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Instantons and Kähler Manifolds

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Abstract. It is shown that for two-dimensional Euclidean chiral models of the field theory with values in arbitrary Kähler manifold "duality equations" reduce to the Cauchy-Riemann equations on this manifold. A class of models is described possessing such type solutions, the so called instanton solutions.

1. In the last few years a considerable progress has been achieved in studying both pseudoeuclidean and Euclidean chiral models of the field theory, i. e. the models for which the field takes the values in nonlinear manifolds (see refs. [1–7]). Note especially the recent results by V. E. Zakharov and A. V. Mikhailov who developed the method of finding the explicit solutions for a certain class of two-dimensional pseudoeuclidean chiral models [7].

In many cases the solutions of field equations can be characterized by topological invariants, the so called topological charges, which allow to estimate the energy (action) of a system from below [1, 2, 8–10]. The solutions of the Euclidean theory equations with a certain topological charge corresponding to the minimum of energy (action), the solutions of the so called "duality equations" are usually called instanton solutions. For the case when the field takes the values in the two-dimensional sphere S^2, or, that is the same, in the one-dimensional complex projective space CP^1 such a problem has been solved in the paper by Belavin and Polyakov [2] while for the case of $CP^n(n>1)$ in the paper [10].

In this work which is an elaboration of the investigation started in [10] we show that just in [2] and in [10], for the two-dimensional chiral models of the field theory with the values in arbitrary compact Kähler manifold[1] the "duality equations" reduce to the Cauchy-Riemann equations on this manifold. The class of manifolds for which such solutions do exist is described.

These are the Kähler homogeneous simply-connected manifolds. Such manifolds can be as well characterized by that they are homogeneous under the

1 For the properties of Kähler manifolds and of more general class complex manifolds as well see the book [11]

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action of a connected compact semisimple Lie group. They are completely classified in the paper by A. Borel [12]. Note that the models considered in paper [10] belong to this class.

2. Let M be a connected simply connected compact complex n-dimensional manifold with local coordinates w^μ in the neighbourhood of the point $w^\mu=0$ and let the field $\varphi(x)(x=(x_1,x_2)\in\mathbb{R}^2)$ take the values in M. We will consider only the field possessing a certain limit: $\varphi(x)\to\varphi_0$ at $|x|\to\infty$. In this case one may assume that such field determines the map of the two-dimensional sphere S^2 which we consider as a compactified plane $\mathbb{R}^2(S^2=\mathbb{R}^2\cup\{\infty\})$ into M.

$$\varphi:S^2\to M. \tag{1}$$

If now $\pi_2(M)$ (the second homotopic group of the manifold M)[2] that is isomorphic due to the Hurevic theorem to the second homology group $H_2(M)$ and, correspondingly, to the second cohomology group $H^2(M)$, is nontrivial, then to each field $\varphi(x)$ a "topological charge" with integral representation

$$Q=c^{-1}\int_{\mathbb{R}^2}\tilde{\omega} \tag{2}$$

can be prescribed, where c is some constant,

$$\tilde{\omega}=\varphi^*\omega, \tag{3}$$

$\omega\in H^2(M)$, while $\varphi^*:H^2(M)\to H^2(S^2)$ is the mapping induced by the map φ.

Let us endow M in the neighbourhood of the point $w^\mu=0$ with Hermitian metric

$$ds^2=h_{\mu\bar{\nu}}dw^\mu d\bar{w}^\nu, \qquad \mu,\nu=1,\dots,n. \tag{4}$$

Then the system under consideration is described by the Euler equations corresponding to the condition $\delta S=0$ for the functional of the energy (action)

$$S=\tfrac{1}{2}\int h_{\mu\bar{\nu}}\partial_j w^\mu\partial_j\bar{w}^\nu d^2x, \qquad j=1,2. \tag{5}$$

Substituting into (5) $\partial_j w^\mu\pm i\varepsilon_{jk}\partial_k w^\mu$ instead of $\partial_j w^\mu$ we come to inequality

$$S\geqq c|Q|, Q=c^{-1}\int_{\mathbb{R}^2}\tilde{\omega}=\frac{i}{2}c^{-1}\int\varepsilon_{jk}h_{\mu\bar{\nu}}\partial_j w^\mu\partial_k\bar{w}^\nu d^2x \tag{6}$$

where $\tilde{\omega}=\varphi^*\omega$, $\omega=\frac{i}{2}h_{\mu\bar{\nu}}dw^\mu\wedge d\bar{w}^\nu$ is the imaginary part of Hermitian form $h_{\mu\bar{\nu}}$. The sign of equality in (6) is achieved only for the fields satisfying the "duality" equations

$$\partial_j w^\mu\pm i\varepsilon_{jk}\partial_k w^\mu=0 \tag{7}$$

or, going over to the complex coordinate $z=x_1+ix_2$,

$$\partial=\partial/\partial z=\tfrac{1}{2}(\partial_1-i\partial_2), \bar{\partial}=\partial/\partial\bar{z}=\tfrac{1}{2}(\partial_1+i\partial_2)$$

we get

$$\bar{\partial}w^\mu=0 \quad (\text{or} \quad \partial w^\mu=0). \tag{8}$$

2 All the topological notations used in the text can be found in book [13]

The local solution of these equations is $w^\mu = f^\mu(z)$ (or $w^\mu = f^\mu(\bar{z})$). Thus, the equality $S = cQ$ (resp. $S = -cQ$) can be achieved for the holomorphic (resp., antiholomorphic) maps of the compactified plane z (which can be considered as a one-dimensional complex projective space CP^1) into the manifold M.

Unfortunately, Q is not in general topplogical invariant and changes when $w^\mu(x)$ deforms. So, one cannot state in the general case that the solution of the duality equations is that of unitial Euler equations.

3. Let now M be the Kähler manifold, i.e. the complex manifold for which the imaginary part ω of the Hermitian form h is closed nondegenerate 2-form. Note that such a manifold is symplectic $\omega \in H^2(M)$ and that the condition as to the form ω is closed is equivalent to conditions

$$\frac{\partial h_{\mu\bar{\nu}}}{\partial w^\lambda} = \frac{\partial h_{\lambda\bar{\nu}}}{\partial w^\mu} \quad \text{or} \quad \frac{\partial h_{\mu\bar{\nu}}}{\partial \bar{w}^\lambda} = \frac{\partial h_{\mu\bar{\lambda}}}{\partial \bar{w}^\nu}. \tag{9}$$

The map $\varphi : S^2 \to M$ determines the two-dimensional cycle in M. The quantity cQ is in this case the integral of ω over this cycle and because of the closeness of the form ω, depends only on the homology class to which this cycle belongs. And in the given class Q is constant and the functional of the action S is equal to its minimum value provided the conditions (8) to be fulfilled, i.e. for the holomorphic maps $\varphi : S^2 = CP^1 \to M$. These maps, if they do exist, give the solutions of the "duality" equations (7) and are usually called "instanton" solutions. Note, if the manifold M is algebraic, i.e. analytical submanifold without singularities in the complex projective space CP^N for a certain N, then there exists such Kähler metric (the so called Hodge metric) on it that Q will be always integer.

Note more, that when going over from variables x_1 and x_2 to variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ the expressions for the energy (action) and topological charge Q take the form

$$S = \int h_{\mu\bar{\nu}}(\partial w^\mu \bar{\partial}\bar{w}^\nu + \bar{\partial} w^\mu \partial \bar{w}^\nu) d^2x, \tag{10}$$

$$Q = c^{-1} \int h_{\mu\bar{\nu}}(\partial w^\mu \bar{\partial}\bar{w}^\nu - \bar{\partial} w^\mu \partial \bar{w}^\nu) d^2x. \tag{11}$$

The coincidence of S with $c|Q|$ for holomorphic and antiholomorphic fields is now obviously seen.

The "equations of motion" are obtained, as usual, from the condition $\delta S = 0$. Accounting for the Kähler properties of the manifold M expressed by conditions (9) we get

$$h_{\mu\bar{\nu}}(\partial\bar{\partial} w^\mu) + \frac{\partial h_{\mu\bar{\nu}}}{\partial w^\lambda} \bar{\partial} w^\mu \partial w^\lambda = 0 \tag{12}$$

and the equation complex-conjugated to it.

Let us now multiply the left-handed part of Eq. (12) by $\partial \bar{w}^\nu$ and sum over ν and add to it the left-handed part of the equation complex-conjugated to (12) multiplied by ∂w^ν and summed over ν. Making use of relations (9) the expression obtained can be transformed to the form

$$\bar{\partial}(h_{\mu\bar{\nu}} \partial w^\mu \partial \bar{w}^\nu) = 0. \tag{13}$$

Whence we obtain

$$h_{\mu\bar{\nu}}\partial w^{\mu}\partial\bar{w}^{\nu}=f(z) \tag{14}$$

and analogously

$$h_{\mu\bar{\nu}}\bar{\partial}w^{\mu}\bar{\partial}\bar{w}^{\nu}=f(\bar{z}). \tag{15}$$

5. Let us now consider the solutions of the "duality equations" (8). The duality equations are of the form of the Cauchy-Riemann equations but because of the compactness of the manifold M, the global solution, or, that is the same, the holomorphic map $\varphi: CP^1 \to M$ is far from existing always. Thus for instance, if M is two-dimensional compact manifold of genre $g(g=0, M=S^2=CP^1; g=1,$ $M-$ is the two-dimensional torus), i.e. the Riemann surface, such a map exists only if $M=CP^1$. (This recalls the case considered in the paper by Belavin and Polyakov [2]. The map with the topological charge $Q=n$ depends here on $4n$ real parameters.)

Such a map exists however if $M=CP^n$ is the complex projective space (this case is considered in paper [10]). Unlike the preceding case the solution with the topological charge $Q=n$ depends on larger number of the parameters.

Note also that the image CP^1 in M under the map φ is an algebraic manifold N [14].

6. Let us now turn to consideration of an important class of the Kähler manifolds M for which the holomorphic maps $CP^1 \to M$ exist. These are the simply connected compact homogeneous Kähler manifolds[3]. From [12, 15] it follows that all of them have the form G/H where G is the compact connected semisimple Lie group with the trivial center, H is the centralizer of some torus in G. It can be readily seen for these spaces to be considered as orbits of adjoint representation of the compact semisimple Lie group in the Lie algebra of this group. All of these spaces are not only algebraic but rational as well[4] [16]. All of them admit complex-analytical cellular decomposition [12, 15].

It appears further that on M acts transitively not only the real group G but also corresponding to it complex group G^c. So M may be also represented as $M=G/H$ $=G^c/P$, where P is a parabolic subgroup, i.e. the subgroup of G^c containing the maximal connected solvable subgroup. Here $H=P\cap G$ [15].

It is known [15] that any such subgroup P is constructed in a canonic way over the subsystem I of simple roots of Lie algebra of group G^c.

Let R_I be a subset of positive roots consisting of linear combinations of elements I. Let G_I be a subgroup of G generated by H and by subgroups N_γ $=\{\exp tE_\gamma | t\in\mathbb{C}\}$ for $\gamma\in\mathbb{R}_I\cup\{-\mathbb{R}_I\}$, $P_I=G_I N_I$. As is known, each parabolic subgroup is conjugated in G^c to one of such subgroups.

7. Let us give the A. Borel construction of invariant Kähler metrics [12]. They are constructed by means of left-invariant forms, the so called Maurer-Cartan

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3 All the compact homogeneous Kähler manifolds (not necessarily simply connected) are also known. Namely: any connected compact homogeneous Kähler manifold is a direct product of a complex torus and algebraic rational manifold [17]

4 The manifold M^n is called rational if the field of meromorphic functions on it is isomorphic to that of rational functions from n complex variables

forms. Consider the simplest case G/T (T maximal torus). Let ω^α be the left-invariant Maurer-Cartan forms on G^c which induce on the Lie algebra the basis dual to X_α and are orthogonal to H^c. Using the Maurer-Cartan equations and well-known properties of structure constants, one shows that

$$\omega = \frac{i}{2} \sum_{\alpha>0} c_\alpha \omega^\alpha \wedge \omega^{-\alpha} \tag{16}$$

is closed if and only if

$$c_\alpha + c_\beta = c_\gamma \quad \text{if} \quad \alpha+\beta=\gamma. \tag{17}$$

The form ω is therefore determined by the constants c_α for simple roots α, which are arbitrary ($c_\alpha=(h,\alpha)$). Its restriction on G is left-invariant under G, right-invariant under T and represents a form on G/T which is of the type (1, 1) because $\omega^{-\alpha}$ corresponds to ω^α in the complex structure used in this construction. For real c_α this form is real-values and its real cohomology class may be shown to be the image by transgression of the element $h\in H^{(1)}(T)$ for which $(\alpha,h)=c_\alpha$ (α is a simple root). If h belongs to the interior of the positive Weyl chamber, all the $c_\alpha>0$ and

$$ds^2 = \sum_{\alpha>0} c_\alpha \omega^\alpha \bar{\omega}^\alpha \tag{18}$$

is a Kählerian metrics on G/T.

If, moreover, $h\in H^{(1)}(T;Z)$ then corresponding orbit is integer and its image by transgression is an integer class, the corresponding metric is a Hodge metric, and G/T is algebraic by a result of Kodaira. Note also that in Reference [16] it is proved that G/H is a rational algebraic manifold.

This fact is of special importance for us because in this case there do exist the nonconstant holomorphic maps $CP^1 \to \Phi = G/H$ (Yu. I. Manin, private communication), or, that is the same, the nontrivial instanton solutions of corresponding chiral theories.

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