_2}\delta_{\gamma_2\nu_2}$ yields

$$\begin{aligned}F_3F'_1G''_1+F_3B'H''_1=F_3G'_1F''_1+F_2B'F''_3+NE_4F'_1E''_5\\+(F_2+H_2+F_3)F'_1E''_5+E_4F'_1(F''_1+H''_2+F''_3).\end{aligned} \tag{16}$$

Then

$F_3=0\Rightarrow EF'E''=0$.

Now

$F=0=F_3\Rightarrow E=0\Rightarrow$ solution (7) or (8),

and

$E=0, \quad F=-\hat{F}\neq 0$ contradicts unitarity.

This concludes the proof.

Acknowledgement. We thank M. Karowski, V. Kurak, and B. Schroer for discussions.

Appendix A

Starting from unitarity (3f) we prove Eqs. (10) and (11) for the invariant amplitudes. We have

$${}_{ij}S_{kl}=\lambda^i_{\alpha_2\alpha_1}\lambda^j_{\beta_2\beta_1}\lambda^k_{\gamma_1\gamma_2}\lambda^l_{\delta_1\delta_2}\,{}_{\alpha_1\alpha_2\beta_1\beta_2}S_{\gamma_1\gamma_2\delta_1\delta_2}. \tag{A.1}$$

Using

$$\operatorname{tr}\lambda^i\lambda^j\lambda^k\lambda^l=\frac{4}{N}\delta_{ij}\delta_{kl}+2(d_{ijn}+if_{ijn})(d_{kln}+if_{kln}) \tag{A.2}$$

and (2), (4), and (6) we obtain

$$\begin{aligned}{}_{ij}S_{kl}&=S_1\delta_{ij}\delta_{kl}+S_2\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})\\&\quad+S_3(d_{ikn}d_{jln}+d_{jkn}d_{iln})+S_4d_{ijn}d_{kln}\\&\quad+A_1\tfrac{1}{2}(\delta_{ik}\delta_{jl}-\delta_{jk}\delta_{il})+A_2f_{ijn}f_{kln}\end{aligned} \tag{A.3}$$

with

$$\begin{aligned} S_1 &= -NC - 4E - \frac{4}{N}\hat{F} \\ S_2 &= -N\hat{C} - ND - 4E + \frac{4}{N}\hat{F} \\ S_3 &= \hat{F} \\ S_4 &= -NE - 2\hat{F} \\ A_1 &= N\hat{C} - ND \\ A_2 &= NE + 2F + \hat{F}. \end{aligned} \tag{A.4}$$

The unitarity relation (5a) together with the identities

$$\begin{aligned} & f_{piq}f_{qjr}f_{rkp} = -\frac{N}{2}f_{ijk} \\ & d_{piq}f_{qjr}f_{rkp} = -\frac{N}{2}d_{ijk} \\ & d_{piq}d_{qjr}f_{rkp} = \frac{(N^2-4)}{2N}f_{ijk} \\ & d_{piq}d_{qjr}d_{rkp} = \frac{(N^2-12)}{2N}d_{ijk} \\ & f_{ijk}f_{ljk} = N\delta_{il},\ d_{ijk}d_{ljk} = \frac{(N^2-4)}{N}\delta_{il},\ \delta_{ii} = N^2-1 \\ & d_{ikm}d_{jlm} - d_{ilm}d_{jkm} = f_{ijm}f_{klm} - \frac{2}{N}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ & d_{ikp}d_{jlp}(d_{kmq}d_{lnq} + d_{knq}d_{lmq}) = \frac{2(N^2-4)}{N^2}(\delta_{ij}\delta_{mn} + \tfrac{1}{2}[\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}]) \\ & -\frac{4}{N}(d_{imq}d_{jnq} + d_{inq}d_{jmq}) + \frac{(N^2-16)}{2N}d_{ijq}d_{mnq} \end{aligned} \tag{A.5}$$

now yields the invariant amplitudes (11) U_i, $i=1,\ldots,6$ corresponding to the SU(N) representations occuring in the product of two adjoints:

$\frac{1}{4}(N^2-1)(N^2-4) \oplus \overline{\frac{1}{4}(N^2-1)(N^2-4)}$, $\quad N^2-1$ (antisymmetric)

$\frac{1}{4}N^2(N-3)(N+1)$, $\quad \frac{1}{4}N^2(N+3)(N-1)$, $\quad N^2-1$ (symmetric), (singlet)

respectively.

To make sure that we have done no algebraic error we have checked the final result with the algebraic computer program [14].

Appendix B

We demonstrate the calculation of simple δ-function coefficients from the factorisation Eq. (5b) for

$$\delta_{\alpha_1\mu_1}\delta_{\beta_1\nu_1}\delta_{\alpha_2\gamma_1}\delta_{\delta_1\nu_2}\delta_{\beta_2\mu_2}\delta_{\gamma_2\delta_2}.$$

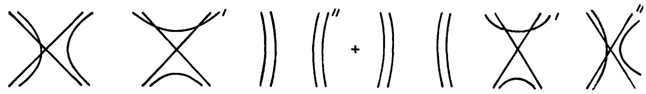

Fig. 3

Let us choose the indices to be ($N \geqq 6$)

$$\alpha_1 = \mu_1 = 1, \quad \beta_1 = \nu_1 = 2, \quad \gamma_1 = \alpha_2 = 3, \quad \delta_1 = \nu_2 = 4, \quad \beta_2 = \mu_2 = 5, \quad \gamma_2 = \delta_2 = 6.$$

The Eq. (5b) reads

$$_{1325}S_{a_1a_2b_1b_2}\,{}_{a_1a_236}S'_{46c_1c_2}\,{}_{b_1b_2c_1c_2}S''_{1524} = {}_{c_1c_2a_1a_2}S_{4615}\,{}_{13b_1b_2}S'_{c_1c_224}\,{}_{2536}S''_{a_1a_2b_1b_2}.$$

Inspection of the first and last factors shows that contributions to the left hand side can only come when simultaneously $\{a_1, b_1\} = \{1, 2\}$, $\{a_2, b_2\} = \{3, 5\}$, $\{b_1, c_1\} = \{1, 2\}$, $\{b_2, c_2\} = \{4, 5\}$. This implies $b_2 = 5$, $a_2 = 3$, $c_2 = 4$ and $b_1 = 1$, $a_1 = c_1 = 2$ or $b_1 = 2$, $a_1 = c_1 = 1$.

Thus the left hand side is given by

$$_{1325}S_{2315}\,{}_{2336}S'_{4624}\,{}_{1524}S''_{1524} + {}_{1325}S_{1325}\,{}_{1336}S'_{4614}\,{}_{2514}S''_{1524}$$

or diagramatically by Fig. 3. Similarly the rhs can only give non-zero contributions if simultaneously

$$\{a_1, c_1\} = \{1, 4\}, \qquad \{a_2, c_2\} = \{5, 6\}$$
$$\{a_1, b_1\} = \{2, 3\}, \qquad \{a_2, b_2\} = \{5, 6\}$$

has solutions, which is obviously not the case. Hence we obtain the restrictive Eq. (12).

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Communicated by E. Brézin

Received October 16, 1978