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Generic Instability of Rotating Relativistic Stars

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Abstract. All rotating perfect fluid configurations having two-parameter equations of state are shown to be dynamically unstable to nonaxisymmetric perturbations in the framework of general relativity. Perturbations of an equilibrium fluid are described by means of a Lagrangian displacement, and an action for the linearized field equations is obtained, in terms of which the symplectic product and canonical energy of the system can be expressed. Previous criteria governing stability were based on the sign of the canonical energy, but this functional fails to be invariant under the gauge freedom associated with a class of trivial Lagrangian displacements, whose existence was first pointed out by Schutz and Sorkin [12]. In order to regain a stability criterion, one must eliminate the trivials, and this is accomplished by restricting consideration to a class of "canonical" displacements, orthogonal to the trivials with respect to the symplectic product. There nevertheless remain perturbations having angular dependence $e^{im\phi}$ (ϕ the azimuthal angle) which, for sufficiently large m, make the canonical energy negative; consequently, even slowly rotating stars are unstable to short wavelength perturbations. To show strict instability, it is necessary to assume that time-dependent nonaxisymmetric perturbations radiate energy to null infinity. As a byproduct of the work, the relativistic generalization of Ertel's theorem (conservation of vorticity in constant entropy surfaces) is obtained and shown to be Noether-related to the symmetry associated with the trivial displacements.

I. Introduction

In the introduction to their 1970 paper on cosmological singularities, Hawking and Penrose [1] noted that gravity is an "essentially unstable" force. For small concentrations of mass, the instability is masked by enormously larger short range forces. But when the density of matter is sufficiently large or its mass sufficiently great, gravity becomes dominant and collapse inevitable. From this instability to collapse arises the theoretical expectation of black holes; and the strongest observational argument in their favor is provided by the associated upper limit on

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the mass of dense spherical stars. Rotation can delay the onset of instability to radial pulsation and in this way raise the upper mass limit. Remarkably, however, gravity seems to provide a conspiracy of instabilities: we will find that any rotating, self-gravitating perfect fluid, is unstable to nonaxisymmetric perturbations, which (presumably) radiate its angular momentum until it settles down to a non-rotating star.

Dynamical instability of rotating Newtonian stars was first understood for the Maclaurin sequence of uniformly rotating, uniform density ellipsoids (see [2]). For sufficiently rapid rotation, the sequence becomes unstable to nonaxisymmetric deformations, via a mode having angular dependence (in the linear theory) $e^{2i\phi}$, where ϕ is the angle about the symmetry axis. When viscosity is present, instability sets in earlier – for smaller angular momentum, but again via an $m=2$ mode [3, 4]. A parallel situation in relativity was first considered by Chandrasekhar [5]. Using a post-Newtonian treatment, he found that in the presence of radiation reaction, an $m=2$ mode again becomes unstable at precisely the same point along the sequence that marks the onset of viscosity-induced instability. It has subsequently been widely assumed that the Maclaurin sequence, and rotating fluids in general, are stable in relativity for small values of the angular momentum as is the case in the strictly Newtonian theory. Recently, however, Schutz and I [6] showed, in a post-Newtonian framework, that the radiation reaction induced instability sets in first via short wavelength oscillations and that even for slowly rotating configurations, there are, for sufficiently large m, unstable modes having angular dependence $e^{im\phi}$; and Comins [7] has explicitly found the corresponding unstable modes of the slowly rotating Maclaurin ellipsoids. The present paper extends these post-Newtonian results to the exact theory, showing that all rotating axisymmetric perfect fluid configurations are unstable to perturbations having angular dependence of the form $e^{im\phi}$ for all integers m greater than some m_0 [1]. Strictly, we establish the existence of at best marginally stable perturbations. A proof of instability requires the additional assumption that time dependent, nonaxisymmetric oscillations of an axisymmetric star radiate, at least when m is large – presumably when $m>1$; although the assumption may appear obvious, in that the multipole moments must change at null infinity, there is as yet no formal proof.

A side issue in the work involves the clarification of an oversight that plagued nearly all recent studies involving the stability of rotating stars (see [6] and [8]), and which arises in the following way. In phrasing a stability criterion, one introduces a canonical energy [9–11], obtained from the action that governs the linear perturbation equations [2]. The existence of an action is predicated on a description of fluid perturbations in terms of a Lagrangian displacement, a vector field connecting fluid elements in the perturbed and unperturbed flows (equivalently, one must single out a "comoving frame"). But such a description is not unique: Schutz and Sorkin [12] have pointed out the existence of a class of trivial displacements that leave invariant all physical quantities, so that a given physical

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[1] The work here is restricted to fluids having two parameter equations of state. The generic instability for strictly isentropic fluids differs somewhat in its physical features and its mathematical details

[2] In references [9] and [10], the stability functionals are equivalent to the canonical energy of time-independent initial data, although they are not so identified

perturbation can be described by more than one Lagrangian displacement. Unfortunately, the canonical energy is not invariant under this additional "gauge" freedom, and a way to eliminate the spurious displacements is needed.

It turns out to be possible to remove the freedom in a natural way – that is, in a way provided by the mathematical structure of the theory. One introduces the symplectic product of the perturbation equations and singles out as a preferred class the set of displacements orthogonal to all trivials with respect to that product. By restricting consideration to the preferred displacements, one obtains a unique value for the canonical energy of each physical perturbation and thereby regains a criterion governing stability.

The generic instability found here substantially alters the previous picture of the region of stable perfect fluid configurations. Its physical implications, however, are less dramatic. For slowly rotating stars, only the shortest wavelength modes will be unstable, and their growth times will be inordinately long: only for stars which are unstable to low m modes can one expect a timescale short compared to that of stellar evolution. In addition, work by Lindblom and Detweiler [13] suggests that in imperfect fluids, where dissipation due to viscosity is comparable to the loss of energy to gravitational radiation, viscosity can damp out the radiation induced instability; and if that were the case, only rapidly rotating configurations would be unstable in any realistic model.

The plan of the paper is as follows. Section II reviews the treatment of perfect fluid perturbations in terms of a Lagrangian displacement. The class of trivial displacements is defined and an explicit form for the generic trivial is obtained. In §III, a conserved symplectic product associated with the perturbation equations is introduced and is first used to obtain an expression for the canonical energy E_c. Then a dynamically preserved class of "canonical" displacements is defined as the subspace orthogonal to the trivials with respect to the symplectic product. Canonical displacements turn out to preserve a generalized vorticity in surfaces of constant entropy; as an associated result, we obtain the relativistic generalization of Ertel's theorem (the conservation of vorticity in uniform entropy surfaces) and use the symplectic product to show that this conservation law is Noether-related to the symmetry associated with the trivial displacements.

In §IV, the formalism developed in §§II and III is used to phrase a criterion governing the stability of self-gravitating fluids to nonaxisymmetric perturbations: we show that if the energy functional E_c is negative for some canonical initial data on a hypersurface S, then the corresponding physical perturbation cannot settle down to a time independent state. If one assumes that time dependent, non-axisymmetric perturbations of an axisymmetric equilibrium necessarily radiate, it then follows that E_c will decrease without bound, and that the system is unstable. Finally, by means of the criterion thus obtained we show that any rotating, self-gravitating perfect fluid is unstable to perturbations having angular dependence $e^{im\phi}$ for all integers m greater than some m_0.

II. Lagrangian Perturbation Theory of Relativistic Fluids

This section develops a formalism for perturbations of a stationary self-gravitating perfect fluid in terms of a Lagrangian displacement. The preliminary formulae have been derived previously in Friedman and Schutz [11] (see also [14, 15] and [10]), but the treatment of trivial displacements is new and extends to relativistic fluids the description obtained by Friedman and Schutz [6] in the Newtonian theory. The framework for our discussion will be an asymptotically flat (topologically Euclidean) spacetime M with metric g_{ab} whose only source is a perfect fluid characterized by the energy momentum tensor

$$T^{ab}=\varepsilon u^a u^b + p q^{ab}. \tag{1}$$

Here the vector field u^a is the fluid's velocity,

$$u^a u_a = -1, \tag{2}$$

while the tensor q^{ab} is the projection operator orthogonal to u^a,

$$q^{ab}=g^{ab}+u^a u^b ; \tag{3}$$

the scalars ε and p are the energy density and pressure, and are assumed to satisfy a two parameter equation of state, which (without loss of generality) has the form

$$\begin{aligned} \varepsilon &= \varepsilon(n,s) \\ p &= p(n,s), \end{aligned} \tag{4}$$

with n and s the baryon density and the entropy per baryon, respectively. In terms of the scalars n and s, the second law takes the form

$$d\varepsilon = T ds + \frac{\varepsilon+p}{n} dn, \tag{5}$$

and conservation of baryons is expressed by

$$\nabla_a(nu^a)=0. \tag{6}$$

Strictly speaking, by introducing the scalar n one adds nothing new; the three fluid variables ε, p, and s together with an equation of state $\varepsilon=\varepsilon(p,s)$ and Condition (11) below describe the same physics. But the formalism is simpler when written in terms of n and s.

Finally, we have the field equations,

$$G_{ab}=8\pi T_{ab}, \tag{7}$$

and the implied equation of motion

$$\nabla_b T^{ab}=0. \tag{8}$$

By projecting Eq. (8) orthogonal to the velocity u^a, one obtains

$$u^b \nabla_b u_a = -\frac{1}{\varepsilon+p} q_a{}^b \nabla_b p, \tag{9}$$

and the projection along u^a gives

$$u^a \nabla_a \varepsilon = -(\varepsilon+p)\nabla_a u^a. \tag{10}$$

From Eq. (10), together with the second law (5) and baryon conservation (6), it follows that the entropy of each fluid element is conserved:

$$u^a \nabla_a s = 0 . \tag{11}$$

By an equilibrium configuration we mean a stationary solution $\{g_{ab}, n, s, u^a\}$ to the field Eq. (7) together with the equation of state (4) and the conservation laws (6) and (11). That is, there is to be an asymptotically timelike Killing vector t^a,

$$£_t g_{ab} = \nabla_a t_b + \nabla_b t_a = 0 , \tag{12}$$

which also Lie derives the equilibrium fluid variables

$$£_t n = £_t s = £_t u^a = 0 . \tag{13}$$

In discussing stability, one is interested in the time evolution of nearby configurations having the same baryon number and the same total entropy, configurations that can be viewed as deformations of the original equilibrium. Formally, we introduce a family of (time dependent) solutions

$$Q(\lambda) = \{g_{ab}(\lambda), u^a(\lambda), n(\lambda), s(\lambda)\} \tag{14}$$

to Eqs. (4), (6), (7), and (11), indexed by a parameter λ, and compare, to first order in λ, the perturbed variables $Q(\lambda)$ with their equilibrium values, $Q(0)$. We further suppose that the family of solutions $Q(\lambda)$ is such that each member can be reached by an adiabatic deformation of the equilibrium $Q(0)$. That is, there is to be a family of diffeomorphisms χ_λ from the support of the equilibrium fluid in the solution $Q(0)$ to its support in the solution $Q(\lambda)$ which has the following properties:

i) χ_λ takes fluid trajectories to fluid trajectories;

ii) the entropy of each fluid element is preserved,

$$s(\lambda) \circ \chi_\lambda^{-1} = s(0) ;$$

iii) the baryon number of each fluid element is preserved. Condition iii) may be written in terms of the volume element

$$\varepsilon_{abc} = \varepsilon_{abcd} u^d \tag{15}$$

orthogonal to u^a in the manner

$$\chi_\lambda [\varepsilon_{abc}(\lambda) n(\lambda)] = \varepsilon_{abc}(0) n(0) , \tag{16}$$

where χ_λ acts by the differential map.

First order departures from equilibrium can be described in two ways. The Eulerian perturbations in the quantities $Q(\lambda)$ are defined by

$$\delta Q = \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0} \tag{17}$$

and compare values of Q at the same point of the spacetime.

In the region occupied by the original fluid, one can also introduce the Lagrangian perturbations

$$\Delta Q = \frac{d}{d\lambda}[\chi_{-\lambda} Q(\lambda)]|_{\lambda=0}$$
$$= (\delta + \pounds_\xi) Q \,, \tag{18}$$

where ξ^a is the generator of the family of diffeomorphisms χ_λ [that is, the curve $\lambda \to \chi_\lambda(P)$ has tangent $\xi^a(P)$ at the point P]. The field ξ^a is termed a Lagrangian displacement and may be regarded as the connecting vector from fluid elements in the unperturbed configuration to the corresponding elements in the perturbed spacetime.

The first order changes in the variables Q can be expressed in terms of the displacement ξ^a and the Eulerian change in the metric

$$h_{ab} = \delta g_{ab} \tag{19}$$

by means of the linearized version of conditions i)–iii) above ([11, 14, 15]). Requiring that world lines of the unperturbed configuration are mapped by χ_λ to world lines of the perturbed fluid implies

$$\Delta u^a = \tfrac{1}{2} u^a u^b u^c \Delta g_{bc} \,; \tag{20}$$

that the deformation be adiabatic means

$$\Delta s = 0 \,; \tag{21}$$

and that it locally conserve baryons implies

$$\Delta(\varepsilon_{abc} n) = 0 \,. \tag{22}$$

Now

$$\Delta g_{ab} = \delta g_{ab} + \pounds_\xi g_{ab}$$
$$= h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a \tag{23}$$

and

$$\Delta \varepsilon_{abcd} = \delta \varepsilon_{abcd} + \pounds_\xi \varepsilon_{abcd}$$
$$= \varepsilon_{abcd}(\tfrac{1}{2} h + \nabla_e \xi^e)$$
$$= \tfrac{1}{2} \varepsilon_{abcd} g^{ef} \Delta g_{ef} \,. \tag{24}$$

Then by (20) and (24), we have

$$\Delta \varepsilon_{abc} = \varepsilon_{abc} \tfrac{1}{2} q^{ef} \Delta g_{ef} \,, \tag{25}$$

and baryon conservation (22) takes the form

$$\frac{\Delta n}{n} = -\tfrac{1}{2} q^{ab} \Delta g_{ab} \,. \tag{26}$$

Equations (20), (21), (23), and (26) express ΔQ in terms of ξ^a and h_{ab}. The dependent variables Δp and $\Delta\varepsilon$ may be similarly expressed by

$$\frac{\Delta p}{p}=\gamma\frac{\Delta n}{n} \tag{27}$$

and [by Eq. (5)]

$$\frac{\Delta\varepsilon}{\varepsilon+p}=\frac{\Delta n}{n}, \tag{28}$$

where the adiabatic index, γ, is defined by

$$\gamma=\frac{n}{p}\frac{\partial}{\partial n}p(n,s)=\frac{\varepsilon+p}{p}\frac{\partial}{\partial\varepsilon}p(\varepsilon,s). \tag{29}$$

Relation (28), written in the form

$$\frac{\Delta\varepsilon}{\varepsilon+p}=-\tfrac{1}{2}q^{ab}\Delta g_{ab}, \tag{30}$$

is a first integral of the Lagrangian variation of Eq. (10),

$$\Delta[u^a\nabla_a\varepsilon+(\varepsilon+p)\nabla_a u^a]=0 \tag{31}$$

(and could be obtained in this way had the baryon density n not been introduced). Finally, from Eqs. (20) and (26)–(28), the Lagrangian change in the energy momentum tensor takes the form

$$(-g)^{1/2}\Delta[(-g)^{1/2}T^{ab}]=W^{abcd}\Delta g_{cd}, \tag{32}$$

where

$$W^{abcd}=\tfrac{1}{2}(\varepsilon+p)u^au^bu^cu^d+\tfrac{1}{2}p(g^{ab}g^{cd}-g^{ac}g^{bd}-g^{ad}g^{bc})-\tfrac{1}{2}\gamma pq^{ab}q^{cd}. \tag{33}$$

We will also need the expressions for Eulerian perturbations that follow from Eqs. (20), (21), and (26), namely

$$\delta u^a=\tfrac{1}{2}u^au^bu^ch_{bc}+q^a{}_b\pounds_u\xi^b, \tag{34}$$

$$\delta s=-\xi^a\nabla_a s, \tag{35}$$

$$\delta n=-\frac{n}{2}q^{ab}h_{ab}-q^a{}_b\nabla_a(n\xi^b). \tag{36}$$

By thus writing the perturbed fluid variables in terms of a Lagrangian displacement, one automatically satisfies the linearized conservation equations

$$\Delta(u^a\nabla_a s)=0 \tag{37}$$

and

$$\Delta[\nabla_a(nu^a)]=0. \tag{38}$$

As a result, as we will see in § III, one acquires an unconstrained action for the linearized equations

$$\delta(G^{ab} - 8\pi T^{ab}) = 0 \tag{39}$$

and

$$\delta(\nabla_b T^{ab}) = 0, \tag{40}$$

and one also acquires a conserved canonical energy used to analyze stability. Moreover, the restriction to adiabatic perturbations ($\Delta s = 0$) amounts to little beyond the assumption that one is dealing with a perfect fluid: any perturbation δQ can be characterized by some displacement ξ^a (together with a metric perturbation h_{ab}) provided that the *total* mass and entropy of the configuration remain unchanged and that $\frac{\delta s}{|\nabla s|} \neq 0$ (in particular $|\nabla s|$ must not vanish too fast near the symmetry axis of an axisymmetric equilibrium). For initially isentropic stars, where $\nabla s = 0$, however, one is restricted to perturbations that keep the star isentropic.

But for introducing a "potential" ξ^a to describe the fluid perturbations, one pays a price in the form of an additional freedom that complicates the theory. Before considering this new freedom, it is helpful first to recall the gauge freedom that one always has in treating perturbations of a spacetime with tensor fields Q. Suppose that δQ are the perturbed tensor fields tangent to the family $Q(\lambda)$ and let Ψ_λ be a family of diffeomorphisms generated by a vector field ζ^a with Ψ_0 the identity. Then the family of fields $\Psi_\lambda Q(\lambda)$ is physically indistinguishable from the original family $Q(\lambda)$, and the corresponding Eulerian variations,

$$\hat{\delta} Q = \frac{d}{d\lambda} [\Psi_\lambda Q(\lambda)]_{\lambda=0} = \delta Q - £_\zeta Q, \tag{41}$$

are physically equivalent to the Eulerian variations δQ. Now, however, the map connecting fluid elements in the unperturbed spacetime to corresponding fluid elements in the perturbed spacetime [described by tensor fields $\psi_\lambda Q(\lambda)$] is $\psi_\lambda \circ \chi_\lambda$ and the new Lagrangian displacement is $\hat{\xi}^a = \xi^a + \zeta^a$. In particular, the pair $(\hat{h}_{ab}, \hat{\xi}^a) = (h_{ab} - £_\zeta g_{ab}, \xi^a + \zeta^a)$. Note that the Lagrangian variations remain unchanged:

$$\hat{\Delta} Q = \frac{d}{d\lambda} [(\psi_\lambda \circ \chi_\lambda)^{-1} \psi_\lambda Q(\lambda)]|_{\lambda=0} = \hat{\delta} Q + £_{\xi+\zeta} Q = \Delta Q,$$

and by choosing $\zeta^a = -\xi^a$, one describes the fluid perturbations in a "comoving" frame in which $\hat{\xi}^a = 0$ and $\hat{\Delta} = \hat{\delta} = \Delta$.

The additional freedom in the case of a fluid arises from a class of trivial displacements whose existence was first out by Schutz and Sorkin [12]. For these displacements, the Eulerian change in all fluid variables vanishes. Two pairs,

(h_{ab}, ξ^a) and $(h_{ab}, \hat{\xi}^a)$, thus correspond to the same physical perturbation if and only if the displacements ξ^a and $\hat{\xi}^a$ differ by a trivial displacement

$$\hat{\xi}^a = \xi^a + \eta^a ,$$

where η^a satisfies the equations obtained from Eqs. (31)–(33) by setting $\delta Q = 0$:

$$q^a{}_b \pounds_u \eta^b = 0 , \tag{42}$$

$$\eta^b \nabla_b s = 0 , \tag{43}$$

$$q^b{}_c \nabla_b (n\eta^c) = 0 . \tag{44}$$

Because Eqs. (36) and (37) involve only Eulerian changes in the variables Q, and trivial displacement satisfies them, and therefore satisfies the full set of perturbed equations.

In what follows we will be primarily concerned with perturbations of equilibrium configurations whose velocity fields are divergence free,

$$\nabla_a u^a = 0 , \tag{45}$$

or, equivalently whose baryon density is constant along fluid lines,

$$u^a \nabla_a n = 0 . \tag{46}$$

This condition is automatically satisfied by models of rotating axisymmetric stars, for the vector u^a is at each point tangent to a Killing vector. More generally, one would expect Eq. (46) to hold in a realistic fluid equilibrium; otherwise the resulting energy production from bulk viscosity would quickly disrupt the equilibrium. Where the entropy per baryon s is not constant, the general solution to Eqs. (42)–(44) is then

$$\eta^a = \frac{1}{n} \varepsilon^{abc} \nabla_b s \nabla_c f + g u^a , \tag{47}$$

where g is an arbitrary scalar and f is any scalar constant along fluid trajectories,

$$u^a \nabla_a f = 0 . \tag{48}$$

To include regions where $\nabla_a s$ vanishes, the generic trivial can be written

$$\eta^a = \frac{1}{n} \varepsilon^{abc} \nabla_b h \nabla_c f + g u^a \tag{49}$$

with h any scalar for which $u^a \nabla_a h = 0$ and $\varepsilon^{abc} \nabla_b h \nabla_c s = 0$.

In verifying that Eq. (49) provides the generic trivial and also for use in § III below, it will be convenient to introduce the following three dimensional formalism. Consider the manifold $\mathscr{M}$ of fluid trajectories in the background spacetime. Tensor fields on $\mathscr{M}$ can be identified with tensor fields $T^{a\dots b}{}_{c\dots d}$ on the spacetime which are orthogonal to the velocity u^a,

$$O = T^{a\dots b}{}_{c\dots d} u_a = \dots = T^{a\dots b}{}_{c\dots d} u^d \tag{50}$$

and whose "convective derivative" [15] vanishes:

$$q^a{}_m \cdots q^b{}_n q_c{}^r \cdots q_d{}^s \pounds_u T^{m\ldots n}{}_{r\ldots s} = 0. \tag{51}$$

In particular, the form ε_{abc} is a tensor on $\mathscr{M}$ because

$$\varepsilon_{abc} u^c = \varepsilon_{abcd} u^c u^d = 0 \tag{52}$$

and

$$q_a{}^m q_b{}^n q_c{}^r \pounds_u(\varepsilon_{mnrs}) = -\nabla_m u^m \varepsilon_{abc} = 0, \tag{53}$$

by Eq. (45); the quantity ε_{abc} is the volume 3-form on $\mathscr{M}$. If we define a derivative operator D_a by

$$D_e T^{a\ldots b}{}_{c\ldots d} = q^a{}_m \cdots q^b{}_n q_c{}^r \cdots q_d{}^s q_e{}^p \nabla_p T^{m\ldots n}{}_{r\ldots s}, \tag{54}$$

it follows that

$$D_a q_{bc} = 0 = D_a \varepsilon_{bcd} \tag{55}$$

and, from (53), that $D_a \alpha^a$ is a scalar on $\mathscr{M}$ whenever α^a is a vector on $\mathscr{M}$.

Consider now solutions η^a to Eqs. (42)–(44). It is easy to see that any vector of the form gu^a with g an arbitrary scalar satisfies the three equations by virtue of Eqs. (11) and (45). We can therefore retrict consideration to vectors η^a orthogonal to u^a. Equation (42) then means that η^a is a vector on $\mathscr{M}$ and Eqs. (43) and (44) have the form

$$\eta^a D_a s = 0, \tag{56}$$

$$D_a(n\eta^a) = 0, \tag{57}$$

where we have used the fact that n and s are themselves scalars on $\mathscr{M}$. In other words, within $\mathscr{M}$, the quantity $n\eta^a$ is a divergence-free vector field lying in surfaces of constant entropy per baryon, s. Equivalently, restricting consideration to a particular (2-dimensional) constant entropy surface s_0, and regarding $n\eta_a$ as a 1-form on s_0, Equation (57) means that

$$d^*(\chi^{-1} n\eta) = 0, \tag{58}$$

where

$$\chi = \nabla_b s \nabla^b s. \tag{59}$$

Thus (assuming that constant entropy surfaces are simply connected), the form $^*(\chi^{-1} n\eta)$ must be exact: in index notation,

$$^*(n\eta\chi^{-1})_a = \varepsilon_{abc} \nabla^b s \chi^{-1} n\eta^c = \nabla_a f,$$

or

$$\eta^a = \frac{1}{n} \varepsilon^{abc} \nabla_b s \nabla_c f, \tag{60}$$

where f is a scalar on $\mathscr{M}$. When the gradient of s vanishes, Eq. (43) is automatically satisfied, while (42) and (44) imply only that $n\eta^a = \varepsilon^{abc}\nabla_b h \nabla_c f$, where f and h are arbitrary scalars on $\mathscr{M}$ with vanishing gradient at the fluid surface.

There are, then, two types of trivial displacements. Those of the form $\eta^a = gu^a$ push the fluid along its own worldlines and so simply map each fluid element to its position at a slightly later time in the background flow. Because the unperturbed flow preserves both s and n, the map leaves the fluid unchanged. Trivial displacements of the form

$$\eta^a = \frac{1}{n}\varepsilon^{abc}\nabla_b h \nabla_c f \tag{61}$$

are permutations of fluid elements within surfaces of constant entropy that preserve the volume of each element. They amount to a relabeling of particles; and the requirement (42) that η^a be convectively derived by the fluid means that the initial relabeling is simply carried along by the unperturbed motion of the star.

As noted above, the introduction of a conserved baryon density n is a formal convenience, not a logical necessity. To describe the trivials without recourse to n, one replaces Eq. (45) by

$$\delta\varepsilon = -\nabla_a[(\varepsilon + p)\eta^a] = 0$$

and writes the general trivial orthogonal to u^a in the form

$$\eta^a = \frac{e^H}{\varepsilon + p}\varepsilon^{abc}\nabla_b h \nabla_c f, \tag{62}$$

where

$$H = \int^p \frac{dp}{\varepsilon(p, s) + p} \tag{63}$$

is the specific enthalpy of the fluid.

III. A Symplectic Product, Conserved Quantities, and Canonical Displacements

Having introduced a Lagrangian displacement, we can construct an action

$$I = \int L(\xi, h)\,d\tau \tag{64}$$

for the perturbation Eqs. (39) and (40), [10]. The functional $L(\xi, h)$ has the form

$$L(\xi, h) = \tfrac{1}{2}\mathscr{L}(\xi, h; \xi, h), \tag{65}$$

where $\mathscr{L}(\hat{\xi}, \hat{h}; \xi, h)$ is an operator linear in (ξ, h) and in $(\hat{\xi}, \hat{h})$ and symmetric under interchange of the two pairs. Explicitly

$$\begin{aligned}\mathscr{L}(\hat{\xi}, \hat{h}, \xi, h) = {} & U^{abcd} \nabla_a \hat{\xi}_b \nabla_c \xi_d + V^{abcd}(\hat{h}_{ab} \nabla_c \xi_d + h_{ab} \nabla_c \hat{\xi}_d) \\ & - \frac{1}{32\pi} \varepsilon^{aceg} \varepsilon^{bdf}{}_g \nabla_c \hat{h}_{ab} \nabla_d h_{ef} \\ & - T^{ab} R_{acbd} \hat{\xi}^c \xi^d + \left(\tfrac{1}{2} W^{abcd} - \frac{1}{16\pi} G^{abcd} \right) \hat{h}_{ab} h_{cd} \\ & - \tfrac{1}{2} \nabla_c T^{ab} (\hat{h}_{ab} \xi^c + h_{ab} \hat{\xi}^c), \end{aligned} \tag{66}$$

where

$$\begin{aligned} G^{abcd} = {} & \tfrac{1}{2} R^{a(cd)b} + \tfrac{1}{4}(2R^{ab} g^{cd} + 2R^{cd} g^{ab} - 3R^{a(c} g^{d)b} - 3R^{b(c} g^{d)a}) \\ & + \tfrac{1}{4} R (g^{ac} g^{bd} + g^{ad} g^{bc} - g^{ab} g^{cd}), \end{aligned} \tag{67}$$

$$U^{abcd} = (\varepsilon + p) u^a u^c q^{bd} + p(g^{ab} g^{cd} - g^{ad} g^{bc}) - \gamma p q^{ab} q^{cd}, \tag{68}$$

and

$$2V^{abcd} = (\varepsilon + p)(u^a u^c q^{bd} + u^b u^c q^{ad} - u^a u^b q^{cd}) - \gamma p q^{ab} q^{cd}. \tag{69}$$

The operator $\mathscr{L}$ satisfies

$$\hat{\xi}_b \Delta(\nabla_c T^{bc}) + \frac{1}{16\pi} \hat{h}_{bc} \delta(G^{bc} - 8\pi T^{bc}) = -\mathscr{L}(\hat{\xi}, \hat{h}; \xi, h) + \nabla_b R^b, \tag{70}$$

where

$$R^a(\hat{\xi}, \hat{h}; \xi, h) = U^{abcd} \hat{\xi}_b \nabla_c \xi_d + V^{cdab} h_{cd} \hat{\xi}_b - \frac{1}{32\pi} \varepsilon^{aceg} \varepsilon^{bdf}{}_g \hat{h}_{cd} \nabla_b h_{ef}, \tag{71}$$

from which it follows that

$$\frac{d}{d\lambda} L(\xi + \lambda \hat{\xi}, h + \lambda \hat{h})|_{\lambda=0} = -\hat{\xi}_b \Delta(\nabla_c T^{bc}) - \hat{h}_{bc} \delta(G^{bc} - 8\pi T^{bc}) + \nabla_b R^b(\hat{\xi}, \hat{h}; \xi, h). \tag{72}$$

In other words, demanding that $\int_M L(\xi, h) d\tau$ be stationary to all perturbations of ξ^a and h_{ab} that have compact support in M implies

$$\delta(G_{ab} - 8\pi T_{ab}) = 0 \quad \text{and} \quad \Delta(\nabla_b T^{ab}) = 0$$

on M.

a) Symplectic Product and Conserved Quantities

If we introduce a shorthand

$$y_A = (\xi^a, h_{ab}) \tag{73}$$

for the pair of tensor fields that describe fluid perturbations, Eq. (70) can be written in the form

$$\hat{y}_A F^A(y) = -\mathscr{L}(\hat{y}, y) + \nabla_a R^a(\hat{y}, y), \tag{74a}$$

where $F^a(y)=0$ are the field Eqs. (39) and (40). With y and $\hat{y}$ interchanged we have[3]

$$y_A F^A(\hat{y}) = -\mathscr{L}(\hat{y}, y) + \nabla_a R^a(\hat{y}, y). \tag{74b}$$

Because the operator $\mathscr{L}$ is symmetric,

$$\mathscr{L}(y, \hat{y}) = \mathscr{L}(\hat{y}, y),$$

it follows from (74) that when y_A and $\hat{y}_A$ satisfy

$$F^A(y) = 0 = F^A(\hat{y}), \tag{75}$$

the current

$$W^a(\hat{y}, y) = R^a(\hat{y}, y) - R^a(y, \hat{y}) \tag{76}$$

is conserved:

$$\nabla_a W^a = 0. \tag{77}$$

One thus obtains a conserved antisymmetric inner product

$$W(\hat{y}, y) = \int_S W^a(\hat{y}, y) dS_a, \tag{78}$$

independent of the spacelike hypersurface S if the surface integral at spatial infinity should vanish. We will call W the symplectic product of y and $\hat{y}$. (The symplectic structure associated with L degenerates to the product W when the configuration space of fields y_A and its tangent space at each point are identified.) In the case where L is the Lagrangian density of a scalar field, W is the Klein Gordon inner product.

When the product W is conserved it is also gauge invariant under asymptotically well behaved gauge transformations, a fact that can be seen in the following way. Given a solution (ξ^a, h_{ab}) to the field equations, one obtains from a gauge transformation generated by the gauge vector ζ^a new fields $(\xi^a - \zeta^a, h_{ab} + \pounds_\zeta g_{ab})$ which again satisfy the linearized field equations. Now the expression for $W(\xi, h; \hat{\xi}, \hat{h})$ depends only on the values of (ξ^a, h_{ab}), $(\hat{\xi}^a, \hat{h}_{ab})$, and their derivatives on S. But without changing the value of ζ^a near S, we are free to let ζ^a vanish on a neighborhood of some S' to the future of S. We then have

$$W(\xi, h; \hat{\xi}, \hat{h})|_S = W(\xi, h; \hat{\xi}, \hat{h})|_{S'} = W(\xi - \zeta, h + \pounds_\zeta g; \hat{\xi}, \hat{h})|_{S'}$$

because near S', $\zeta^a = 0$. Furthermore,

$$W(\xi - \zeta, h + \pounds_\zeta g; \hat{\xi}, \hat{h})|_{S'} = W(\xi - \zeta, h + \pounds_\zeta g; \hat{\xi}, \hat{h})|_S,$$

[3] With the exception of comments on gauge invariance, the following discussion is valid for any linear Lagrangian field theory

because W is independent of hypersurface when the gauge transformed fields satisfy the field equations and are asymptotically well behaved (i.e. give vanishing surface integral of spatial infinity). Thus

$$W(\xi, h; \hat{\xi}, \hat{h})|_S = W(\xi - \zeta, h + \pounds_\zeta g; \hat{\xi}, \hat{h})|_S; \tag{79}$$

that is, W is invariant under asymptotically nice gauge transformations, as claimed.

We will ultimately use the symplectic product W to eliminate the trivial displacements. In the meantime, the product affords us a simple construction of the conserved canonical energy associated with the asymptotically timelike Killing vector of the background spacetime. A Killing vector t^a generates a family of diffeomorphisms T_λ which commute with the operator $\mathscr{L}$ and with R^a:

$$T_\lambda \mathscr{L}(y, y) = \mathscr{L}(T_\lambda y, T_\lambda y), \tag{80}$$

$$T_\lambda R^a(y, y) = R^a(T_\lambda y, T_\lambda y), \tag{81}$$

where T_λ acts on tensor fields by the differential map $T_{\lambda *}$.

Differentiating Eqs. (80) and (81) with respect to λ, we have

$$\pounds_t \mathscr{L}(\hat{y}, y) = \mathscr{L}(\pounds_t \hat{y}, y) + \mathscr{L}(\hat{y}, \pounds_t y), \tag{82}$$

$$\pounds_t R^a(\hat{y}, y) = R^a(\pounds_t \hat{y}, y) + R^a(\hat{y}, \pounds_t y), \tag{83}$$

and it follows from (74a) that if y_A is a solution to the field equations

$$F^A(y) = 0,$$

its Lie derivative, $\pounds_t y_A$, is again a solution:

$$F^A(\pounds_t y_A) = 0. \tag{84}$$

By Eq. (77), one thus acquires, for every solution y_A to the field equations, a conserved current $W^a(\dot{y}, y)$ and associated conserved canonical momentum conjugate to t^a,

$$P_t(y) = \tfrac{1}{2} W(\dot{y}, y) = \tfrac{1}{2} \int_S W^a(\dot{y}, y) dS_a, \tag{85}$$

where the dot $(\cdot)$ is used to mean the Lie derivative $\pounds_t$. An equivalent form for the conjugate momentum P_t is

$$P_t(y) = \int_S [R^a(\dot{y}, y) - t^a L(y)] dS_a, \tag{86}$$

which can be derived using the identity

$$\int_S \nabla_a A^a t^b dS_b = \int_S \dot{A}^a dS_a, \tag{87}$$

valid for any vector field A^a that vanishes on ∂S; one takes as the vector A^a the quantity $R^a(y, y)$, and, from Eq. (74a) with $F^a(y) = 0$, obtains

$$\int_S t^a \mathscr{L}(y, y) dS_a = \int [R^a(\dot{y}, y) + R^a(y, \dot{y})] dS_a.$$

Equation (86) then follows from the defining Eq. (76) for W^a.

In this way one finds via the symplectic product the conserved quantities associated by Noether's theorem with the symmetry vector t^a. In the present case, where the vector t^a is asymptotically timelike, the conserved quantity is the canonical energy

$$E_c(y) = \tfrac{1}{2} W(\dot{y}, y). \tag{88}$$

Explicitly,

$$\begin{aligned} E_c = \int \Big\{ & U^{ibcd}\dot{\xi}_b \nabla_c \xi_d + V^{cdib} h_{cd}\dot{\xi}_b - \frac{1}{32\pi}\varepsilon^{iceg}\varepsilon^{bdf}{}_g \dot{h}_{cd}\nabla_b h_{ef} \\ & - \tfrac{1}{2} t^i \Big[U^{abcd}\nabla_a \xi_b \nabla_c \xi_d + 2V^{cdab} h_{cd}\nabla_a \xi_b - \frac{1}{32\pi}\varepsilon^{aceg}\varepsilon^{bdf}{}_g \nabla_a h_{cd}\nabla_b h_{ef} \\ & - T^{ab}R_{acbd}\xi^c\xi^d + \tfrac{1}{2}\Big(W^{abcd} - \frac{1}{16\pi} G^{abcd}\Big) h_{ab}h_{cd} \\ & - \nabla_c T^{ab} h_{ab}\xi^c \Big]\Big\} dS_i . \end{aligned} \tag{89}$$

When the background is axisymmetric as well, with axial Killing vector ϕ^a, the canonical angular momentum

$$J_c = -W(\pounds_\phi y, y) \tag{90}$$

is also conserved. If the hypersurface S is taken to be asymptotically null, rather than spacelike, the canonical energy and momentum are no longer exactly conserved. Instead, they change in time due to the radiation of energy and angular momentum at future null infinity, expressed in this context by a nonvanishing contribution from the surface integral of W^a at null infinity. In particular, we will see in §IV that the canonical energy decreases monotonically from one asymptotically null hypersurface, S, to any other, S', in its future.

From the gauge invariance of the symplectic product, it follows that the quantities E_c and J_c are also invariant under asymptotically well behaved gauge transformations. Unfortunately, however, these canonical conserved quantities are not invariant under the additional freedom associated with trivial displacements. In particular, it is not difficult to show (see Appendix A) that $E_c(\eta)$ will in general be nonzero for trivial displacements η^a, and thus

$$\begin{aligned} E_c(\xi+\eta) &= E_c(\xi) + W(\pounds_t \xi, \eta) + W(\pounds_t \eta, \xi) + E_c(\eta) \\ &\neq E_c(\xi). \end{aligned}$$

b) Ertel's Theorem

The symplectic product provides, in addition to the canonical energy and momentum, a conserved quantity associated with the trivial displacements. This turns out to be a relativistic generalization of Ertel's theorem [16], the conservation of circulation in constant entropy surfaces. After first obtaining the relativistic version of Ertel's theorem in a direct way, we will see that for the perturbed fluid it is related by the symplectic product to the trivial displacements.

For an isentropic fluid, the generalization of circulation conservation has the well known form [17]

$$\pounds_u \omega_{ab} = 0\,, \tag{91}$$

where

$$\omega_{ab} = \nabla_a\left(\frac{\varepsilon+p}{n}u_b\right) - \nabla_b\left(\frac{\varepsilon+p}{n}u_a\right) \tag{92}$$

is an extension of the Newtonian vorticity. An equivalent form (see below) is the statement that the integral

$$\int_{c_\lambda} \frac{\varepsilon+p}{n} v_a dl^a$$

is constant along a family of closed curves c_λ obtained by dragging the first curve c_0 with the fluid. The generalization of Ertel's theorem – which applies to nonisentropic fluids having two parameter equations of state – has a similar form, namely[4]

$$\pounds_u(\omega_{[ab}\nabla_{c]}s) = 0\,. \tag{93}$$

To verify Eq. (93), first use Eqs. (6) and (10) to cast the equation of motion (9) in the form

$$\pounds_u\left(\frac{\varepsilon+p}{n}u_a\right) = -\frac{\nabla_a p}{\varepsilon+p}\,.$$

Then

$$\begin{aligned}\nabla_{[a}\left\{\pounds_u\left(\frac{\varepsilon+p}{n}u_{b]}\right)\right\} &= \pounds_u\nabla_{[a}\left(\frac{\varepsilon+p}{n}u_{b]}\right)\\ &= \frac{\nabla_{[a}\varepsilon\nabla_{b]}p}{(\varepsilon+p)^2}\,,\end{aligned}$$

and we have

$$\begin{aligned}\pounds_u\left\{\nabla_{[a}\left(\frac{\varepsilon+p}{n}u_b\right)\nabla_{c]}s\right\} &= \pounds_u\nabla_{[a}\left(\frac{\varepsilon+p}{n}u_b\right)\nabla_{c]}s\\ &= (\varepsilon+p)^{-2}\nabla_{[a}\varepsilon\nabla_b p\nabla_{c]}s\\ &= 0\,,\end{aligned}$$

as claimed, where the relation

$$\pounds_u\nabla_c s = \nabla_c\pounds_u s = 0$$

was used.

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4 One can avoid use of the baryon density n by redefining ω_{ab} in the manner $\omega_{ab} = \nabla_{[a}(u_{b]}\exp H)$, with H the specific enthalpy introduced in Eq. (63)

An integral form of the relation follows by writing Eq. (93) in the manner

$$d\pounds_u v = 0, \tag{94}$$

where v is the pullback to a surface of constant entropy $\mathscr{S}$ of the form $\frac{\varepsilon+p}{n} u_a$. Integrating the 2-form $d\pounds_u v$ over a 2-surface in $\mathscr{S}$ bounded by c_0 and using Stokes' theorem, we have

$$\int_{c_0} \pounds_u v = 0 .$$

Thus if ψ_λ is the family of diffeos generated by the velocity u^a,

$$0 = \int_{c_0} \pounds_u v = -\frac{d}{d\lambda} \int_{c_0} \psi_\lambda v = -\frac{d}{d\lambda} \int_{\psi_\lambda(c_0)} v .$$

In other words, the integral

$$\int_{c_\lambda} \frac{\varepsilon+p}{n} u_a dl^a \tag{95}$$

is constant along any closed path that lies in a surface of constant entropy and moves with the fluid:

$$c_\lambda = \psi_\lambda c_0 .$$

A final alternative form of the conservation theorem is

$$\pounds_u \alpha = 0, \tag{96}$$

where the scalar α is defined by

$$\alpha = \frac{1}{n} \varepsilon^{abc} \nabla_a h \omega_{bc} . \tag{97}$$

For the perturbed fluid we have

$$0 = \Delta(\pounds_u \alpha) = \pounds_{\Delta u} \alpha + \pounds_u \Delta\alpha ;$$

and from Eqs. (96) and (20),

$$\pounds_{\Delta u} \alpha = \tfrac{1}{2} u^b u^c \Delta g_{bc} \pounds_u \alpha = 0 ,$$

whence

$$\pounds_u \Delta\alpha = 0 . \tag{98}$$

Equivalently,

$$\Delta \int_c \frac{\varepsilon+p}{n} u_a dl^a$$

is preserved by the fluid flow, where, as before, c is a closed curve in a surface of constant entropy carried with the fluid.

A correspondence between Ertel's theorem and the trivial displacements arises in the following way. If the pair (ξ^a, h_{ab}) satisfy the linearized field equations, then the quantity $W(\eta, 0; \xi, h)$ is conserved for any trivial displacement η^a, since trivials satisfy the perturbation equations. In the next paragraph we will see that by writing the trivial displacement η^a in the form (61), the quantity W can be brought to the form

$$W(\eta, 0; \xi, h) = \int_S f \Delta_\xi \alpha u^a dS_a . \tag{99}$$

Because Eq. (99) holds for arbitrary scalars f, Lie derived by the fluid velocity, conservation of the product W is equivalent to the linearized version of Ertel's theorem,

$$£_u \Delta_\xi \alpha = 0 .$$

Because the product W is invariant under gauge transformations of either argument, we are free to evaluate it in a comoving gauge for the pair (ξ^a, h_{ab}) – that is, in a gauge for which $\xi^a = 0$. From expression (71), it then follows that the integrand in expression (76) for W^a has the form

$$\begin{aligned} R^a(\eta, 0; 0, h) - R^a(0, h; \eta, 0) &= V^{cdab} h_{cd} \eta_b \\ &= (\varepsilon + p)(u^a u^b \eta^c h_{bc} - \tfrac{1}{2} \eta^a u^b u^c h_{bc}) - \tfrac{1}{2} \gamma p \eta^a q^{bc} h_{bc} \\ &= (\varepsilon + p)(u^a \eta^b \Delta u_b - \eta^a u^b \Delta u_b) + \eta^a \Delta p , \end{aligned} \tag{100}$$

where Eq. (69) was used to obtain the second equality, and Eqs. (20), (23), and (27) were used to find the final expression.

It will be useful in the manipulations that follow to introduce a scalar t for which S is a surface of constant t and with δt defined to vanish. Then, defining the volume element $d\sigma$ by

$$dS_a = \nabla_a t d\sigma ,$$

we have, from Eqs. (78) and (100),

$$W(0, h; \eta, 0) = \int_S \frac{1}{n} \varepsilon^{abcd} \nabla_b s \nabla_c f u_d [(\varepsilon + p)(u^e \Delta u_a \nabla_e t - u^e \Delta u_e \nabla_a t) + \Delta p \nabla_a t] d\sigma$$

After an integration by parts and some algebra, we obtain the relations

$$\begin{aligned} &\int_S \frac{\varepsilon + p}{n} \varepsilon^{abcd} \nabla_b s \nabla_c f u_d [u^e \Delta u_a \nabla_e t - u^e \Delta u_e \nabla_a t] d\sigma \\ &= \int_S f \varepsilon^{abcd} \nabla_b s \nabla_d \left(\frac{\varepsilon + p}{n} \Delta u_c \right) dS_a \end{aligned} \tag{101}$$

and

$$\int_S \varepsilon^{abcd} \nabla_b s \nabla_c f u_d \frac{\Delta p}{n} dS_a = - \int_S f \varepsilon^{abcd} \nabla_b s \nabla_c \left(u_d \frac{\Delta p}{n} \right) dS_a . \tag{102}$$

Combining the expressions on the right hand sides of Eqs. (101) and (102), and making use of the relation

$$\Delta\left(\frac{\varepsilon+p}{n}\right)=\frac{\Delta p}{n},$$

we have

$$W=-\int_S f\varepsilon^{abcd}\nabla_b s\nabla_c\Delta\left(\frac{\varepsilon+p}{n}u_d\right)dS_a.$$

In our comoving gauge,

$$\Delta\nabla_a t=\delta\nabla_a t=0,$$

whence, from Eqs. (20), (24), and (26), we obtain

$$\Delta\left[\frac{1}{nu^e\nabla_e t}\varepsilon^{abcd}\right]=0 \tag{103}$$

and

$$W=-\int_S fnu^e\nabla_e t\Delta\left[\frac{1}{nu^f\nabla_f t}\varepsilon^{abcd}\nabla_b s\nabla_c\left(\frac{\varepsilon+p}{n}u_d\right)\right]dS_a.$$

Now the bracketed expression in Eq. (97) has vanishing projection orthogonal to u^a:

$$\begin{aligned}q^a{}_e\varepsilon^{ebcd}\nabla_b s\nabla_c\left(\frac{\varepsilon+p}{n}u_d\right)&=q^a{}_e\varepsilon^{ebcd}\nabla_b s\left(\frac{1}{n}\nabla_c p u_d-\frac{\varepsilon+p}{n}\nabla_d u_c\right)\\&=\frac{1}{n}q^a{}_e\varepsilon^{ebcd}\nabla_b s u_d[D_c p-(\varepsilon+p)u^f\nabla_f u_c]\\&=0\end{aligned}$$

by Eq. (10). Thus, finally,

$$\begin{aligned}\left[\frac{1}{nu^f\nabla_f t}\varepsilon^{abcd}\nabla_b s\nabla_c\left(\frac{\varepsilon+p}{n}u_d\right)\right]\nabla_a t&=-\left[\frac{1}{nu^f\nabla_f t}\varepsilon^{abcd}\nabla_b s\nabla_c\left(\frac{\varepsilon+p}{n}u_d\right)\right]u_a u^e\nabla_e t\\&=-\frac{1}{n}\varepsilon^{abc}\nabla_a s\nabla_b\left(\frac{\varepsilon+p}{n}u_c\right)\\&=-\alpha,\end{aligned} \tag{104}$$

and

$$\begin{aligned}W&=-\int_S fnu^e\nabla_e t\Delta\left[\frac{1}{nu^f\nabla_f t}\varepsilon^{abcd}\nabla_b s\nabla_c\left(\frac{\varepsilon+p}{n}u_d\right)\nabla_a t\right]d\sigma\\&=\int_S fn\Delta\alpha u^a dS_a.\end{aligned} \tag{105}$$

The connection between the set of trivial displacements and Ertel's theorem can be obtained in a more familiar context. In particular, the trivials generate families of transformations

$$(\xi^a,h_{ab})\rightarrow(\xi^a+\lambda\eta^a,h_{ab})$$

that change the Lagrangian density L only by a divergence:

$$\begin{aligned} L(\xi+\lambda\eta,h) &= \tfrac{1}{2}\mathscr{L}(\xi+\lambda\eta,\xi+\lambda\eta) \\ &= \tfrac{1}{2}\mathscr{L}(\xi,\xi)-\lambda\nabla_a R^a(\eta,\xi)-\tfrac{1}{2}\lambda^2\nabla_a R^a(\eta,\eta) \\ &= L(\xi)-\nabla_a[\lambda R^a(\eta,\xi)+\tfrac{1}{2}\lambda^2 R^a(\eta,\eta)]\,. \end{aligned} \tag{106}$$

These transformations are therefore what Trautman, in his extension of Noether's theorem, calls generalized invariant transformations [18]; and one can show directly that the associated conserved quantity is the product W given in Eq. (105). Thus Ertel's theorem arises via Noether's theorem as the conservation law associated with the trivial displacements. Our calculation, which obtained the relation by means of the symplectic product, also suggests a way to pick a set of displacements that eliminates the spurious freedom arising from the trivials.

c) *Canonical Displacements*

We noted in a) that the canonical energy E_c can have different values for pairs (ξ^a, h_{ab}) and $(\xi^a+\eta^a, h_{ab})$ that describe the same physical perturbation. As we shall see in §IV below, however, one would like to test stability by asking whether E_c is positive definite. In order that such a procedure make sense, it is first necessary to restrict the class of allowed displacements in such a way that the functional E_c have a unique value for each physical perturbation. That is, one would like to find a class of "canonical" pairs (ξ^a, h_{ab}) for which

i) the time evolution of a canonical pair is canonical and

ii) if (ζ^a, h_{ab}) and $(\hat{\xi}^a, \hat{h}_{ab})$ are canonical displacements corresponding to the same physical perturbation, then $E_c(\xi,h)=E_c(\hat{\xi},\hat{h})$.

In particular, condition ii) implies that $E_c(\xi,h)=0$ only if (ξ,h) is trivial. One can often do better than this in that it is often possible to find a unique canonical pair corresponding to each physical perturbation; this property appears not to be universally true, however, and will not be required for the proof of generic instability in § IV.

Let us now define a canonical pair as a solution (ξ^a, h_{ab}) to the linearized equations which is orthogonal to all trivial displacements with respect to the product W:

$$W(\xi,h;\eta,0)=0\,,$$

for all trivial η. By Eq. (99), this condition is simply the requirement

$$\Delta_{(\xi,h)}\alpha=0\,; \tag{107}$$

in other words, a canonical identification of fluid elements from the unperturbed to the perturbed spacetime preserves their (generalized) vorticity in surfaces of constant entropy. Equivalently,

$$\Delta\int_c \frac{\varepsilon+p}{n}u_a dl^a=0\,, \tag{108}$$

where c lies in any constant entropy surface.

Conditions i) and ii) are easily seen to be satisfied by the class of canonical pairs just defined. Condition i) is an immediate consequence of Ertel's theorem, Eq. (98). Condition ii) follows from the fact that if (ξ, h) and $(\hat{\xi}, \hat{h})$ correspond to the same physical perturbation, then $(\xi - \hat{\xi}, h - \hat{h})$ is trivial – that is, equal within a gauge transformation to a trivial displacement $(\eta, 0)$. Thus $(\hat{\xi}, \hat{h})$ is equal within a gauge transformation to $(\xi + \eta, h)$ and by the gauge invariance of the functional E_c,

$$E_c(\hat{\xi}, \hat{h}) = E_c(\xi + \eta, h) = \tfrac{1}{2} W(\dot{\xi} + \dot{\eta}, \dot{h}; \xi + \eta, h).$$

Then

$$W(\dot{\xi} + \dot{\eta}, \dot{h}; \xi + \eta, h) = W(\dot{\xi}, \dot{h}; \xi, h) + W(\dot{\eta}, 0; \xi, h) + W(\dot{\xi}, \dot{h}; \eta, 0) + W(\dot{\eta}, 0; \eta, 0).$$

Now if η is trivial, so also is $\dot{\eta}$ and if (ξ, h) is canonical, so also is $(\dot{\xi}, \dot{h})$; thus $W(\dot{\eta}, 0; \xi, h) = 0$ and $W(\dot{\xi}, \dot{h}; \eta, 0) = 0$. Finally, because (ξ, h) and $(\hat{\xi}, \hat{h})$ are orthogonal to all trivials, so is $(\xi - \hat{\xi}, h - \hat{h})$, and so also, therefore, is $(\eta, 0)$. Thus $W(\dot{\eta}, 0; \eta, 0) = 0$, and

$$E_c(\hat{\xi}, \hat{h}) = \tfrac{1}{2} W(\dot{\xi}, \dot{h}; \xi, h) = E_c(\xi, h).$$

Under what circumstances can one find a canonical pair $(\hat{\xi}, \hat{h})$ corresponding to a physical perturbation described by an arbitrary pair (ξ, h)? Without loss of generality one can assume for the canonical pair the form

$$(\hat{\xi}, \hat{h}) = (\xi + \eta, h),$$

where η is trivial,

$$\eta^a = \frac{1}{n} \varepsilon^{abc} \nabla_b h \nabla_c f.$$

From the canonical condition

$$\Delta_{(\hat{\xi}, \hat{h})} \alpha = 0;$$

we then have

$$\Delta_\eta \alpha = -\Delta_{(\xi, h)} \alpha$$

or

$$\frac{1}{n} \varepsilon^{abcd} \nabla_a \alpha \nabla_b s \nabla_c f u_d = -\Delta_{(\xi, h)} \alpha. \tag{109}$$

For a rotating star, the fluid velocity has the form

$$u^a = \mu(t^a + \Omega \phi^a),$$

where t^a and ϕ^a are the translational and rotational Killing vectors and we have

$$£_u f = 0 \Rightarrow f = f(\alpha, s, \phi - \Omega t),$$

where ϕ and t are scalars satisfying

$$\phi^a \nabla_a \phi = t^a \nabla_a t = 1, \phi^a \nabla_a t, t^a \nabla_a \phi = 0.$$

Equation (109) has the solution

$$\pounds_\phi f = -\frac{\mu n \Delta_{(\xi,h)}\alpha}{\varepsilon^{abcd}\nabla_a\alpha\nabla_b s\nabla_c t\nabla_d\phi}. \tag{110}$$

For nonaxisymmetric perturbations, canonical pairs corresponding to each physical perturbation will exist when

$$|\nabla\alpha\times\nabla s|\neq 0$$

away from the axis of symmtery, and when $|\nabla\alpha\times\nabla s|\to 0$ near the axis no faster than $(\phi^a\phi_a)^{m/2}$ for a perturbation with angular dependence $e^{\mathrm{i}m\phi}$.

IV. Generic Instability of Rotating Stars

a) Canonical Energy of Asymptotically Null Hypersurfaces

Before phrasing a stability criterion, we need to show that the canonical energy of a perturbation, evaluated on a sequence of asymptotically null hypersurfaces, is a decreasing function of time. Let S_1 and S_2 be two asymptotically null hypersurfaces and let (u, r, θ, ϕ) with $-\infty < u < \infty$ and $r > r_0$ be a standard null chart for M outside a bounded region. In particular, where the chart is defined, $S_1(S_2)$ is taken to be a surface $u = u_1(u = u_2)$; lines of constant r, θ, and ϕ are trajectories of the Killing vector t^a; and the metric has the asymptotic form given, for example, in Newman and Unti [19] and characteristic of a stationary geometry:

$$g^{uu}=0, g^{ur}=-1, g^{u\theta}=0, g^{u\phi}=0,$$

$$g^{rr}=1-\frac{2M}{r}+O(r^{-2}), g_r{}^{\theta}=O(r^{-3}), g_r{}^{\phi}=O(r^{-3})$$

$$g^{\theta\theta}=\frac{1}{r^2}+O(r^{-4}), g^{\theta\phi}=O(r^{-5}), g^{\phi\phi}=\frac{1}{r^2\sin^2\theta}+O(r^{-4}). \tag{111}$$

Let us consider a region R bounded by the surfaces S_1, S_2, and by an $r=$constant cylinder. We have

$$O=\int_R \nabla_a W^a = \int_{\partial R} W^a dS_a. \tag{112}$$

Thus

$$E_c(u_2)-E_c(u_1) = -\lim_{r\to\infty}\int_{u_1}^{u_2} W^r r^2 d\Omega du. \tag{113}$$

An asymptotic regularity condition is now required to define E_c uniquely and to ensure that the expression on the right hand side of Eq. (113) be the energy radiated to null infinity between S_1 and S_2. One obtains a sufficient condition by demanding that in the asymptotic chart (111), h_{ab} have the asymptotic behavior [19]

$$\begin{aligned} &h^{uu}=h^{ur}=h^{u\theta}=h^{u\phi}=0\,,\\ &h^{rr}=O(r^{-1}), h^{r\theta}=O(r^{-2}), h^{r\phi}=O(r^{-2})\,,\\ &h^{\theta\theta}=-\frac{1}{r^3}2\,\mathrm{Re}\,\sigma^0+O(r^{-4})=-h^{\phi\phi}\sin^2\theta\,,\\ &h^{\theta\phi}=-\frac{2}{r^3\sin\theta}\mathrm{Im}\,\sigma^0+O(r^{-5})\,, \end{aligned} \tag{114}$$

where σ^0 is the leading term in the shear of the $u, \theta, \phi=$constant null geodesics. In this case one finds (see below)

$$J^r=\frac{1}{4\pi r^2}|\dot{\sigma}^0|^2 \tag{115}$$

and

$$E_c(u_2)-E_c(u_1)=-\frac{1}{4\pi}\int_{u_1}^{u_2}|\dot{\sigma}^0|^2 d\Omega du\,. \tag{116}$$

The change in E_c from S_1 to S_2 is thus the Bondi mass radiated between the two hypersurfaces.

The asymptotic gauge Condition (114) is, however, unnecessarily restrictive. In fact, we will see that $\lim\limits_{r\to\infty} r^2 J^r$ is invariant under any change of gauge

$$h_{ab}\to h_{ab}+2\nabla_{(a}\eta_{b)} \tag{117}$$

as long as the physical components $\eta^{(i)}$ of the gauge vector (that is, the components in an orthonormal frame) vanish as $r\to\infty$, and in addition $\partial_r\eta^{(i)}=o(r^{-1})$ and $\dot{\eta}^u=o(r^{-1})$. In particular, the expression

$$\lim_{r\to\infty}\tfrac{1}{2}\int_{u_1}^{u_2} W^r r^2 d\Omega du \tag{118}$$

will have the same value (will be the radiated energy between u_1 and u_2) in all gauges for which the physical components $h^{(i)(j)}$ are $O(r^{-1})$ and for which $\dot{h}^{u(i)} -\frac{1}{2}g^{u(i)}\dot{h}=o(r^{-1})$. It follows that the value of the canonical energy E_u will also be unique for gauges satisfying the above regularity conditions.

Finally, we observe that the de Donder gauge, which will be used below to facilitate our initial-value discussion is in this class of asymptotically regular gauges. That is, writing

$$\gamma_{ab}=h_{ab}-\tfrac{1}{2}g_{ab}h\,, \tag{119}$$

we have that outside the source

$$\nabla_c\nabla^c\gamma_{ab}-\tfrac{1}{2}R_{acbd}\gamma^{cd}=0\,,$$

whence

$$\gamma_{(i)(j)} = O(r^{-1}) \quad \text{and} \quad \partial_r \gamma^{(i)(j)} = o(r^{-1});$$

further, the gauge condition $\nabla_b \gamma^{ab} = 0$ implies that $\dot{\gamma}^{u(i)} = \partial_u \gamma^{u(i)} = o(r^{-1})$. The identification of the expression $\lim_{r\to\infty} \frac{1}{2} \int W^r r^2 d\Omega du$ with the radiated energy and its invariance under gauge transformations can be seen in this way. Outside the star the current W^a has a the simplified form

$$\tfrac{1}{2} W^a = t^a \left(\frac{1}{64\pi} \varepsilon^{bdfn} \varepsilon^{ceg}{}_n \nabla_b h_{de} \nabla_c h_{fg} \right) - \frac{1}{32\pi} \varepsilon^{acen} \varepsilon^{bdf}{}_n \dot{h}_{cd} \nabla_b h_{ef}; \tag{120}$$

after some algebra, the component $W^r = W^a \nabla_a r$ can be written in terms of the tensor γ_{ab} in the manner

$$W^r = \frac{1}{16\pi} (-\dot{\gamma}_{bc} \nabla^a \gamma^{bc} + \dot{\gamma}_{bc} \nabla^b \gamma^{ac} + \dot{\gamma}^a{}_b \nabla_c \gamma^{bc} + \tfrac{1}{2} \dot{\gamma} \nabla^a \gamma) \nabla_a r\,. \tag{121}$$

Now if the gauge is regular, so that $\dot{\gamma}^{u(i)} = o(r^{-1})$ and $\partial_r \gamma^{u(i)} = o(r^{-1})$, then $\dot{\gamma}_r{}^{(i)} = o(r^{-1})$ as well and we have

$$W^r = \frac{1}{16\pi} |\dot{\gamma}_{ab} m^a m^b|^2 + o(r^{-2}), \tag{122}$$

where m^a has the (u, r, θ, ϕ) components

$$2^{-1/2} r^{-1} \left(0, 0, 1, \frac{i}{\sin\theta}\right). \tag{123}$$

In the Newman-Unti gauge (114), the expression is just $\frac{1}{2} W^r = \frac{1}{4\pi r^2} |\dot{\sigma}^0|^2$, as expected. Moreover, under an asymptotically regular gauge transformation, J^r retains the form (80), and

$$\gamma_{ab} m^a m^b \to \gamma_{ab} m^a m^b + 2\nabla_a \eta_b m^a m^b - m^a m_a \nabla_b \eta^b = \gamma_{ab} m^a m^b + o(r^{-1})\,. \tag{124}$$

Thus $\lim_{r\to\infty} r^2 W^r$ is unchanged, and so the expression (118) for the radiated energy and the quantity E_c are independent of gauge (up to asymptotic regularity).

b) Stability Criteria

We now have the machinery required to state criteria governing the stability of rotating fluids to nonaxisymmetric perturbations. Consider a spacelike hypersurface S with unit normal n_a along the gradient $\nabla_a t$

$$n_a = e^{-\nu} \nabla_a t \tag{125a}$$

where

$$e^{-2\nu} = -\nabla_b t \nabla^b t\,. \tag{125b}$$

By a canonical initial data set $D=(\xi^a, h_{ab}; £_n\xi^a, £_n h_{ab})$ on S will be meant a solution to the initial value equation

$$\delta(G^b{}_a - 8\pi T^b{}_a)\nabla_b t|_S = 0, \tag{126}$$

set in a de Donder gauge

$$\nabla_b(h^{ab} - \tfrac{1}{2}g^{ab}h)|_S = 0, \tag{127}$$

and satisfying the additional condition

$$\Delta_{(\xi,h)}\alpha|_S = 0. \tag{128}$$

The surface S is to be asymptotically spacelike, not null, to simplify an analysis of the initial value equations below. In Appendix B, however, we show that the canonical energy can be negative for canonical data on a strictly spacelike hypersurface S, only if it is negative for some canonical data on an asymptotically null hypersurface S'. Thus one is free to consider data on either S or S'.

A sufficient condition for instability may now be stated in the following manner.

i) If $E_c(D) < 0$ for some canonical initial data D, then the configuration is unstable or marginally unstable: there exist nonaxisymmetric perturbations which do not die away in time.

Similarly, for stars in which $\varepsilon^{abc}\nabla_b s \nabla_c \alpha \neq 0$, a sufficient condition for stability is

ii) If $E_c(D) \geqq 0$ for all canonical data D, the configuration is stable in the sense that for any perturbation, the magnitude of E_c is bounded in time and only finite energy can be radiated.

Ideally, one would like to show that when $E_c < 0$ for some canonical data, the configuration is strictly unstable, that within the linearized theory the time evolved data radiates infinite energy and that $|E_c|$ becomes infinite along a sequence of asymptotically null hypersurfaces. There is as yet no formal proof of this conjecture, but it is easy to show that if $E_c(D) < 0$, the time derivatives $£_t\xi^a$ and $£_t h_{ab}$ must remain infinitely large forever. Thus a configuration with $E_c < 0$ will be strictly unstable unless it admits nonaxisymmetric perturbations which are time dependent but *non-radiative.* The key fact here, that the perturbation's time derivatives are bounded away from zero (in an integral norm) when $E_c(D) < 0$, follows immediately from the expression (88) for E_c in terms of the symplectic product. That is, because E_c is always bounded above by its initial negative value, $E_c|_S$, the pair $(£_t\xi, £_t h)$ is also bounded away from zero by the relation

$$\tfrac{1}{2}W(\xi, h; £_t\xi, £_t h) = E_c \leqq E_c|_S < 0. \tag{129}$$

c) Generic Instability

The aim of this final section is to establish that all rotating stars are unstable to nonaxisymmetric perturbations in the sense of i) above. That is, there always exists canonical initial data having angular dependence of the form $(\xi^a, h_{ab}) = \mathrm{Re}(\hat{\xi}^a e^{\mathrm{i}m\phi}, \hat{h}_{ab} e^{\mathrm{i}m\phi})$, where $£_\phi\hat{\xi}^a = £_\phi\hat{h}_{ab} = 0$, and for which $E_c < 0$; such initial data can be found for all m greater than some m_0. It is easier to work with complex perturbations than with their real parts, and so we will really calculate an expression that gives the sum of the canonical energies corresponding to the real and imaginary parts of a complex perturbation having angular dependence $e^{\mathrm{im}\phi}$.

Suppose, first, that for each integer m one can find canonical initial data on S having the following properties:

$$(\xi^a, h_{ab}; \pounds_n\xi^a, \pounds_n h_{ab}) = (\hat{\xi}^a e^{\mathrm{im}\phi}, \hat{h}_{ab} e^{\mathrm{im}\phi}; \pounds_n \hat{\xi}^a e^{\mathrm{im}\phi}, \pounds_n \hat{h}^{ab} e^{\mathrm{im}\phi}), \tag{130}$$

where

$$\pounds_\phi \hat{\xi}^a = \ldots = \pounds_\phi \pounds_n \hat{h}_{ab} = 0\,;$$

$$\|\xi^a\| < k, \|\nabla_a \hat{\xi}^b\| < k, \tag{131}$$

$$\|\nabla_a h_{bc}\| < k, \left\|\frac{1}{r} h_{ab}\right\| < k, \tag{132}$$

where k is a constant independent of the integer m, r is asymptotically a radial coordinate, and where by $\|T^{a\ldots b}{}_{c\ldots d}\|$ is meant the L^2 norm on S of the components of $T^{a\ldots b}{}_{c\ldots d}$ with respect to a field of Lorentz frames; finally

$$\xi^a \phi_a = \xi^a t_a = 0\,. \tag{133}$$

Expression (89) for the canonical energy then involves three types of terms. Terms having no ϕ-derivatives of the displacement vector ξ^a and at most one ϕ-derivative of h_{ab} are bounded for all data of the type specified above by a constant independent of m – that is, they are of order m^0. Terms involving one ϕ-derivative of ξ^a are of order m, and terms involving two ϕ-derivatives of ξ^a are of order m^2. For sufficiently large m, if ξ^a remains finitely large, the terms of order m^2 will dominate the functional E_c. (Because $\|\nabla_a h_{bc}\| < k$ even terms involving $\pounds_\phi h_{ab}$ are only of order m.) The only term in expression (89) in which two ϕ-derivatives of ξ^a occur is

$$\begin{aligned} &-\tfrac{1}{2}\int U^{abcd}\nabla_a\xi_b^*\nabla_c\xi_d t^e dS_e \\ &= -\tfrac{1}{2}\int [(\varepsilon+p)u^a u^c q^{bd} + p(g^{ab}g^{cd} - g^{ad}g^{bc}) - \gamma p q^{ab} q^{cd}]\, \nabla_a\xi_b^*\nabla_c\xi_d t^e dS_e \\ &= -\frac{m^2}{2}\int (\varepsilon+p)(u^a\nabla_a\phi)^2 \hat{\xi}^{a*}\hat{\xi}_a t^e dS_e + O(m)\,. \end{aligned} \tag{134}$$

Thus the only term of order m^2 is negative definite, and if, for example, $\hat{\xi}^a$ is independent of m, then for sufficiently large m, $E_c < 0$.

It remains to be shown that canonical data satisfying Conditions (130)–(133) and for which ξ^a is independent of m can be found for all sufficiently large m. The difficulty here lies in solving the simultaneous constraint Eqs. (126)–(128), and it will be helpful to introduce a derivative operator $\mathscr{D}_a$, identical for tensors on S (tensors orthogonal in all indices to n_a) to the covariant derivative induced on S by the metric on M [20]:

$$\mathscr{D}_e T^{a\ldots b}{}_{c\ldots d} = j^a{}_m \cdots j^b{}_n j_c{}^r \cdots j_d{}^s j_e{}^f \nabla_f T^{m\ldots n}{}_{r\ldots s}\,. \tag{135}$$

If one now chooses for the perturbed metric on S a tensor h_{ab} whose form is given by the equation

$$h_{ab} - \tfrac{1}{2} g_{ab} h = \sigma n_a n_b + \tau_a n_b + \tau_b n_a , \tag{136}$$

with

$$\tau^a n_a = 0 , \tag{137}$$

the de Donder condition takes the form

$$\pounds_n \sigma = -\pi\sigma + \mathscr{D}_b \tau^b + 2\mathscr{D}_b \nu \tau^b \tag{138}$$

$$\pounds_n \tau^a = -\mathscr{D}^a \nu\sigma - (2\pi^a{}_b + \pi j^a{}_b + n^a \mathscr{D}_b \nu)\tau^b , \tag{139}$$

where

$$\pi_{ab} = j_a{}^c \nabla_c n_b \tag{140}$$

is the extrinsic curvature of S. When Eqs. (138) and (139) are used to replace first time derivatives of the quantities σ and τ^a by their spatial derivatives, the initial value Eq. (126) becomes two spatial constraints of the form

$$\begin{aligned} 8\pi\delta T^{ab} n_a n_b = {} & \mathscr{D}_b \mathscr{D}^b \sigma + A^b \mathscr{D}_b \sigma + B^{bc} \mathscr{D}_b \tau_c \\ & + E\sigma + F^b \tau_b , \end{aligned} \tag{141}$$

$$\begin{aligned} 8\pi\delta T^{cb} n_b j^a{}_c = {} & \mathscr{D}_b \mathscr{D}^b \tau^a + \hat{A}^{ab} \mathscr{D}_b \sigma + \hat{B}^{abc} \mathscr{D}_b \tau_c \\ & + \hat{E}^a \sigma + \hat{F}^{ab} \tau_b , \end{aligned} \tag{142}$$

where $A^a, \ldots, \hat{F}^{ab}$ are tensors on S constructed from n_a and the metric g_{ab}.

In considering the requirement that the perturbation be canonical, we will use the form of $\Delta\alpha$ given by Eq. (104), instead of the simpler form (97), because the former expression involves only first derivatives of ξ^a orthogonal to S (the alternative form involves only first derivatives along u^a, but second derivatives along n^a). We have

$$\Delta\alpha = -\frac{1}{n u^f \nabla_f t} \varepsilon^{abcd} \nabla_a t \nabla_b s \nabla_c \Delta \left(\frac{\varepsilon + p}{n} u_d \right) = 0 .$$

Using the fact that the indices c and d must be orthogonal to $\nabla_a t$ and $\nabla_b s$, we obtain, for perturbations with angular dependence $e^{\mathrm{i}m\phi}$,

$$0 = \varepsilon^{abcd} \nabla_a t \nabla_b s \nabla_c \phi \left\{ \mathrm{i}m \Delta \left(\frac{\varepsilon + p}{n} u_d \right) - \nabla_d \left[\phi^e \Delta \left(\frac{\varepsilon + p}{n} u_e \right) \right] \right\} . \tag{143}$$

Then if $r_a{}^b = \delta_a{}^b - t^b \nabla_a t - \phi^b \nabla_a \phi$ denotes the projection operator orthogonal to t^a and ϕ^a, a sufficient condition for the vanishing of $\Delta\alpha$ is that

$$r_a{}^b \left\{ \mathrm{i}m \Delta \left(\frac{\varepsilon + p}{n} u_b \right) - \nabla_b \left[\phi^c \Delta \left(\frac{\varepsilon + p}{n} u_c \right) \right] \right\} = 0 . \tag{144}$$

If ξ^a is chosen orthogonal to t^a and ϕ^a

$$r_a{}^b \xi_b = \xi_a , \tag{145}$$

we find, using Eqs. (20), (23), and (26)–(28), together with the relations

$$q^{bc}\nabla_b\xi_c = e^{-\nu}\mathscr{D}_b(e^{\nu}\xi^b) + \frac{1}{\varepsilon+p}\xi^b\mathscr{D}_p p\,,$$

$$u^b u^c \nabla_b \xi_c = \frac{1}{\varepsilon+p}\xi^b\mathscr{D}_b p\,,$$

$$\phi^b u^c \nabla_{(b}\xi_{c)} = -\xi^b\phi^c\nabla_c u_b\,,$$

that Eq. (144) takes the form

$$\pounds_u\xi_a = 2r_a{}^c\nabla_c u_b\xi^b - r_a{}^b u^c h_{bc} + \frac{1}{im}r_a{}^c\mathscr{D}_c\left[\phi^b\Delta\left(u_b\frac{\varepsilon+p}{n}\right)\right], \tag{146}$$

where

$$\phi^b\Delta\left(u_b\frac{\varepsilon+p}{n}\right) = -\frac{\gamma p}{n}\phi^b u_b e^{-\nu}\mathscr{D}_c(e^{\nu}\xi^c) + \left[\frac{\varepsilon+p-\gamma p}{n(\varepsilon+p)}\mathscr{D}_b p - 2\frac{\varepsilon+p}{n}\phi^c\nabla_c u_b\right]\xi^b$$
$$+\left[-\tfrac{1}{2}\frac{\gamma p}{n}\phi^d u_c q^{bc} + \tfrac{1}{2}\frac{\varepsilon+p}{n}\phi^d u_d u^b u^c + \frac{\varepsilon+p}{n}\phi^b u^c\right]h_{bc}\,. \tag{147}$$

Together, Eqs. (146) and (147) express the fluid derivative $\pounds_u\xi_a|_S$ in terms of spatial derivatives of the displacement $\xi_a|_S$ and first spatial derivatives of the metric perturbation h_{ab}.

It is now not difficult to show that initial data satisfying Conditions (130)–(133) exists. We need a solution $(\xi^a, h_{ab}; \pounds_n\xi^a, \pounds_n h_{ab})$ to Eqs. (138), (139), (141), (142), and (146). First note that when Eqs. (141), (142) and (146) are used to eliminate $\pounds_n h_{ab}$ and $\pounds_u\xi^a$ in favor of $h_{ab}|_S$, $\mathscr{D}_c h_{ab}|_S$, $\xi_a|_S$, $\mathscr{D}_b\xi_a|_S$, and $\mathscr{D}_c\mathscr{D}_b\xi_a|_S$, the remaining Eqs. (138) and (139) are an elliptic system for $\sigma|_S$ and $\tau^a|_S$ whose source involves only $\xi_a|_S$ and its spatial derivatives. Furthermore, only single ϕ-derivatives of ξ_a occur in the expression for $\pounds_u\xi_a$ and in $\delta T^a{}_b$, so by choosing $\xi^a = \hat{\xi}^a e^{\mathrm{i}m\phi}$ with $\hat{\xi}^a$ independent of m, the source terms in the elliptic system are bounded by km, with k a constant independent of m. Fortunately the elliptic system conforms to the conditions of a theorem recently proved by Cantor for elliptic operators on R^n [21]. In Appendix B the theorem is applied to show that the elliptic system is invertible for sufficiently large m and that the resulting metric perturbations satisfy Conditions (131) and (132). Because $\xi^a|_S$ can be chosen arbitrarily, we impose Condition (133) by fiat and similarly require $\xi^a = \hat{\xi}^a e^{\mathrm{i}m\phi}$ with $\hat{\xi}^a$ independent of m. Then [with $\pounds_u\xi^a$ determined by Eq. (146)], Conditions (130)–(133) are met, and the canonical energy will be negative for sufficiently large m.

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Appendix A: Canonical Energy of a Trivial Displacement

From Eqs. (88) and (99), the canonical energy of a trivial displacement η has the form

$$E_c(\eta) = \tfrac{1}{2}W(\pounds_t\eta, \eta) = \int_S \pounds_t f u^a \Delta_\eta \alpha\, dS_a\,, \tag{A.1}$$

where

$$\eta^a = \frac{1}{n}\varepsilon^{abc}\nabla_b h \nabla_c f\,. \tag{A.2}$$

Because η^a is trivial, $\delta_\eta \alpha = 0$ and

$$\Delta_\eta \alpha = £_\eta \alpha = \frac{1}{n}\varepsilon^{abc}\nabla_b h \nabla_c f\,\nabla_a \alpha\,. \tag{A.3}$$

By choosing f to be scalar of the form

$$f = g\cos m(\phi - \Omega t)\,, \tag{A.4}$$

with $£_t g = £_\phi g = 0$, and using for the velocity the explicit form

$$u^a = \mu(t^a + \Omega\phi^a)\,,$$

we find

$$\nabla_{[a}\cos m(\phi - \Omega t)u_{b]} = -\frac{m}{\mu}\sin m(\phi - \Omega t)\nabla_{[a}t\nabla_{b]}\phi\,. \tag{A.5}$$

Using Eqs. (A.4)–(A.6), we can rewrite the expression on the right of Eq. (A.1) in the form

$$E_c(\eta) = \tfrac{1}{2}\int_S [-m\Omega g\sin m(\phi - \Omega t)]\varepsilon^{abcd}\nabla_a\alpha\nabla_b h\nabla_c t\nabla_d\phi[-mg\sin(\phi - \Omega t)]\frac{1}{\mu}u^e dS_e$$

$$= \tfrac{1}{2}\pi m^2\int_S \Omega g^2\varepsilon^{abcd}\nabla_a\alpha\nabla_b h\nabla_c t\nabla_d\phi u^e dS_e\,. \tag{A.6}$$

Unless $\nabla\alpha \times \nabla s$ vanishes everywhere, the magnitude of E_c is thus arbitrary; and unless $\varepsilon^{abcd}\nabla_a\alpha\nabla_b s\nabla_c t\nabla_d\phi$ has the same sign everywhere, the sign of E_c is arbitrary as well. [That is, E_c may be freely varied by suitably choosing the functions g and (where $\nabla_a s = 0$)h.]

Appendix B: Initial Data

Two results are obtained here. The first relates initial data on spacelike hypersurfaces to data on asymptotically null hypersurfaces, and the second concerns the existence of solutions to the initial value equations.

Proposition 1. *Canonical intial data for which $E_c < 0$ exists on an asymptotically null spacelike hypersurface if such data exists on a strictly spacelike hypersurface.*

Let S be the strictly spacelike hypersurface with initial data set $D = (\xi^a, h_{ab}; £_n\xi^a, £_n h_{ab})$ for which $E_c < 0$. An asymptotically null hypersurface S' will be chosen in this way: let B be a compact region containing the fluid and pick S' so that within B, S' coincides with S, and outside of B, S' lies in the domain of dependence of $S - B$. Let $\Phi_\lambda, \lambda \in [0, \infty)$ be a family of diffeomorphisms that smoothly contract the spacetime so that $\Phi_\lambda g_{ab}$ and $\Phi_\lambda h_{ab}$ converge in a second derivative norm to a flat metric η_{ab} on $S - B$ as $\lambda \to \infty$. Now the initial data sets $\Phi_\lambda D$

on S with background fields $\Phi_\lambda Q$ are equivalent to the data D on S with background field Q; therefore $\Phi_\lambda D$ has the same canonical energy when E_c is constructed from the background fields $\Phi_\lambda Q$ as has D with E_c constructed from Q. But by the stability of the Cauchy problem on spacelike surfaces, the data $\Phi_\lambda D$ time evolved to S' from $S-B$ has norm that approaches zero as the norm of $\Phi_\lambda D$ approaches zero on $S-B$. Thus the contribution to E_c of the data on $S-B$ and $S'-B$ becomes negligible as $\lambda\to\infty$: only the data on B (where S and S' coincide) contributes. Whence $E_c(S)\to E_c(S')$ as $\lambda\to\infty$, and for sufficient large λ, the data $\Phi_\lambda D$, time evolved to S' has canonical energy $E_c(S')<0$.

In order to state the second proposition, definitions of some weighted Sobolev spaces are necessary. Given $f^i:\mathbb{R}^3\to\mathbb{R}^4$, the L^p norm of f is

$$|f|_p=[\int|f|^p]^{1/p}. \tag{B.1}$$

To demand asymptotic regularity one employs the function

$$\sigma(x)=(1+r^2)^{1/2} \tag{B.2}$$

on $\mathbb{R}^3$; and additional smoothness is incorporated by bounding the derivatives

$$D^\alpha f\equiv\partial_1^{\alpha_1}\ldots\partial_k^{\alpha_k}f$$

of f. Thus one defines the weighted norm $|\;|_{p,s,\delta}$ by

$$|f|_{p,s,\delta}=\sum_{|\alpha|\leqq s}|\sigma^{\delta+|\alpha|}D^\alpha f|_p, \tag{B.3}$$

where $|\alpha|=\sum_i\alpha_i$, for integers $s\geqq 0$ and real numbers δ. The corresponding weighted Sobolev space $M^p_{s,\delta}$ is the completion of C_0^∞ functions from $\mathbb{R}^3\to\mathbb{R}^4$ with respect to the norm $\|_{p,s,\delta}$. Finally, we note that $M^p_{s,\delta}$ can be divided into a set of closed subspaces S_m invariant under $\pounds_\phi$, where ϕ^a is a rotational Killing vector of $\mathbb{R}^3$ and $\mathbb{R}^4\supset\mathbb{R}^3$ (in other words, S_m contains functions of the form $f=\mathrm{Re}\hat{f}$, where $\hat{f}$ has values in $\mathbb{C}^4$ and $\pounds_\phi\hat{f}=\pm\mathrm{im}\hat{f}$). Given an integer m_0, we write

$$\hat{M}^p_{s,\delta}\equiv\bigcup_{m>m_0}S_m, \tag{B.4}$$

and note that $\hat{M}^p_{s,\delta}$ and S_m are Banach spaces with norm $|\;|_{p,s,\delta}$.

Proposition 2. *Let A be an elliptic operator on $M^p_{s,\delta}$, axisymmetric*

$$[\pounds_\phi, A]=0, \tag{B.5}$$

and satisfying

$$a_\alpha\in M^p_{s-2,2-|\alpha|}\qquad |\alpha|<2, \tag{B.6}$$

$$a_\alpha-\bar{a}_\alpha\in M^p_{s-2,0}\qquad |\alpha|=2, \tag{B.7}$$

where

$$\nabla^2=\sum_{|\alpha|=2}\bar{a}_\alpha D^\alpha.$$

Then for sufficiently large m_0, *the restriction* $\hat{A}$ *of* A *to* $\hat{M}^p_{s,\delta}$ *is an isomorphism* $\hat{A}:\hat{M}^p_{s,\delta}\to\hat{M}^p_{s-2,\delta+2}$ *when* $p>3$, $s>2+\frac{3}{p}$ *and* $\delta<\frac{3(p-1)}{p}-2$.

To establish Proposition 2, we require the following

Lemma. *Let* $\lambda\to L_\lambda$ *be a continuous curve from* $[0,1]$ *to the space of linear Banach space maps* $L_\lambda:\mathscr{A}\to\mathscr{B}$, *and suppose each* L_λ *is an injection with closed range. Then if the initial map* L_0 *is an isomorphism of* $\mathscr{A}$ *onto* $\mathscr{B}$, *so is the final map* L_1.

For the proof of the Lemma, see for example Cantor [21].

Consider now the curve $\lambda\to A_\lambda$ of differential operators on $M^p_{s\delta}$ defined by

$$A_\lambda=(1-\lambda)\nabla^2+\lambda A\,.$$

Each operator A_λ satisfies (B.5)–(B.7), and a theorem due to Cantor [21] then implies that each A_λ is a map $A_\lambda:M^p_{s\delta}\to M^p_{s-2,\delta+2}$ having closed range and finite dimensional kernel. From (B.5) it follows that $[\pounds_\phi, A_\lambda]=0$, and consequently, if $f\in\operatorname{Ker}A_\lambda$, so is the projection f_m of f onto S_m (defined in spherical coordinates by, for example, $f^\theta_m=\operatorname{Re}\left[e^{\mathrm{i}m\phi}\frac{1}{2\pi}\int_0^{2\pi}f^\theta e^{-\mathrm{i}m\phi'}d\phi'\right]$. Since $\operatorname{Ker}A_\lambda$ is finite dimensional there is an integer m_λ large enough that $\operatorname{Ker}A_\lambda$ contains no functions in S_m for $m>m_\lambda$. Thus, if $m_0=\operatorname{lub}m_\lambda$, the restriction $\hat{A}_\lambda$ of A_λ to the subspace $\hat{M}^p_{s\delta}$ is an injection $\hat{M}^p_{s\delta}\to\hat{M}^p_{s-2,\delta+2}$. Furthermore, the range of $\hat{A}_\lambda$ is closed in $\hat{M}^p_{s-1,\delta+2}$ because the range of A_λ is closed in $M^p_{s-2,\delta+2}$. Thus $\lambda\to\hat{A}_\lambda$ is a curve satisfying the conditions of our lemma. But the first map $\hat{A}_0=\nabla^2|_{\hat{M}^p_{s\delta}}$ is an isomorphism, whence, by the Lemma, $\hat{A}_1$ is an isomorphism as well; in other words, A, restricted to $\hat{M}^p_{s\delta}$, is an isomorphism.

Because the spacetime is topologically Euclidean and asymptotically flat, one can pick a global chart in which the initial value Eqs. (141) and (142) together have the form $Af=g$, with $(f^0,f^1,f^2,f^3)=(\sigma,\tau^1,\tau^2,\tau^3)$, and where A is an operator satisfying (B.5)–(B.7). Consequently, for sufficiently large m, the initial value equations with source of the form $g=\operatorname{Re}(\hat{g}e^{\mathrm{i}m\phi})$ can be inverted to give $\sigma|_S$ and $\tau^a|_S$ with components in $M^p_{s\delta}$ when the components of g are in $M^p_{s-2,\delta+2}$. Because $\hat{A}$ is a Banach space isomorphism,

$$|\sigma|^p_{s\delta}<k|g|^p_{s-2,\delta+2}\quad\text{and}\quad|\tau^a|^p_{s\delta}<k|g|^p_{s-2,\delta+2}\,. \tag{B.8}$$

Finally, because the source g involves only first ϕ-derivatives of ξ^a, by choosing ξ^a of the form $\operatorname{Re}[\hat{\xi}^a e^{\mathrm{i}m\phi}]$ with $|\hat{\xi}^a|^p_{s\delta}<k$ for $s=2$, say and satisfying Eq. (133), we have

$$|g|^p_{s-2,\delta+2}<km^{s-1}\,. \tag{B.9}$$

Condition (131) then follows from Eqs. (136), (138), (139), (B.8), and (B.9). In other words, if $\hat{\xi}^a$ is of order m^0, the source g will be of order m. Because the solution (σ,τ^a) to the elliptic system is smoother by two derivatives than the source, h_{ab} and $\nabla_a h_{bc}$ will be of order m^0; and by choosing p sufficiently large, the $M^p_{s\delta}$ norm guarantees the asymptotic condition required by Eq. (132). Thus initial data of the form required in §IV can always be found for sufficiently large integers m.

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