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$\mathbb{C}P^2$ as a Gravitational Instanton

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Abstract. We compare some of the properties of $\mathbb{C}P^2$ with those of the SU(2) Yang-Mills Instanton and conclude that $\mathbb{C}P^2$ may be regarded as a gravitational pseudoparticle surrounded by an event horizon.

1. Introduction

This paper is one of three concerned with Riemannian solutions of the Einstein equations with cosmological constant Λ,

$$R_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0. \qquad (1)$$

The first [1] contains the general theory of such spaces and their role in quantum gravity. The second (this paper) treats a particular example, $\mathbb{C}P^2$. The third [2] deals with generalized spin structures in Riemannian spaces, taking $\mathbb{C}P^2$ as a particular example.

$\mathbb{C}P^2$ is a two dimensional complex manifold which may also be given a Riemannian metric (known to mathematicians as the Fubini-Study metric) which satisfies (1). The fact that $\mathbb{C}P^2$ has non-vanishing Pontrjagin number has led Eguchi and Freund [3] to consider $\mathbb{C}P^2$ as an analogue of the well known "Instanton" solution of the SU(2) Yang-Mills equations [4]. What one calls an instanton outside the domain of SU(2) Yang-Mills theory depends upon which features of the Yang-Mills solutions one is making an analogy with. In this paper we shall point out some of the similarities and the differences between the two cases and relate them to the general discussion of [1]. Before doing so (in Section 6) we shall collect together some properties of $\mathbb{C}P^2$. Most of these are well known to mathematicians but less well known in the physics community. Section 2 contains an account of $\mathbb{C}P^2$ as a complex manifold, together with its standard Kähler structure. In Section 3 we discuss the isometry group $(SU(3)/Z_3)$ and a particular 4-dimensional subgroup. The possession of a 4-dimensional isometry group acting on 3-spheres is characteristic of the Taub-NUT family of solutions of the Einstein equations and we show $\mathbb{C}P^2$ to be a limiting case of the general form.

We also discuss the fixed point sets and relate them to the discussion in [1]. In Section 4 we discuss the geodesics and the spectra of the basic elliptic operators defined over $\mathbb{C}P^2$. In Section 5 we exhibit some solutions of the Maxwell and SU(2) Yang-Mills equations on this background and their connection with generalized spin structure.

Conventions. Greek indices run from 0 to 3 and latin from 1 to 3. The alternating tensor $\varepsilon_{\alpha\beta\gamma\delta}$ is $\sqrt{g}$ if $(\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)$. The Ricci identity is

$$\nabla_{[\alpha} \nabla_{\beta]} K^\delta = \tfrac{1}{2} R^\delta{}_{\varepsilon\alpha\beta} K^\varepsilon .$$

The Ricci tensor is $R_{\alpha\beta} = R^\sigma{}_{\alpha\sigma\beta}$. A connection on a vector bundle whose curvature $F_{\alpha\beta}$ is either

$$\text{self-dual:} \quad F_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu} = *F_{\alpha\beta} \quad \text{or}$$
$$\text{anti self-dual:} \quad F_{\alpha\beta} = -\tfrac{1}{2}\varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu} = -*F_{\alpha\beta}$$

will be called "half flat". "Self-dual" will also be called "left flat". In a two component SU(2)×SU(2) notation undotted indicies correspond to right handed objects. The spinor transcription of a self-dual 2-form corresponds to a symmetric 2 index undotted spinor.

2. The Manifold

$\mathbb{C}P^2$ or complex projective two space or the projective complex plane is defined by identifying the set of triples of complex numbers (Z_1, Z_2, Z_3), not all of which vanish, under the equivalence relation

$$(Z_1, Z_2, Z_3) = (\lambda Z_1, \lambda Z_2, \lambda Z_3), \tag{2}$$

where λ is any non-zero complex number. It may be coordinatized by introducing

$$W_{ij} = Z_i / Z_j . \tag{3}$$

For fixed j, provided $Z_j \neq 0$, W_{ij}, $i \neq j$, are a pair of complex coordinates. As j runs from 1 to 3 we obtain an atlas of 3 charts which cover $\mathbb{C}P^2$ and are holomorphically related to one another. If

$$\zeta^1 = W_{13} = Z_1 / Z_3 , \tag{4}$$

$$\zeta^2 = W_{23} = Z_2 / Z_3 \tag{5}$$

then (ζ^1, ζ^2) cover all points for which $Z_3 \neq 0$. This region is homeomorphic to $\mathbb{C}^2 = \mathbb{R}^4$. The points $Z_3 = 0$ may be regarded as "points at infinity" and are pairs (Z_1, Z_2) identified under

$$(Z_1, Z_2) = (\lambda Z_1, \lambda Z_2), \qquad \lambda \neq 0 . \tag{6}$$

This is just $\mathbb{C}P^1$ or S^2, the familiar Riemann sphere. Thus $\mathbb{C}P^2$ may be thought of as a compactification of $\mathbb{R}^4$ by the addition of a sphere at infinity. Considered as a real 4-dimensional manifold $\mathbb{C}P^2$ is compact and simply connected, with Euler number 3, Pontrjagin number 3 and second Betti number $b_2 = 1$.

$\mathbb{C}P^2$ is given its standard metric by considering firstly the metric induced on the 5-sphere

$$|Z_1|^2+|Z_2|^2+|Z_3|^2=\frac{6}{\Lambda} \tag{7}$$

from the standard metric on $\mathbb{C}^3$ or $\mathbb{R}^6$

$$ds^2=|dZ_1|^2+|dZ_2|^2+|dZ_3|^2. \tag{8}$$

The one-parameter family G of maps of S^5 into itself given by

$$Z_j \to \exp(i\alpha)Z_j \tag{9}$$

is evidently an isometry of the metric on S^5. The orbits of G are homeomorphic to circles and the space of orbits may be identified with $\mathbb{C}P^2$. (In fact the projection of points in S^5 onto orbits is the Hopf fibration of S^5 [5].) Now the orbits may be given a metric by taking that obtained by projecting the metric on S^5 orthogonally to the orbits. In local coordinates ζ^1, ζ^2 this leads to

$$ds^2=\left[1+\frac{\Lambda}{6}(|\zeta^1|^2+|\zeta^2|^2)\right]^{-2}$$

$$\cdot\left[|d\zeta^1|^2+|d\zeta^2|^2+\frac{\Lambda}{6}|d(\zeta^1\overline{\zeta^2})|^2-\frac{\Lambda}{6}d(|\zeta^1|^2)d(|\zeta^2|^2)\right] \tag{10}$$

$$=\frac{\partial^2 K}{\partial\zeta^A\partial\overline{\zeta^A}}d\zeta^A d\overline{\zeta}^{\bar{A}}\cdot A=1,2, \tag{11}$$

$$K=\frac{6}{\Lambda}\log\left[1+\frac{\Lambda}{6}(|\zeta^1|^2+|\zeta^2|^2)\right] \tag{12}$$

which shows that $\mathbb{C}P^2$ has a Kähler structure [6] with Kähler form

$$J=i\partial\bar{\partial}K. \tag{13}$$

ζ^1 and ζ^2 are related to the coordinates of Eguchi and Freund by

$$\tfrac{1}{2}\zeta^1=x+iy, \tag{14}$$

$$\tfrac{1}{2}\zeta^2=z+i\tau, \tag{15}$$

where (x, y, z, τ) are Eguchi and Freund's $\{x^\mu\}$ and

$$\Lambda=\frac{3}{2a^2}. \tag{16}$$

Their matrix C corresponds to the standard complex structure on $\mathbb{C}^2$.

One may easily establish that $\mathbb{C}P^2$ enjoys the following properties:

1. $\mathbb{C}P^2$ is an Einstein space satisfying (1) with total volume $\frac{18\pi^2}{\Lambda^2}$.

2. The Weyl tensor is anti-self-dual

$$C_{\alpha\beta\gamma\delta}=-\tfrac{1}{2}\varepsilon_{\alpha\beta\mu\nu}C^{\mu\nu}{}_{\gamma\delta}. \tag{17}$$

3. The Weyl curvature spinor $\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$ is type D. That is, it may be factored into the symmetrized outer product of 2 one index dotted spinors. Details of spinors and the Petrov classification for complex Riemannian spaces may be found in [6].

The fact that the Weyl tensor is half flat is sufficient to show that $\mathbb{C}P^2$ has no Lorentzian section. Thst is one cannot complexify $\mathbb{C}P^2$, regarded as a real 4-dimensional manifold, to obtain another real section with real metric tensor with Lorentzian signature.

3. The Isometry Group

The group SU(3) of unitary 3×3 matrices with unit determinant acting on $\mathbb{C}^3 - \{0\}$ in the standard way is a subgroup of SO(6), the isometry group of S^5. Since it commutes with arbitrary multiples of the unit matrix it clearly preserves the Hopf fibration of S^5 and the metric on the fibres. However it does not act effectively on $\mathbb{C}P^2$—the set of Hopf fibres. The set of SU(3) matrices of the form ωI where $\omega^3 = 1$ leaves all points in $\mathbb{C}P^2$ fixed. This group is the centre, Z_3, of SU(3). To obtain an effective transitive action on $\mathbb{C}P^2$ which leaves the Riemannian metric invariant we must factor out the Z_3 to obtain SU(3)/Z_3. In fact it is the largest such continuous group. SU(3)/Z_3 is 8-dimensional and groups of dimension 10 are ruled out because they would imply that $\mathbb{C}P^2$ had constant curvature and 9 because it is one less than maximal [13].

SU(3) has a $U(2)$ subgroup which acts transitively on submanifolds of $\mathbb{C}P^2$ which are 3-spheres but for two exceptional orbits—the origin and the 2-sphere at infinity. This may be seen by considering elements of the form

$$\begin{pmatrix} U(2) & 0 \\ 0 & [\det U(2)]^{-1} \end{pmatrix} \tag{18}$$

for arbitrary $U(2)$ matrices. It is easily seen that this subgroup leaves invariant

$$|\zeta^1|^2 + |\zeta^2|^2 \tag{19}$$

and thus acts on 3-spheres unless $\zeta^1 = \zeta^2 = 0$ (the origin) or ζ^1 or $\zeta^2 = \infty$ (the sphere at infinity). $U(2)$ does not act effectively on $\mathbb{C}P^2$ since elements of the form

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \tag{20}$$

where

$$\omega^3 = 1 \tag{21}$$

leave every point of $\mathbb{C}P^2$ fixed, nor is it a subgroup of SU(3)/Z_3. However by considering the previous factoring of SU(3) by Z_3 we could obtain an effective action of the corresponding 4-dimensional subgroup which we shall call G. This group has the same Lie algebra as $U(2)$. It is advantageous to introduce coordinates adapted to the group. This we do by defining Euler angles $(\psi, \vartheta, \varphi)$ and a radial coordinate r by

$$\zeta^1 = r\cos\vartheta/2\, e^{i(\psi+\varphi)/2}, \tag{22}$$

$$\zeta^2 = r\sin\vartheta/2\, e^{i(\psi-\varphi)/2}. \tag{23}$$

If

$$
\begin{aligned}
&0 \leqq \vartheta \leqq \pi \\
&0 \leqq \varphi \leqq 2\pi \\
&0 \leqq \psi \leqq 4\pi \\
&0 \leqq r \leqq \infty
\end{aligned}
\tag{24}
$$

these will cover $\mathbb{R}^4$ except for the obvious trivial coordinate singularities at $r=0$ and $\vartheta=0$ or π. The surfaces $r=\text{constant} \neq 0$ are homeomorphic to S^3 and the curves r, ϑ, $\varphi=\text{constant}$ correspond to the Hopf fibration of S^3. In these coordinates the Fubini-Study metric becomes

$$
\begin{aligned}
ds^2 = &\frac{dr^2}{\left(1+\frac{\Lambda}{6}r^2\right)^2} + \frac{r^2}{4\left(1+\frac{\Lambda}{6}r^2\right)^2}(d\psi + \cos\vartheta d\varphi)^2 \\
&+ \frac{r^2}{4\left(1+\frac{\Lambda}{6}r^2\right)}(d\vartheta^2 + \sin^2\vartheta d\varphi^2).
\end{aligned}
\tag{25}
$$

The points at infinity may be covered by introducing the coordinate $u=\frac{1}{r}$. (25) now takes the form

$$
\begin{aligned}
ds^2 = &\frac{du^2}{\left(u^2+\frac{\Lambda}{6}\right)^2} + \frac{u^2}{4\left(u^2+\frac{\Lambda}{6}\right)^2}(d\psi + \cos\vartheta d\varphi)^2 \\
&+ \frac{1}{4\left(u^2+\frac{\Lambda}{6}\right)}(d\vartheta^2 + \sin^2\vartheta d\varphi^2).
\end{aligned}
\tag{26}
$$

At $r=\infty$ $(u=0)$, which is clearly a 2-sphere of area $A_\infty = \frac{6\pi}{\Lambda}$, these coordinates also break down, however this is rather analogous to the breakdown of plane polar coordinates on $\mathbb{R}^2$. It is easily seen that if one introduces coordinates $x=u\cos\psi/2$, $y=u\sin\psi/2$ the metric is well behaved at $x=y=0$.

The possession of an isometry group with the Lie algebra of $U(2)$ acting on 3-surfaces is characteristic of a family of solutions of (1) usually referred to as Taub-NUT-de Sitter. These in turn may be embedded in the general class of type D solutions [7]. The general form with Taub-NUT symmetry and parameters adjusted to give a Riemannian signature is

$$
\begin{aligned}
ds^2 = &\frac{\varrho^2 - L^2}{\Delta} d\varrho^2 + \frac{4L^2\Delta}{\varrho^2 - L^2}(d\psi + \cos\vartheta d\varphi)^2 \\
&+ (\varrho^2 - L^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),
\end{aligned}
\tag{27}
$$

$$
\Delta = \varrho^2 - 2M\varrho + L^2 + \Lambda(L^4 + 2L^2\varrho^2 - \tfrac{1}{3}\varrho^4).
\tag{28}
$$

$\mathbb{C}P^2$ may be obtained by setting

$$
M = \pm L(1 + \tfrac{4}{3}\Lambda L^2)
\tag{29}
$$

which ensures that the metric has a right (or left) flat Weyl tensor. One then lets $L \to \infty$ at the same time introducing a new radial coordinate r defined by

$$\varrho^2 - L^2 = \frac{r^2}{4\left(1 + \frac{\Lambda}{6} r^2\right)}. \tag{30}$$

The geometrical reason for this procedure is as follows. The general form (27) which is valid only in a coordinate patch for which $\Delta \neq 0$ is invariant under a one-parameter group G_ψ generated by $\partial/\partial\psi$. G_ψ has fixed points at the 4 roots of Δ. We label these in increasing numerical order ϱ_{--}, ϱ_-, ϱ_+, and ϱ_{++}. If the local form (27) is to be successfully extended to produce a non-singular Riemannian space these fixed point sets must be either zero- or two-dimensional. In the notation of [1] these are "nuts" and "bolts" respectively. In general (27) contains 4 bolts. If condition (29) holds ϱ_- and ϱ_+ will coincide and at $\varrho = \varrho_+$ the coefficient of $(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$ will vanish. This is necessary if (27) is to contain an isolated fixed point or nut at $\varrho = \varrho_+$. We must also identify ψ appropriately. If $\Lambda = 0$ (and hence $\varrho_{--} \to -\infty$, $\varrho_{++} \to +\infty$) this would be sufficient to produce a regular solution for $|L| \leqq \varrho < \infty$. This is just Hawking's solution [8]. If on the other hand L were zero the roots at $\varrho_\pm$ would disappear and we would have to take care of the cosmological event horizon at ϱ_{++} or ϱ_{--}. The resulting solution is of course S^4 and has been treated in [9]. One must again identify appropriately. In our case the periodicities of ψ required to treat both ϱ_+ and ϱ_{++} are different in general and only by taking the limiting case $L \to \infty$ is it possible to complete the local form at ϱ_+ and ϱ_{++} preserving the half-flat condition.

Another possibility is to drop the half-flat condition, set $L = 0$ keeping $\tau = 2L\psi$ non-zero and treat the Kottler [10] or Schwarzschild-de Sitter solution

$$ds^2 = \left(1 - \frac{2M}{\varrho} - \frac{\Lambda}{3}\varrho^2\right)d\tau^2 + \frac{d\varrho^2}{\left(1 - \frac{2M}{\varrho} - \frac{\Lambda}{3}\varrho^2\right)} + \varrho^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{31}$$

This has a black hole and 2 cosmological bolts or horizons ($\varrho_- = 0$). A simultaneous completion at ϱ_+ and ϱ_{++} is possible only in the limiting case $9M^2\Lambda = 1$, when ϱ_+ and ϱ_{++} coincide. Again one must introduce a new radial coordinate. The result is $S^2 \times S^2$ with its standard metric

$$ds^2 = (1 - \Lambda r^2)d\tau^2 + \frac{dr^2}{(1 - \Lambda r^2)} + \frac{1}{\Lambda}(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{32}$$

and τ has period $\frac{2\pi}{\sqrt{\Lambda}}$.

These metrics may all be cast in the form

$$ds^2 = d\chi^2 + A(d\psi + \varepsilon\cos\vartheta d\varphi)^2 + B(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$

where A, B, and are given in Table 1.

Table 1. Positive definite Einstein metrics of Taub-NUT type with positive or zero cosmological constant. For comparison we give also the metric on $S^2 \times S^2$ which is not of Taub-NUT type. In the fourth case

$$\chi = M \operatorname{arcosh}\left(\frac{r}{M}\right) + (r^2 - M^2)^{1/2}.$$

Space	A	B	ε	Range of χ
Flat	$\frac{1}{4}\chi^2$	$\frac{1}{4}\chi^2$	1	$[0, \infty]$
S^4	$\frac{1}{4}\left(\frac{3}{\Lambda}\right)\sin^2\left(\chi\sqrt{\frac{\Lambda}{3}}\right)$	$\frac{1}{4}\left(\frac{3}{\Lambda}\right)\sin^2\left(\chi\sqrt{\frac{\Lambda}{3}}\right)$	1	$\left[0, \pi\sqrt{\frac{3}{\Lambda}}\right]$
$\mathbb{C}P^2$	$\frac{1}{4}\left(\frac{6}{\Lambda}\right)\sin^2\left(\chi\sqrt{\frac{\Lambda}{6}}\right)\cos^2\left(\chi\sqrt{\frac{\Lambda}{6}}\right)$	$\frac{1}{4}\left(\frac{6}{\Lambda}\right)\sin^2\left(\chi\sqrt{\frac{\Lambda}{6}}\right)$	1	$\left[0, \frac{\pi}{2}\sqrt{\frac{6}{\Lambda}}\right]$
Taub-NUT	$4M^2\left(\frac{r-M}{r+M}\right)$	$(r^2 - M^2)$	1	$[0, \infty]$
$S^2 \times S^2$	$\frac{1}{4}\left(\frac{1}{\Lambda}\right)\sin^2(\chi\sqrt{\Lambda})$	$\left(\frac{1}{\Lambda}\right)$	0	$\left[0, \pi\sqrt{\frac{1}{\Lambda}}\right]$

4. Geodesics and Spectrum

Since $\mathbb{C}P^2$ is an homogeneous space all geodesics are equivalent to those through any given point—e.g. the origin. But clearly the isotropy subgroup G acts transitively on the initial tangent vectors and so all must be equivalent. Consideration of a special case (e.g. $r = \infty$, $\varphi = 0$) shows that they are all closed curves of length $\pi\sqrt{\frac{6}{\Lambda}}$. This is similar to the S^4 case when all great circles are of length $\pi\sqrt{\frac{12}{\Lambda}}$. In $S^2 \times S^2$ on the other hand, there exists a one-parameter family of distinct geodesics and only for rational values of the parameter are the curves closed.

The behaviour of the geodesics is closely related to the spectrum of various differential operators on $\mathbb{C}P^2$ and to quantum fluctuations on this background. We merely remark here that the various tensor harmonics on $\mathbb{C}P^2$ carry zero triality representations of SU(3) (i.e. in terms of "quarks" the number of quarks minus antiquarks is zero mod 3). The simplest example is the n quark-antiquark representation. This is the same as homogeneous polynomials of degree n in Z_i and $\bar{Z}_i$. These project down onto $\mathbb{C}P^2$ as the $(n+1)^3$ scalar eigenfunctions of the Laplacian with eigenvalues $\frac{2\Lambda}{3}n(n+2)$ [5]. This is very similar to the $\frac{1}{6}(n+1)(n+2)(2n+3)$ eigenfunctions on S^4 with eigenvalues $\frac{\Lambda}{3}n(n+3)$ which carry a representation of $O(5)$.

Using the eigenvalues of the Laplacian on $\mathbb{C}P^2$ we can evaluate the zeta functions $\zeta_D(s)$ corresponding to various differential operators D acting on scalars. If $\{\lambda_n\}$ are the eigenvalues of D then

$$\zeta_D(s) = \sum_{\lambda_n} \lambda_n^{-s}.$$

For the important case of the conformally invariant operator $D = -\nabla_\alpha \nabla^\alpha + \frac{1}{6}R$ we readily find

$$\zeta_D(s) = \left(\frac{2\Lambda}{3}\right)^{-s} \zeta(2s-3),$$

where ζ is the ordinary Riemann zeta function.

In the case of other members of the de Rham complex the situation is more complicated. However we know on general grounds [11] that there are no harmonic one-forms and one harmonic 2-form which must be anti-self-dual. As discussed in [2] $\mathbb{C}P^2$ has no spinor structure. This is related to the fact that $SU(3)/Z_3$ has only faithful and three-valued representations. If $\mathbb{C}P^2$ had spinor structure one could construct spinor eigenfunctions of the Dirac operator and one would expect these to carry a double-valued representation of $SU(3)/Z_3$ because if we keep one point fixed in $\mathbb{C}P^2$, a rotation of 2π should reverse the sign of the spinor. However the only possible representations in SU(3) of this rotation are multiplications by rhe cube roots of unity and we have a contradiction.

5. Electromagnetic and Yang-Mills Fields

In any Kähler manifold the Kähler form J is a solution of Maxwell's equations since it is covariantly constant. In the case of $\mathbb{C}P^2$ it coincides (up to a factor) with the unique anti-self-dual harmonic 2-form. In local coordinates it is

$$J = \frac{r}{\left(1+\frac{\Lambda}{6}r^2\right)^2} dr \wedge (d\psi + \cos\vartheta d\varphi) - \frac{\frac{1}{2}r^2}{\left(1+\frac{\Lambda}{6}r^2\right)} \sin\vartheta d\vartheta \wedge d\varphi . \tag{33}$$

Since J is half-flat it has zero energy momentum tensor and so the pair $(g_{\mu\nu}, 4PJ_{\mu\nu})$ solves the coupled Einstein-Maxwell equations in a trivial way for all P. The value of P may be fixed up to an integer by the requirement that $4PJ_{\mu\nu}$ be a connection on a U(1) bundle over $\mathbb{C}P^2$. That is, by the requirement that one can consistently minimally couple $4PJ_{\mu\nu}$ to a complex scalar field with electric charge e. This leads directly to the Dirac quantization condition

$$2eP = \text{integer}.$$

P may be regarded as the magnetic monopole moment threading the r, ψ surface. This surface is in fact homeomorphic to S^2 and is not homotopic to zero. Using this field one can construct a generalized spin structure over $\mathbb{C}P^2$.

One might also like to consider non-abelian Yang-Mills fields—e.g. SU(2). Charap and Duff [12] have given a local prescription for obtaining half-flat SU(2) Yang-Mills fields from solutions of (1). This amounts to setting

$$A^i = \pm\omega^{0i} + *\omega^{0i}, \tag{34}$$

$$F^i = \pm\Theta^{0i} + *\Theta^{0i}, \tag{35}$$

where $\omega^{\mu\nu}$ and $\Theta^{\mu\nu}$ are the connection forms and curvature forms in a tetrad basis of forms $\{\omega^\mu\}$ and $*$ denotes the dual operator. The upper sign corresponds to the self-dual case. These have Pontrjagin numbers.

$$P_{YM} = \pm \tfrac{1}{2}\chi - \tfrac{3}{4}\tau, \tag{36}$$

where χ and τ are the Euler number and Hirzebruch signature of the manifold. In the present case this leads to

$$P_{YM} = \tfrac{3}{4} \quad \text{or} \quad \tfrac{9}{4}. \tag{37}$$

If we choose the obvious basis $\{\omega^{\mu}\}$ along dr, $(d\psi + \cos\vartheta d\varphi)$, $d\vartheta$ and $d\varphi$ the second case corresponds to the trivial Maxwell solution we discussed above. The first is obviously not globally well behaved if regarded as a connection on an SU(2) bundle since P_{YM} is not an integer. This appears to be the same pathology as is encountered with the consistent definition of spinors [2].

6. Discussion

Outside the strict confines of SU(2) Yang-Mills theory an "Instanton" is usually defined to be a finite action solution of the Euclidean or Riemannian equations. In this sense $\mathbb{C}P^2$ is definitely an instanton since its action is $\frac{9\pi}{4\Lambda}$. The notion of duality still holds for gravity but the half-flat classification is finer than for SU(2) Yang-Mills theory since the Weyl tensor rather than the entire Riemann tensor may be half-flat. If $\Lambda \neq 0$ this is the most one can expect and this is the case for $\mathbb{C}P^2$. The analogy between Yang-Mills instantons and $\mathbb{C}P^2$ (or Hawking's solution) goes rather further. Just as the Yang-Mills solution with $P_{YM} = 1$ may be thought of as a "pseudoparticle" we can view the Hawking half-flat solution as a pseudoparticle immersed in $\mathbb{R}^4$, and $\mathbb{C}P^2$ as a combination of this solution and the conformally flat S^4 or de Sitter solution. That is, we wish to regard $\mathbb{C}P^2$ as a half-flat pseudoparticle surrounded by a cosmological event horizon, the Λ term serving to close up space. These ideas will be given a more precise form in [1]. The basic idea is to consider Riemannian spaces which admit the action of a one-parameter family of isometries G_{ξ}. The nature of the fixed point sets is determined by the quantity

$$L_{\alpha\beta} = \xi_{\alpha;\beta} \tag{38}$$

at the fixed points, where $\partial/\partial\xi = \xi^{\alpha}\partial/\partial x^{\alpha}$ generates the group. Bolts occur when $L_{\alpha\beta}$ is degenerate and nuts when it is not so. Nuts may be self-dual or anti-self-dual. In both the Hawking and $\mathbb{C}P^2$ cases the group generated by $\partial/\partial\psi$ has an anti-self-dual nut at the origin and in addition $\mathbb{C}P^2$ has a bolt at the sphere at infinity. (In fact in Hawking's solution $L_{\alpha\beta}$ is an everywhere anti-self-dual Maxwell field.) One might object that the choice of $\partial/\partial\psi$ is arbitrary. Since $\mathbb{C}P^2$ is an homogeneous space the nut could have been located anywhere in it, using a suitable group transformation. Further, one might have chosen a non conjugately related subgroup—e.g. that generated by $\partial/\partial\varphi$ which in $\mathbb{C}P^2$ has nuts at the origin and at the north and south poles of the sphere at infinity. Nevertheless the numbers of nuts and bolts are constrained by certain topological theorems. For instance the number of nuts plus the sum of the Euler numbers of the bolts equals the Euler number of the manifold. In $\mathbb{C}P^2$ one has 1 nut and a bolt of Euler number 2, or alternatively 3 nuts, giving an Euler number of 3 for the manifold. This will be elaborated in [1].

We have shown that $\mathbb{C}P^2$ shares many of the properties of the Yang-Mills instanton and have put it in a more general setting. What remains to be investigated is its possible role in dominating the functional integral for Quantum Gravity.

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Since writing this paper we have become aware of work by A. Trautman [Int. J. Theor. Phys. **16**, 561—565 (1977)] who also considers the $U(1)$ bundle over $\mathbb{C}P^2$ that we have treated, and its relation to the Hopf fibering of S^5.

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