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Dissipations on von Neumann Algebras

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Abstract. We extend a characterisation by Lindblad of complete normal dissipations on hyperfinite von Neumann algebras to general semifinite von Neumann algebras.

Introduction

The time-development of certain quantum systems can be represented by one-parameter semigroups of completely positive maps on the associated C^*-algebras (see [4] for a discussion of the physical justification for this). When the semigroup is norm-continuous the infinitesimal generator is a bounded linear map on the C^*-algebra, and Lindblad [4] gives a characterisation of those linear maps which are infinitesimal generators of such semigroups. These he calls *complete dissipations*.

If we now take a von Neumann algebra $\mathscr{A}$ and look at complete normal dissipations on $\mathscr{A}$, we would like to prove a result corresponding to the theorem that every derivation on a von Neumann algebra is inner. In [4], Lindblad shows that if $\theta:\mathscr{A}\to\mathscr{A}$ is completely positive then $\gamma_\theta:\mathscr{A}\to\mathscr{A}$ defined by

$$\gamma_\theta(a)=\theta(a)-\tfrac{1}{2}\{\theta(1)a+a\theta(1)\} \tag{1}$$

is a complete dissipation on $\mathscr{A}$, and it is clear that γ_θ is normal if and only if θ is.

Definition. A complete dissipation γ on a C^*-algebra $\mathscr{A}$ is called *inner* if $\gamma-\gamma_\theta$ is an inner derivation for some completely positive map θ on $\mathscr{A}$.

Lindblad shows in [4] that every complete normal dissipation on a hyperfinite von Neumann algebra $\mathscr{A}$ is inner. In [5] he uses the general theory of cohomology of operator algebras to show that the same is true for any type I von Neumann algebra, except that in this case he can only show that the range of the completely positive map θ is contained in $\mathscr{B}(H)$, where $\mathscr{A}$ is considered as a weak-operator closed subalgebra of $\mathscr{B}(H)$ containing the identity map. However, since any type I von Neumann algebra is injective, there is an expectation from $\mathscr{B}(H)$ onto $\mathscr{A}$, so by the remark at the end of [5] we can choose θ with range contained in $\mathscr{A}$.

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We show here that every complete normal dissipation on a semifinite von Neumann algebra is inner. The starting point is Proposition 1 of [5], which we state in the next section for completeness.

1. Preliminary Definitions and Results

Let $\mathscr{A}$ be a C^*-algebra with identity.

Definition. A dissipation on $\mathscr{A}$ is a linear map

$$\gamma : \mathscr{A} \to \mathscr{A}$$

satisfying, for a in $\mathscr{A}$,

1) $\gamma(a^*) = \gamma(a)^*$,
2) $\gamma(1) = 0$,
3) $\gamma(a^*a) \geqq a^*\gamma(a) + \gamma(a^*)a$.

It is called a *complete dissipation* if

$$\gamma_n = \gamma \otimes \mathrm{id} : \mathscr{A} \otimes M_n \to \mathscr{A} \otimes M_n$$

is a dissipation on $\mathscr{A} \otimes M_n$ for every $n = 1, 2, \ldots$, where M_n is the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$ (so $\mathscr{A} \otimes M_n$ can be considered as the C^*-algebra of $n \times n$ matrices over $\mathscr{A}$).

Kishimoto shows in [3] that every dissipation on a C^*-algebra is bounded.

For a dissipation γ on a C^*-algebra $\mathscr{A}$ we define, following Lindblad [5], two related functions, the first from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$ and the second from $\mathscr{A} \times \mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$. They are defined as follows: for a, b, c in $\mathscr{A}$

$$d(a, b) = d_\gamma(a, b) = \gamma(ab) - \gamma(a)b - a\gamma(b)$$

and

$$D(a, b, c) = D_\gamma(a, b, c) = d(ab, c) - ad(b, c).$$

Note that if $\mathscr{A}$ is a von Neumann algebra and γ is ultraweakly continuous, then d and D are separately ultraweakly continuous in each variable. Also

$$d(a, b) = D(a, 1, b) \qquad (a, b \in \mathscr{A}).$$

The following proposition is Proposition 1 of [5], except for the normality of π and V, which is easily verified. We can also deduce the normality of V from [7].

Proposition 1. *If γ is a complete dissipation on a C^*-algebra $\mathscr{A}$ and D is defined as above, and if $\mathscr{A}$ is considered as a norm-closed algebra of operators on a Hilbert space H, containing the identity on H, then there is a *-representation*

$$\pi : \mathscr{A} \to \mathscr{B}(K)$$

of $\mathscr{A}$ on a Hilbert space K and a bounded linear map

$$V : \mathscr{A} \to \mathscr{B}(H, K)$$

such that, for a, b, c in $\mathscr{A}$,

$$D(a, b, c) = V(a^*)^* \pi(b) V(c) \tag{2}$$

and

$$V(ab) = V(a)b + \pi(a)V(b). \tag{3}$$

If $\mathscr{A}$ is a von Neumann algebra and is ultraweakly closed in $\mathscr{B}(H)$ and γ is normal then π and V can be chosen to be normal (i.e. continuous in the ultraweak topologies on $\mathscr{B}(H)$ and $\mathscr{B}(H, K)$).

For the remainder of the paper we assume, unless otherwise stated, that $\mathscr{A}$ is a von Neumann algebra, considered as a weak-operator closed subalgebra of operators on a Hilbert space H, containing the identity on H, and that γ is a complete normal dissipation on $\mathscr{A}$.

Define

$$\Lambda_0 = \{\pi(a)V(b)c : a, b, c \in \mathscr{A}\} \subseteq \mathscr{B}(H, K),$$

where π, V, K are as in Proposition 1. If

$$x = \pi(a_1)V(b_1)c_1, \; y = \pi(a_2)V(b_2)c_2$$

are general elements in Λ_0, then

$$\begin{aligned} y^*x &= c_2^* V(b_2)^* \pi(a_2^*)\pi(a_1)V(b_1)c_1 \\ &= c_2^* V(b_2)^* \pi(a_2^* a_1)V(b_1)c_1 \\ &= c_2^* D(b_2^*, a_2^* a_1, b_1)c_1 \in \mathscr{A}. \end{aligned}$$

Now let Λ be the weak-operator closed linear span of Λ_0.

Lemma 2. (i) *$y^*x \in \mathscr{A}$ for every x, y in Λ;*

(ii) *$\pi(a)x$ and $xa \in \Lambda$ for every $x \in \Lambda$, $a \in \mathscr{A}$.*

Proof. Let Λ_1 be the linear span of Λ_0 and let

$$x = \Sigma_i \lambda_i x_i, \; y = \Sigma_j \mu_j y_j$$

be general elements of $\Lambda_1(\lambda_i, \mu_j \in \mathbb{C}, x_i, y_j \in \Lambda_0)$. Then

$$\begin{aligned} y^*x &= (\Sigma_j \bar{\mu}_j y_j^*)(\Sigma_i \lambda_i x_i) \\ &= \Sigma_{i,j} \lambda_i \bar{\mu}_j y_j^* x_i \in \mathscr{A} \end{aligned}$$

by the previous calculation.

Now let $x, y \in \Lambda$ and choose nets (x_α), (y_β) in Λ_1, converging in the weak-operator topology to x, y respectively. Since $\mathscr{A}$ is weak-operator closed in $\mathscr{B}(H)$ and multiplication is separately weak-operator continuous, fixing β we have

$$y_\beta^* x = \lim_\alpha y_\beta^* x_\alpha \in \mathscr{A}.$$

Now using the fact that the *-operation is weak-operator continuous we obtain

$$y^*x = \lim_\beta y_\beta^* x \in \mathscr{A}.$$

The proof of (ii) is similar and is omitted.

Note. Since Λ is a weak-operator closed subspace of $\mathscr{B}(H, K)$ it has a predual Λ_*, and with respect to this predual it becomes a dual normal $\mathscr{A}$-module in the sense of [8, p. 404]. Further, by (i) of Lemma 2 we can define an $\mathscr{A}$-valued "inner product" on Λ by

$$(x, y) \mapsto y^* x \qquad (x, y \in \Lambda)$$

and Λ thus becomes a right Hilbert $\mathscr{A}$-module [6]. It can be shown that the dual normal module structure on Λ implies that it is a self-dual right Hilbert $\mathscr{A}$-module in the sense of Paschke [6]. By Proposition 1, V ia a derivation of $\mathscr{A}$ into Λ. In what follows we implicitly use the $\mathscr{A}$-module structure on Λ, and in particular the $\mathscr{A}$-valued inner product.

2. The Main Results

The proof of the following proposition is an adaptation of the proof by Johnson and Ringrose ([2] or [9, Theorem 4.1.6]) that every derivation on a von Neumann algebra is inner.

The proof easily generalises to prove that every derivation on a dual normal Hilbert module over a semi-finite von Neumann algebra is inner.

Proposition 3. *Let $\mathscr{A}$ be a semi-finite von Neumann algebra. Then with the same notation and assumptions as in the previous section, there is a $\hat{V} \in \Lambda$ with $\|\hat{V}\| \leqq \|V\|$ such that for a in $\mathscr{A}$,*

$$V(a) = \hat{V} a - \pi(a) \hat{V}.$$

Proof. We write $\mathscr{A}^u$ for the group of unitary elements of $\mathscr{A}$. For u in $\mathscr{A}^u$ define as map

$$T_u : \Lambda \to \Lambda$$

by

$$T_u(x) = \pi(u) x u^* + V(u) u^* \qquad (x \in \Lambda).$$

For u, v in $\mathscr{A}^u$ and x in Λ,

$$\begin{aligned} T_u(T_v(x)) &= T_u[\pi(v) x v^* + V(v) v^*] \\ &= \pi(u)[\pi(v) x v^* + V(v) v^*]\, u^* + V(u) u^* \\ &= \pi(uv) x (uv)^* + [\pi(u) V(v) + V(u) v](uv)^* \\ &= \pi(uv) x (uv)^* + V(uv)(uv)^* \\ &= T_{uv}(x), \end{aligned}$$

so $T_u T_v = T_{uv}$ for u, v in $\mathscr{A}^u$.

Let Δ be the collection of non-empty, weak-operator closed convex sets K of Λ satisfying

1) $T_u(K) \subseteqq K \qquad (u \in \mathscr{A}^u)$,

and

2) $\sup\{\|x\| : x \in K\} \leqq \|V\|$.

For u in $\mathscr{A}^u$, $\|T_u(0)\| = \|V(u)u^*\| \leqq \|V\|$, $T_u(0) \in \Lambda$, and

$$T_u(\{T_v(0) : v \in \mathscr{A}^u\}) = \{T_{uv}(0) : v \in \mathscr{A}^u\} = \{T_v(0) : v \in \mathscr{A}^u\},$$

so the weak-operator closure of the convex hull of $\{T_v(0) : v \in \mathscr{A}^u\}$ is a member of Δ, and Δ is non-empty.

Order Δ by inclusion. Using the fact that weak-operator closed bounded sets in Λ are weak-operator compact [since Λ is weak-operator closed in $\mathscr{B}(H, K)$], we can easily see that each chain in Δ has a lower bound, namely the intersection of all members of the chain. So by Zorn's lemma, Δ has a minimal element K_0.

If $x, y \in K_0$ and $u \in \mathscr{A}^u$

$$\pi(u)(x-y)u^* = T_u(x) - T_u(y) \in K_0 - K_0,$$

so $K_0 - K_0$ is invariant under the mappings

$$\Phi^u : z \mapsto \pi(u)zu^* (z \in \Lambda)$$

for each $u \in \mathscr{A}^u$.

Firstly assume that $\mathscr{A}$ is a finite, countably-decomposable von Neumann algebra and therefore has a faithful tracial state τ. Define an inner product $\langle .,. \rangle_\tau$ on Λ by

$$\langle x, y \rangle_\tau = \tau(y^* x) \qquad (x, y \in \Lambda).$$

This is well-defined by Lemma 2(i). We write

$$\|x\|_\tau = \langle x, x \rangle_\tau^{1/2} \qquad (x \in \Lambda).$$

We want to show that $K_0 - K_0 = \{0\}$, so conversely assume that there is a non-zero $c = a - b$ with a, b in K_0. Let

$$\lambda = \sup\{\|x\|_\tau : x \in K_0\}.$$

If $x \in K_0$, the weak-operator closure of the convex hull of $\{T_u(x) : u \in \mathscr{A}^u\}$ is a member of Δ and contained in K_0 (by the invariance of K_0 relative to the T_u), so by minimality it must be equal to K_0. So taking $x = \frac{1}{2}(a+b)$ and $\varepsilon > 0$ we can find a $u \in \mathscr{A}^u$ such that

$$\left\| T_u\left(\frac{a+b}{2}\right) \right\|_\tau > \lambda - \varepsilon.$$

Since $\|T_u(a)\|_\tau \leqq \lambda$, $\|T_u(b)\|_\tau \leqq \lambda$, by the parallelogram law,

$$\begin{aligned} \|\tfrac{1}{2}(T_u(a) - T_u(b))\|_\tau^2 &= \tfrac{1}{2}(\|T_u(a)\|_\tau^2 + \|T_u(b)\|_\tau^2) \\ &\quad - \|\tfrac{1}{2}(T_u(a) + T_u(b))\|_\tau^2 \\ &\leqq \tfrac{1}{2}(\lambda^2 + \lambda^2) - (\lambda - \varepsilon)^2 \\ &= 2\lambda\varepsilon - \varepsilon^2 \end{aligned}$$

since $\frac{1}{2}(T_u(a) + T_u(b)) = T_u\left(\frac{a+b}{2}\right)$.

But on the other hand

$$\begin{aligned}\|T_u(a)-T_u(b)\|_\tau^2 &= \|\pi(u)(a-b)u^*\|_\tau^2\\ &= \tau(u(a-b)^*(a-b)u^*)\\ &= \tau((a-b)^*(a-b)) \qquad \text{(since } \tau \text{ is tracial)}\\ &= \|a-b\|_\tau^2,\end{aligned}$$

so letting $\varepsilon\to 0$ we get $\|a-b\|_\tau^2=0$ and $a-b=c=0$ (since τ is faithful), a contradiction. Hence $K_0-K_0=\{0\}$, and K_0 consists of a single point $\hat{V}$ say. Since K_0 is invariant under each T_u,

$$[\pi(u)\hat{V}+V(u)]u^*=\hat{V} \qquad (u\in\mathscr{A}^u)$$

and rearranging

$$V(u)=\hat{V}u-\pi(u)\hat{V} \qquad (u\in\mathscr{A}^u).$$

But $\mathscr{A}^u$ linearly generates $\mathscr{A}$, so

$$V(a)=\hat{V}a-\pi(a)\hat{V} \qquad (a\in\mathscr{A}).$$

Note that by construction $\|\hat{V}\|\leqq\|V\|$ and $\hat{V}\in\Lambda$.

Now let $\mathscr{A}$ be any semifinite von Neumann algebra. For a countably-decomposable finite projection e in $\mathscr{A}$ define

$$V_e: e\mathscr{A}e\to\mathscr{B}(eH,\pi(e)K)$$

by

$$V_e(eae)=\pi(e)V(eae)e.$$

$e\mathscr{A}e$ is a finite countably-decomposable von Neumann algebra, so by the first half of the proof there is a $\hat{V}_e\in\pi(e)\Lambda e$ with $\|\hat{V}_e\|\leqq\|V_e\|\leqq\|V\|$ and

$$V_e(eae)=\hat{V}_e eae-\pi(eae)\hat{V}_e \qquad (a\in\mathscr{A}).$$

Now let $(e_\alpha)_{\alpha\in I}$ be an increasing directed set of finite countably decomposable projections with supremum 1 (see corresponding proof in [2] for a proof of the existence of such a net). Then for each $\alpha\in I$ there is a $\hat{V}_\alpha=\hat{V}_{e_\alpha}$ with $\|\hat{V}_\alpha\|\leqq\|V\|$ and

$$V_{e_\alpha}(e_\alpha ae_\alpha)=\hat{V}_\alpha e_\alpha ae_\alpha-\pi(e_\alpha ae_\alpha)\hat{V}_\alpha \qquad (a\in\mathscr{A}).$$

Also $\hat{V}_\alpha\in\pi(e_\alpha)\Lambda e_\alpha\subseteq\Lambda$. By the weak-operator compactness of bounded sets in Λ we can find a cofinal convergent subset of $(\hat{V}_\alpha)_{\alpha\in I}$, and so we may assume that $(\hat{V}_\alpha)_{\alpha\in I}$ is weak-operator convergent to an element $\hat{V}$ in Λ with $\|\hat{V}\|\leqq\|V\|$ (the subnet of projections has supremum 1 since it is cofinal).

If $\beta\leqq\alpha$, $e_\beta\leqq e_\alpha$ so

$$V_{e_\alpha}(e_\beta ae_\beta)=\hat{V}_\alpha e_\beta ae_\beta-\pi(e_\beta ae_\beta)\hat{V}_\alpha \qquad (a\in\mathscr{A}).$$

Letting $\alpha\to\infty$ and noting that $V_{e_\alpha}(e_\beta ae_\beta)=\pi(e_\alpha)V(e_\beta ae_\beta)e_\alpha\to V(e_\beta ae_\beta)$ in the weak-operator topology, we get

$$V(e_\beta ae_\beta)=\hat{V}e_\beta ae_\beta-\pi(e_\beta ae_\beta)\hat{V} \qquad (a\in\mathscr{A}).$$

Now let $\beta \to \infty$, so $e_\beta a e_\beta$ converges to a ultraweakly and in the weak-operator topology (the two topologies coincide on bounded sets); then using the normality of π and V,

$$\begin{aligned} V(a) &= \lim_\beta V(e_\beta a e_\beta) \\ &= \lim_\beta [\hat{V} e_\beta a e_\beta - \pi(e_\beta a e_\beta)\hat{V}] \\ &= \hat{V}a - \pi(a)\hat{V} \qquad (a \in \mathscr{A}) \end{aligned}$$

as required.

We can now easily deduce the main result of the paper.

Theorem 4. *Every complete normal dissipation on a semi-finite von Neumann algebra is inner.*

Proof. With the same notation as before, put

$$\theta(a) = \hat{V}^* \pi(a) \hat{V} \qquad (a \in \mathscr{A}).$$

By Lemma 2 $\theta(\mathscr{A}) \subseteq \mathscr{A}$ and by [10] θ is completely positive. A straightforward calculation gives

$$\begin{aligned} d_{\gamma_\theta}(a,b) &= (\hat{V}a^* - \pi(a^*)\hat{V})^*(\hat{V}b - \pi(b)\hat{V}) \\ &= V(a^*)^* V(b) \\ &= D_\gamma(a,1,b) \\ &= d_\gamma(a,b) \qquad (a,b \in \mathscr{A}), \end{aligned}$$

so

$$d_{(\gamma-\gamma_\theta)}(a,b) = d_\gamma(a,b) - d_{\gamma_\theta}(a,b) = 0 \qquad (a,b \in \mathscr{A}),$$

that is, $\gamma - \gamma_\theta$ is a derivation on $\mathscr{A}$. But every derivation on a von Neumann algebra is inner, so γ is inner.

Note. If $\mathscr{A}$ is a non-hyperfinite type III von Neumann algebra we do not know whether every complete normal dissipation on $\mathscr{A}$ is inner. However Christensen has proved in [1] that if $\mathscr{A}$ is considered as a weakly-closed subalgebra of $\mathscr{B}(H)$, containing the identity on H, and $V: \mathscr{A} \to \mathscr{B}(H)$ is a derivation, then there is a $\hat{V} \in \mathscr{B}(H)$ such that

$$V(a) = \hat{V}a - a\hat{V} \qquad (a \in \mathscr{A}).$$

Using this result we can easily deduce that if $\pi: \mathscr{A} \to \mathscr{B}(K)$ is a normal *-representation of $\mathscr{A}$ on a Hilbert space K and $V: \mathscr{A} \to \mathscr{B}(H,K)$ is a derivation (where $\mathscr{B}(H,K)$ is an $\mathscr{A}$-module in the obvious way), then there is a $\hat{V} \in \mathscr{B}(H,K)$ such that

$$V(a) = \hat{V}a - \pi(a)\hat{V}$$

[consider $\mathscr{B}(H, K)$ as a submodule of $\mathscr{B}(H \oplus K)$ in the obvious way]. Combining this result with Proposition 1 we obtain the following:

If γ is a complete normal dissipation on a type III von Neumann algebra $\mathscr{A} \subseteq \mathscr{B}(H)$ then there is a completely positive map $\theta: \mathscr{A} \to \mathscr{B}(H)$ such that γ_θ is a complete normal dissipation on $\mathscr{A}$ and $\gamma - \gamma_\theta$ is a derivation on $\mathscr{A}$ (which is therefore inner).

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