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Causality Criteria

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Abstract. By "causality of matter" one means its property not to admit *superluminal* excitations, i.e. excitations that propagate faster than the vacuum speed of light c. In discussing the propagation of small excitations, one has to distinguish between *phase velocities* ω_j/k, $(1 \leqq j \leqq g = \text{number of dispersion branches})$, *group velocities* $d\omega_j/dk$, a front velocity $v_f := \max_j \lim_{k\to\infty} (\omega_j/k)$, and the propagation speed $v_q := (dp/d\varrho)^{1/2}$ of isotropic quasistatic (small) perturbations. We discuss some of their properties. In particular, the (maximal) speed v_s of small signals is not smaller than v_f, and equals v_f whenever the dispersion branches $\omega_j(k)$ behave reasonably at infinity of the complex k-plane. In essence stronger conditions guarantee $v_q < v_f$ (in which case $v_q \geqq c$ would imply superluminal behaviour).

1. Introduction

Superluminal propagation velocities of perturbations have already been discussed by Sommerfeld and Brillouin [1, 2] in application to ordinary matter governed by Maxwell's equations, and have received renewed interest in the physics of matter at extreme densities, e.g. in the cores of neutron stars, cf. [3–6]. Very often in the literature one can find discussions of superluminal behaviour based on the velocity $(dp/d\varrho)^{1/2}$ which is directly formed from an equation of state $p = p(\varrho) =$ pressure as a function of mass-energy density. Though at first sight unrelated, such a procedure will receive some justification by our subsequent analysis (see in particular Proposition 5).

It is our intention to discuss and relate the fundamental velocities introduced in the abstract. To this end, our concern will be small excitations of an arbitrary extended physical system. Here the assumption "small" stands synonymously for a linearized spacetime dependence so that the solutions are superposable, and harmonic plane waves form a basis of elementary solutions. (Soliton solutions are disregarded.) Then "dispersion branches" $\omega_j = \omega_j(\boldsymbol{k})$, $(1 \leqq j \leqq g)$, govern the wave vector dependence of (angular) frequencies. This assumption does not exclude

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statistical or quantized models: (even) a quantized N-body system can be (rigorously, cf. [7, 8]) described by phase space functions whose small perturbations have a basis of harmonic plane waves (w.r.t. space and time).

In Section 2, we shall discuss the roles of a signal velocity, phase velocity and group velocity, and the wave vector dependence of the Fourier transformed perturbations. Four typical examples may serve to appreciate the generality of our assumptions.

In Section 3, the front velocity v_f will be (defined and) shown to equal v_s for a large class of physical models containing all the relativistic ones of which we are aware. The class contains all models in which ω/k is a (power of a) generalized susceptibility, like in electrodynamics. A counter example shows, however, that not *all* physical models belong to this class. Note that whenever $v_f = v_s$ holds, causality means $v_f \leqq c$. We perform our analysis in wave number space (rather than in frequency space) because we regard the initial value problem as more transparent than the one-sided boundary value problem.

Finally, in Section 4, the Kramers-Kronig relations are applied in order to prove $v_q < v_f$ under stronger conditions (cum grano salis) than needed for the proof of $v_f = v_s$.

2. Propagation of Small Signals

What is a signal? In a broad sense, we mean by a signal a localized excitation propagating between two spacetime points. Clearly, a signal cannot be described by analytic, or quasi-analytic [9] functions because they have infinite support, i.e. are non-local. On the other hand, any departure from quasi-analytic behaviour can be used as a signal, which vanishes for $x \geqq 0$, say, but not for $x < 0$. For instance, a signal can be realized by the threshold-crossing of a field excitation and of a finite number of its space-time derivatives. The maximum speed v_s of a small signal will be called "signal velocity". An algorithm to calculate v_s needs some preparatory analysis with which we begin now.

We consider small excitations of a physical system which can be described (by assumption) by a linear homogeneous system $\mathscr{L}$ of equations for the Fourier coefficients $g_a(\boldsymbol{k}, \omega; p_\nu)$ of a finite number of excitations (=varied physical variables) $f_a(x_\mu; p_\nu)$, where the index $a = 1, \ldots, f$ counts independent components, $k_\mu := (\boldsymbol{k}, \omega/c)$ is the wave-number 4-vector, $x_\mu := (\boldsymbol{x}, ct)$ is the position 4-vector, and p_ν stands for further independent parameters such as momentum variables of phase space distributions which will not be mentioned in the sequel. Solvability of $\mathscr{L}$ implies a dispersion relation $D(\boldsymbol{k}, \omega) = 0$, whose solutions $\omega_j = \omega_j(\boldsymbol{k})$, $1 \leqq j \leqq g$, are called "dispersion branches". Solvability of the general Cauchy problem implies that the number g of branches equals the order (in ∂_t, or ω) of the system; e.g. $g = 2f$ for a second order system. For example, if the governing field equations are differential equations then $D(\boldsymbol{k}, \omega)$ is the determinant of the Fourier transform of the varied system, hence a polynomial in $(\boldsymbol{k}, \omega)$, and $\omega_j(\boldsymbol{k})$ are its roots. But if they are kinetic equations for phase space distribution functions, $D(\boldsymbol{k}, \omega)$ ceases to be rational. For isotropic, Lorentz invariant equations, $D(\boldsymbol{k}, \omega)$ must be a function of $k_\mu k^\mu$ and $k_\mu u^\mu$ where u^μ is the center-of-mass 4-velocity, (because there are no further Lorentz invariants).

In any case, the general excitation $f_a(x_\mu)$ can be expanded (for any reasonable model) as

$$f_a(\boldsymbol{x}, t) = \sum_j \int d^3k c_j(\boldsymbol{k}) g_{aj}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_j(\boldsymbol{k})t)}, \tag{1}$$

where $\{g_{aj}(\boldsymbol{k})\}$ is a basis of normalized solution vectors of $\mathscr{L}$, g_{aj} belonging to branch "j", and $c_j(\boldsymbol{k})$ are suitable expansion coefficients. Here "generality" is equivalent with the fact that $f_a(\boldsymbol{x}, t)$ can take arbitrary initial (Cauchy) data:

$$f_a(\boldsymbol{x}, 0) = \int d^3k \sum_j c_j(\boldsymbol{k}) g_{aj}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2}$$

$$\dot{f}_a(\boldsymbol{x}, 0) = -i \int d^3k \sum_j c_j(\boldsymbol{k}) \omega_j(\boldsymbol{k}) g_{aj}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3}$$

i.e. that Equations (2) and (3) can be inverted: The solution $c_j(\boldsymbol{k})$, or rather $c_j g_{aj}$, is of the form

$$c_{aj}(\boldsymbol{k}) := c_j(\boldsymbol{k}) g_{aj}(\boldsymbol{k}) = \int d^3x\, l_{aj}(\boldsymbol{x}; \boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4}$$

where the $l_{aj}(\boldsymbol{x}; \boldsymbol{k})$ are linear in the initial data $f_a(\boldsymbol{x})$, $\dot{f}_a(\boldsymbol{x})$ with rational coefficients in $g_{aj}(\boldsymbol{k})$, $\omega_j(\boldsymbol{k})$, homogeneous of degree zero in the g_{aj} and of degree zero or minus one in the ω_j. All that matters is the fact that the $\boldsymbol{x}$-support of the $l_{aj}(\boldsymbol{x}; \boldsymbol{k})$ equals that of the initial data, and that their $\boldsymbol{k}$-dependence is implicit via a rational dependence on the coefficients of $\mathscr{L}$ and on $\omega_j(\boldsymbol{k})$. This $\boldsymbol{k}$-dependence is algebraic if $\mathscr{L}$ depends algebraically on $\boldsymbol{k}$. If so, the $c_{aj}(\boldsymbol{k})$ in Equation (4) grow (at most) like $\exp\{\pm \mathrm{Im}(k)\cdot x_0\}$ for $k \to \infty$ in the complex k-plane, where $k := (\boldsymbol{k}^2)^{1/2}$, and where x_0 is a bound on the $\boldsymbol{x}$-support in $\boldsymbol{k}$-direction. This growth property will be needed below, and will be made an assumption in case $\mathscr{L}$ does not depend algebraically on $\boldsymbol{k}$.

From now on we restrict considerations to one component f of the (small) perturbation $f_a(x_\mu)$, and write Equations (1) and (4) as:

$$f(\boldsymbol{x}, t) = \int d^3k \sum_j \tilde{f}_j(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_j(\boldsymbol{k})t)}. \tag{5}$$

Here $\tilde{f}_j(\boldsymbol{k})$ is the Fourier amplitude of the j^{th} excited mode, whose phase and group velocity are $\omega_j/\boldsymbol{k}$ and $\partial\omega_j/\partial\boldsymbol{k}$ respectively. Despite its great importance for practical purposes, the group velocity cannot be used for a rigorous discussion of propagation speeds: it has only approximate character, and loses its meaning in domains of strong dispersion. This can be seen by remembering that a wave group propagates with $\partial\omega_j/\partial\boldsymbol{k}$ iff ω_j in the exponent of Equation (5) can be approximated by its Taylor series up to the first term, i.e. iff higher order terms can be neglected. They can certainly not be neglected in domains of anomalous dispersion where $|\partial\omega_j/\partial\boldsymbol{k}|$ exceeds c even within everyday applications of Maxwell's theory.

Next we reduce the 3-dim problem to that in one space dimension by restricting considerations to the propagation along one straight line through the (arbitrarily chosen) origin of space coordinates, spanned by the unit vector $\boldsymbol{e}$. Writing $\boldsymbol{k} =: k_{\|}\boldsymbol{e} + \boldsymbol{k}_\perp$, $x := \boldsymbol{x}\cdot\boldsymbol{e}$, we obtain from Equation (5)

$$g(x, t) := f(x\boldsymbol{e}, t) = \int dk_{\|} \int d^2k_\perp \sum_j \tilde{f}_j(k_{\|}, \boldsymbol{k}_\perp) e^{i(k_{\|}x - \omega_j(k_{\|}, \boldsymbol{k}_\perp)t)} \tag{6}$$

which is of the form:

$$g(x,t) = \int d^2 l \int_{-\infty}^{\infty} dk \tilde{g}(k,l) e^{i(kx - \omega(k,l)t)}, \tag{7}$$

where l stands short for $(\boldsymbol{k}_\perp, j)$. Thus the 3-dim problem differs from that in one space dimension formally by having a continuous number of dispersion branches $\omega(k;l)$ rather than a finite number of branches $\omega_j(k)$.

We conclude this section with four examples of dispersion relations $D(\boldsymbol{k},\omega)=0$. For an unquantized, multicomponent, cold, magnetized plasma, D reads [10]:

$$\begin{aligned} D = {} & (\varepsilon_0 - n_\perp^2)\{\varepsilon_+\varepsilon_- + [n_{\shortparallel}^2 - (\varepsilon_+ + \varepsilon_-)/2](n^2 + n_{\shortparallel}^2) - n_{\shortparallel}^4\} \\ & + n_{\shortparallel}^2 n_\perp^2 [n^2 - (\varepsilon_+ + \varepsilon_-)/2] \end{aligned} \tag{8}$$

with

$$\boldsymbol{n} := c\boldsymbol{k}/\omega, \quad \varepsilon_0 := 1 - \sum_\alpha (\omega_\alpha/\omega)^2, \quad \varepsilon_\pm := 1 - \sum_\alpha \omega_\alpha^2/\omega(\omega \mp \Omega_\alpha),$$

where the index α numbers different components, ω_α, Ω_α are the plasma and gyro frequencies respectively, and where the indices $\shortparallel, \perp$ denote parallel and perpendicular components w.r.t. the magnetic field. If the latter vanishes, D simplifies to $D = \varepsilon(\varepsilon - n^2)^2$, with $\varepsilon := \varepsilon_0 = \varepsilon_\pm$. In any case, $D = D(\boldsymbol{k}, \omega^2) = 0$ leads to algebraic dispersion branches $\pm\omega_j(\boldsymbol{k})$. The latter are real for real $\boldsymbol{k}$, have no infinities for complex, finite $\boldsymbol{k}$, and vanish at most at $\boldsymbol{k} = 0$.

For a warm, one-component, non-magnetized plasma, the Vlasov equation leads to [11]:

$$D = 1 + (n\omega_p^2/c^2k^2) \int_{-\infty}^{\infty} d\beta f'(\beta)/(1 - n\beta) \quad \text{for} \quad \mathrm{Im}(n) < 0, \tag{9}$$

where $f(\beta) := mc \int d^2 p_\perp \cdot f_0(mc\beta, \boldsymbol{p}_\perp)$ is the effective unperturbed velocity distribution density in $\boldsymbol{k}$-direction, $\boldsymbol{p}_\perp \perp \boldsymbol{k}$, ω_p = plasma frequency, and where f_0 is normalized such that $\int d^3 p f_0(\boldsymbol{p}) = 1$. As a result, $n(k)$ is no longer algebraic.

Thirdly, water surface waves have the (only) dispersion branch [12]

$$\omega(k) = \{gk \tanh(kh)[1 + k^2\sigma/\varrho g]\}^{1/2}, \tag{10}$$

where g = gravity acceleration, h = water height, and σ/ϱ is the specific surface tension. Note that $\omega(k)$ has an essential singularity at complex $k = \infty$, and that $n(k) := ck/\omega \to 0$ or ∞ for $\sigma \neq 0$ or $\sigma = 0$ respectively. This example will show that our results below do not all follow from first physical principles.

A fourth example is the non-relativistic heat conduction equation. Its dispersion branch $\omega \sim k^2$ has $v_f = \infty$, manifesting an unrealistic description in the limit as $k \to \infty$.

3. Front Velocity and Signal Velocity

Equation (6) suggests that the velocity

$$v_f := \sup_{j, \boldsymbol{k}_\perp} \mathrm{Re} \lim_{k_{\shortparallel} \to \infty} [\omega_j(k_{\shortparallel}, \boldsymbol{k}_\perp)/k_{\shortparallel}], (\boldsymbol{k} \text{ real}), \tag{11}$$

may control the propagation speed (in $\boldsymbol{e}$-direction) of small excitations of an extended physical system because the Fourier transform of a (non-analytic!) signal must have essential contributions from large wave numbers. We call it the "front velocity", and prove that v_f equals the signal velocity v_s if a certain number of reasonable assumptions are satisfied. Though all plausible physical models have a finite v_f, it helps to know that $v_f = v_s$ holds more generally. We therefore start with

Proposition 1. *$v_f = v_s$ holds for $v_f = \infty$.*

Proof. Remember first that the Fourier transform $\tilde{f}(\boldsymbol{k})$ of a function of compact support $f(\boldsymbol{x})$ is regular analytic for complex $\boldsymbol{k}$, and behaves exponentially at complex infinity:

$$\tilde{f}(\boldsymbol{k}) = \int d^3x f(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sim e^{\mathrm{Im}(\boldsymbol{k}\cdot\boldsymbol{x}_0)} \tag{12}$$

for $|\mathrm{Im}(\boldsymbol{k}\cdot\boldsymbol{x}_0)| \to \infty$; i.e. $|\tilde{f}|$ grows exponentially, or shrinks exponentially to zero, depending on the sign of $\mathrm{Im}(\boldsymbol{k}\cdot\boldsymbol{x}_0)$. Now the spatial Fourier transform $\tilde{e}(\boldsymbol{k};t)$ of $f(\boldsymbol{x},t)$ [Eq. (5)] is of the form $\tilde{e}(\boldsymbol{k};t) := \sum_j \tilde{f}_j(\boldsymbol{k}) \exp\{-i\omega_j(\boldsymbol{k})t\}$ where $\tilde{f}_j(\boldsymbol{k}) = \int d^3x\, l_j(\boldsymbol{x};\boldsymbol{k})$ is a linear combination of functions of the form (12) with rational coefficients in $g_{aj}(\boldsymbol{k})$, $\omega_j(\boldsymbol{k})$. $\tilde{e}(\boldsymbol{k};t)$ cannot be the Fourier transform of a function of compact support if one of the $\omega_j/k_{\parallel}$ is unbounded for real $\boldsymbol{k}$: the $\omega_j(\boldsymbol{k})$ would have to be analytic (because $\tilde{e}$ and $\tilde{f}_j$ have this property), and their imaginary parts would have to be bounded by $C\cdot|\boldsymbol{k}|$ near complex infinity [because of the exponential growth law in Equation (12) which has to hold identically in t and for all initial data of compact support]. But an unbounded real part of $\omega_j/k_{\parallel}$ for real $\boldsymbol{k}$ implies an imaginary part of ω_j that grows faster than $|k_{\parallel}|$ for complex $k_{\parallel} \to \infty$, as can be found from Cauchy's integral theorem for regular analytic functions applied to a suitable contour near ∞. As a result, $f(\boldsymbol{x},t)$ cannot be of compact support for $t > 0$, even when the initial data at $t = 0$ vanish outside some finite domain. This means an infinite signal velocity. □

For the sake of illumination we offer yet another proof which works when $\omega \sim k^{\alpha}$. In this case, a point source $g(x,0) = \delta(x)$, $\dot{g}(x,0) = 0$, gives rise to a signal $g(x,t) = x^{-3}h(x^{\alpha}/t)$, and the curves of constant $x^3 g(x,t)$ are given by $x^{\alpha} = Ct$. Whenever $\alpha > 1$ and $h(y)$ is non-zero in a neighbourhood of $y = \infty$ (which it has to be according to the preceding proof), such δ-shape signals spread with arbitrary speed (as $C \to \infty$). But even without this knowledge, their speed at $t = +0$ can be seen to be infinite for $\alpha > 1$. □

We now treat the more interesting case of an asymptotically dispersion-free theory:

Proposition 2. *$v_f = v_s$ holds for $v_f < \infty$ if:*

A) *the dispersion relation $D(\boldsymbol{k},\omega) = 0$ is analytic;*

B) *$\omega_j/k_{\parallel}$ has a limit for complex $k_{\parallel} \to \infty$ (not just for real $k_{\parallel}$);*

C) $\mathrm{Im}(\omega_j) \to 0$ *for real* $|k_{\parallel}| \to \infty$;

D) *for Cauchy data of compact support, the Fourier amplitudes $l_{aj}(\boldsymbol{x};\boldsymbol{k})$ in Equation (4) grow less than exponentially for complex $k_{\parallel} \to \infty$ ($\boldsymbol{x}$ fixed);*

E) *the Cauchy problem for the excitations is uniquely solvable.*

Before we proceed to the proof, let us explain why we consider assumptions A)–E) physically reasonable: The analyticity assumption A) is natural even though it loses its direct interpretation for very large k; it holds for all models of interest, and is an unavoidable tool in most proofs. Regularity assumption B) (at infinity) seems natural at least for relativistic models; and follows if $k_{||}/\omega_j$ is a (power of a) generalized susceptibility, like in electrodynamics [14]; but counter example (10) of water surface waves shows that it does not follow from first principles alone. Assumption C) should hold for every realistic physical system: the imaginary part of ω describes damping, and damping should go to zero for frequencies above all resonances of a system. Morever, C) follows from B) plus the space reflection symmetry F) discussed below. Even more so, assumption E) looks like a law of nature. Only assumption D) is ad hoc and unpleasant. It is automatically satisfied for all systems deriving from differential equations, as mentioned below Equation (4), and may well hold generally for realistic systems. A violation of assumption D) would imply $v_s = \infty$ because $f(\boldsymbol{x}, t)$ would not be of compact support for $t > 0$ (see proof of Proposition 1), hence $v_q < v_s$ in this case.

Proof of Proposition 2. i) It has to be shown (first) that $v_f \geqq v_s$. This means that the solution $g(x, t)$ in Equation (7) vanishes for all points $\boldsymbol{x} = x\boldsymbol{e}$ whose spatial distance from the initial support at $t = 0$ is larger than $v_f t$. It suffices to show that the integral over k vanishes for these points, which are characterized by

$$\pm \operatorname{Im}[k \cdot (x - x_0) - \omega(k, l) \cdot t] > 0 \text{ for all } l \text{ and large } |\operatorname{Im}(k)|, \tag{13}$$

where x_0 is a bound on the support of the initial data. But this condition and assumptions A)–D) guarantee that the path of integration from $-\infty$ to ∞ over k can be closed in the upper or lower complex k-half-plane, depending on the sign of Im [...].

The integral therefore vanishes iff the sum of possible residues plus contour integrals around branch cuts vanishes in the respective half-plane. Residues could result from zeros in the denominator determinants of the $l_{aj}(\boldsymbol{x}; \boldsymbol{k})$ in Equation (4). These would, however, give rise to non-trivial solutions of $\mathscr{L}$ for zero initial data, in contradiction to assumption E). Contour integrals would result from poles and/or branch cuts of $\omega(k, l)$. In such cases, the path of integration in the complex k-plane could be chosen to deviate from the real axis by some detour above or below all of the (finite number of) poles and/or branch points. Such detouring would not change the initial data at $t = 0$, whose Fourier transforms [of shape (4) and (12)] are regular for all $\boldsymbol{k}$; (only the sums over j enter). And it could not change the value of the integral for finite t either because the integral solves $\mathscr{L}$ for any path, and takes the correct initial data for any finite excursion from the real axis, hence solves the Cauchy problem and is thereby unique [ass. E)]. It will be argued below, however, that such irregularities of the Fourier amplitudes are even ruled out by assumption E).

In any case, we have shown that $g(x, t)$ vanishes whenever condition (13) is satisfied. It can be rewritten as: $|\Delta x/\Delta t| > |\operatorname{Im}(\omega_j)/\operatorname{Im}(k)|$ for large $|\operatorname{Im}(k)|$. Assumptions B) and C) imply that forming the imaginary part is not necessary, so that the right hand side can be replaced by v_f from Definition (11), and $v_s \leqq v_f$ is proven.

ii) A demonstration of $v_s \geqq v_f$ for $0 < v_f < \infty$ asks for a detailed evaluation of Fourier integrals, with the result that small signals can propagate with any speed smaller than v_f. Such an evaluation has been performed in [1,2]. Suffice it to remark that for large wave numbers, the theory gets dispersion-free, with v_f = phase velocity = group velocity, so that $v_s = v_f$ is plausible without detailed calculation. □

Proposition 2 implies that no small signal can propagate if $v_f = 0$ [and conditions A)–E) are satisfied]: an initial excitation stays where it is. We are not aware of any realistic model with this property, but if such existed, it would probably develop discontinuities at the edges of the support which would have to be treated by the full non-linear theory, and would most likely admit (large) signals with $v_s > 0$. Note that example (10) with $\sigma = 0$ has $v_f = 0$, but violates condition B). One finds $v_s = \infty$ for this model, along the same lines as in the proof of Proposition 1.

In the proof of Proposition 2, we allowed for the possible existence of singularities of the Fourier amplitudes.

$$\tilde{e}(k) := \sum_j \tilde{f}_j(k) e^{-i\omega_j(k)t} \tag{14}$$

in the complex k-plane. Such singularities are, however, ruled out by assumption E): Whereas the function $k(\omega)$ can have poles (and branch points) in the complex ω-plane, cf. [2], the functions $\omega_j(k)$ are finite in the whole complex k-plane, and their possible branches are such that $\tilde{e}(k)$ is unbranched in the complex k-plane. (For instance, when ω_j have branch points but ω_j^2 are unbranched, $\tilde{e}$ is an even function of the ω_j.) Such an asymmetry between k and ω mirrors the asymmetry between boundary value problems and the initial value problem, apart from the fact that k-space is in general 3-dimensional. We prove

Proposition 3. *Assumptions* A) *and* E) *of Proposition 2 imply the regularity in the complex k-plane of the Fourier amplitudes* $\tilde{e}(k)$, *Equation* (14), *occurring in Equations* (5) *or* (7), *for all initial data of compact support. In particular,* $\omega_j(k)$ *are finite for all complex k.*

Proof. According to our discussion under Equation (12), the Fourier amplitudes of the (localized) initial data are everywhere regular. Compared with them, the Fourier amplitudes $\tilde{e}(k)$ [in Eq. (14)] at $t > 0$ involve the exponential factors $\exp\{-i\omega_j(k)t\}$, which moreover separate the terms belonging to different branches j. Quite generally, if an $\tilde{e}(k)$ had an irregularity with non-vanishing contour integral around it, one could construct different solutions for a given Cauchy problem by once choosing the path of integration in Equation (7) along the real k-axis, another time by pushing it beyond the irregularity (compare proof of Proposition 2). Condition E) rules out that such non-vanishing contour integrals can occur. But this implies the absence of any irregularity of $\tilde{e}(k)$ because the contour integrals vanish for all (regular) Fourier transforms $\tilde{f}_j(k)$ of functions with compact support. At the same time, it forbids the presence of poles of $\omega_j(k)$: such poles would e.g. show up as residues in the repeated time derivatives of $\tilde{e}(k)$ at $t = 0$, again in conflict with assumption (*E*). □

For the sake of completeness, we finally mention the (trivial) inequality

Proposition 4. $v_f \leqq v_s$.

It can be verified by combining Proposition 1 with the second part of the proof of Proposition 2.

4. $dp/d\varrho$ and Dispersion Branches

For an isotropic equation of state $p = p(\varrho)$, $(dp/d\varrho)^{1/2}$ is the propagation speed of quasistatic small perturbations. It will therefore coincide with $\mathrm{Re} \lim_{\omega_j \to 0} (\omega_j/k)$ for some j, namely with the low-frequency-limit $v(0)$ of the phase velocity of some longitudinal mode (or dispersion branch). In order for this limit to be finite, k must tend to zero as well, so that:

$$v_q := (dp/d\varrho)^{1/2} = \mathrm{Re} \lim_{k \to 0} (\omega_j/k) =: v_j(0) \tag{15}$$

for some j.

So far, v_q looks unrelated to v_s, and to discussions of causality. As pointed out by Ruderman [4], however, the Kramers-Kronig relations establish a relationship. They can be applied to certain analytic functions which are regular in the upper half-plane and have a finite limit at infinity (in this half-plane). A candidate for such a function is $n(\omega)$; a comparison of $n(0)$ with $n(\infty)$ will give the desired result. Note that the function $\omega_j(k)$ cannot be used in the Kramers-Kronig relation because it tends to have branch points in both k-half-planes. Also, the analytic continuation of one branch $\omega_j(k)$ generally includes several branches of which all but the "highest" end in a resonance. In other words: the analytic function $\omega_j(k)$ tends to be multivalued whereas its inverse $k(\omega)$ tends to be singlevalued in the upper ω-half-plane for each polarization state. [Typically, on the real ω-axis, $\mathrm{Re}\, k(\omega)$ is similar to $\tan \omega$ for not too large $|\omega|$, and to $C\omega$ for large $|\omega|$.] In order to get a finite limit for $|\omega| \to \infty$, one has to pass from $k(\omega)$ to $n(\omega)$.

Proposition 5. *$v_q < v_f \leqq v_s$ holds if the dispersion relation $D(\boldsymbol{k}, \omega) = 0$ satisfies assumptions* A) *and* B) *of Proposition 2, and if moreover:*

F) *$\omega(\boldsymbol{k})$ satisfies the symmetry relation $-\omega^*(-\boldsymbol{k}^*) = \omega(\boldsymbol{k})$ (where we have dropped the index j, and a star denotes complex conjugation);*

G) $\mathrm{Im}\,(\omega(\boldsymbol{k})) \leqq 0$ *holds for real* $\boldsymbol{k}$*;*

H) *$n_{||}(\omega) := ck_{||}/\omega$ is regular in the upper complex ω-half-plane, including the real axis.*

For a better understanding of the assumptions, note that B) and F) imply C), so that only conditions D) and E) of Proposition 2 can be spared. Of these, D) may be automatically satisfied whereas E) is fulfilled by every reasonable physical model. The assumptions of Proposition 2 are therefore more or less contained in those of the present one.

Concerning the physical significance of conditions F)–H) we mention that F) is the (analytic continuation of the) symmetry under space reflection: invariant speed and damping. Note that relation (8) is expressed in the rest system of the medium; its Lorentz invariant form can be found e.g. in [13]. Condition G) excludes anti-damping, a property satisfied at least for excitations from the ground state which ought to be stable. Not equally transparent is condition H): it holds if $n(\omega)$ is a (power of a) generalized susceptibility, like in electrodynamics [14]. And it is plausible: poles of $n(\omega)$ mean resonances of the system which should be damped (rather than anti-damped), hence occur for negative imaginary part of ω; moreover, single-valuedness of $n(\omega)$ in a neighbourhood of the real axis means a well-defined propagation speed and damping rate (of that mode). Note that assumption H) cannot be derived from conditions A)–G): The (probably unrealistic) dispersion relation

$$n^2 = 1 + \omega_0^2/[(\omega + i\alpha)^2 - \Omega^2] \tag{16}$$

satisfies A)–F) and even G) for $\alpha < 0$ but has its (unbounded) branch points in the upper ω-half-plane. [It is of the form of example (8) with imaginary gyro frequency $\omega_p = i\omega_0$, i.e. corresponds to negative particle masses, and to negative damping for $\alpha < 0$.] Condition H) is therefore independent and crucial for our proof of $v_q < v_f$. It is expected to hold at least in the rest system of the medium.

Proof of Proposition 5. In view of Proposition 4, the proof reduces to that of $v_q < v_f$, or $v(0) < v(\infty)$, or $n(0) > n(\infty)$.

For an analytic function $f(z)$ in the upper half-plane which is regular on the real axis and vanishes at infinity, Cauchy's integral theorem in the upper half-plane implies [15, 14]:

$$f(x) = \frac{1}{i\pi}\left(\fint_{-\infty}^{\infty} \frac{dz f(z)}{z - x} - \int_C \frac{dz f(z)}{z - x} \right), \quad x \text{ real}, \tag{17}$$

where $\fint$ denotes Cauchy's principal value, and C is a contour in the upper half-plane that encloses all possible singularities and branch cuts of $f(z)$. Taking the real part and adding $i \cdot \mathrm{Im} f(x)$ leads to the (generalized) Hilbert relation

$$f(x) = \frac{1}{\pi}\left(\int_{-\infty}^{\infty} \frac{dz \, \mathrm{Im} f(z)}{z - x - i0} - \mathrm{Im} \int_C \frac{dz f(z)}{z - x} \right), \tag{18}$$

where "$i0$" stands short for "$\lim (i\varepsilon)$ as $\varepsilon \to +0$".

If, moreover, $f(z)$ satisfies $f^*(-z^*) = f(z)$, $\mathrm{Im} f(z)$ vanishes at the origin, and $f(0)$ can be formally subtracted:

$$f(x) - f(0) = \frac{x}{\pi}\left(\int_{-\infty}^{\infty} \frac{dz \, \mathrm{Im} f(z)}{z(z - x - i0)} - \mathrm{Im} \int_C \frac{dz f(z)}{z(z - x)} \right). \tag{19}$$

Again using $f^*(-z^*) = f(z)$, we obtain a (generalized) Kramers-Kronig relation [15] by re-expressing the integral from $-\infty$ to 0 as an integral from 0 to ∞ and adding:

$$\begin{aligned} f(x) - f(0) &= \frac{2x^2}{\pi} \int_0^{\infty} \frac{dz \, \mathrm{Im} f(z)}{z(z^2 - x^2 - i0)} - \frac{x}{\pi} \mathrm{Im} \int_C \frac{dz f(z)}{z(z - x)} \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{dy \, \mathrm{Im} f(xy)}{y(y^2 - 1 - i0)} + \mathscr{I}(x), \end{aligned} \tag{20}$$

where, unfortunately, nothing can be said in general about $\mathscr{I}(\infty)$. If, however, $f(z)$ is regular in the upper half-plane, $\mathscr{I}(x)$ vanishes, and we get in particular

$$\mathrm{Re}\,[f(\infty) - f(0)] = \lim_{x \to \infty} \frac{2}{\pi} \fint_0^{\infty} \frac{dy \, \mathrm{Im} f(xy)}{y(y^2 - 1)}. \tag{21}$$

We now verify that $n(\omega)-n(\infty)$ satisfies all the assumptions made about $f(z)$, because $\operatorname{Im} n(\infty)$ vanishes [conditions B) and F)]. In addition, we find $\operatorname{Im} n(\omega) \geqq 0$ for $\omega \geqq 0$ as a consequence of G), because the map $k \to \omega$ sends the real axis into the lower half-plane and therefore its inverse $\omega \to k$ sends the real ω-axis into the upper k-half-plane, and correspondingly for $\omega \to n$. We therefore obtain from Equation (21):

$$c[v^{-1}(\infty)-v^{-1}(0)] = \operatorname{Re}[n(\infty)-n(0)] = \lim_{x\to\infty} \frac{2}{\pi} \int_0^\infty \frac{dy \operatorname{Im} n(xy)}{y(y^2-1)} \leqq 0 . \qquad (22)$$

Under realistic conditions, $\operatorname{Im} n(\omega)$ does not vanish identically for $\omega > 0$, and we get $v(\infty) > v(0)$ as claimed. □

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