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On Trace Representation of Linear Functionals on Unbounded Operator Algebras

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Abstract. In this paper we consider the following problem: Given a $*$-algebra $\mathscr{A}$ of unbounded operators, under what conditions is every strongly positive linear functional f on $\mathscr{A}$ a trace functional, i.e. of the form $f(a)=\mathrm{Tr}\, ta$, $a\in\mathscr{A}$, where t is an appropriate positive nuclear operator. Further, the linear functionals f on $\mathscr{A}$ which can be represented as $f(a)=\mathrm{Tr}\, ta$ (f and t not necessarily positive) are characterized by their continuity in a certain topology. Some applications (canonical commutation relations on the Schwartz space, integrable representations of enveloping algebras) are discussed.

Introduction

In the algebraical frame of quantum theory the observables are symmetric elements of a $*$-algebra of (in general unbounded) operators in a Hilbert space. The states are usually considered as positive linear functionals on this algebra. Many important examples of states in quantum physics (for instance, the Gibbs states for free Bose gas) are trace functionals, i.e. they are of the form $f(a)=\mathrm{Tr}\, ta$ with a certain density matrix t. In this paper we are dealing with trace representation of strongly positive linear functionals and more generally of arbitrary linear functionals. To be more precise, we will study the following problems.

Problem 1. Under what conditions is every strongly positive linear functional f on an $Op*$-algebra $\mathscr{A}$ a trace functional $f(a)=\mathrm{Tr}\, ta$, $a\in\mathscr{A}$, t an appropriate positive nuclear operator.

Problem 2. Characterize the (not necessarily positive) linear functionals on $\mathscr{A}$ which can be represented as $f(a)=\mathrm{Tr}\, ta$. Here the nuclear operator t is in general not positive.

Problem 1 has already been studied by several authors [16, 21, 4, 18]. For Problem 1 Sherman [16] proved this to be the case for each countably generated closed $Op*$-algebra which contains the restriction of the inverse of a completely continuous operator. Woronowicz [21] has shown that the algebra $L^+(\mathscr{S})$,

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$\mathcal{S}=\mathcal{S}(R_1)$ being the Schwartz space, and the $Op*$-algebra generated by the position and momentum operators also have this property. Lassner and Timmermann [4, 18] obtained results on the continuity of trace functionals. It is not difficult to see that every strongly positive linear functional on a $*$-algebra $\mathcal{A}$ with Fréchet graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ can be extended to a strongly positive linear functional on $L^+(\mathcal{D})$. This suggests the problem to characterize the closed domains $\mathcal{D}$ having the property that all strongly positive linear functionals on $L^+(\mathcal{D})$ are trace functionals with positive operator t. For Fréchet domains we give a complete characterization of these domains. Before discussing our results let us note that in the case of an infinite dimensional Hilbert space $\mathcal{D}=\mathcal{H}$ there are always positive linear functionals on $B(\mathcal{H})$ which are not trace functionals [15]. A simple example of such a functional can be obtained by extending a character $g_s(x):=x(s)$, $s\in[0,1]$, on the commutative C^*-algebra $C(0,1)$ to $B(L_2(0,1))$.

In this paper Problem 1 is studied in Sects. 2 and 3, while Problem 2 is treated in Sects. 4 and 5. We will prove that all strongly positive linear functionals on a self-adjoint $Op*$-algebra $\mathcal{A}$ on $\mathcal{D}$ are trace functionals with positive densities t if $\mathcal{A}$ contains the restriction to $\mathcal{D}$ of the inverse of a completely continuous operator (Sect. 2). This extends the corresponding results of Sherman, Woronowicz, Lassner and Timmermann. For domains $\mathcal{D}$ with Fréchet graph topology t_+ it is shown in Sect. 3 that all strongly positive linear functionals on $L^+(\mathcal{D})$ (or equivalently, on *each* $Op*$-algebra $\mathcal{A}$ on $\mathcal{D}$ with $t_{\mathcal{A}}=t_+$) are trace functionals if and only if $\mathcal{D}[t_+]$ is a Montel space. Note that the Montel property of t_+ is weaker than the existence of an operator in $L^+(\mathcal{D})$ which is the inverse of a completely continuous operator. If the Fréchet space $\mathcal{D}[t_+]$ has an unconditional basis, $\mathcal{D}[t_+]$ is a Montel space iff the domain does not contain an infinite dimensional Hilbert space as a subspace. These domains are called domains of class I in [5]. Section 4 is devoted to a topological characterization of the trace functionals. We prove that an arbitrary (not necessarily positive) linear functional f on a closed $Op*$-algebra with metrizable graph topology has a trace representation $f(a)=\operatorname{Tr} ta$ if and only if it is continuous with respect to a certain topology $\tau^c_{\mathcal{D}}$. In Sect. 5 we give some applications of this theorem. For example, it is shown that each linear functional f on the $Op*$-algebra generated by the position and momentum operators q_j, p_j, $j=1,\dots,n$, on the Schwartz space $\mathcal{S}(R_n)$ is a trace functional $f(a)=\operatorname{Tr} ta$.

The definitions and notations used in the following are collected in Sect. 1.

1. Preliminaries

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ a vector space of linear operators on $\mathcal{D}$. We call $\mathcal{A}$ *$*$-invariant* if for each $a\in\mathcal{A}$ the adjoint operator a^* is also defined on $\mathcal{D}$ and the restriction $a^+:=a^*\upharpoonright\mathcal{D}$ is in $\mathcal{A}$. For $*$-invariant vector spaces $\mathcal{A}$ the following notations are useful:

$$\hat{\mathcal{D}}(\mathcal{A}):=\bigcap_{a\in\mathcal{A}}\mathcal{D}(\bar{a}),\quad \mathcal{D}_*(\mathcal{A}):=\bigcap_{a\in\mathcal{A}}\mathcal{D}(a^*),\quad \mathcal{A}_h:=\{a\in\mathcal{A}: a=a^+\},$$

$$\mathcal{A}_+:=\{a\in\mathcal{A}:\langle a\phi,\phi\rangle\geqq 0\quad\text{for all}\quad \phi\in\mathcal{D}\}$$

and $a\geqq b$ iff $a-b\in\mathcal{A}_+$. $\mathcal{A}$ is said to be *closed* on $\mathcal{D}$ if $\mathcal{D}=\hat{\mathcal{D}}(\mathcal{A})$ and *self-adjoint* on $\mathcal{D}$ if $\mathcal{D}=\mathcal{D}_*(\mathcal{A})$ [20, 9]. By a *strongly positive linear functional* f on $\mathcal{A}$ we mean a linear functional f on $\mathscr{A}$ with $f(a)\geqq 0$ for all $a\in\mathscr{A}_+$. Notice that for unbounded operator algebras this requirement is considerably stronger than the condition $f(a^+a)\geqq 0\ \forall a\in\mathscr{A}$ which usually defines the positive linear functionals on $\mathscr{A}$ (see in this connection [12]).

Suppose now $\mathscr{A}$ is a $*$-invariant vector space of operators on $\mathscr{D}$ containing the identity $I=I_{\mathscr{D}}$. (It is not assumed that the operators $a\in\mathscr{A}$ leave $\mathscr{D}$ invariant.) We define

$$_1\mathfrak{S}(\mathscr{A})=\{t\in B(\mathscr{H}):\ \text{the closures of } ta \text{ and } t^*a \text{ are of trace class for all } a\in\mathscr{A}\},$$
$$\mathfrak{S}_1(\mathscr{A})=\{t\in {}_1\mathfrak{S}(\mathscr{A}):\ t\mathscr{H}\subseteqq\mathscr{D},\ t^*\mathscr{H}\subseteqq\mathscr{D}\},$$
$$_1\mathfrak{S}(\mathscr{A})_+=\{t\in {}_1\mathfrak{S}(\mathscr{A}):\ t\geqq 0\},\quad \mathfrak{S}_1(\mathscr{A})_+=\{t\in\mathfrak{S}_1(\mathscr{A}):\ t\geqq 0\}.$$

Further let us write $\mathfrak{S}_1(\mathscr{D})_+$ instead of $\mathfrak{S}_1(L^+(\mathscr{D}))_+$.

It is clear from the definition that $t\in {}_1\mathfrak{S}(\mathscr{A})$ (or $t\in\mathfrak{S}_1(\mathscr{A})$) implies $t^*\in {}_1\mathfrak{S}(\mathscr{A})$ (or $t^*\in\mathfrak{S}_1(\mathscr{A})$). Because $I\in\mathscr{A}$, all operators $t\in {}_1\mathfrak{S}(\mathscr{A})$ are of trace class on $\mathscr{H}$. For $t\in {}_1\mathfrak{S}(\mathscr{A})$ we define $_tf(x)=\mathrm{Tr}\,tx$ and $f_t(x)=\mathrm{Tr}(x^+)^*t$, $x\in\mathscr{A}$ (cf. [4], p. 298). Since a^*t is of trace class for each $t\in {}_1\mathfrak{S}(\mathscr{A})$ and $a\in\mathscr{A}$ and $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$ (see the lemma below), the definition of $f_t(x)$ make sense. Some basic properties of these notions are collected in Lemma 1. For simplicity we suppose that $\mathscr{H}$ is a separable Hilbert space.

Lemma 1. (1) *For each* $t\in {}_1\mathfrak{S}(\mathscr{A})$ *the operator* a^*t *is of trace class and* $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$.

(2) *If* $\mathscr{A}$ *is self-adjoint, then* $_1\mathfrak{S}(\mathscr{A})=\mathfrak{S}_1(\mathscr{A})$ *and* $_1\mathfrak{S}(\mathscr{A})_+=\mathfrak{S}_1(\mathscr{A})_+$.

(3) *If* $t\in\mathfrak{S}_1(\mathscr{A})$, *then* $_tf(x)=\mathrm{Tr}\,tx=\mathrm{Tr}\,xt=f_t(x)$ *for all* $x\in\mathscr{A}$.

(4) *For each* $t\in\mathfrak{S}_1(\mathscr{A})_+$ $f_t(x)=\mathrm{Tr}\,xt$ $(=\mathrm{Tr}\,tx)$ *is a strongly positive linear functional on* $\mathscr{A}$.

(5) *Suppose* $\mathscr{A}$ *is an Op$*$-algebra. Then every* $t\in\mathfrak{S}_1(\mathscr{A})$ *can be written as* $t=t_1-t_2+i(t_3-t_4)$ *whereby* $t_j\in\mathfrak{S}_1(\mathscr{A})_+$, $j=1,\dots,4$.

We shall sketch the proof of the lemma.

Let $t\in {}_1\mathfrak{S}(\mathscr{A})$ and $a\in\mathscr{A}$. Since $t^*\in {}_1\mathfrak{S}(\mathscr{A})$, $\overline{t^*a}$ is an operator of trace class. Because of $(\overline{t^*a})^*\supseteqq a^*t$, $\overline{a^*t}$ is of trace class. $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$ is an immediate consequence of the boundedness of ta. This proves (1). (2) is obvious because $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$.

The proof of (3) and (4) is similar to the bounded case. Here we carry out the proof of (4). Suppose $t\in\mathfrak{S}_1(\mathscr{A})_+$ and $a\in\mathscr{A}_+$. We have $\mathrm{Tr}\,ta=\sum_n\langle\overline{ta}\phi_n,\phi_n\rangle$ for each orthonormal basis $\{\phi_n\}_{n\in N}$ of $\mathscr{H}$. We choose an orthonormal basis consisting of eigenvectors ϕ'_m to the non-zero eigenvalues λ_m of t and of elements $\phi''_k\in\mathrm{Lin}(\phi'_m, m\in N)^\perp$. Since the vectors $\phi'_m=\lambda_m^{-1}t\phi'_m$ are contained in $t\mathscr{H}$ and $t\mathscr{H}\subseteqq\mathscr{D}$ for $t\in\mathfrak{S}_1(\mathscr{A})$, it follows $\mathrm{Tr}\,ta=\sum_m\langle\overline{ta}\phi'_m,\phi'_m\rangle+\sum_k\langle\overline{ta}\phi''_k,\phi''_k\rangle=\sum_m\lambda_m\langle a\phi'_m,\phi'_m\rangle\geqq 0$ because the eigenvalues λ_m are positive. Thus (4) is proved.

Next we show (5). By the $*$-invariance of $\mathfrak{S}_1(\mathscr{A})$ we may assume that t is hermitian. As already used in (4) we have $\mathrm{Tr}\,ta=\sum_m\lambda_m\langle a\phi_m,\phi_m\rangle$. Denote by λ_m^+

(λ_m^-) the positive (negative) eigenvalues of t and by ϕ_m^+ (ϕ_m^-) the corresponding eigenvectors. Putting

$$t_+\phi = \sum_m \lambda_m^+ \langle \phi, \phi_m^+ \rangle \phi_m^+ \quad \text{and} \quad t_-\phi = \sum_m -\lambda_m^- \langle \phi, \phi_m^- \rangle \phi_m^-,$$

we have $t = t_+ - t_-$. Because $t \in \mathfrak{S}_1(\mathscr{A})$ the series $\sum_n |\langle \overline{ta}\phi_n, \phi_n \rangle| = \sum_m |\lambda_m \langle a\phi_m', \phi_m' \rangle|$, hence also $\sum_m \lambda_m^+ |\langle a\phi_m^+, \phi_m^+ \rangle|$, are convergent for all $a \in \mathscr{A}$. Putting $a = xx^+$, $x \in \mathscr{A}$, ($\mathscr{A}$ was assumed to be an $Op*$-algebra!) we see that $\sum_m \lambda_m^+ \langle xx^+ \phi_m^+, \phi_m^+ \rangle$ $= \sum_m \|x^+ \sqrt{t_+}\, \phi_m^+\|^2 = \sum_n \|x^+ \sqrt{t_+}\, \phi_n\|^2 < \infty$. This means that $x^+ \sqrt{t_+}$ is an operator of the Hilbert-Schmidt class for each $x \in \mathscr{A}$. Hence $\sqrt{t_+}\, x \subseteq (x^+ \sqrt{t_+})^*$ and $\sqrt{t_+}$ (because $I \in \mathscr{A}$) are also of the Hilbert-Schmidt class. Thus, $t_+ x = \sqrt{t_+}\, \sqrt{t_+}\, x$ is of trace class for all $x \in \mathscr{A}$, i.e. $t_+ \in \mathfrak{S}_1(\mathscr{A})_+$.

Similarly, $t_- \in \mathfrak{S}_1(\mathscr{A})_+$ which finishes the proof of (5).

Remark. For each operator $t \in {}_1\mathfrak{S}(\mathscr{A})$ the functionals ${}_tf(x) = \operatorname{Tr} tx$ and $f_t(x) = \operatorname{Tr}(x^+)^* t$ are well-defined linear functionals on $\mathscr{A}$. But without the additional assumptions $t\mathscr{H} \subseteq \mathscr{D}$, $t^*\mathscr{H} \subseteq \mathscr{D}$, the assertions of Lemma 1, (3) and (4), are no longer true even if $\mathscr{A}$ is a closed $Op*$-algebra. We include a simple counterexample.

Consider the differential operator $x = -\dfrac{d^2}{ds^2}$ on the invariant domain $\mathscr{D}_1$ of all infinitely differentiable functions with support strictly inside the interval $[0,1]$ in the Hilbert space $\mathscr{H} = L_2(0,1)$. Let $\mathscr{P}(x)$ be the $Op*$-algebra on $\mathscr{D}_1$ consisting of all polynomials $p(x)$ in x. The extension $p(x) \to \overline{p(x)} \upharpoonright \mathscr{D}$ of $\mathscr{P}(x)$ to $D := \bigcap_{p \in \mathscr{P}(x)} \mathscr{D}(\overline{p(x)})$ is a closed $Op*$-algebra $\mathscr{A}$ on $\mathscr{D}$. For the function $\psi(s) := \exp\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) s \in C^\infty(0,1)$ we have $x^*\psi = i\psi$. Hence, $\psi \in \mathscr{D}_*(\mathscr{A})$. Clearly, $\psi \notin \mathscr{D}$. Now take the one dimensional operator $P_\psi \xi = \langle \xi, \psi \rangle \psi$ on $\mathscr{H}$. Since $P_\psi a\phi = \langle a\phi, \psi \rangle \psi = \langle \phi, a^*\psi \rangle \psi$ for $a \in \mathscr{A}$, $\phi \in \mathscr{D}$, it is clear that $P_\psi a$ is nuclear for all $a \in \mathscr{A}$. Thus, $P_\psi \in {}_1\mathfrak{S}(\mathscr{A})_+$. We see that $\operatorname{Tr} P_\psi a = \langle \psi, a^*\psi \rangle$ and $\operatorname{Tr}(a^+)^* P_\psi = \langle (a^+)^*\psi, \psi \rangle$ for all $a \in \mathscr{A}$. Putting $a = x$ we get $\operatorname{Tr} P_\psi x = -i\|\psi\|^2$ and $\operatorname{Tr} x^* P_\psi = i\|\psi\|^2$; hence $\operatorname{Tr} P_\psi x \neq \operatorname{Tr} x^* P_\psi$. Therefore neither $\operatorname{Tr} P_\psi a$ nor $\operatorname{Tr}(a^+)^* P_\psi$ are hermitian linear functionals on $\mathscr{A}$. In particular, they are not strongly positive because $x \in \mathscr{A}_+$.

Finally, we collect some definitions about unbounded operator algebras (cf. [3]). $L^+(\mathscr{D})$ is the set of all linear operators a which are together with the adjoint operator a^* defined on $\mathscr{D}$ and leave $\mathscr{D}$ invariant. $L^+(\mathscr{D})$ is a $*$-algebra with the multiplication $(ab)\phi := a(b\phi)$, $\phi \in \mathscr{D}$, and the involution $a \to a^+ := a^* \upharpoonright \mathscr{D}$.

An *$Op*$-algebra* $\mathscr{A}$ on $\mathscr{D}$ is a $*$-subalgebra of $L^+(\mathscr{D})$ which contains the identity $I = I_{\mathscr{D}}$. The locally convex topology $t_{\mathscr{A}}$ on $\mathscr{D}$ defined by the seminorms $\|\phi\|_a := \|a\phi\|$, $a \in \mathscr{A}$, is called the *graph topology* of $\mathscr{A}$ on $\mathscr{D}$. For brevity we write t_+ instead of $t_{L^+(\mathscr{D})}$. An $Op*$-algebra $\mathscr{A}$ is closed if and only if the space $\mathscr{D}[t_{\mathscr{A}}]$ is complete. The domain $\mathscr{D}$ is said to be closed if $L^+(\mathscr{D})$ is closed on $\mathscr{D}$.

By $\mathscr{F}(\mathscr{H})$ we always denote the $*$-invariant vector space of all bounded operators a on $\mathscr{H}$ whose range is contained in a finite dimensional subspace of $\mathscr{H}$. $\mathscr{F}(\mathscr{D})$ is the $*$-invariant subspace of all $a \in \mathscr{F}(\mathscr{H})$ with range $a \subseteqq \mathscr{D}$.

2. *Op*∗-Algebras with a Compact Embedding

Let $\mathscr{A}$ be an $Op*$-algebra on $\mathscr{D}$ and $\mathscr{F}$ a $*$-invariant linear subspace of $\mathscr{F}(\mathscr{H})$ with $\mathscr{F} \supseteqq \mathscr{F}(\mathscr{D})$. By $\mathscr{A}_{\mathscr{F}}$ we denote the linear span of $\mathscr{A}$ and $\mathscr{F}$ regarded as operators on the dense domain $\mathscr{D}$. Clearly, $\mathscr{A}_{\mathscr{F}}$ is $*$-invariant.

Lemma 1. *Let f be a strongly positive linear functional on $\mathscr{A}_{\mathscr{F}}$, i.e. $f(a) \geqq 0$ for all $a \in \mathscr{A}_{\mathscr{F}_+}$. Suppose the Hilbert space $\mathscr{H}$ is separable.*

Then there exists an unique operator $t \in {}_1\mathfrak{S}(\mathscr{A})_+$ such that $f(a) = \operatorname{Tr} ta$ for all $a \in \mathscr{F}$. Furthermore, we have $\operatorname{Tr} a^* ta \leqq f(a^+ a)$ for all $a \in \mathscr{A}$.

Proof. Since all operators $a \in \mathscr{F}(\mathscr{H})$ are bounded and $I \in \mathscr{A}$, $\mathscr{A}_{\mathscr{F}_h}$ is cofinal in the ordered vector space $\mathscr{A}_{\mathscr{F}(\mathscr{H})_h}$. By the classical Krein-Rutman theorem the functional f can be extended to a strongly positive linear functional on $\mathscr{A}_{\mathscr{F}(\mathscr{H})_h}$ and by linearity on $\mathscr{A}_{\mathscr{F}(\mathscr{H})}$. This means that we may assume without loss of generality that $\mathscr{F} = \mathscr{F}(\mathscr{H})$.

Let $P_{\phi,\psi}$ be the one dimensional operator on $\mathscr{D}$ defined by $P_{\phi,\psi}\xi = \langle \xi, \psi \rangle \phi$ for $\phi, \psi \in \mathscr{H}$. Then $P_{\phi,\psi} \in \mathscr{F}(\mathscr{H}) \subseteqq \mathscr{A}_{\mathscr{F}(\mathscr{H})}$ for all $\phi, \psi \in \mathscr{H}$. Let $B(\phi, \psi) := f(P_{\phi,\psi})$ for $\phi, \psi \in \mathscr{H}$. Since $0 \leqq P_{\phi,\phi} \leqq \|\phi\|^2 I$, the strong positivity of f implies $0 \leqq B(\phi, \phi) \equiv f(P_{\phi,\phi}) \leqq \|\phi\|^2 f(I)$. The usual polarization decomposition of $B(\phi, \psi)$ gives us $|B(\phi, \psi)| \leqq C\|\phi\| \, \|\psi\| \; \forall \phi, \psi \in \mathscr{H}$. Hence $B(\phi, \psi)$ is a bounded quadratic form on $\mathscr{H}$. Thus there is a bounded operator $t \in B(\mathscr{H})$ such that $\langle t\phi, \psi \rangle = B(\phi, \psi) \; \forall \phi, \psi \in \mathscr{H}$. Because $B(\phi, \phi) \geqq 0 \; \forall \phi \in \mathscr{H}$, t is a positive (hence self-adjoint) operator.

Our next step is to prove that $t \in {}_1\mathfrak{S}(\mathscr{A})_+$. Suppose that $a \in \mathscr{A}$. By the separability of $\mathscr{H}$ we can choose an orthonormal basis $\{\phi_n\}_{n \in N}$ for $\mathscr{H}$ of vectors $\phi_n \in \mathscr{D}$. Let us consider the operator $b_k := \sum_{j=1}^{k} P_{a\phi_n, a\phi_n} \in \mathscr{F}(\mathscr{D})$. Then

$$\langle b_k \psi, \psi \rangle = \sum_{j=1}^{k} \langle a\phi_n, \psi \rangle \langle \psi, a\phi_n \rangle \leqq \sum_{j=1}^{\infty} |\langle \phi_n, a^+ \psi \rangle|^2 = \|a^+ \psi\|^2 = \langle aa^+ \psi, \psi \rangle$$

for all $\psi \in \mathscr{D}$ which means that $b_k \leqq aa^+$. Using the strong positivity of f we get

$$f(b_k) = \sum_{j=1}^{k} f(P_{a\phi_n, a\phi_n}) = \sum_{j=1}^{k} \langle ta\phi_n, a\phi_n \rangle \leqq f(aa^+)$$

and

$$\sum_{j=1}^{\infty} \langle ta\phi_n, a\phi_n \rangle = \sum_{j=1}^{\infty} \|\sqrt{t}\, a\phi_n\|^2 \leqq f(aa^+). \tag{1}$$

Hence $\sqrt{t}\, a$ is a Hilbert-Schmidt class operator for all $a \in \mathscr{A}$. In particular, $\sqrt{t}$ is of Hilbert-Schmidt type because $I \in \mathscr{A}$. Consequently, $ta = \sqrt{t}\, \sqrt{t}\, a$ is of trace class which proves $t \in {}_1\mathfrak{S}(\mathscr{A})_+$.

Now we verify that $f(a)=\mathrm{Tr}\,ta$ for $a\in\mathscr{F}$. By $f(P_{\phi,\psi})=\langle t\phi,\psi\rangle=\mathrm{Tr}\,tP_{\phi,\psi}$ this is true for the one dimensional operator $P_{\phi,\psi}$. Since each operator $a\in\mathscr{F}(\mathscr{H})$ is a linear combination of one dimensional projections we get $f(a)=\mathrm{Tr}\,ta$ for all $a\in\mathscr{F}(\mathscr{H})$. Obviously, the bounded operator t is uniquely determined by the requirement $\langle t\phi,\psi\rangle=f(P_{\phi,\psi})$ for all vectors ϕ,ψ of the dense domain $\mathscr{D}$. Because $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$, (1) gives us $\mathrm{Tr}\,a^*ta=\sum_{j=1}^{\infty}\langle a^*ta\phi_n,\phi_n\rangle\leqq f(aa^+)$. Now the proof of the lemma is complete.

Remarks. 1. Lemma 1 (in a different form) is due to Uhlmann [19].

2. Let $\mathscr{A}=L^+(\mathscr{D})$ and $\mathscr{F}=\mathscr{F}(\mathscr{D})$. Then $\mathscr{A}_{\mathscr{F}}=L^+(\mathscr{D})$. If f is a strongly positive linear functional on $L^+(\mathscr{D})$, then according to Lemma 1 there is an unique trace functional $g(a)=\mathrm{Tr}\,ta$, $t\in{}_1\mathfrak{S}(\mathscr{A})_+$, on $L^+(\mathscr{D})$ such that f and g coincide for all finite dimensional operators $a\in\mathscr{F}(\mathscr{D})$. We call g the trace part of f.

The main result in this section is

Theorem 2. *Let $\mathscr{A}$ be a self-adjoint Op*-algebra on $\mathscr{D}$ and f a strongly positive linear functional on $\mathscr{A}$. Suppose, there is an operator $c\in\mathscr{A}$ such that the embedding map $i_c:\mathscr{D}(\bar{c})\to\mathscr{H}$ is completely continuous.*

Then f is a trace functional on $\mathscr{A}$ with positive density matrix t, i.e. $f(a)=\mathrm{Tr}\,ta$ with $t\in\mathfrak{S}_1(\mathscr{A})_+$ for all $a\in\mathscr{A}$.

First we note a simple lemma the proof of which is easy and will be omitted.

Lemma 3. *Let c be a closable densely defined linear operator in the Hilbert space $\mathscr{H}$.*

Then the embedding map $i_c:\mathscr{D}(\bar{c})\to\mathscr{H}$ is completely continuous if and only if $(c^*\bar{c}+I)^{-1}$ is completely continuous in $\mathscr{H}$.

Here $\mathscr{D}(\bar{c})$ denotes the domain of the closure $\bar{c}$ of the operator c endowed with the scalar product $\langle\phi,\psi\rangle_c=\langle\bar{c}\phi,\bar{c}\psi\rangle+\langle\phi,\psi\rangle$. In the proof of Theorem 2 we extend an argument due to Woronowicz [21].

Proof of Theorem 2. First let us note that the Hilbert space $\mathscr{H}$ is separable because the dense domain $\mathscr{D}(\bar{c})$ is the range of the completely continuous operator i_c. Since $\mathscr{A}_h$ is cofinal in $\mathscr{A}_{\mathscr{F}(\mathscr{H})_h}$, the functional f can be extended to a strongly positive linear functional on $\mathscr{A}_{\mathscr{F}(\mathscr{H})}$. We will denote this extension by the same symbol f. In virtue of Lemma 1 there is a trace class operator $t\in{}_1\mathfrak{S}(\mathscr{A})_+$ such that $f(a)=\mathrm{Tr}\,ta$ for all $a\in\mathscr{F}(\mathscr{H})$. Since $t\in{}_1\mathfrak{S}(\mathscr{A})_+$ we know that $t\mathscr{H}\subseteqq\mathscr{D}_*(\mathscr{A})$. Because $\mathscr{A}$ was assumed to be self-adjoint, we have $t\mathscr{H}\subseteqq\mathscr{D}$ and hence $t\in\mathfrak{S}_1(\mathscr{A})_+$ (see Lemma 1.1).

By assumption and Lemma 3 the operator $(c^*\bar{c}+I)^{-1}$ is completely continuous. Denote by $\{\lambda_n\}_{n\in N}$ the eigenvalues of this operator (taken with multiplicity) and by $\{\phi_n\}_{n\in N}$ an orthonormal system of the corresponding eigenvectors. Without loss of generality we may assume that $\lambda_n\geqq\lambda_{n+1}$ for all $n\in N$. $\{\phi_n\}_{n\in N}$ is an orthonormal basis because $(c^*\bar{c}+I)^{-1}$ is invertible. Thus we have $(c^*\bar{c}+I)^{-1}\phi=\sum_n\lambda_n\langle\phi,\phi_n\rangle\phi_n$ for each $\phi\in\mathscr{H}$ [10]. Putting $\phi=(c^+c+I)\psi$, $\psi\in\mathscr{D}$, we get

$$\psi=\sum_n\lambda_n\langle(c^+c+I)\psi,\phi_n\rangle\phi_n.$$

We wish to show that $f(x)=\mathrm{Tr}\,tx$ for all $x\in\mathscr{A}$. It is enough to prove this for hermitian elements $x=x^+\in\mathscr{A}$. Let us regard the finite dimensional operator y_k defined by

$$y_k\psi=\sum_{j=1}^{k}\lambda_n\langle(c^+c+I)x\psi,\phi_n\rangle\phi_n.$$

Since we could replace c^+c+I by $c^+c^+y^+y+I$ (which has also compact inverse) and $\mathscr{A}$ is the linear hull of the elements of the form c^+c+y^+y+I, $y\in\mathscr{A}$, it suffices to consider $x=c^+c+I$. Then y_k is a bounded hermitian operator, i.e. $y_k\in\mathscr{F}(\mathscr{H})_h$. For $\psi\in\mathscr{D}$ we get

$$\begin{aligned}|\langle(x-y_k)\psi,\psi\rangle| &=\left|\sum_{n=k+1}^{\infty}\lambda_n\langle(c^+c+I)x\psi,\phi_n\rangle\langle\phi_n,\psi\rangle\right|\\ &\leqq\left(\sup_{n\geqq k+1}\lambda_n\right)\left\{\sum_{n=k+1}^{\infty}|\langle(c^+c+I)x\psi,\phi_n\rangle\langle\phi_n,\psi\rangle|\right\}\\ &\leqq\lambda_{k+1}\left\{\sum_{j=1}^{\infty}|\langle(c^+c+I)x\psi,\phi_n\rangle|^2\right\}^{\frac{1}{2}}\left\{\sum_{j=1}^{\infty}|\langle\phi_n,\psi\rangle|^2\right\}^{\frac{1}{2}}\\ &=\lambda_{k+1}\|(c^+c+I)x\psi\|\,\|\psi\|\leqq\lambda_{k+1}[\|(c^+c+I)x\psi\|^2+\|\psi\|^2]\\ &=\lambda_{k+1}\langle[x(c^+c+I)^2x+I]\psi,\psi\rangle.\end{aligned}$$

Hence $\pm(x-y_k)\leqq[x(c^+c+I)^2x+I]$. By the strong positivity of f on $\mathscr{A}_{\mathscr{F}(\mathscr{H})}$ it follows that

$$|f(x)-f(y_k)|\leqq\lambda_{k+1}f(x(c^+c+I)^2x+I). \tag{2}$$

Since $t\in\mathfrak{S}_1(\mathscr{A})_+$, $\mathrm{Tr}\,ta$ is a strongly positive linear functional on $\mathscr{A}_{\mathscr{F}(\mathscr{H})}$ by Lemma 1.1, (4). Therefore by the same reason we have

$$|\mathrm{Tr}\,tx-\mathrm{Tr}\,ty_k|\leqq\lambda_{k+1}\,\mathrm{Tr}\,t(x(c^+c+I)^2x+I). \tag{3}$$

(2) and (3) together give

$$\begin{aligned}|f(x)-\mathrm{Tr}\,tx| &\leqq|f(x)-f(y_k)|+|f(y_k)-\mathrm{Tr}\,ty_k|+|\mathrm{Tr}\,tx-\mathrm{Tr}\,ty_k|\\ &\leqq\lambda_{k+1}f(x(c^+c+I)^2x+I)+\mathrm{Tr}\,t(x(c^+c+I)^2x+I)=\lambda_{k+1}\cdot\mathrm{const.}.\end{aligned}$$

Here we applied Lemma 1 which gives $f(y_k)=\mathrm{Tr}\,ty_k$. Since $(c^*\bar{c}+I)^{-1}$ is completely continuous, $\lim_{k\to\infty}\lambda_k=0$. Consequently, $f(x)=\mathrm{Tr}\,tx$. This completes the proof of Theorem 2.

Remarks. 1. Sherman [14] has made the assumption that there is an operator in $\mathscr{A}$ which is the restriction to $\mathscr{D}$ of the inverse of a completely continuous operator. This condition is equivalent to our assumption of a completely continuous embedding $i_c:\mathscr{D}(\bar{c})\to\mathscr{H}$ for an operator $c\in\mathscr{A}$ (see Lemma 3).

2. Suppose in Lemma 1 that the $Op*$-algebra $\mathscr{A}$ is self-adjoint. Then $f(a)=\mathrm{Tr}\,ta$ is true for all operators $a\in\mathscr{A}_{\mathscr{F}}$ whose real and imaginary parts are in the τ_0-closure of $\mathscr{F}_h$. Here τ_0 denotes the order topology of the real ordered vector space $\mathscr{A}_{\mathscr{F}_h}$. This statement follows immediately from Lemma 1 and the fact that all

strongly positive linear functionals (in particular, $f(x)$ and $\mathrm{Tr}\,tx$) are τ_0-continuous on $\mathscr{A}_{\mathscr{F}_h}$.

In the preceding proof of Theorem 2 it was shown that (under the assumptions of the theorem) $\mathscr{F}(\mathscr{H})_h$ is τ_0-dense in $\mathscr{A}_{\mathscr{F}(\mathscr{H})_h}$.

3. There are some analogies with the classical moment problem. Each positive linear form on $C_0(X)$, X a locally compact Hausdorff space, is given by a positive Borel measure. Every strongly positive linear functional on $\mathscr{F}(\mathscr{H})$ is given by a trace class operator. In the classical problem of moments each positive linear functional on an adapted vector subspace $\mathscr{A}$ of $C(X)$ can be represented by a positive measure [22]. An equivalent definition of an adapted vector space $\mathscr{A}$ is that $C_0(X)$ is dense in $C_0(X)+\mathscr{A}$ with respect to the order topology τ_0. Calling an $Op*$-algebra $\mathscr{A}$ adapted if $\mathscr{F}(\mathscr{H})$ is τ_0-dense on $\mathscr{A}_{\mathscr{F}(\mathscr{H})}$, then similarly each strongly positive linear form on an adapted $Op*$-algebra is a trace functional.

4. If we replace $\mathfrak{S}_1(\mathscr{A})_+$ by ${}_1\mathfrak{S}(\mathscr{A})_+$ in Theorem 2, then the assertion of this theorem is valid without the assumption of selfadjointness of $\mathscr{A}$. This can be shown by using some results about operator idaels in $Op*$-algebras.

3. *Op**-Algebras on Montel Domains

In the preceding section it was proved that all strongly positive linear functionals on a self-adjoint $Op*$-algebra $\mathscr{A}$ are trace functionals if there is an operator in $\mathscr{A}$ which is the inverse of a completely continuous operator. We shall see below that (even for $L^+(\mathscr{D})$) this condition is not necessary. However, it is often applicable in quantum physics. The most important physical application which was already covered by the results of Woronowicz and Sherman is the $Op*$-algebra generated by the Schrödinger representation on the Schwartz space $\mathscr{S}(R_n)$ of the canonical commutation relations for a finite number of degrees of freedom. Here the number operator is the inverse of a compact operator. This example is a particular case of a more general one which arises from the representation theory of Lie groups. Let U be a strongly continuous unitary representation of a Lie group G. Suppose that the operator $U(f)$ is completely continuous for all functions $f\in L_1(G)$. (Note that this is fulfilled by definition if G is a CCR group in the sense of Kaplansky and U is irreducible.) Then the associated representation dU of the enveloping algebra $\mathscr{E}(G)$ of the corresponding Lie algebra satisfies the assumptions of Theorem 2. If Δ denotes the Nelson Laplacian in $\mathscr{E}(G)$, then $dU(\Delta-1)$ has a compact inverse by a theorem of Nelson and Stinespring ([7], Theorem 4.1). Clearly, $dU(\mathscr{E}(G))$ is self-adjoint on $\mathscr{D}=\bigcap_{n\in N}\mathscr{D}(dU(\Delta-1)^n)$.

In the present section we turn to the characterization of the domains $\mathscr{D}$ with the property that all strongly positive linear functionals on $L^+(\mathscr{D})$ can be given by density matrices. Our main results are contained in the following theorem.

Theorem 1. *Let $\mathscr{D}$ be a dense domain in a Hilbert space $\mathscr{H}$. Suppose $\mathscr{D}[t_+]$ is a Fréchet space. The following are equivalent:*

(1) *$\mathscr{D}[t_+]$ is a Montel space.*

(2) *For each $Op*$-algebra $\mathscr{A}$ on $\mathscr{D}$ with $t_{\mathscr{A}}=t_+$ all strongly positive linear functionals are trace functionals* $\mathrm{Tr}\,ta$ *with $t\in\mathfrak{S}_1(\mathscr{D})_+$.*

(3) *Every strongly positive linear functional on* $L^+(\mathscr{D})$ *is a trace functional* $\mathrm{Tr}\, ta$ *with* $t \in \mathfrak{S}_1(\mathscr{D})_+$.

If $\mathscr{D}[t_+]$ *admits an unconditional basis, then each of these conditions is equivalent to* (4).

(4) $\mathscr{D}$ *contains no infinite dimensional Hilbert space as a subspace, i.e.* $\mathscr{D}$ *is of class I in the sense of* [5].

For the sake of completeness we recall some notions used in the theorem. A system $\{\phi_n\}_{n\in N}$ of elements of a locally convex space $E[\tau]$ is called a basis of E, if each element $\phi \in E$ can be represented in the form $\phi = \sum_{n=1}^{\infty} f_n(\phi)\phi_n$ with uniquely determined coefficients $f_n(\phi)$. The basis $\{\phi_n\}$ is said to be unconditional if the series is unconditionally convergent for each $\phi \in E$. A barreled locally convex space is called Montel space ([2], p. 372) if each bounded set is relatively compact. Since Fréchet spaces are always barreled, in our case the Montel property of $\mathscr{D}[t_+]$ is equivalent to the requirement that the bounded sets are relatively compact.

The proof of the theorem will be given in several steps.

Statement 2. *Suppose* $\mathscr{A}$ *is an* $Op*$*-algebra on a domain* $\mathscr{D}$ *and* $\mathscr{D}[t_{\mathscr{A}}]$ *is a Fréchet Montel space. Then all strongly positive linear functionals on* $\mathscr{A}$ *are of the form* $\mathrm{Tr}\, ta$ *with* $t \in \mathfrak{S}_1(\mathscr{A})_+$.

Further, we have $t_{\mathscr{A}} = t_+$. *Hence the implication* (1)→(2) *in Theorem 1 is true.*

Proof. Sherman [16] has shown that all strongly positive linear functionals on a closed $Op*$-algebra $\mathscr{A}$ of countable dimension are trace functionals $\mathrm{Tr}\, ta$ with $t \in \mathfrak{S}_1(\mathscr{A})_+$ if the following condition is fulfilled:

There is an operator in $\mathscr{A}$ which is the restriction to $\mathscr{D}$ of the inverse of a completely continuous operator on $\mathscr{H}$. $(+)$

By a closer examination of Sherman's proof one can see that the same assertion is true if the weaker condition $(++)$ is satisfied.

$\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space. $(++)$

Let us verify this. The assumption $(+)$ was only used in Sect. 4 of Sherman's paper at two points. Firstly, $(+)$ was used to conclude that the underlying Hilbert space $\mathscr{H}$ is separable ([16], p. 305). Since the Fréchet Montel space $\mathscr{D}[t_{\mathscr{A}}]$ is separable ([2], p. 373) and the topology $t_{\mathscr{A}}$ is stronger than the Hilbert space norm topology (because $I \in \mathscr{A}$), $(++)$ also implies the separability of $\mathscr{H}$. Secondly, $(+)$ is applied in proving Lemma 16 in [16]. (This is the essential point in applying $(+)$). Here $(+)$ is used in order to conclude that if a sequence $\{\psi_n\}_{n\in N}$, $\psi_n \in \mathscr{D}$, converges weakly in $\mathscr{H}$ to $\psi \in \mathscr{D}$ and the set $\{\psi_n\}$ is $t_{\mathscr{A}}$-bounded, then $\{\psi_n\}$ converges to ψ in the topology $t_{\mathscr{A}}$. This is already true if each bounded set in $\mathscr{D}[t_{\mathscr{A}}]$ is relatively compact. Hence the Montel property of $\mathscr{D}[t_{\mathscr{A}}]$ is sufficient for this argument and for the whole proof of Sherman's result.

Our next aim is to show how we can drop the assumption that the $Op*$-algebra $\mathscr{A}$ is of countable dimension. Now suppose $\mathscr{A}$ is an $Op*$-algebra on $\mathscr{D}$ and $\mathscr{D}[t_{\mathscr{A}}]$

is a Fréchet Montel space. Let f be a strongly positive linear functional on $\mathscr{A}$. By the closed graph theorem we have $t_+ = t_{\mathscr{A}}$ because $\mathscr{D}[t_{\mathscr{A}}]$ is a Fréchet space. In particular, it follows that for each $b = b^+ \in L^+(\mathscr{D})$ there is an operator $a \in \mathscr{A}$ such that $\langle b\phi, \phi\rangle \leqq \|b\phi\| \, \|\phi\| \leqq \|a\phi\| \, \|\phi\| \leqq \langle (a^+ a + I)\phi, \phi\rangle$ for all $\phi \in \mathscr{D}$. This means that $\mathscr{A}_h$ is cofinal in $L^+(\mathscr{D})_h$. Hence f can be extended to a strongly positive linear functional on $L^+(\mathscr{D})$. We prove that there is an operator $t \in \mathfrak{S}_1(\mathscr{A})_+$ with $f(a) = \mathrm{Tr}\, ta$ for all $a \in \mathscr{A}$ (this is actually true for all $a \in L^+(\mathscr{D})$).

Suppose the topology of the Fréchet space $\mathscr{D}[t_{\mathscr{A}}]$ is defined by the countable system of seminorms $\|\cdot\|_{a_n}$, $a_n \in \mathscr{A}$, $n \in N$. Fix an orthonormal basis $\{\phi_n\}_{n \in N}$ of vectors $\phi_n \in \mathscr{D}$ which is possible by the separability of $\mathscr{H}$. Take an arbitrary $x \in \mathscr{A}$. By $\mathscr{A}_x$ we denote the $*$-algebra generated by $a_n, P_{\phi_n, \phi_m}, n, m \in N$, and x. Clearly, $\mathscr{A}_x$ is a closed $Op*$-algebra of countable dimension. Since $t_{\mathscr{A}} = t_{\mathscr{A}_x}$, $(+\,+)$ is fulfilled. Therefore by our modified version of Sherman's result there is an operator $t = t_x \in \mathfrak{S}_1(\mathscr{A}_x)_+$ such that $f(a) = \mathrm{Tr}\, ta$ for all $a \in \mathscr{A}_x$. We claim that t_x actually depends only on f and $\mathscr{A}$ but not on x. If $x_1, x_2 \in \mathscr{A}$, then $\mathrm{Tr}\, t_{x_1} P_{\phi_n, \phi_m} = \mathrm{Tr}\, t_{x_2} P_{\phi_n, \phi_m}$ because $P_{\phi_n, \phi_m} \in \mathscr{A}_{x_1} \cap \mathscr{A}_{x_2}$. This means that $\langle t_{x_1}\phi_n, \phi_m\rangle = \langle t_{x_2}\phi_n, \phi_m\rangle$ for all $n, m \in N$. Since the operators t_{x_1}, t_{x_2} are bounded, this implies $t_{x_1} = t_{x_2}$. Hence t_x doesn't depend on x and $f(x) = \mathrm{Tr}\, tx$ for all $x \in \mathscr{A}$.

(2)→(3) is obvious.

Statement 3. (3)→(1).

Proof. Assume the contrary, $\mathscr{D}[t_+]$ is not a Montel space. Then there exists a bounded set $\mathscr{M}$ in $\mathscr{D}[t_+]$ which is not relatively compact in $\mathscr{D}[t_+]$. Hence $\mathscr{M}$ contains a sequence $\{\psi_n\}_{n \in N}$ which has no cluster point in $\mathscr{D}[t_+]$. Since $\{\psi_n\}$ is bounded in the Hilbert space norm (because the identity is in $L^+(\mathscr{D})$), $\{\psi_n\}$ has a weakly convergent subsequence in $\mathscr{H}$. For simplicity suppose that $\{\psi_n\}$ is weakly convergent to $\psi \in \mathscr{H}$. We want to verify that $\psi \in \mathscr{D}$. Let $a \in L^+(\mathscr{D})$. Since $\{a\psi_n\}$ is bounded in $\mathscr{H}$, there is a subsequence $\{a\psi_{n_k}\}$ which is weakly convergent to $\phi \in \mathscr{H}$. For $\eta \in \mathscr{D}(a^*)$ we get $\langle a\psi_{n_k}, \eta\rangle = \langle \psi_{n_k}, a^*\eta\rangle \to \langle \phi, \eta\rangle = \langle \psi, a^*\eta\rangle$. Hence $\psi \in \mathscr{D}(a^{**}) \equiv \mathscr{D}(\bar{a})$. This implies $\psi \in \mathscr{D}$ since the $Op*$-algebra $L^+(\mathscr{D})$ is closed.

The set $\mathscr{K} = \{\psi_n, n \in N\}$ endowed with the induced topology by $\mathscr{D}[t_+]$ is a Tychonoff space. Hence there exists the Stone-Czech compactification $\beta(\mathscr{K})$ of $\mathscr{K}$ ([1], p. 153). The functions $h_a(\phi) = \langle a\phi, \phi\rangle$ for $a \in L^+(\mathscr{D})$ are continuous bounded functions on the topological space $\mathscr{K}$. Thus they can be extended uniquely to continuous functions $h_a(\cdot)$ on $\beta(\mathscr{K})$. The set $\beta(\mathscr{K}) | \mathscr{K}$ is not empty because $\mathscr{K}$ is not compact. Let $s \in \beta(\mathscr{K}) | \mathscr{K}$. We define a linear functional on $L^+(\mathscr{D})$ by setting $f(a) := h_a(s)$. If $a \in L^+(\mathscr{D})_+$, then $h_a(\phi) = \langle a\phi, \phi\rangle \geqq 0$ for all $\phi \in \mathscr{K}$ and hence $h_a(s) \geqq 0$. Consequently, f is a strongly positive linear functional. s is a cluster point of $\mathscr{K}$ because $\mathscr{K}$ is dense in $\beta(\mathscr{K})$. Since the functions $h_a(\cdot)$ are continuous on $\beta(\mathscr{K})$, $f(a) = h_a(s)$ is a cluster point of the set $\{\langle a\psi_n, \psi_n\rangle, n \in N\}$ for each $a \in L^+(\mathscr{D})$. By the assumption f is a trace functional, i.e. $f(a) = \mathrm{Tr}\, ta$ for a certain $t \in \mathfrak{S}_1(\mathscr{D})_+$. Suppose $\xi, \eta \in \mathscr{D}$ and $\varepsilon > 0$. Since $\psi_n \rightharpoonup \psi$, we have $|\langle \psi_n, \eta\rangle \langle \xi, \psi_n\rangle - \langle \psi, \eta\rangle \langle \xi, \psi\rangle| < \frac{\varepsilon}{2}$ for $n \geqq n_0(\varepsilon)$. On the other hand, there is a $k \geqq n_0(\varepsilon)$ such that $|f(P_{\xi,\eta}) - \langle P_{\xi,\eta}\psi_k, \psi_k\rangle| < \frac{\varepsilon}{2}$ because $f(P_{\xi,\eta})$ is a cluster point of $\{\langle P_{\xi,\eta}\psi_n, \psi_n\rangle, n \in N\}$. Combining these inequalities we get $|f(P_{\xi,\eta})-\langle\psi,\eta\rangle\ \langle\xi,\psi\rangle|<\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\langle\psi,\eta\rangle\ \langle\xi,\psi\rangle=f(P_{\xi,\eta})=\langle t\xi,\eta\rangle$. This means that the bounded quadratic forms $\langle t\xi,\eta\rangle$ and $\langle\psi,\eta\rangle\ \langle\xi,\psi\rangle$ coincide on $\mathscr{D}\times\mathscr{D}$. Hence $t=P_\psi$ because $\mathscr{D}$ is dense in $\mathscr{H}$. Now we claim that $\psi\in\mathscr{D}$ is a cluster point of $\mathscr{K}=\{\psi_n, n\in N\}$ in $\mathscr{D}[\ell_+]$. Take $a\in L^+(\mathscr{D})$ and $\varepsilon>0$. Then $\psi_n\rightharpoonup\psi$ implies that $|\langle a^+a\psi,\psi\rangle-\langle a^+a\psi,\psi_n\rangle|<\frac{\varepsilon}{3}$ for all $n\geqq n_1(\varepsilon)$ with a suitable number $n_1(\varepsilon)$. Further choose an integer $k\geqq n_1(\varepsilon)$ such that $|f(a^+a)-\langle a^+a\psi_k,\psi_k\rangle|<\frac{\varepsilon}{3}$. Since $f(a^+a)=\operatorname{Tr}P_\psi a^+a=\|a\psi\|^2$ and $\|a(\psi_k-\psi)\|^2=\|a\psi_k\|^2-\langle a^+a\psi_k,\psi\rangle-\langle a^+a\psi,\psi_k\rangle+\|a\psi\|^2$, this implies $\|a(\psi_k-\psi)\|^2<\varepsilon$. Consequently, ψ is a cluster point of $\mathscr{K}$ in $\mathscr{D}[\ell_+]$ which is a contradiction. This completes the proof.

Lassner and Timmermann [5] proposed a classification of domains of unbounded operator algebras. They called a domain $\mathscr{D}$ of class I if $\mathscr{D}$ contains no infinite dimensional Hilbert space as a subspace. These domains are closely connected with the Montel property of $\mathscr{D}[\ell_+]$ as we shall see by the following lemma.

Lemma 4. *Consider the following properties of a dense linear subspace $\mathscr{D}$ in the Hilbert space $\mathscr{H}$:*

(a) *$\mathscr{D}[\ell_+]$ is a Montel space.*

(b) *$\mathscr{D}$ contains no infinite dimensional linear subspace $\mathscr{D}_1$ such that the topology ℓ_+ on $\mathscr{D}_1$ is normable.*

(c) *$\mathscr{D}$ is of class I.*

Then we have (a)→(c) *and* (b)→(c). *If $\mathscr{D}$ is closed (i.e. $L^+(\mathscr{D})$ is closed), then* (c)→(b). *If $\mathscr{D}[\ell_+]$ is a Fréchet space with an unconditional basis, then* (b)→(a) *and hence* (a), (b), (c) *are equivalent.*

Proof. (a)→(c) and (b)→(c): Assume that $\mathscr{D}$ is not of class I, i.e. $\mathscr{D}$ contains an infinite dimensional Hilbert space $\mathscr{H}_1$ (endowed with the scalar product induced by $\mathscr{H}$!) as a subspace. Let S_1 be the unit ball of $\mathscr{H}_1$. By the closed graph theorem all operators $a\in L^+(\mathscr{D})$ are bounded on $\mathscr{H}_1$. Therefore the restriction of ℓ_+ to $\mathscr{H}_1$ coincides with the usual norm topology of the Hilbert space $\mathscr{H}_1$. This contradicts (b). Thus (b)→(c) is proved. Further, S_1 is a bounded set which is not relatively compact in $\mathscr{D}[\ell_+]$. This is a contradiction to the Montel property of $\mathscr{D}[\ell_+]$. Hence we have (a)→(c).

Suppose now that the domain $\mathscr{D}$ is closed. We show (c)→(b). Assume the contrary of (b). Let $\mathscr{D}_1[\ell_+]$ be an infinite dimensional topological linear subspace of $\mathscr{D}[\ell_+]$ such that the topology ℓ_+ on $\mathscr{D}_1$ can be defined by a norm $\|\cdot\|'$. Since $\ell_{\|\cdot\|'}\subseteqq\ell_+$, there is an operator $a_0\in L^+(\mathscr{D})$ with $\|\phi\|'\leqq\|a_0\phi\|$ for all $\phi\in\mathscr{D}_1$. Conversely, $\ell_+\subseteqq\ell_{\|\cdot\|'}$ implies that $\|a_0\phi\|\leqq C\|\phi\|'$ on $\mathscr{D}_1$. Hence the norm $\|\phi\|_{a_0}:=\|a_0\phi\|$ defines the topology ℓ_+ on $\mathscr{D}_1$. Let $(\mathscr{H}_1,\|\cdot\|_{a_0})$ be the completion of $(\mathscr{D}_1,\|\cdot\|_{a_0})$. Since $\mathscr{D}[\ell_+]$ is a complete space, $\mathscr{H}_1$ can be identified with the topological closure of $\mathscr{D}_1$ in $\mathscr{D}[\ell_+]$. Thus $\mathscr{H}_1\subseteqq\mathscr{D}$. Clearly, $(\mathscr{H}_1,\|\cdot\|_{a_0})$ is a Hilbert space. Therefore $\mathscr{H}_2:=a_0\mathscr{H}_1$ equipped with the Hilbert space norm $\|\cdot\|$ is an infinite dimensional Hilbert space. Since $\mathscr{H}_1\subseteqq\mathscr{D}$ and $a_0\in L^+(\mathscr{D})$, we have $\mathscr{H}_2\subseteqq\mathscr{D}$ which is a contradiction to (c).

Next we show that if $\mathscr{D}[t_+]$ is a Fréchet space with an unconditional basis, then (b)→(a). Let $\{\phi_n\}_{n\in N}$ be an unconditional basis of $\mathscr{D}[t_+]$. The topology t_+ can be given by a countable system of Hilbert space norms $\|\cdot\|_{a_k}$, $k\in N$, $a_k\in L^+(\mathscr{D})$. According to ([6], Prop. 1, p. 117), the space $\mathscr{D}$ is isomorphic to the Köthe space

$$1_2(\langle\phi_n,\phi_n\rangle_{a_k})=\left\{(\alpha_n):\sum_{n=1}^{\infty}|\alpha_n|^2\langle\phi_n,\phi_n\rangle_{a_k}=:q_k((\alpha_n))^2<\infty\right\}.$$

and the seminorms q_k, $k\in N$, define the graph topology t_+. Thus $\mathscr{D}[t_+]$ is a "Stufenraum" of order $p=2$ in the sense of Köthe [2]. Suppose (b) is fulfilled. In particular, there is no subspace $\mathscr{D}_1$ of $\mathscr{D}$ such that $\mathscr{D}_1[t_+]$ is topologically isomorphic to the Hilbert space 1_2. By a theorem of Köthe ([2], p. 424) this implies that $\mathscr{D}[t_+]$ is a Montel space?[1]

The statements and the lemma together prove the theorem.

Problem. Suppose the domain $\mathscr{D}$ is of class I and $\mathscr{D}[t_+]$ is a Fréchet space. Can we conclude that $\mathscr{D}[t_+]$ is always a Montel space?[1]

Remarks. 1. The assumption that $\mathscr{D}[t_+]$ is a Montel space is properly weaker than the existence of an operator $c\in L^+(\mathscr{D})$ with compact embedding map $i_c:\mathscr{D}(\bar{c})\to\mathscr{H}$. First we claim that a Fréchet domain $\mathscr{D}[t_+]$ is a Montel space if there is an operator $c\in L^+(\mathscr{D})$ such that the map i_c is compact. Indeed, let $\mathscr{M}$ be a bounded subset of $\mathscr{D}[t_+]$ and $a\in L^+(\mathscr{D})$. Then, in particular, the set $(c^+c+I)a\mathscr{M}$ is normbounded in the Hilbert space $\mathscr{H}$. Since the operator $(c^*\bar{c}+I)^{-1}$ is completely continuous (cf. Lemma 2.3), the set $a\mathscr{M}$ is relatively compact in the Hilbert space $\mathscr{H}$. Using the fact that the domain $\mathscr{D}$ is closed we see that $\mathscr{M}$ is a relatively compact subset of $\mathscr{D}[t_+]$.

The following example showing that the converse implication is not true is a slight modification of an example due to Köthe ([2], p. 436). Let us consider the infinite matrices

$$x^{(k)}=(y_1^{(k)},\dots,y_{k-1}^{(k)};k^k e,k^{k+1}e,\dots)$$

where

$$y_j^{(k)}=(1,2^k,3^k,\dots),e=(1,1,1,\dots),j=1,\dots,k-1,k\in N.$$

By a diagonal procedure we write each matrix $x^{(k)}$ as a sequence. Then $x^{(k)}$ corresponds a diagonal operator a_k in the Hilbert space 1_2. Let

$$\mathscr{D}=\bigcap_{r,k_j,n_j\in N}\mathscr{D}(a_{k_1}^{n_1}a_{k_2}^{n_2}\dots a_{k_r}^{n_r})$$

be the intersection of the domains of all finite products of operators a_k, $k\in N$. Clearly, the operators $a_{k_1}^{n_1}\dots a_{k_r}^{n_r}$ are in $L^+(\mathscr{D})$. This implies that $\mathscr{D}[t_+]$ is a Fréchet space. From the definition it is clear that for all operators $a_{k_1}^{n_1}\dots a_{k_r}^{n_r}$ the embedding map $i_{a_{k_1}^{n_1}\dots a_{k_r}^{n_r}}$ is not completely continuous. Using the closed graph theorem, it follows that the embedding map $i_a:\mathscr{D}(\bar{a})\to\mathscr{H}$ is not compact for each $a\in L^+(\mathscr{D})$.

$\mathscr{D}[t_+]$ is a Fréchet Montel space by Köthe's criterion ([2], p. 424). Hence $\mathscr{D}$ is of class I according to Lemma 4, (a)→(c). But there is no operator $c\in L^+(\mathscr{D})$ such that $\mathscr{D}(\bar{c})$ is of class I. Otherwise we would have the existence of a compact

1 Added in proof. The answer to this question is affirmative (P. Kröger, oral communication). Hence the basis assumption in Theorem 3.1 can be dropped

embedding (cf. [5], Prop. 1, p. 160) which is impossible according to the preceding discussion.

2. Notice that for domains of the form $\mathscr{D} = \bigcap_{n \in N} \mathscr{D}(a^n)$, a selfadjoint operator, $\mathscr{D}[t_+]$ has always an unconditional basis. This is a consequence of the spectral theorem. In this case the following conditions are equivalent:

(a) The embedding map $i_a : \mathscr{D}(a) \to \mathscr{H}$ is compact.

(b) $(a^2 + I)^{-1}$ is completely continuous.

(c) $\mathscr{D}[t_+]$ is a Montel space.

(d) The spectrum of a consists of a countable set of eigenvalues λ_n with $\lim_{n \to \infty} |\lambda_n| = +\infty$.

(e) $\mathscr{D}$ is of class I.

Let us add few remarks concerning the proof. (a)$\leftrightarrow$(b) is clear by Lemma 2.3. (c)$\leftrightarrow$(d) was proved by Pietsch [8]. Finally, (b)$\leftrightarrow$(d) is well-known from operator theory in a Hilbert space.

4. On the Continuity of Trace Functionals

The main purpose of the present section is to characterize the (not necessarily positive) trace functionals on an $Op*$-algebra $\mathscr{A}$ by the continuity in a certain locally convex topology $\tau_{\mathscr{D}}^c$ on $\mathscr{A}$. First we shall define the topologies under discussion. The uniform topology $\tau_{\mathscr{D}}$ on an $Op*$-algebra $\mathscr{A}$ (see [3]) is given by the seminorms $p_{\mathscr{M}}(a) := \sup_{\phi, \psi \in \mathscr{M}} |\langle a\phi, \psi \rangle|$ taken for all bounded sets $\mathscr{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$. The well-known decomposition

$$\begin{aligned}\langle a\phi, \psi \rangle = \tfrac{1}{4}\{&\langle a(\phi+\psi), \phi+\psi \rangle - \langle a(\phi-\psi), \phi-\psi \rangle \\ &- i\langle a(\phi+i\psi), \phi+i\psi \rangle + i\langle a(\phi-i\psi), \phi-i\psi \rangle\}\end{aligned}$$

implies that $\tau_{\mathscr{D}}$ can also be defined by the equivalent system of seminorms $p'_{\mathscr{M}}(a) := \sup_{\phi \in \mathscr{M}} |\langle a\phi, \phi \rangle|$. If we restrict ourselves to relatively compact (bounded) subsets $\mathscr{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$, then the seminorms $p_{\mathscr{M}}(a)$ (or equivalently, the seminorms $p'_{\mathscr{M}}(a)$) define a locally convex topology denoted by $\tau_{\mathscr{D}}^c$. Clearly, $\tau_{\mathscr{D}} \supseteqq \tau_{\mathscr{D}}^c$. If $\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space, then we have $\tau_{\mathscr{D}}^c = \tau_{\mathscr{D}}$.

Since the image $a\mathscr{M}$ of a relatively compact subset $\mathscr{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$ is relatively compact, the right and left multiplications in $\mathscr{A}$ are $\tau_{\mathscr{D}}^c$-continuous. The continuity of the involution is trivial. Therefore, $\mathscr{A}[\tau_{\mathscr{D}}^c]$ is a topological $*$-algebra.

Now we are in position to establish our results. We assume in the following that the underlying Hilbert space is separable.

Proposition 1. *Let $\mathscr{A}$ be a closed $Op*$-algebra on the dense domain $\mathscr{D}$. For each continuous linear functional f on $\mathscr{A}[\tau_{\mathscr{D}}^c]$ there exists an operator $t \in \mathfrak{S}_1(\mathscr{A})$ such that $f(a) = \operatorname{Tr} ta$ for all $a \in \mathscr{A}$.*

Proof. Since $f \in \mathscr{A}[\tau_{\mathscr{D}}^c]'$, there is a relatively compact set $\mathscr{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$ such that $|f(a)| \leqq p'_{\mathscr{M}}(a) \equiv \sup_{\phi \in \mathscr{M}} |\langle a\phi, \phi \rangle|$. Without loss of generality we may assume that $\mathscr{M}$ is closed in $\mathscr{D}[t_{\mathscr{A}}]$. Then $\mathscr{M}$ endowed with the topology $t_{\mathscr{A}}$ is a compact Hausdorff space. Each operator $a \in \mathscr{A}$ corresponds to a function $h_a(\phi) := \langle a\phi, \phi \rangle$ on $\mathscr{M}$. These functions are continuous on $\mathcal{M}[t_{\mathcal{A}}]$ for all $a\in\mathcal{A}$. By putting $F(h_a)=f(a)$ we define a linear functional on the vector space $\mathcal{V}$ of the functions h_a, $a\in\mathcal{A}$. This definition is correct because $h_a(\phi)=0$ for all $\phi\in\mathcal{M}$ implies $f(a)=0$ by $|f(a)|\leqq p'_{\mathcal{M}}(a)$. By the Hahn-Banach theorem we extend F to a continuous linear functional (also denoted by F) on the C^*-algebra $C(\mathcal{M})$ of all continuous functions on the compact Hausdorff space $\mathcal{M}$. F can be written as a linear combination $F=F_1-F_2+i(F_3-F_4)$ of positive linear functionals $F_1,\dots,F_4$ on the C^*-algebra $C(\mathcal{M})$. Obviously, for all elements $\xi,\eta\in\mathcal{H}$ the functions $h_{P_{\xi,\eta}}(\phi)=\langle\xi,\phi\rangle\,\langle\phi,\eta\rangle$ are $t_{\mathcal{A}}$-continuous on $\mathcal{M}$ (they are even continuous in the Hilbert space norm). Since each $a\in\mathcal{F}(\mathcal{H})$ is a linear combination of operators $P_{\xi,\eta}$, where $\xi,\eta\in\mathcal{H}$, $C(\mathcal{M})$ contains all functions $h_a(\phi)=\langle a\phi,\phi\rangle$ for the operators $a\in\mathcal{F}(\mathcal{H})$ and hence for $a\in\mathcal{A}_{\mathcal{F}(\mathcal{H})}$. Therefore F_j, $j=1,\dots,4$, induce strongly positive linear functionals f_j on $\mathcal{A}_{\mathcal{F}(\mathcal{H})}$ by the definition $f_j(a)=F_j(h_a)$, $a\in\mathcal{A}_{\mathcal{F}(\mathcal{H})}$. It is sufficient to prove the assertion for the strongly positive linear functionals f_j, $j=1,\dots,4$, because $f=f_1-f_2+i(f_3-f_4)$ and $\mathfrak{S}_1(\mathcal{A})$ is a vector space.

By the Riesz representation theorem there exists a positive Borel measure μ_j on the compact space $\mathcal{M}$ such that

$$F_j(h)=\int_{\mathcal{M}} h(\phi)d\mu_j(\phi)$$

for all functions $h\in C(\mathcal{M})$. In particular, for the functions h_a, $a\in\mathcal{A}_{\mathcal{F}(\mathcal{H})}$, it means that

$$F_j(h_a)=f_j(a)=\int_{\mathcal{M}}\langle a\phi,\phi\rangle d\mu_j(\phi).$$

In virtue of Lemma 2.1 there is an operator $t_j\in{}_1\mathfrak{S}(\mathcal{A})_+$ such that $f_j(a)=\operatorname{Tr}t_ja$ for all $a\in\mathcal{F}(\mathcal{H})$. We claim that $f_j(a)=\operatorname{Tr}t_ja$ for each operator $a\in\mathcal{A}$, $j=1,\dots,4$.

Take an orthonormal basis $\{\phi_n\}_{n\in N}$ on $\mathcal{H}$ of vectors $\phi_n\in\mathcal{D}$. Suppose $a\in\mathcal{A}$. Then we have

$$\begin{aligned}\operatorname{Tr}t_ja&=\sum_{n=1}^{\infty}\langle t_ja\phi_n,\phi_n\rangle=\sum_{n=1}^{\infty}\operatorname{Tr}t_jP_{a\phi_n,\phi_n}\\&=\sum_{n=1}^{\infty}f_j(P_{a\phi_n,\phi_n})=\sum_{n=1}^{\infty}\int_{\mathcal{M}}\langle P_{a\phi_n,\phi_n}\phi,\phi\rangle d\mu_j(\phi)\\&=\sum_{n=1}^{\infty}\int_{\mathcal{M}}\langle a\phi_n,\phi\rangle\langle\phi,\phi_n\rangle d\mu_j(\phi)=\int_{\mathcal{M}}\sum_{n=1}^{\infty}\langle a\phi_n,\phi\rangle\langle\phi,\phi_n\rangle d\mu_j(\phi)\\&=\int_{\mathcal{M}}\langle\phi,a^+\phi\rangle d\mu_j(\phi)=f_j(a).\end{aligned}$$

To interchange the summation and integration we could apply the Lebesgue theorem, because

$$\sum_{n=1}^{\infty}|\langle a\phi_n,\phi\rangle\langle\phi,\phi_n\rangle|\leqq\|a^+\phi\|\,\|\phi\|\leqq\langle(aa^++I)\phi,\phi\rangle$$

and the function $h_{aa^++I}(\phi)=\langle(aa^++I)\phi,\phi\rangle$ is μ_j-integrable.

It remains to show that the t_j is in $\mathfrak{S}_1(\mathcal{A})$ for $j=1,\dots,4$ but not only in ${}_1\mathfrak{S}(\mathcal{A})$. [Notice that for self-adjoint $Op*$-algebras this is automatically fulfilled by Lemma 1.1, (2).]

Take an arbitrary vector $\xi \in \mathscr{H}$. Let $a \in \mathscr{A}$. It is enough to check that $t_j \xi \in \mathscr{D}(a^{**})$ because $\mathscr{A}$ was assumed to be closed on $\mathscr{D}$ and hence $\mathscr{D} = \bigcap_{a \in \mathscr{A}} \mathscr{D}(\bar{a}) \equiv \bigcap_{a \in \mathscr{A}} \mathscr{D}(a^{**})$. Suppose $\eta \in \mathscr{D}(a^*)$. Since $h_{P_{\xi, a^* \eta}} \in C(\mathscr{M})$, we have

$$\langle t_j \xi, a^* \eta \rangle = f_j(P_{\xi, a^* \eta}) = \int_{\mathscr{M}} \langle \xi, \phi \rangle \langle \phi, a^* \eta \rangle d\mu_j(\phi)$$
$$= \int_{\mathscr{M}} \langle \xi, \phi \rangle \langle a\phi, \eta \rangle d\mu_j(\phi).$$

Now we estimate

$$|\langle t_j \xi, a^* \eta \rangle| \leqq \int_{\mathscr{M}} |\langle \xi, \phi \rangle \langle a\phi, \eta \rangle| d\mu_j(\phi)$$
$$\leqq \|\xi\| \, \|\eta\| \int_{\mathscr{M}} \|a\phi\| \, \|\phi\| \, d\mu_j(\phi) \leqq \|\xi\| \, \|\eta\| \int_{\mathscr{M}} \langle (a^+ a + I)\phi, \phi \rangle d\mu_j(\phi)$$
$$= \text{const.} \cdot \|\eta\|.$$

This means that $t_j \xi \in \mathscr{D}(a^{**})$. Therefore $t_j \mathscr{H} \subseteqq \mathscr{D}$ and hence $t \in \mathfrak{S}_1(\mathscr{A})$.

This finishes the proof.

For Montel spaces $\mathscr{D}[t_{\mathscr{A}}]$ the topologies $\tau_{\mathscr{D}}$ and $\tau_{\mathscr{D}}^c$ coincide. Thus we get the following corollary which generalizes Theorem 3 in [4] because the existence of the inverse of a nuclear operator in $\mathscr{A}$ implies the Montel property of $\mathscr{D}[t_{\mathscr{A}}]$ (see Remark 1 in the preceding section).

Corollary 2. *Let $\mathscr{A}$ be a closed Op*-algebra on $\mathscr{D}$. Suppose $\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space.*

Then all uniformly continuous linear functionals f an $\mathscr{A}$ (i.e. $f \in \mathscr{A}[\tau_{\mathscr{D}}]'$) are of the form $f(a) = \mathrm{Tr}\, ta$ with $t \in \mathfrak{S}_1(\mathscr{A})$.

The following proposition deals with the converse problem.

Proposition 3. *Suppose $\mathscr{A}$ is an Op*-algebra with metrizable graph topology $t_{\mathscr{A}}$.*

If $t \in \mathfrak{S}_1(\mathscr{A})$, then the linear functional $f(a) = \mathrm{Tr}\, ta$ is $\tau_{\mathscr{D}}^c$-continuous on $\mathscr{A}$.

The proof of Proposition 3 is a modification of the argument used in proving Theorem 2 in [4].

Proof. Since each operator $t \in \mathfrak{S}_1(\mathscr{A})$ is a linear combination of positive operators $t_j \in \mathfrak{S}_1(\mathscr{A})_+$ [Lemma 1.1, (5)], it only remains to prove the $\tau_{\mathscr{D}}^c$-continuity of $f(a) = \mathrm{Tr}\, ta$ for positive operators $t \in \mathfrak{S}_1(\mathscr{A})_+$. Suppose the topology $t_{\mathscr{A}}$ is defined by the countable set of seminorms $\|\cdot\|_{a_k}$, $a_k \in \mathscr{A}$, $k \in N$. Since $t \in \mathfrak{S}_1(\mathscr{A})_+$ there are an orthonormal system of vectors $\phi_n \in \mathscr{D}$ and positive numbers λ_n such that $f(a) = \mathrm{Tr}\, ta = \sum_n \lambda_n \langle a\phi_n, \phi_n \rangle$ for all $a \in \mathscr{A}$.

We show that there is a sequence $\{\alpha_n\}_{n \in N}$ of positive numbers such that

(1) $\lim_{n \to \infty} \alpha_n \|a_k \phi_n\|^2 = 0$ for all $k \in N$

and

(2) $\sum_n \lambda_n \alpha_n^{-1} < \infty$.

For the sake of continuity we shall verify the existence of such a sequence below. Using this sequence, let us consider the set $\mathscr{M} = \{\sqrt{\alpha_n}\, \phi_n, n \in N\}$. Clearly, $\mathscr{M}$

is relatively $t_{\mathscr{A}}$-compact because each countable subset of $\mathscr{M}$ is converging to zero in $\mathscr{D}[t_{\mathscr{A}}]$. We have

$$|f(a)| = \left|\sum_n \lambda_n \langle a\phi_n, \phi_n\rangle\right| = \left|\sum_n \lambda_n \alpha_n^{-1} \alpha_n \langle a\phi_n, \phi_n\rangle\right|$$
$$\leqq \left(\sum_n \lambda_n \alpha_n^{-1}\right) \sup_n |\langle a\sqrt{\alpha_n}\phi_n, \sqrt{\alpha_n}\phi_n\rangle| \equiv C p'_{\mathscr{M}}(a) \quad \text{for all} \quad a \in \mathscr{A}.$$

Hence f is $\tau^c_{\mathscr{D}}$-continuous.

Now we construct the sequence $\{\alpha_n\}_{n\in N}$ by induction on n. Put

$$\beta_{k,r} = \max\{1; \langle (a_1^+ a_1 + a_2^+ a_2 + \ldots + a_r^+ a_r)\phi_k, \phi_k\rangle\}.$$

Let $\alpha_1 = 2^{-1}\beta_{1,1}^{-1}$, $\alpha_2 = 2^{-1}\beta_{2,1}^{-1}, \ldots, \alpha_{n_2-1} = 2^{-1}\beta_{n_2-1,1}^{-1}$ where the number $n_2 \in N$ is chosen so large that $\sum_{r \geqq n_2} \lambda_r \beta_{r2} \leqq 2^{-2\cdot 2}$. This is possible because $f(a_1^+ a_1 + a_2^+ a_2) = \sum_n \lambda_n \langle (a_1^+ a_1 + a_2^+ a_2)\phi_n, \phi_n\rangle \leqq \sum_n \lambda_n \beta_{n,2} < \infty$. Further, put

$$\alpha_{n_2} = 2^{-2}\beta_{n_2,2}^{-1}, \ldots, \alpha_{n_3-1} = 2^{-2}\beta_{n_3-1,2}^{-1}.$$

n_3 will be chosen such that $\sum_{r \geqq n_3} \lambda_r \beta_{r,3} \leqq 2^{-2\cdot 3}$. Continuing this construction, we obtain a sequence $\{\alpha_n\}_{n\in N}$. We check the conditions (1) and (2). First we get

$$\sum_n \lambda_n \alpha_n^{-1} = \lambda_1 2^1 \beta_{1,1} + \lambda_2 2^1 \beta_{2,1} + \ldots + \lambda_{n_2-1} 2^1 \beta_{n_2-1,1} + \sum_{s=2}^{\infty} \left\{ \sum_{n_s \leqq k \leqq n_{s+1}-1} \lambda_k 2^s \beta_{ks} \right\}$$
$$\leqq \text{const.} + \sum_{s=2}^{\infty} 2^s \{2^{-2\cdot s}\} < \infty$$

which gives (2).

Next we show (1). By the construction we see that for $j = 1, \ldots, s$ $\beta_{n_s+k,s} \geqq \langle a_j^+ a_j \phi_{n_s+k}, \phi_{n_s+k}\rangle$ and hence

$$\alpha_{n_s+k} \|a_j \phi_{n_s+k}\|^2 = 2^{-k} \beta_{n_s+k,s}^{-1} \|a_j \phi_{n_s+k}\|^2 \leqq 2^{-k}.$$

Obviously this implies (1).

Propositions 1 and 3 together give us Theorem 4.

Theorem 4. *Suppose $\mathscr{A}$ is a closed $Op*$-algebra on $\mathscr{D}$ with metrizable graph topology $t_{\mathscr{A}}$. Suppose the Hilbert space $\mathscr{H}$ is separable.*

A linear functional f on $\mathscr{A}$ is of the form $f(a) = \operatorname{Tr} ta$ with $t \in \mathfrak{S}_1(\mathscr{A})$ if and only if it is $\tau^c_{\mathscr{D}}$-continuous on $\mathscr{A}$. If $\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space, then the $\tau_{\mathscr{D}}$-continuity of a linear functional on $\mathscr{A}$ is a necessary and sufficient condition that f is a trace functional $f(a) = \operatorname{Tr} ta$ whereby $t \in \mathfrak{S}_1(\mathscr{A})$.

We note a corollary of the preceding results.

Corollary 5. *Suppose $\mathscr{A}$ is an $Op*$-algebra on $\mathscr{D}$ such that $\mathscr{D}[t_{\mathscr{A}}]$ is a Fréchet Montel space. Then each strongly positive linear functional f on $\mathscr{A}$ is $\tau_{\mathscr{D}}$-continuous.*

According to Theorem 3.1 (more precisely, Lemma 3.2) $f(a)$ is a trace functional $\operatorname{Tr} ta$ with $t \in \mathfrak{S}_1(\mathscr{A})_+$. Thus Proposition 3 gives the $\tau_{\mathscr{D}}$-continuity.

5. Some Applications

In this section we apply the results of the preceding section to some concrete $Op*$-algebras. Furthermore, we want to demonstrate how one can get results on the structure of the state space and of the linear functionals on unbounded operator algebras by topological methods.

Example 1. *Let $\mathscr{A}$ be the $Op*$-algebra generated by the position and momentum operators q_j, p_j, $j=1,\ldots,n$, on the Schwartz space $\mathscr{S}(R_n)$.*

Then each linear functional f on $\mathscr{A}$ is a trace functional $f(a)=\operatorname{Tr} ta$, $a\in\mathscr{A}$, whereby $t\in\mathfrak{S}_1(\mathscr{A})$.

First Proof. The uniform topology $\tau_{\mathscr{D}}$ on $\mathscr{A}$ is equal to the strongest locally convex topology τ_{st} on $\mathscr{A}$. This was first shown in [17]; another proof is contained in [14]. The graph topology $t_{\mathscr{A}}$ is the usual topology of the space $\mathscr{S}(R_n)$. Since the Schwartz space is a Montel space, we have $\tau_{\mathscr{D}}=\tau_{\mathscr{D}}^c$ and hence $\tau_{\mathscr{D}}^c=\tau_{st}$. Now the assertion follows from Theorem 4.4 or from Proposition 4.1.

To illustrate how the topological method works we include a second proof which avoids the application of Theorem 4.4 or Proposition 4.1.

Second Proof. Since $\tau_{\mathscr{D}}=\tau_{st}$, the cone $\mathscr{A}_+$ is τ_{st}-normal and hence each linear functional on $\mathscr{A}$ is a linear combination of strongly positive linear functionals on $\mathscr{A}$ [11]. Thus it is enough to show the assertion for strongly positive linear functionals. But in this case we can apply Theorem 2.2.

Example 2. *Let G be a compact connected Lie group and $\mathscr{E}(G)$ the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Let dU_r be the realization of $\mathscr{E}(G)$ as a closed $Op*$-algebra of left invariant differential operators on G with the domain $\mathscr{D}=C^\infty(G)\cap L_2(G,\mu)$; μ denotes the Haar measure of G.*

Then every linear functional f on $dU_r(\mathscr{E}(G))$ is a trace functional $f(a)=\operatorname{Tr} ta$ with $t\in\mathfrak{S}_1(dU_r(\mathscr{E}(G)))$. If f is strongly positive on the $Op*$-algebra $dU_r(\mathscr{E}(G))$, then there is an operator $t\in\mathfrak{S}_1(dU_r(\mathscr{E}(G)))_+$ such that $f(a)=\operatorname{Tr} ta$.

Proof. Let $x_1,\ldots,x_n$ be a basis of $\mathfrak{g}$ and $\Delta=x_1^2+\ldots+x_n^2$. Since the group G is compact, the operator $dU_r(\Delta-1)$ has a compact inverse. Hence $\mathscr{D}[t_{dU_r(\mathscr{E}(G))}]$ is a Montel space. By ([13], Theorem 1), the uniform topology $\tau_{\mathscr{D}}$ is equal to the strongest locally convex topology τ_{st} on $dU_r(\mathscr{E}(G))$. Therefore $\tau_{\mathscr{D}}^c=\tau_{st}$ because $\tau_{\mathscr{D}}=\tau_{\mathscr{D}}^c$ for graph topologies with Montel property. From Theorem 4.4 follows that $f(a)=\operatorname{Tr} ta\ \forall a\in\mathscr{A}$ where $t\in\mathfrak{S}_1(dU_r(\mathscr{E}(G)))$. If f is strongly positive, then we obtain the assertion from Theorem 2.2.

Both examples are contained in the following general theorem. Using Lie group representations or differential operators on R^n or on manifolds it is possible to derive further applications from this theorem.

Theorem 3. *Let $\mathscr{A}$ be a countably generated, closed $Op*$-algebra on $\mathscr{D}$. Suppose the following assumptions are satisfied:*

(1) *$\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space.*

(2) *For every $x\in\mathscr{A}$ the vector space*

$$\mathscr{N}_x:=\{a\in\mathscr{A}:|\langle a\phi,\phi\rangle|\leqq C_{a,x}\|x\phi\|^2\ \forall\phi\in\mathscr{D}\}$$

is finite dimensional.

Then for each linear functional f on $\mathscr{A}$ there is an operator $t \in \mathfrak{S}_1(\mathscr{A})$ such that $f(a) = \operatorname{Tr} ta \; \forall a \in \mathscr{A}$.

Proof. According to Theorem 1 in [14], condition (2) is equivalent to $\tau_{\mathscr{D}} = \tau_{st}$. Since $\tau_{\mathscr{D}} = \tau_{\mathscr{D}}^c$ by the Montel property of $t_{\mathscr{A}}$, this implies $\tau_{\mathscr{D}}^c = \tau_{st}$. Further, the Fréchet Montel space $\mathscr{D}[t_{\mathscr{A}}]$ is separable [2]. Hence the underlying Hilbert space must be separable. Now the assertion is an immediate consequence of Theorem 4.4.

Remarks. 1. Clearly, it is enough to check condition (2) for a sequence $\{x_n\}$ of operators in $\mathscr{A}$ for which the seminorms $\|\phi\|_{x_n} := \|x_n \phi\|$ already define the topology $t_{\mathscr{A}}$ on $\mathscr{D}$.

2. By considering examples it is not difficult to see that neither condition (1) nor (2) can be dropped in Theorem 3.

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