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On the Analyticity in the Potential in Classical Statistical Mechanics

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Abstract. Some general results on strong cluster properties of connected or partially connected correlations, and their links with analyticity properties with respect to the potential or to classes of perturbations of the potential are presented.

1. Introduction

In earlier papers [1–3] the notion of strong cluster properties of correlations was introduced; these properties were proved for totally connected correlations in various situations and their links with the analyticity of thermodynamic and correlation functions with respect to the activity or magnetic field were established.

In the present work we present more general links between cluster properties and analyticity. Namely an equivalence is exhibited between analyticity with respect to general classes of perturbations of the potential, and corresponding strong cluster properties of totally or partly connected correlations. As an example, analyticity under perturbations of the two-body potential is linked to clustering with respect to subsets of two points.

A large part of the results has been previously reported in the thesis of one of the authors [4], where the details of some proofs, omitted here, will be found.

The systems considered here will be lattice gases. By the Lee-Yang isomorphism, they can be turned into spin $\frac{1}{2}$ systems and the results can be adapted to this case with only minor modifications (conversion formulae will be found in [4]). As in previous papers, it should also be possible to obtain results of the same type for continuous systems, but this will not be treated. Finally, the results apply mainly to potentials that decrease at least like an inverse power s of the distance, $s>\nu$, or $s>2\nu$, where ν is the space dimension. For simplicity, some of the results will be stated only in the case of finite range or exponentially decaying potentials, but they can be adapted to the above-mentioned case of power decay.

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The organization of the paper is the following. Preliminary definitions and results on the various systems of totally or partly connected correlations considered, and on some of their links, are given in Section 2. Corresponding strong cluster properties are then introduced in Section 3. These properties are proved at low activity in Section 4. Extension theorems, which allow one to extend them from small regions to larger domains in which analyticity properties are known, are presented, together with some applications, in Section 5. Finally it is shown in Section 6 that conversely strong cluster properties at real points imply analyticity properties, and equivalence theorems follow. Final remarks are indicated in Section 7.

In the following, analyticity with respect to the potential is to be understood as analyticity on a Banach space. It is equivalent to analyticity in any direction of the space, together with a local boundedness condition [5].

2. Preliminary Definitions and Results

A) Correlations

The following families of correlation functions, which are all equivalent for the definition of the physical states, will be considered. Being given any subset X of points $x_1, \ldots, x_N$ of the lattice, $\varrho(X)$ denotes the probability that each point of X is occupied by a particle. For reasons which appear later, it will also be convenient to consider functions $\hat{\varrho}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ that denote the probability that each point of X_i is occupied by a particle, $i=1, \ldots, M$. Here each X_i is, as previously, defined as a subset of points of the lattice, and not as a multiplet of points of the lattice, i.e. the points of X_i cannot coincide and their ordering is irrelevant. In view of the definitions, $\hat{\varrho}(X) \equiv \varrho(X)$. On the other hand some points of a subset X_i may coincide with some points of other subsets X_j, $j \neq i$, in which case $\hat{\varrho}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ does not necessarily vanish (for instance $\hat{\varrho}(\underset{\sim}{X}, \underset{\sim}{X}) \equiv \varrho(X)$ by definition. For general expressions of the functions $\hat{\varrho}$ in terms of the function ϱ, see [1,4]).

The underscripts $\sim$ in $\hat{\varrho}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ and later in $\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$, $L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)$, etc. are intended to specify in a completely unambiguous way the subsets X_i that are considered.

B) Partially Connected Correlations

Being given a family of functions $X_1, \ldots, X_M \to f(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$, where each X_i is a set of points in $\mathbb{Z}^\nu$ and M is any positive integer ($M \geqq 1$), a corresponding family of functions f^T that are partially connected with respect to the sets $X_1, \ldots, X_M$ is defined, by induction, through the formula:

$$f^T(\underset{\sim}{X}) = f(X), \tag{1}$$

$$f^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M) = f(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M) - \sum_{\substack{\pi_1, \ldots, \pi_k \\ k>1}}^{1 \ldots M} \prod_{j=1}^{k} f^T(\{\underset{\sim}{X}_i\}; i \in \pi_j), \tag{2}$$

where the sum $\sum$ in the right-hand side, runs over all non trivial ($k>1$) partitions of $1, \ldots, M$. The totally connected functions $x_1, \ldots, x_N \to f^T(x_1, \ldots, x_N)$ are the particular case of (1) and (2) obtained when all sets X_i have one point. For a presentation of some links between totally and partly connected correlations, see [1,4].

For reasons which appear below, the most convenient functions will be the functions $\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ obtained from the function $\hat{\varrho}$ by the formulae (1) and (2)[1]. The correlation functions $\hat{\varrho}^T_\Lambda$ relative to a finite box $\Lambda \subset \mathbb{Z}^\nu$, for given boundary conditions, are defined similarly.

Finally some of the sets $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ *may be identical. Given any set* $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ of M subsets, $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_p$ will denote, in the following, the p different subsets appearing among the M sets X_i, with multiplicity $M_1, \ldots, M_p$ $\left(\sum_{i=1}^{p} M_i = M\right)$. If we consider $M+1$ sets $\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$, then $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_p$ will refer to the previous sets Y_i defined for $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ and $\underset{\sim}{Y}_0$ will be identified to $\underset{\sim}{X}_0$.

C) Derivation Relations

The explicit definition of the correlations in classical statistical mechanics yield moreover the following derivation relation (see details of the proof, for instance, in [4]):

$$\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M; \Phi) = \frac{\partial^{M+1}}{\partial\lambda_0 \partial\lambda_1 \ldots \partial\lambda_M} \log Z_\Lambda\left(\Phi - \sum_{i=0}^{M} \lambda_i \delta_{X_i}\right)\bigg|_{\substack{\lambda_0=\lambda_1=\ldots\\=\lambda_M=0}}$$

$$= \frac{\partial^{M+1}}{\partial\lambda_0 \partial\lambda_1^{M_1} \ldots \partial\lambda_p^{M_p}} \log Z_\Lambda\left(\Phi - \sum_{i=0}^{p} \lambda_i \delta_{Y_i}\right)\bigg|_{\substack{\lambda_0=\lambda_1=\ldots\\=\lambda_p=0}}, \tag{3}$$

$$\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M; \Phi) = \frac{\partial^M}{\partial\lambda_1 \ldots \partial\lambda_M} \varrho_\Lambda\left(X_0; \Phi - \sum_{i=1}^{M} \lambda_i \delta_{X_i}\right)\bigg|_{\substack{\lambda_1=\ldots=\lambda_M\\=0}}$$

$$= \frac{\partial^M}{\partial\lambda_1^{M_1} \ldots \partial\lambda_p^{M_p}} \varrho_\Lambda\left(X_0; \Phi - \sum_{i=1}^{p} \lambda_i \delta_{Y_i}\right)\bigg|_{\substack{\lambda_1=\ldots=\lambda_p\\=0}}, \tag{3'}$$

where Z_Λ is the partition function and Φ denotes a potential $B \to \Phi(B)$ acting on the subsets B of points of the lattice. By convention it contains the one-body part, i.e., the chemical potential, which is obtained when B is composed of only one point b and is allowed in general to depend on b, and it includes the reciprocal temperature β. I.e., Z_Λ and the correlations ϱ_Λ are defined (for free boundary *conditions) by the relations:*

$$Z_\Lambda(\Phi) = \sum_{Y \subset \Lambda} \exp\left[-\sum_{B \subset Y} \Phi(B)\right], \tag{4}$$

$$\varrho_\Lambda(X; \Phi) = [Z_\Lambda(\Phi)]^{-1} \sum_{Y \subset \Lambda \backslash X} \exp\left[-\sum_{B \subset X \cup Y} \Phi(B)\right], \tag{4'}$$

where Λ/X *is the set of points of* Λ *minus the points of* X.

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[1] The totally or partially connected functions $\varrho^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ will not be used in this paper. They are defined from the functions ϱ by formulae (1) and (2) together with the convention $\varrho(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M) \equiv 0$ if some points in a subset X_i coincide with points of a subset X_j, $j \neq i$

Finally, by definition, $\delta_{X_i}(B)=1$ if $X_i \equiv B$, $\delta_{X_i}(B)=0$, otherwise.

An infinite system will be said to satisfy the *derivation relations* (A) for a given value Φ_0 of the potential, and with respect to the perturbations Ψ of the potential that belong to a given Banach space E, if

i) the function $\hat{\varrho}(X_0; \Phi_0+\Psi)$, $\Psi \in E$, are infinitely differentiable with respect to Ψ at $\Psi=0$; i.e. the derivatives $D^n\hat{\varrho}/\Phi_0$ exist, as multilinear continuous functionals, for all positive integers n.

ii) For all $\Psi_1, \ldots, \Psi_n \in E$

$$D^n\hat{\varrho}/\Phi_0(X_0; \Psi_1, \ldots, \Psi_n) = \sum_{B_1,\ldots,B_n} \left[\prod_{i=1}^{n} \Psi_i(B_i)\right] \cdot \hat{\varrho}^T(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n; \Phi_0)$$

whenever the sum in the right-hand side is convergent.

Such derivation relations are equivalent for finite Λ to the previous relation (3′). For infinite systems they are an extension of the latter and as a matter of fact still imply relations similar to (3′), as can be seen by choosing $\Psi_i=\delta_{X_i}$, $i=1, \ldots, M$. If we consider an infinite system obtained as the limit of a sequence of finite systems when $\Lambda\to\infty$, they will be a consequence of the relation (3′) if for instance the strong cluster properties hold.

3. Strong Cluster Properties (SCP)

Being given M subsets of points $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ and a distance function δ in $\mathbb{Z}^\nu$, $L_\delta(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ will denote the minimal length, with respect to the distance δ, of all graphs that can be constructed on the points of the sets $X_1, \ldots, X_M$ and on possibly arbitrary other points $y_1, \ldots, y_q$, $q \geqq 0$, and are connected with respect to the sets $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ and the points $y_1, \ldots, y_q$: i.e., if each set $\underset{\sim}{X}_i$ or point y_j is represented in an auxiliary topological space by a point, then the graph obtained in this space is connected. The length of a graph is the sum of the lengths of its lines. By definition $L_\delta(X) = L_\delta(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_M)$ if $X = x_1, \ldots, x_M$.

A tree $\mathcal{T}$ on $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ is a graph of $M-1$ lines, connected with respect to the sets $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$. The length $L_\delta(\mathcal{T})$ is the sum of the lengths of its lines.

The distances δ considered will be of the form:

$$\delta(x-x') = \chi|x-x'|, \qquad \chi>0 \tag{5}$$

or

$$\delta(x-x') = s \times \log(1+\alpha|x-x'|), \qquad \alpha>0,\ s>\nu, \tag{6}$$

where $|x-x'|$ is the usual Euclidean distance, and ν is the space dimension. Hence the function $e^{-\delta(x-x')}$ will be either equal to $e^{-\chi|x-x'|}$ (case of exponentially decaying or finite-range interactions), or equal to $1/(1+\alpha|x-x'|)^s$ (case of potentials decaying with a power law).

Let $\mathcal{B}_i$ be a class of subsets of $\mathbb{Z}^\nu$, $i=1, 2, \ldots$. We shall say that *the functions* f^T *satisfy a strong cluster property, or* SCP, of Type 1, resp. of Type 2, with respect to the classes $\mathcal{B}_i$, if there exists for each $i=1, 2, \ldots$ a function g_i defined on the sets of the class $\mathscr{B}_i$ and a distance δ satisfying

$$\sum_{x\in\mathbb{Z}^\nu} e^{-\delta(0,x)} < \infty$$

resp. a positive function u satisfying

$$\sum_{x\in\mathbb{Z}^\nu} u(|x|) < \infty$$

such that the following bounds (7) or (8) respectively, are satisfied:

$$|f^T(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)| < \left[\mathcal{N}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)\prod_{i=0}^{M} g_i(X_i)\right] \times e^{-L\delta(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)}, \tag{7}$$

$$|f^T(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)| < \prod_{i=0}^{M} g_i(X_i) \times \sum_{\mathscr{T}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)} \prod_{\ell\in\mathscr{T}} u(\ell). \tag{8}$$

The function u considered in Equation (8) will most often be of the form $u(\ell)=e^{-\delta(\ell)}$, in which case the term $\prod_{\ell\in\mathscr{T}} u(\ell)$ is equal to $e^{-L\delta(\mathscr{T})}$. The sum $\sum$ in Equation (8) runs over all trees that can be constructed on $\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M$. The factor $\mathcal{N}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)$ in Equation (7) is equal to one if all sets $\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M$ are disjoint from each other (no common point in X_i and X_j if $i\neq j$). It is equal to $M_1!\ldots M_p!$ if the sets $\underset{\sim}{X}_i$ are p disjoint sets $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_p$ occurring $M_1,\ldots,M_p$ times. It is more generally equal, if the sets $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_p$ are different but not disjoint, to:

$$\mathcal{N}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M) = M_1!\ldots M_p!\ \operatorname*{Inf}_{\substack{\{\lambda_i\}\\ \lambda_i>0\\ i=0,\ldots,p}} \frac{1}{\prod_{i=0}^{p}\lambda_i^{M_i}} \operatorname*{Sup}_{x\in\mathbb{Z}^\nu}\left(\sum_{i\in I_x}\lambda_i\right)^M \tag{9}$$

where I_x is the family of sets $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_p$ that contain x.

The function f_Λ^T will be said to satisfy a SCP *of Type 1 or 2* if they satisfy bounds of the Type (7) or (8) with functions g_i and δ or u that are independent of Λ.

As will appear later in Sections 4, 5, SCP of Type 1 or 2 will appear naturally in various situations. We show below that the Forms 1 and 2 are essentially equivalent. This fact will be needed in order to get equivalence theorems between analyticity properties and SCP.

The SCP express both a decrease taking into account the separation of all clusters with respect to each other and precise bounds, with some uniformity with respect to M and to the configurations of the sets in the classes $\mathscr{B}_i$. A discussion of the nature of these bounds in simpler situations has already been presented in [1, 2].

Before outlining the links between the Forms 1 and 2 of the SCP, [Eqs. (7) and (8)], we first make some remarks.

1. The strong cluster properties on the *totally* connected functions imply, as easily checked, bounds on the partially connected functions that do contain the strong decrease factor

$$e^{-L\delta(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)} \quad\text{or}\quad \sum_{\mathscr{T}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)} \prod_{\ell\in\mathscr{T}} u(\ell).$$

Moreover, it can be proved (see Section 7C) that they imply the bounds (7) or (8) when the sets $\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M$ are disjoint [in which case $\mathcal{N}(\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M) = 1$]. However, it is not known so far to us if they also imply precise bounds of the form (7) or (8), when the sets $(\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M)$ are not disjoint.

2. An alternative useful form of the bound (7) is:

$$|f^T(\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M)| < \mathcal{N}(\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_M) \times \prod_{i=0}^{M} g_i(X_i) \times e^{-L_\delta(X_0, \dots, X_M)}. \tag{10}$$

In view of the definition (9), one has:

$$2^{-(M+1)} \mathcal{N}(\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M) \leqq \mathcal{N}(\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_M) \leqq \mathcal{N}(\underset{\sim}{X}_0, \dots, \underset{\sim}{X}_M). \tag{11}$$

Hence (10) implies (7). Conversely (7) implies (10) with each g_i being replaced by $2g_i$.

3. There is no restriction in principle on the classes $\mathcal{B}_i$ that can be considered. However, they will usually be chosen translation invariant and the functions g_i will then also be chosen, or assumed, translation invariant.

The classes $\mathcal{B}_i$ of interest are, for instance, the class of all sets X in $\mathbb{Z}^\nu$, the class of all sets with a given number n of points ($|X| = n$), certain subclasses of the latter (for instance, when $n = 2$, the class of two-point nearest neighbour sets), and finally the class of all sets X such that $|X| \leqq n$.

We now indicate the links between the two forms of the SCP, i.e., the bounds (7) or (10) and the bound (8). For simplicity, we restrict our attention to the case when all classes $\mathcal{B}_i$ and all functions g_i, $i = 1, 2, \dots$ are identical. The results described extend the analogous results on the totally connected correlations given in [2, 3]. In the following, being given a function Ψ defined on the configurations X in $\mathbb{Z}^\nu$, the norm of Ψ is defined by

$$\|\Psi\| = \sup_{x \in \mathbb{Z}^\nu} \sum_{\substack{X \subset \mathbb{Z}^\nu \\ X \ni x}} |\Psi(X)|$$

and $\|\Psi\|_{\mathcal{B}}$ will denote the norm restricted to the sets X in the class $\mathcal{B}$. The subscript $\mathcal{B}$ will be left implicit if there is no possible confusion.

Lemma 1. *The bound* (7) *implies a bound of the form* (8) *with* $u = e^{-\delta'}$, *where*

$$\delta'(x - x') = \frac{\chi}{2}|x - x'| \quad \text{if} \quad \delta(x - x') = \chi|x - x'|$$

$$\delta'(x - x') = s \log\left(1 + \frac{\alpha}{2}|x - x'|\right) \quad \text{if} \quad \delta(x - x') = s \log(1 + \alpha|x - x'|)$$

and with g replaced by $g' = 2gh$, *where h is any (arbitrary) function on* $\mathcal{B}$ *such that* $\|h^{-1}\|_{\mathcal{B}} \leqq 1$.

Remark. The function u obtained is always integrable when δ is of the form (5) or (6) (with $s > \nu$ in this latter case).

Proof. Lemma 1 is a direct consequence of the following inequalities:

$$\underset{\mathscr{T}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)}{\mathrm{Min}} L_{\delta'}(\mathscr{T}) \leqq L_\delta(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M), \tag{12}$$

$$\mathscr{N}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M) \leqq M_1!\ldots M_p! \prod_{i=1}^{M} 2h(X_i), \tag{13}$$

and of the fact that the number of trees on M_i points is always larger than $\frac{1}{2}M_i!$.

The inequality (12) is proved by methods analogous to those of [1], resp. [3], when δ is of the form (5), resp. of the form (6).

The inequality (13) is obtained by replacing in the right-hand side of (9) each λ_i by $1/h(X_i)$.

Remark. If the class $\mathscr{B}$ contains sets X such that the distance between the points of X can be arbitrarily large, the function h $(X \to h(X))$ necessarily increases when the points of X are separated, since

$$\sum_{X \in \mathscr{B}} h^{-1}(X) < \infty .$$

The function h may, however, be chosen such that this increase is slow.

This factor arises from the form (13) of the bound obtained above on $\mathscr{N}$. It is conjectured that $\mathscr{N}$ satisfies, when $L(X_1, \ldots, X_M) = 0$, the more refined bound:

$$\mathscr{N}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M) \leqq M_1!\ldots M_p!\, C^{\sum_{i=1}^{p}|X_i|} N(\mathscr{T}; L(\mathscr{T})=0), \tag{14}$$

where C is independent of $M, X_1, \ldots, X_M$ and $N(\mathscr{T}; L(\mathscr{T})=0)$ is the number of trees on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M$ of zero length, i.e., such that all lines of $\mathscr{T}$ join two identical points.

This result has been checked in a number of cases, but it is not proved in general so far. If proved, it would allow one to obtain a stronger version of Lemma 1 in which $g' = Cg$, where C is a constant, instead of $g' = hg$. This would allow one to work with strong cluster properties of the form (7), (8), (10) in which the functions $X \to g(X)$ would only depend on the number $|X|$ of points in X, and would yield in turn simplifications and improvements of some of the results described in Sections 4–6.

Lemma 2. *Let* $N(\mathscr{B}, \mathscr{B}_0) = \underset{B \in \mathscr{B}, \mathscr{B}_0}{\mathrm{Max}} |B|$ *be finite, and let* δ *be a given distance function. Then: the bounds* (8) *with* $u(\ell) = e^{-\delta(\ell)}$ *imply bounds of the form* (7) *or* (10) *with* δ *replaced by any distance function* δ'' *such that* $e^{-(\delta-\delta'')}$ *is integrable*

$$\sum_{x \in \mathbb{Z}^\nu} e^{-(\delta(x,0) - \delta''(x,0))} < \infty$$

and with g, g_0 *replaced by* Cg, $C_0 g_0$ *where* C *and* C_0 *are constants independent of* $M, X_0, \ldots, X_M$.

Proof. Let: $d(x) = e^{-(\delta(x) - \delta''(x))}$.

For any tree $\mathscr{T}$ on $\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M$, one has clearly:

$$e^{-L_\delta(\mathscr{T})} \leqq e^{-L_{\delta''}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)} \prod_{\ell \in \mathscr{T}} d(\ell). \tag{15}$$

Lemma 2 is then a direct consequence of the inequality:

$$\sum_{\mathcal{T}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)} \prod_{\ell\in\mathcal{T}} d(\ell) \leqq \mathcal{N}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)D^M, \tag{16}$$

where D is a constant independent of $M, X_0, \ldots, X_M$. The inequality (16) is itself the particular case $n=0$ of a more general inequality that will be needed later and that we now state.

Lemma 3. *Let $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$ be classes of subsets of $\mathbb{Z}^\nu$ and let*

$$N(\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2) = \operatorname*{Max}_{B\in\mathcal{B}_0,\mathcal{B}_1,\mathcal{B}_2} |B| = N < \infty .$$

Let Ψ be any function on $\mathcal{B}_2$ with $\|\Psi\|_{\mathcal{B}_2} < \infty$, and d be any positive integrable function. Then for any $X_0 \in \mathcal{B}_0$, $X_1, \ldots, X_M \in \mathcal{B}_1$:

$$\sum_{B_1,\ldots,B_n\in\mathcal{B}_2} \sum_{\mathcal{T}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M,\underset{\sim}{B}_1,\ldots,\underset{\sim}{B}_n)} \prod_{i=1}^{n} |\Psi(B_i)| \prod_{\ell\in\mathcal{T}} d(\ell)$$
$$\leqq C^M \mathcal{N}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)(C\|\Psi\|)^n n!, \tag{17}$$

where $C \leqq 2e\left(\sum_{x\in\mathbb{Z}^\nu} d(x)\right)N$.

Proof. We first show that given any classes $\mathcal{B}_0$, $\mathcal{B}_1, \ldots, \mathcal{B}_m$ such that $N(\mathcal{B}_k) \leqq N$, $k=0,\ldots,m$ and given functions Ψ_k on $\mathcal{B}_k$ with $\|\Psi_k\|_{\mathcal{B}_k} < \infty$, one has:

$$\sum_{\substack{B'_1,\ldots,B'_m \\ B'_k\in\mathcal{B}_k}} \sum_{\mathcal{T}(\underset{\sim}{X}_0,\underset{\sim}{B}'_1,\ldots,\underset{\sim}{B}'_m)} \prod_{k=1}^{m} |\Psi_k(B'_k)| \prod_{\ell\in\mathcal{T}} d(\ell)$$
$$< (m+1)^{m-1}\left[N\left(\sum_{x\in\mathbb{Z}^\nu} d(x)\right)\right]^m \prod_{i=1}^{m} \|\Psi_k\| . \tag{18}$$

In fact, the left-hand side of (18) can be written as the sum over all trees $\mathcal{T}(0,\ldots,m)$ constructed on the *points* $(0,\ldots,m)$ of

$$\sum_{B'_1,\ldots,B'_m} \prod_{k=1}^{m} |\Psi_k(B'_k)| \sum_{\mathcal{T}'(\underset{\sim}{X}_0,\underset{\sim}{B}'_1,\ldots,\underset{\sim}{B}'_m)} \prod_{\ell\in\mathcal{T}'} d(\ell),$$

where the last sum runs over all trees $\mathcal{T}'(\underset{\sim}{X}_0,\ldots,\underset{\sim}{B}'_m)$ that reduce to the given tree $\mathcal{T}(0,\ldots,m)$ in the auxiliary topological space where the sets $\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m$ are replaced by the respective points $0,\ldots,m$. For any given tree $\mathcal{T}(0,\ldots,m)$ this latter expression is bounded by

$$\left[N\left(\sum_{x\in\mathbb{Z}^\nu} d(x)\right)\right]^m \prod_{k=1}^{m} \|\Psi_k\|$$

as can be seen by induction on m, integrating at each step over one of the points of the tree which is an end point of only one line of $\mathcal{T}$.

Equation (18) is then a consequence of the fact that the number of trees $\mathcal{T}(0,\ldots,m)$ is $(m+1)^{m-1}$.

The bound (17) follows from the bound (18) if one chooses $m=M+n$, $B'_k \equiv B_{k-M}$, $k=M+1,\ldots,M+n$, $\Psi_k \equiv \Psi$, $k=M+1,\ldots,M+n$ and $\Psi_k=\sum_{i=1}^{M}\lambda_i\delta_{X_i}$, $k=1,\ldots,M$, together with the remark that the left-hand side of Equation (18) is then larger than:

$$\frac{M!}{\prod_{j=1}^{p} M_j!}\prod_{j=1}^{p}\lambda_j^{M_j}\sum_{B_1,\ldots,B_n}\prod_{i=1}^{n}|\Psi(B_i)|\sum_{\mathcal{T}(\underset{\sim}{X}_0,\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M,\underset{\sim}{B}_1,\ldots,\underset{\sim}{B}_n)}\prod_{\ell\in\mathcal{T}} d(\ell)$$

and that

$$(M+n+1)^{M+n-1}<(2e)^{m+n}M!n!\,.$$

Remark. The above results can be slightly improved by making use of the following formula which gives the exact number $N(p_1,\ldots,p_M)$ of trees on M sets of respectively $p_1,\ldots,p_m$ points (see Appendix):

$$N(p_1,\ldots,p_M)=\prod_{i=1}^{M}p_i\left(\sum_{i=1}^{M}p_i\right)^{M-2}. \tag{19}$$

4. Low Activity Results

In this section we prove the strong cluster properties of partially connected correlations at low activity.

We recall that the potential Φ includes the one-body part, the chemical potential; $\Phi^{(1)}$ will denote the potential without this one-body part and z will be the activity function: $x\in\mathbb{Z}^\nu\to z_x$, with $\|z\|=\sup_{x\in\mathbb{Z}^\nu}|z_x|$.

Theorem 1. *Given a lattice gas with potential* $\Phi^{(1)}$ *and a distance* δ *of the form* (5) *or* (6) *such that:*

$$\|\Phi^{(1)}e^{L\delta}\|<\infty \tag{20}$$

there exists $z_0(\delta)>0$ *such that the correlation functions* $\hat{\varrho}_\Lambda^T$ *satisfy for* $\|z\|<z_0(\delta)$ *and all* Λ *finite or infinite the strong cluster property:*

$$|\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M;z,\Phi^{(1)})|<C^{|X_0|+M}\mathcal{N}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)\cdot e^{-L\delta(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)}, \tag{21}$$

where C *is a constant independent of* Λ, M, $X_0,\ldots,X_M$.

Remark. In the case of a two-body potential $\Phi^{(1)}$, the condition (20) is always satisfied with distance functions $\delta(x)=\chi'|x|$, $\chi'<\chi$ or $\delta(x)=(s-\nu-\varepsilon)\log(1+\alpha|x|)$, $\varepsilon>0$ if $\Phi^{(1)}$ decreases at infinity at least like $e^{-\chi r}$ or $1/r^s$, $s>\nu$.

Proof. A complete proof of Theorem 1, which involves long technical parts, has been obtained in [6] through the study of non-integrated Kirkwood-Salzburg equations for the functions $\hat{\varrho}_\Lambda^T$ and of their kernel.

We present here a different, shorter proof, which provides, however, so far only a slightly weaker form of Theorem 1. In the case of classes of subsets and of potentials whose maximal number of points is finite, it would provide Theorem 1 itself if the conjecture (14) on the factor $\mathcal{N}$ was proved. In contrast to the method of [6], this method does not require a detailed knowledge of series expansions of the functions and it directly makes use of analyticity properties that are known independently [whereas these properties are rederived in the same time as the cluster properties in [6]). It can therefore be adapted to a larger class of problems.

The slightly weaker form proved here is the following. First, instead of (20), we shall assume that $\Phi^{(1)}$ and δ satisfy

$$\|\Phi^{(1)}e^{L_\delta}g\| < \infty, \tag{22}$$

where g is some positive function on the configurations of $\mathbb{Z}^\nu$ such that $\|g^{-1}\| < \infty$, and it is also assumed that

$$N(\mathscr{B}, \mathscr{B}_0) = \underset{B \in \mathscr{B}, \mathscr{B}_0}{\operatorname{Max}} |B|$$

is finite and that $\Phi^{(1)} \equiv 0$ if $|B| > N(\Phi)$, $N(\Phi) < \infty$. Second, the result obtained will be, instead of (21), the following SCP:

$$|\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M; z, \Phi^{(1)}| < C^{|X_0| + M} \mathcal{N}(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M) \prod_{i=1}^{M} g(X_i)$$
$$\cdot e^{-L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)}, \tag{23}$$

where C is, as above, independent of Λ, M, $X_0, \ldots, X_M$.

The proof has two steps. First, it is proved that for z given and $\Phi^{(1)} \equiv 0$ the functions $\hat{\varrho}_\Lambda^T$ satisfy a SCP. Second, it is shown that a SCP at $\Phi^{(1)} \equiv 0$ with a nice dependence of the bounds with respect to the parameters implies a SCP of the $\hat{\varrho}_\Lambda^T$ for any $\Phi^{(1)}$ satisfying (22) and for z small enough. This second step can be considered in some sense as a particular case of Corollary 2 of Section 6.

Let us consider for any finite Λ, the following series expansions, derived from (3), of $\hat{\varrho}_\Lambda^T$ with respect to the potential:

$$\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M; z, \Phi^{(1)}) = \sum_{n \geqq 0} \frac{1}{n!} \sum_{\substack{B_1, \ldots, B_n \\ |B_i| \geqq 2}} \prod_{i=1}^{n} \Phi^{(1)}(B_i)$$
$$\cdot \hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n; z, \Phi^{(1)} \equiv 0). \tag{24}$$

Let now $\underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m$ stand for $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$, $\underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n$. First it can be seen that at $\Phi^{(1)} \equiv 0$ one has:

$$\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m; z, \Phi^{(1)} \equiv 0) = 0 \quad \text{if} \quad L(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m) \neq 0. \tag{25}$$

This result follows from the relations between the function $\varrho^T(x_1, \ldots, x_N)$ $\varrho^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ and $\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)$ which arise from the definitions, and from the fact that $\varrho^T(x_1, \ldots, x_N; z; \Phi^{(1)} \equiv 0) = 0$ if the points $x_1, \ldots, x_N$ do not all coincide with a common point x. This latter fact can be checked through the usual series expansions of ϱ^T with respect to z at $\Phi^{(1)} \equiv 0$. Alternatively one can use simple factorization properties of correlations that occur when $\Phi^{(1)} \equiv 0$.

When $L(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m) = 0$, the following bounds moreover hold:

$$|\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m; z, \Phi^{(1)} \equiv 0) < \mathcal{N}(\underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m) C'(z)^m C''(z)^{|X_0|}, \tag{26}$$

where $C'(z)$, $C''(z)$ are independent of $\Lambda, \ldots, X_0$, $B'_1, \ldots, B'_m$, and where $C'(z)$ can be chosen arbitrarily small if $\|z\|$ is chosen sufficiently small.

The bound (26) is obtained by observing first that, in view of the derivation relations (3′) one has:

$$\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m; z, \Phi^{(1)} \equiv 0)$$
$$= \frac{\partial^m}{\partial\mu_1 \ldots \partial\mu_m} \varrho_\Lambda(X_0; \{ze^{-\Sigma'\mu_i\delta_{B'_i}}\}, \Phi^{(1)} \equiv -\Sigma''\mu_j\delta_{B'_j})_{|\{\mu_i,\mu_j\}=0}, \tag{27}$$

where the sums Σ' and Σ'' run over the sets $\underset{\sim}{B}'_i$ of $\underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m$ with respectively only one point and more than one point.

Well-known low activity results [7–9] and straightforward adaptations of these results for complex potentials ensure that for any z there exists $R(z) > 0$ and $C''(z)$ independent of $\Lambda, m, X_0, B'_1, \ldots, B'_m$ such that the function $\varrho_\Lambda(X_0)$ in the right-hand side of (27) is analytic with respect to the variables μ_k, $k = 1, \ldots, m$, in the region $|\mu_k| < R(z)$ if k is associated with a set of only one point and $\|\Sigma''\mu_k\delta_{B'_k}\| < R(z)$ otherwise, and satisfy in that region the bound

$$|\varrho_\Lambda(X_0)| < C''(z)^{|X_0|}. \tag{28}$$

Moreover, $R(z)$ and $C''(z)$ can be chosen uniform in any region of the form $\|z\| < z_0$, and $\underset{\|z\|<z_0}{\mathrm{Max}}\ R(z)$ is arbitrarily large if z_0 is chosen sufficiently small.

The bound (26) then follows from a Cauchy formula expressing $(\partial^m/(\partial\mu_1 \ldots \partial\mu_m))\varrho_\Lambda$ as an integral of $\varrho_\Lambda(X_0)$ over contours $|\mu_k| = \lambda_k$ chosen in the region of analyticity described above, and from the definition of $\mathcal{N}$.

Now Equations (25) and (26) ensure that the functions $\hat{\varrho}^T_\Lambda$ satisfy, when $\Phi^{(1)} \equiv 0$, the following SCP:

$$|\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m; z, \Phi^{(1)} \equiv 0)| < \mathcal{N}(B'_1, \ldots, B'_m) C''(z)^{|X_0|} C'(z)^m$$
$$\cdot e^{-L_{\delta'}(\underset{\sim}{X}_0, \underset{\sim}{B}'_1, \ldots, \underset{\sim}{B}'_m)}, \tag{29}$$

where δ' is an arbitrary distance function.

By using Lemma 1 of Section 3 and noting that

$$L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M) \leqq L_\delta(\mathcal{T}) + \sum_{i=1}^{n} L_\delta(B_i)$$

for any tree $\mathcal{T}$ constructed on $\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M$, $\underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n$, one obtains from the Equations (24) and (29):

$$|\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M; z, \Phi^{(1)})| \leqq e^{-L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)}$$
$$\cdot \prod_{i=1}^{M} g(X_i) C'(z)^M C''(z)^{|X_0|}$$
$$\cdot \sum_{n \geqq 0} \frac{C'(z)^n}{n!} \sum_{\substack{B_1, \ldots, B_n \\ |B_i| \geqq 2}} \prod_{i=1}^{n} |\Phi^{(1)}(B_i) e^{L_\delta(B_i)} g(B_i)|$$
$$\cdot \sum_{\mathcal{T}(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n)} \prod_{\ell \in \mathcal{T}} u(\ell), \tag{30}$$

where $u(\ell)=e^{-(\delta'-\delta)(\ell)}$ and where the distance function δ' can be chosen arbitrarily large.

Finally Lemma 3 of Section 3 and the fact that $C'(z)$ is arbitrarily small if $\|z\|$ is sufficiently small provide the announced result. Q.E.D.

5. Extension Theorems

The following theorem allows one to extend strong cluster properties from some region where they are known to the whole domain of analyticity. It is a general form of previous particular results given in [10, 11] (in the case of "weak" decay properties and of finite range interactions) and in [2]. Examples of applications will be given later in this section. We note that the theorem follows from the study of interpolating bounds for families of analytic functions and that the method can be applied to more general problems.

Let ξ be a one-dimensional complex variable, which will be identified later for applications with a complexified variable of a path variable in the space of real potentials. Let $\mathscr{U}$ be a complex neighbourhood of a point ξ_0 and let $\mathscr{D}$ be a simply connected domain in $\mathbb{C}$ containing $\mathscr{U}$. Let $\xi \to t(\xi)$ be a conformal mapping from $\mathscr{D}$ onto the unit circle $|t|<1$ chosen such that $t(\xi_0)=0$. Finally let $d>0$ be such that the domain $|t|<d$ belongs to $t(\mathscr{U})$. Then one has, either for the infinite system, or alternatively uniformly in Λ if the assumptions are made uniformly in Λ:

Theorem 2. *Let $f^T(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M; \xi)$, $X_i \in \mathscr{B}_i$, $i=0, \ldots, M$, be a family of functions f^T depending on a complex parameter ξ such that:*

i) *the functions f^T satisfy the* SCP (7) *for all ξ in $\mathscr{U}$ (uniformly with respect to ξ);*

ii) *they are analytic with respect to ξ in $\mathscr{D}$ and satisfy there bounds of the form:*

$$|f^T(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M; \xi)| < \prod_{i=0}^{M} g_i'(X_i) \mathscr{N}(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M), \tag{31}$$

where the functions g_i' are independent of $M, X_0, \ldots, X_M$.

Then the following bounds are satisfied for all ξ in $\mathscr{D}$ such that $|t(\xi)|>d$:

$$|f^T(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M; \xi)| < \mathscr{N}(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M) \prod_{i=1}^{M} g_i''(X_i) C_0(\xi)$$
$$\cdot \exp\left[-\frac{\log|t(\xi)|}{\log d} L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)\right], \tag{32}$$

where $C_0(\xi)$ is independent of $M, X_0, \ldots, X_M$ and $g_i''(X_i)=\mathrm{Max}(g_i(X_i), g_i'(X_i))$.

The proof is similar to that of the second work of [2]: one considers the series expansion $f^T = \sum_{n \geqq 0} t^n \gamma_n$ of f^T with respect to t, whose convergence follows from the analyticity assumption for $|t|<1$. Let $n_0(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)$ be the first integer larger than $(1/|\log d|) L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)$. The coefficients γ_n are then directly bounded for $n<n_0$ or $n \geqq n_0$, respectively, via a Cauchy formula, by using the bounds (7) of the hypothesis i) and the analyticity for $|t|<d$ in the first case, the bounds (31) and the analyticity for

$|t|<1$ in the second case. The resummation yields the result (32) with

$$C_0(\xi) \leqq \frac{|t(\xi)|}{|t(\xi)|-d} + \frac{1}{1-|t(\xi)|}.$$

Remarks. i) The bounds (32) are SCP at any point ξ in $\mathscr{D}$, with $\delta'=((\log|t(\xi)|)/\log d))\delta$, if δ is of the form (5). If δ is of the form (6), they are SCP when $e^{-\delta'}$ is integrable, i.e., when $(\log|t(\xi)|/\log d)s>v$.

ii) In the case of potentials that decrease like r^{-s_0}, the use of a domain $\mathscr{D}$ of analyticity in the activity z alone cannot provide strong cluster properties, and hence satisfactory integrability properties of the functions f^T, at all points z of $\mathscr{D}$. In fact, $(\log|t(z)|)/\log d)s_0$ always tends to zero when z comes close to the boundary of $\mathscr{D}$. However, the use of an analytic continuation in the space of potentials allows one, in some situations, to link a given system that corresponds to a given z (and to a potential decreasing like r^{-s_0}), to a system whose interaction potential at small values of $|t(\xi)|$ decreases much quicker than r^{-s_0}. This in turn allows one in some cases to reobtain SCP for the given system even at activities that were excluded above: see example in [12], where corresponding results on the decay of correlations for ferromagnets are given.

iii) The use of a series expansion in the proof of Theorem 2 in order to get interpolating bounds can be alternatively replaced by the use of results on subharmonic functions [13], or on holomorphy envelopes [14]. These methods would provide analogous results. The method based on subharmonic functions allows one to restrict the bounds (7) of the hypothesis i) to the points ξ of some arbitrarily small arc in $\mathscr{C}$, not necessarily closed. This extension is, however, not required in applications in the present paper. The method of [14] yields on the other hand an improvement of the constant $C_0(\xi)$, but does not improve the rate of decay.

iv) As a matter of fact, the rate of decay obtained in Equation (32) is, at least in some situations, the best possible one, according to the assumptions. This can easily be checked directly [14], when $(1/|\log d|)L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)$ is an integer, by considering the analytic function

$$\left[\mathscr{N}(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M) \prod_{i=0}^{M} g_i''(X_i)\right] \times t^{L_\delta(\underset{\sim}{X}_0, \ldots, \underset{\sim}{X}_M)/|\log d|}.$$

Example of Applications. In physical situations, it is sometimes possible, being given $\{z, \Phi^{(1)}\}$, to consider a family of potentials and activities $\xi \to \{z(\xi), \Phi^{(1)}(\xi)\}$ depending on a complex parameter $\xi, \xi \in \mathscr{D}$ such that $\{z(\xi_0), \Phi^{(1)}(\xi_0)\} \equiv \{z, \Phi^{(1)}\}$ for a given ξ_0 and such that:

i) for small values of ξ, the system belongs to its low activity region, and the results of Section 4 then ensure condition i) of Theorem 2;

ii) for $\xi \in \mathscr{D}$, certain properties of localization of the zeros of the partition function under appropriate perturbations of the potential, or alternatively analyticity properties and bounds on the correlations, are satisfied.

Various particular examples of such situations will be found in [10, 15, 16, 2, 17, 12]. In connection with ii) and hypothesis ii) of Theorem 2, we mention here some general results. It will appear in particular that bounds of the form (31) are linked with the analyticity of the system with respect to perturbations of the potential.

Let $E(\mathscr{B})$ denote the space of potentials on the class $\mathscr{B}$ ($\Psi \in E(\mathscr{B}) \rightleftarrows \Psi(X)=0$ if $X \notin \mathscr{B}$), let $\xi \to \Phi'(\xi)$ be an analytic path from $[0,1]$ to $E(\mathscr{B})$ and let $\{\Phi(\xi)\}$ be the set of potentials $\Phi_0 + \Phi'(\xi)$, where Φ_0 is a given, arbitrary potential. Then one has:

Theorem 3. *If, for all ξ real in the closed segment $[0,1]$, all X_0 in a given class $\mathscr{B}_0$ and all Λ (respectively at Λ infinite if the derivation relations (A) hold), the functions $\varrho_\Lambda(X_0, \Phi(\xi)+\Psi)$ are analytic with respect to Ψ in $E(\mathscr{B})$ in the region $\|\Psi\| < \varepsilon(\xi)$, $\varepsilon(\xi)>0$, and are bounded there in modulus by $C(\xi)^{|X_0|}$, then the functions $\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_M; \Phi(\xi))$ are analytic with respect to ξ in some complex neighbourhood of $[0,1]$ and satisfy there a bound of the form*

$$|\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_M; \Phi(\xi))| < \mathcal{N}(\underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_M) \mathrm{C}^{|X_0|} D^M, \tag{33}$$

where C and D are independent of $\Lambda, M, X_0, B_1, \ldots, B_M$ (respectively of M, X_0, $B_1, \ldots, B_M$).

Before giving the proof, we first state:

Corollary 1. *If the partition function $Z_\Lambda(\Phi(\xi)+\Psi-\lambda\delta_{X_0})$ is different from zero for any ξ real, $\xi \in [0,1]$, $\Psi \in E(\mathscr{B})$, $\|\Psi\| \leqq \varepsilon(\xi)$, $X_0 \in \mathscr{B}_0$ and $|\lambda| < \alpha(\xi)$, where α is some strictly positive function, then the functions $\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_M; \Phi(\xi))$ satisfy the same properties as in Theorem 3.*

Remark. If the analyticity properties of $\varrho_\Lambda(X_0)$, or the localization properties of the zeroes of Z_Λ are known only in the region $\|g\Psi\| < \varepsilon(\xi)$, the same conclusing still hold in Theorem 3 and Corollary 1 except that an extra factor $\prod_{i=1}^{M} g(B_i)$ has to be included in the bounds (33). This type of condition is the one that arises as a matter of fact in some situations, for instance from the results of [17].

Proof of Theorem 3 and Corollary 1. We first prove Theorem 3. Let ξ_0 be a real point in $[0,1]$ and let us consider

$$\Psi(\xi, \{\lambda_i\}) = \Phi(\xi) - \Phi(\xi_0) + \sum_{i=1}^{M} \lambda_i \delta_{B_i},$$

where ξ can be complex.

The assumption of the theorem ensures that the functions $\varrho_\Lambda(X_0; \Phi(\xi_0)+\Psi)$ are analytic with respect to the variables λ_i and ξ in the domain:

$$\left\| \sum_{i=1}^{M} |\lambda_i| \delta_{B_i} \right\| < \varepsilon(\xi_0)(1-\eta),$$

$$\|\Phi(\xi) - \Phi(\xi_0)\| < \varepsilon(\xi_0)\eta,$$

where $0<\eta<1$.

From the second domain, it follows that the functions $\hat{\varrho}^T_\Lambda(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_M; \Phi(\xi))$ are analytic with respect to ξ in some neighbourhood of ξ_0. From the first one a bound of the type (33) follows for all ξ in the previous neighbourhood of ξ_0, through the derivation relation (3), or the derivation relation (A) if one works only with the infinite system, and through a Cauchy formula and the use of the definition of $\mathcal{N}$.

The compacity of the segment $[0,1]$ then allows one to obtain Theorem 3.

Concerning Corollary 1, analyticity follows directly from the hypothesis and hence the result follows from Theorem 3 if the needed bound on $\varrho_\Lambda(X_0;\Phi(\xi)+\Psi)$ can be proved.

The partition function $Z_\Lambda(\Phi(\xi_0)+\Psi-\lambda\delta_{X_0})$ can be written [see Eq. (4)] in the form $ae^\lambda+b$, where a and b depend on X_0, $\Phi(\xi_0)$, Ψ and Λ, and Z_Λ is by assumption different from zero when $|\lambda|<\alpha$. Since $\varrho_\Lambda(X_0;\Phi(\xi_0)+\Psi)$ is the derivative of $\log Z_\Lambda(\Phi(\xi_0)+\Psi-\lambda\delta_{X_0})$ with respect to λ at $\lambda=0$, it can be rewritten as:

$$\varrho_\Lambda(X_0;\Phi(\xi_0)+\Psi)=\frac{1}{2i\pi}\oint_{|\lambda|=\varepsilon\alpha}\frac{ae^\lambda}{ae^\lambda+b}\frac{d\lambda}{\lambda},\qquad \varepsilon<1, \tag{34}$$

where we have used:

$$\oint\frac{f(z)}{z^n}dz=\oint\frac{f'(z)}{z^{n-1}}dz.$$

Let λ_0 be chosen such that $e^{-\lambda_0}=-b/a$, $|\mathrm{Im}(\lambda-\lambda_0)|\leqq\pi$. One has $(ae^\lambda/ae^\lambda+b)=(1/1-e^{(\lambda_0-\lambda)})$. On the other hand $Z_\Lambda(\lambda_0)=ae^{\lambda_0}+b=0$ and hence $|\lambda_0|\geqq\alpha$. Equation (34) therefore provides a bound on $\varrho_\Lambda(X_0;\Phi(\xi_0)+\Psi)$ that is clearly independent of X_0, Φ, Ψ, Λ and depends only on the distance of λ in the integration contour to the boundary of the domain $|\lambda|\leqq\alpha$, that is $(1-\varepsilon)\alpha(\xi_0)$.

6. Strong Cluster Properties and Analyticity—General Results

In the previous Sections 4 and 5, SCP were proved in various situations. We intend here to show that conversely strong cluster properties at a *real* point or in the *physical region* imply analyticity of the system with respect to the potential or to classes of perturbation of the potential. They moreover imply SCP in appropriate neighbourhoods of the physical region. These results will then lead to statements of equivalence between analyticity and SCP.

For simplicity, we shall consider in the following the case when $\mathscr{B}_0$ is the class of all subsets in $\mathbb{Z}^\nu$ with N points or less, and $\mathscr{B}$ is the class of all subsets with p points, $p\leqq N$ or is the union of such classes. Otherwise the results would hold for the correlations $\varrho(X_0)$, $X_0\in\mathscr{B}_0$ and for partly connected correlations $\hat{\varrho}^T(\underset{\sim}{X}_0,\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)$, $X_0\in\mathscr{B}_0$, $X_i\in\mathscr{B}$, $i=1,\ldots,M$ but they would not necessarily hold for all correlations.

Let g_0, g be functions defined on $\mathscr{B}_0,\mathscr{B}$ and let $E(\mathscr{B},g)$ be the Banach space of potentials Φ defined on $\mathscr{B}$, including possibly the chemical potential, and satisfying: $\|\Phi\|_{g,\mathscr{B}}\equiv\|\Phi g\|_{\mathscr{B}}<\infty$. Then one has:

Theorem 4. *Let Φ_0 be a given potential.*

1) *If the correlations have a thermodynamic limit at* Φ_0 *and if the partly connected correlations* $\hat{\varrho}_\Lambda^T$ *satisfy at* Φ_0 *for all* Λ *a SCP of the form* (8) *with respect to* $\mathscr{B}_0, \mathscr{B}$, *then the correlations have a thermodynamic limit in some complex neighbourhood of* Φ_0 *in* $\{\Phi_0 + E(\mathscr{B}, g)\}$ *and this limit is analytic with respect to the potential in this neighbourhood.*

2) *If the correlations of the infinite system satisfy the derivation relation* (A) *in a real neighbourhood of* Φ_0 *in* $\{\Phi_0 + E(\mathscr{B}, g)\}$ *and verify there a* SCP *with respect to* $\mathscr{B}_0, \mathscr{B}$, *then the correlations are analytic with respect to the potential in some complex neighbourhood of* Φ_0 *in* $\{\Phi_0 + E(\mathscr{B}, g)\}$.

Proof. We first prove Part 1 of the theorem. We consider the series expansion

$$\varrho_\Lambda(X_0; \Phi_0 + \Psi) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{B_1, \ldots, B_n \\ B_i \in \mathscr{B}, B_i \subset \Lambda}} \prod_{i=1}^{n} \Psi(B_i) \cdot \hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n; \Phi_0) \tag{35}$$

which, as already mentioned, follows from the derivation relations (3). Lemma 3 ensures the absolute convergence of that series and the analyticity with respect to Ψ of $\varrho_\Lambda(X_0; \Phi_0 + \Psi)$ in the region $\|\Psi\|_{\mathscr{B},g} < \varepsilon$ for all finite Λ; ε is strictly positive and independent of Λ.

It follows, on the other hand, from the hypothesis that the functions $\hat{\varrho}_\Lambda^T(\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n; \Phi_0)$ have a thermodynamic limit $\hat{\varrho}^T$ for all $\underset{\sim}{X}_0, \underset{\sim}{B}_1, \ldots, \underset{\sim}{B}_n$, $B_i \in \mathscr{B}$. The SCP at Φ_0 then ensure again the absolute convergence of the series at the limit, and one checks, in view of the definition of analyticity on Banach spaces [5][2] that this series defines also an analytic function in the region $\|\Psi\|_{\mathscr{B},g} < \varepsilon$.

Finally, one shows that $\varrho_\Lambda(X_0; \Phi_0 + \Psi)$ tends indeed in the $\Lambda \to \infty$ limit to the function defined by the sum of this latter series at Λ infinite. In fact, for any given α, the SCP and Lemma 3 allow one to choose $\mathscr{L}$ and M_0 such that, for each one of the series obtained at Λ finite or infinite, the sum of all terms corresponding either to $n > M_0$ or $L(X_0, B_1, \ldots, B_n) > \mathscr{L}$, where $L(X_0, B_1, \ldots, B_0)$ is the distance between all points of the sets $X_0, B_1, \ldots, B_n$, is less than $\alpha/4$. On the other hand, the difference of the respective sums of the remaining terms, whose number is finite, is bounded in modulus by $\alpha/2$ for Λ large enough, by virtue of the convergence of $\hat{\varrho}_\Lambda^T$ to $\hat{\varrho}^T$ at Φ_0.

These results hold so far for the correlations $\varrho(X_0), X_0 \in \mathscr{B}_0$. They can be proved in a similar way for the functions $\hat{\varrho}^T(X_0, B_1, \ldots, B_n)$, $X_0 \in \mathscr{B}_0$, $B_i \in \mathscr{B}$. The relations between the correlations and partly connected correlations then yield the results for all correlations.

The second part of theorem is obtained as follows. Let us consider the expression:

$$\sum_{\substack{n \geq 0 \\ n \leq p}} \frac{1}{n!} \sum_{\substack{B_1, \ldots, B_n \\ B_i \in \mathscr{B}}} \prod_{i=1}^{n} \Psi(B_i) \hat{\varrho}^T(\underset{\sim}{X}_0, \ldots, \underset{\sim}{B}_n; \Phi_0). \tag{36}$$

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[2] Being given a formal series $\sum_{n \geq 0} \frac{1}{n!} A_n(\Psi, \ldots, \Psi)$ where each A_n is a multilinear continuous functional $\Psi_1, \ldots, \Psi_n \to A_n(\Psi_1, \ldots, \Psi_n)$, this series defines an analytic function in a neighborhood of $\Psi = 0$ if $\|A_n\| < C^n n!$, where $\|A_n\| = \sup_{\Psi_1, \ldots, \Psi_n; \|\Psi_i\| < 1} |A_n(\Psi_1, \ldots, \Psi_n)|$

The SCP at Φ_0 and Lemma 3 imply, as previously, its absolute convergence when $\|\Psi\|_{\mathscr{B},g}<\varepsilon$, and the analyticity of the function thus defined. On the other hand the derivation relations and the SCP in a real neighbourhood of Φ_0 ensure, in view of the Lagrange version of Taylor series [18], applied here to $\varrho(X_0,\Phi_0+\lambda\Psi)$, that the expression (36) is equal to $\varrho(X_0;\Phi_0+\Psi)$ on the real.

Finally the result is extended to all correlations as in the previous case. Q.E.D.

We now consider a distance function δ' of the form (5) or (6) and the Banach space $E(\mathscr{B},g,\delta')$ of potentials Ψ on the class $\mathscr{B}$ such that: $\|\Phi\|_{\mathscr{B},g,\delta'}=\|\Phi g e^{L_{\delta'}}\|<\infty$. Then one has:

Corollary 2. *Let Φ_0 be a given potential and let the hypothesis of Part 1 or of Part 2 of Theorem 4 hold. If moreover the function $u(\ell)$ in the* SCP *is of the form $u(\ell)=e^{-\delta(\ell)}$, then for all distances δ' such that*

$$\sum_{\ell\in\mathbb{Z}_\nu} e^{-(\delta(\ell)-\delta'(\ell))}<\infty$$

there exists a neighbourhood $\mathscr{U}$ of Φ_0 in $\{\Phi_0+E(\mathscr{B},g,\delta')\}$ and a constant C such that, for all Λ finite and infinite, resp. for Λ infinite, the functions $\hat\varrho^T_\Lambda$ satisfy for all Φ in $\mathscr{U}$ the SCP:

$$|\hat\varrho^T_\Lambda(\underset{\sim}{X}_0,\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M;\Phi)|<g_0(X_0)\prod_{i=1}^{M} g(X_i)\mathscr{N}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)$$
$$\cdot\; C^{M+|X_0|}e^{-L_{\delta'}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)}\,.$$

Proof. This result is proved by the same methods as Theorem 4. It is sufficient to remark that, in the domain considered, one can first extract a common factor $e^{-L_{\delta'}(\underset{\sim}{X}_0,\ldots,\underset{\sim}{X}_M)}$ of all terms of the series. Q.E.D.

We are now in a position to state the equivalence theorem between SCP and analyticity that follows from the results of Sections 4 and 5 and the previous results of this section.

For simplicity we shall restrict our attention to exponentially decreasing potentials, the corresponding distances δ being of the form (5). As already mentioned, the functions involved below can, on the other hand, be chosen with slow increase [more precisely $|g(B)|<C(|B|)e^{L_{\delta''}(B)}$ where δ'' is a distance function of the form (6)].

Let Φ_0 be a real exponentially decaying potential, i.e., such that $\|\Phi_0 e^{L_\delta}\|<\infty$ for a certain δ of the form (5). Let $\mathscr{B}$ be a class of sets of $\mathbb{Z}^\nu$, $N(\mathscr{B})=N<\infty$. Let $\mathscr{E}$ be a real subset of the space of potentials such that, for all Φ in $\mathscr{E}$, there exists an analytic path $\xi\in[0,1]\to\Phi(\xi)\in\mathscr{E}$ satisfying:

i) $\Phi(0)=\Phi_0$, $\Phi(1)=\Phi$;

ii) $\forall\xi_0\in[0,1]$, there exists a complex neighbourhood $\mathscr{U}(\xi_0)$ of ξ_0 and a distance function δ_{ξ_0} of the form (5) such that, for any ξ in $\mathscr{U}(\xi_0)$, $\|\Phi(\xi)e^{\delta_{\xi_0}}\|<\infty$.

(The most simple example of such a set $\mathscr{E}$ is the set of potentials $\{(1+\lambda)\Phi_0\}$ where λ belongs to a real open segment including $[0,1]$.)

Then the following equivalence theorem holds, either uniformly for all Λ finite or infinite, if the hypothesis hold uniformly for all finite Λ and if there exists a thermodynamic limit of correlations on $\mathscr{E}$, or alternatively only for the infinite system if the hypotheses hold only for the infinite system and if the derivation relations (A) are satisfied in $\mathscr{E}$.

Theorem 5. *The two following properties* i) *and* ii) *are equivalent:*

i) *for all Φ in the real set $\mathscr{E}$, the functions $\hat{\varrho}^T$ satisfy a* SCP *with respect to the classes $\mathscr{B}_0, \mathscr{B}$;*

ii) a) *the functions $\hat{\varrho}^T$ satisfy a* SCP *at the point Φ_0 with respect to the classes $\mathscr{B}_0, \mathscr{B}$;*

b) *there exists g', g'_0 such that, for all Φ in $\mathscr{E}$, the correlations $\varrho(X, \Phi + \Psi)$ are analytic with respect to Ψ in a region $\|\Psi g'\| < \varepsilon(\Phi)$, $\varepsilon(\Phi) > 0$, in $E(\mathscr{B}, g')$, and satisfy there for $X_0 \in \mathscr{B}_0$ the bound $|\varrho(X_0; \Phi + \Psi)| < g'_0(X_0)$.*

Remark. If $\mathscr{E}$ intersects a region where strong cluster properties have been proved (such as the low activity region), then condition a) of property ii) is automatically satisfied and can be removed from the statement of the theorem.

Proof. The proof follows from previous results or from simple adaptations of these results and we therefore only briefly outline it.

Property i) contains condition a) of property ii) and it implies condition b) of property ii) in view of Theorem 4. The bounds on correlations are obtained by a resummation of the series used in the proof of Theorem 4.

Conversely, let Φ be a point in $\mathscr{E}$ and let $\xi \to \Phi(\xi)$ be the path previously introduced in the definition of $\mathscr{E}$. The assumption a) of property ii) implies, in view of Corollary 2 of Section 6, that the functions $\hat{\varrho}^T$ satisfy a SCP at all points $\Phi(\xi)$, $|\xi| < \varepsilon$ for some $\varepsilon > 0$. On the other hand, Theorem 3 ensures, in view of assumption b), the analyticity of the correlations with respect to ξ in a complex neighbourhood of $[0, 1]$ and appropriate bounds on the functions $\hat{\varrho}^T$. Theorem 2 then allows one to obtain property i).

7. General Remarks

A) The strong cluster properties yield also, as a matter of fact, the analyticity of the pressure and not only of correlations.

B) At phase transition points, it can be checked that strong cluster properties cannot hold uniformly in Λ with respect to sets of the class $\mathscr{B}$ corresponding to the perturbations of the potential which are at the origin of the phase transition.

On the other hand, the strong cluster properties with respect to sets of this class might a priori still hold for the infinite system itself in pure phases. If the derivation relations hold for the infinite system at least in a given part of a neighbourhood of a phase transition point (e.g., on a given side of the phase transition region), a slight adaptation of the previous results then would again yield analyticity properties at the phase transition point considered and hence the existence of analytic continuations through the phase transition regions. Whether such analyticity properties hold or not is today an open question.

C) An open question of interest in the framework of the present paper is to know whether the strong cluster properties of the totally connected correlations imply the strong cluster properties of partially connected correlations. If this was true, the strong cluster properties of totally connected correlations, which are linked with analyticity with respect to the one-body potential, or magnetic field (see [2] or Theorem 4 of the present paper), would be sufficient to yield analyticity with respect to the potential. This question is related to a result of [10], according to which analyticity with respect to the activity is sufficient, under certain conditions, to imply also the analyticity with respect to the reciprocal temperature β.

We show in Appendix 2 that the strong cluster properties of totally connected correlations do imply the strong cluster properties of partially connected correlations when all subsets $X_1, \dots, X_M$ are disjoint ($X_i \cap X_j = \emptyset$ if $i \neq j$, $i,j = 1, \dots, M$). When the subsets $X_1, \dots, X_M$ are not disjoint, we do not know, however, so far if the same result holds in general, although it can still be checked in some particular cases. The absence of the precise bounds (7) or (8) in these situations prevents one from deriving corresponding analyticity properties.

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Appendix 1

We present here the proof of the following Lemma, which has been obtained by M. Duneau and one of the present authors (B.S.).

Lemma. *The number $N(p_1, \dots, p_q)$ of trees on q sets of respectively $p_1, \dots, p_q$ points is given by the formula:*

$$N(p_1, \dots ; p_q) = \prod_{i=1}^{q} p_i \left(\sum_{i=1}^{q} p_i \right)^{q-2} . \tag{37}$$

Proof. The proof is obtained by induction on the number q of sets. If $q=1$, $N(p_1)$ is set equal to 1 by convention. This is compatible with Equation (37). If $q=2$, the result is trivially checked.

The following induction relation then holds:

$$N(p_0, p_1, \dots, p_q) = \sum_{\substack{I \subset (1, \dots, q) \\ I \neq \emptyset}} p_0^{|I|} \prod_{i \in I} p_i \cdot N\left(\sum_{i \in I} p_i ; \{p_j\}, j \in \complement I \right) . \tag{38}$$

It is obtained from the definition of trees, taking apart the sets i, $i \in I$ which are connected to the first set of p_0 points directly by a line. Since $I \neq \emptyset$, there are at most q blocs in the factor N in the right-hand side of Equation (38).

Let $M = \sum_{i=0}^{q} p_i$; then:

$$\begin{aligned} N(p_0, p_1, \dots, p_q) &= \sum_{\substack{I \subset (1, \dots, q) \\ I \neq \emptyset}} p_0^{|I|} \prod_{j \in I} p_i \prod_{j \in \complement I} p_j \left(\sum_{i \in I} p_i \right) (M - p_0)^{q - |I| - 1} \\ &= \prod_{i=1}^{q} p_i \sum_{k=1}^{q} p_0^k (M - p_0)^{q-k-1} \sum_{\substack{I \subset (1, \dots, q) \\ |I| = k}} \left(\sum_{i \in I} p_i \right) . \end{aligned}$$

On the other hand:

$$\sum_{\substack{I\subset(1,\ldots,q)\\|I|=k}}\left(\sum_{i\in I}p_i\right)=\left(\sum_{i=1}^{q}p_i\right)\frac{(q-1)!}{(k-1)!\,(q-k)!}$$

Hence:

$$N(p_0,\ldots,p_q)=\prod_{i=0}^{q}p_i\sum_{k=1}^{q}p_0^{k-1}(M-p_0)^{q-k}\frac{(q-1)!}{(k-1)!\,(q-k)!}$$
$$=\left(\prod_{i=0}^{q}p_i\right)(M-p_0+p_0)^{q-1}\,.$$

Appendix 2

The totally or partially connected correlations in the following lemma are considered at a given, common, potential Φ_0 that will be left implicit. For simplicity we consider only the case of exponential decay. The possible index Λ is left implicit.

Lemma. *If the totally connected correlations* $\hat{\varrho}^T(X)\equiv\hat{\varrho}^T(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N)$ *satisfy the following* SCP:

$$|\hat{\varrho}^T(X)|<C^{|X|}e^{-\chi L(x_1,\ldots,x_N)} \tag{39}$$

when all points $x_1,\ldots,x_N$ *are different from each other, then the partially connected correlations satisfy, for any given* $\chi'<\chi$, *the bounds:*

$$|\hat{\varrho}^T(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)|<C'^{\sum_{i=1}^{M}|X_i|}\,e^{-\chi' L(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)} \tag{40}$$

when the subsets $X_1,\ldots,X_M$ *are disjoint. [The constant* C' *in the bounds* (40) *may depend on the choice of* χ' *but is independent of* $X_1,\ldots,X_M$ *and* M.]

The bounds (40) *are* SCP *in the sense of Section* 3 *when the subsets* $X_1,\ldots,X_M$ *are disjoint* [3], *if we restrict for instance our attention to subsets* X_i *whose maximal number of points is less than some fixed integer* N. *The functions* g_i *are then equal to* C'^N.

Proof. The partially connected correlations can be expressed as follows in terms of totally connected correlations when the subsets $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M$ are disjoint:

$$\hat{\varrho}^T(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)=\sum_{(\pi_1,\ldots,\pi_k)^c/\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M}^{\bigcup_i X_i}\ \prod_{j=1}^{k}\hat{\varrho}^T(X(\pi_j)) \tag{41}$$

where the sum $\sum$ in the right-hand side runs over all partitions $\pi_1,\ldots,\pi_k$ of the set[4] $\bigcup_{i=1,\ldots,M}X_i$ into subsets $X(\pi_1),\ldots,X(\pi_k)$ that are connected with respect to the given

[3] We recall that one has in this case $\mathcal{N}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)=1$

[4] $\bigcup_{i=1,\ldots,M}X_i$ is the set of all points of the lattice that belong to one of the subsets X_i

subsets $X_1, \ldots, X_M$[5]; the functions $\hat{\varrho}^T(X(\pi_j))$ are here totally connected with respect to the points of the sets $X(\pi_j)$.

Being given any positive ξ, Equation (41) can be likewise written in the form:

$$\xi^{\Sigma|X_i|}\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M) = \sum_{(\pi_1,\ldots,\pi_k)^c/X_1,\ldots,X_M}^{\bigcup_i X_i} \prod_{j=1}^{k} [\hat{\varrho}^T(X(\pi_j)\xi^{|X(\pi_j)|}]. \tag{42}$$

Hence:

$$|\xi^{\Sigma|X_i|}\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)| < \sum_{(\pi_1,\ldots,\pi_k)^c/X_1,\ldots,X_M}^{\bigcup_i X_i} e^{-\chi'\sum_j L(X(\pi_j))}$$

$$\cdot \prod_{j=1}^{k} |\hat{\varrho}^T(X(\pi_j)e^{\chi' L(X(\pi_j))}\xi^{|X(\pi_j)|}|. \tag{43}$$

In Equation (43), $L(X(\pi_j))$ is the minimal length between all points of the set $X(\pi_j)$ as defined is Section 3, and it can be checked that for any partition $\pi_1, \ldots, \pi_k$ that is connected with respect to $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$, one has:

$$e^{-\chi'\sum_j L(X(\pi_j))} \leqq e^{-\chi' L(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)}. \tag{44}$$

Hence it follows from Equation (43) that

$$|\xi^{\Sigma|X_i|}\hat{\varrho}^T(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M)| < e^{-\chi' L(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_M)}$$

$$\cdot \sum_{\pi_1,\ldots,\pi_k}^{\bigcup_i X_i} \prod_{j=1}^{k} |\hat{\varrho}^T(X(\pi_j))e^{\chi' L(X(\pi_j))}\xi^{|X(\pi_j)|}|, \tag{45}$$

where the remaining sum $\sum$ has been extended, as is possible, to all partitions of $\bigcup_i X_i$. To prove the bounds (40) it is therefore sufficient to show that, for an appropriate choice of the constant ξ, this sum is bounded by $C''^{\Sigma|X_i|}$, where C'' is independent of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_M$ and M.

The proof of this last result can be obtained by using a method communicated to us by H. Kunz. Let us put below for simplicity $X = \bigcup_i X_i$ and $f(Y) = |\hat{\varrho}^T(Y)e^{\chi' L(Y)}\xi^{|Y|}|$. One has:

$$\sum_{\pi_1,\ldots,\pi_k}^{X} \prod_{j=1}^{k} f(X(\pi_j)) \leqq \sum_{k \geqq 1} \frac{1}{k!}\left[\sum_{Y \subset X} f(Y)\right]^k \tag{46}$$

$$\leqq \sum_{k \geqq 1} \frac{1}{k!}\left[|X| \operatorname*{Sup}_x \sum_{Y \ni x} f(Y)\right]^k, \tag{47}$$

where the sums $\sum$ in the brackets of (46) and (47) run over all subsets Y of points of the lattice that are included in X or contain a point x. The fact that all points of X are different from each other has been used.

5 I.e. such that a *connected* diagram of M vertices is obtained when each X_t is identified with one vertex ($t = 1, \ldots, M$) and when X_{t_1}, X_{t_2} are joined by a line whenever one (or more) x_{t_1} in X_{t_1} and one (or more) x_{t_2} in X_{t_2} belong to a common $\pi_j (j = 1, \ldots, k)$

The SCP (39) ensure for any given integer n that

$$\operatorname*{Sup}_x \left| \sum_{\substack{Y \ni x \\ |Y|=n}} \hat{\varrho}^T(Y) e^{\chi' L(Y)} \right| \leqq (C''')^n,$$

where C''' is independent of $y_1, \ldots, y_n$ and of n and depends only on C and on the choice of $\chi' < \chi$. Hence, if ξ is chosen such that $C'''\xi < 1$, one obtains:

$$\operatorname*{Sup}_x \sum_{Y \ni x} f(Y) \leqq \sum_{n \geqq 0} (C'''\xi)^n = \frac{1}{1 - C'''\xi}. \tag{48}$$

The announced result follows from the remark that the right-hand side of (47) is $\exp\left[|X| \operatorname*{Sup}_x \sum_{Y \ni x} f(Y)\right]$, which leads to $C' = \xi^{-1} \exp\left[\frac{1}{1 - C'''\xi}\right]$ in Equation (40).

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