Commun. math. Phys. 59, 213—233 (1978)

Communications in
Mathematical
Physics

The Particle Structure of ν-Dimensional Ising Models at Low Temperatures

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Abstract. In this work we study the ν-dimensional Ising model at low temperatures and establish the existence of an upper gap in the energy-momentum spectrum of the two-point function for $\nu \geqq 3$. For $\nu = 2$, it is known that this gap is absent.

1. Introduction

The low energy structure of the energy-momentum spectrum of a Quantum Field Theory is to a large extent determined by the asymptotic behavior of pair correlation functions. It is also connected with the particle structure of the theory. Results in this direction have been obtained for some nontrivial models in two and three space-time dimensions [7, 15, 1]. While four dimensional models have not yet been constructed, we can study other physical systems which are simple enough to be realized in any dimension and yet have some resemblance to a field theory. One example is the ν-dimensional Ising model, to be compared with the $(\nu-1)$ (space) $+1$ (imaginary time) Euclidean field theory. In this work, we study the particle structure of that model and obtain results that indicate the existence of isolated one particle states when $\nu > 2$ for sufficiently low temperatures. Aside from its connection with field theory, the problem is interesting in itself because of the remarkable difference between the cases $\nu = 2$ and $\nu > 2$. This is manifest in the asymptotic behavior of the pair correlation function, which for $\nu = 2$ violates its expected decay rate at infinity (Orstein-Zernike prediction). The Orstein-Zernike prediction is a consequence of the particle structure we derive here together with further properties of the dispersion relations for one particle states [11], which in principle could be studied by the methods developed in this work. The failure of the Orstein-Zernike behavior for $\nu = 2$ at low temperatures is due to a failure of the required particle structure and is related to the existence of solitons for $\nu = 2$ only, [14]. In Section III, we will give a simple geometric picture to explain the

* Supported in part by Conselho Nacional de Pesquisas (CNPq-Brazil), Universidade Federal de Minas Gerais (Brazil) and the National Science Foundation under Grant PHY76-17191

0010-3616/78/0059/0213/$04.20

difference between the $\nu=2$, $\nu>2$ low temperature particle structure. We now define the model and give a brief descriptions of the chapters.

The ν-dimensional Ising model is a system of spins σ_i located at the points of the lattice $\mathbb{Z}^\nu$ and taking the values ± 1. The interaction energy is only between nearest-neighbor spins, and has the form

$$U(\sigma_i,\sigma_j)=-J\sigma_i\sigma_j\,;\qquad |i-j|=1\,,\qquad J>0\,.$$

Thermal averages are calculated using the Gibbs canonical ensemble. Thus, if the spins in the boundary of a finite set $\Lambda\subset Z^\nu$ are fixed, the average value of $\sigma_A=\prod\limits_{i\in A}\sigma_i(A\subset\Lambda)$ at inverse temperature β is

$$\langle\sigma_A\rangle_{\Lambda,b_\Lambda,\beta}=\frac{1}{Z_{\Lambda,b_\Lambda,\beta}}\sum_{\vec{\sigma}}\sigma_A\exp\left(\beta J\sum_{|i-j|=1}{}'\sigma_i\sigma_j\right) \tag{1.1}$$

where $\sum\limits_{\vec{\sigma}}$ represents a sum over all spin configurations in Λ and the summation in the exponent is restricted to nearest neighbor pairs $\{i,j\}$ such that i or j belongs to Λ. b_Λ specifies the boundary condition and $Z_{\Lambda,b_\Lambda,\beta}$ (the partition function) is a normalization factor, defined so that $\langle 1\rangle_{\Lambda,b_\Lambda,\beta}=1$.

The thermodynamic state of the system is characterized by the $\Lambda\uparrow\infty$ limit of the correlation functions 1.1. At low temperatures, this limit depends on the boundary conditions b_Λ, [12].

In Section II, we describe the physical Hilbert space associated with the thermodynamic state and introduce the energy-momentum operators. The lower gap in the two-point energy momentum spectrum is implied by the exponential decay of the truncated pair correlation function while the upper gap follows from a stronger exponential decay of its convolution inverse.

In Section III, we establish estimates that prove the existence of the lower and upper gap in the (two-point) energy-momentum spectrum when $\nu>2$ for sufficiently low temperatures.

Fisher and Camp in a series of papers [4, 3] considered this problem from the point of view of perturbation theory. Their results are similar to ours.

II. One-Particle States

a) The Transfer Matrix

We start by describing briefly the infinite volume transfer matrix for the Ising model. The construction is analogous to one in Quantum Field Theory [10, 6] and is based on the Osterwalder-Schrader positivity of Gibbs measure. A detailed treatment can be found in [14].

Consider the state of the ν-dimensional Ising model whose correlation functions $\langle\sigma_A\rangle, A\subset\mathbb{Z}^\nu$ finite are obtained as limits of $\langle\sigma_A\rangle_{\Lambda,+}$ [as defined in (1.1) with $+$ boundary conditions] when $\Lambda\uparrow Z^\nu$. This state can be described by a probability measure μ on the space of configurations $\mathscr{X}=\{-1,1\}^{-\nu}$ of the system, such that

$$\langle\sigma_A\rangle=\int\sigma_A(y)d\mu(y)\,;\qquad A\subset Z^\nu\,,\qquad A\text{ finite}\,.$$

In this formula, $\sigma_A = \prod_{i\in A} \sigma_i$, and $\sigma_i : \mathscr{X} \to \{-1, 1\}$ is the projection along the i^{th} component. Lattice translations and reflections are implemented on $\mathscr{X}$ by $\sigma_i(T_j y) = \sigma_{i-j}(y)$ (T_j represents translation by $j \in Z^\nu$) and $\sigma_i(\theta y) = \sigma_{\theta_i}(y)$, ($\theta_i$ is the reflection of i across the $i_1 = 0$ hyperplane) and μ is invariant under these operations. In addition, μ has the Osterwalder-Schrader positivity:

$$\langle \theta f, f\rangle \equiv \int (\overline{\theta f}) f \, d\mu \geqq 0; \quad f \in L^2(\mathscr{X}, \Sigma_+, d\mu).$$

where Σ_+ is the σ-algebra generated by the family $\{\sigma_i : i_1 \geqq 0\}$ and θ is the unitary operator on $L^2(\mathscr{X}, \Sigma, d\mu)$ (Σ = Borel σ-algebra) given by $(\theta f)(y) = f(\theta^{-1} y)$.

Let b be the sesquilinear form on $L^2(\mathscr{X}, \Sigma_+, d\mu)$ given by $b(f, g) = \langle \theta f, g\rangle$, and let

$$\mathscr{N} = \{f \in L^2(\mathscr{X}, \Sigma_+, d\mu) : b(f, f) = 0\}.$$

b lifts to a positive definite scalar product [denoted $(\cdot, \cdot)$] on the quotient $L^2(\mathscr{X}, \Sigma_+, d\mu)/\mathscr{N}$. We define $\mathscr{H}$ (the physical Hilbert space) to be the completion of this quotient with respect to $(\cdot, \cdot)$ and denote by π the natural injection of $L^2(\mathscr{X}, \Sigma_+, d\mu)$ onto $L^2(\mathscr{X}, \Sigma_+ d\mu)/\mathscr{N}$. Thus, for any $f_1, f_2 \in L^2(\mathscr{X}, \Sigma_+, d\mu)$,

$$(\pi f_1, \pi f_2) = b(f_1, f_2) = \langle \theta f_1, f_2\rangle.$$

Next, we consider the problem of representing lattice translations in $\mathscr{H}$. In $L^2(\mathscr{X}, \Sigma, d\mu)$ the operator $\hat{T}_1$ (representing translation by $(1, 0, \ldots, 0)$) is defined by $(\hat{T}_1 f)(y) = f(T_{-e_1} y)$, where T_{-e_1} is translation by $(-1, 0, \ldots, 0)$ in $\mathscr{X}$. $\hat{T}_1$ is unitary and maps $L^2(\mathscr{X}, \Sigma_+, d\mu)$ into itself. One can also verify that $\hat{T}_1$ leaves $\mathscr{N}$ invariant and therefore can be lifted to the quotient $L^2(\mathscr{X}, \Sigma_+, d\mu)/\mathscr{N}$ by setting

$$T_1 \pi f = \pi \hat{T}_1 f \quad \forall f \in L^2(\mathscr{X}, \Sigma_+, d\mu).$$

Notice that

$$(T_1 \pi f, \pi g) = \langle \theta \hat{T}_1 f, g\rangle = \langle \theta f, \hat{T}_1 g\rangle = (\pi f, T_1 \pi g),$$

where we have used the fact $\theta \hat{T}_1 = \hat{T}_1^{-1} \theta$. It is also possible to show that $\| T_1 \pi f\| \leqq \|\pi f\|$, so that T_1 extends to $\mathscr{H}$ as a bounded self adjoint operator with $\|T\| = 1$ ($T_1 \pi 1 = \pi 1$). The operator T_1 is called the infinite volume transfer matrix associated with the state described by μ. T_1 has the additional property of being a positive operator.

We may apply similar considerations to the other translation operators $\hat{T}_\alpha (2 \leqq \alpha \leqq \nu)$. The major difference is that $\theta \hat{T}_\alpha = \hat{T}_\alpha \theta$. This implies that the corresponding T_α's extend to unitary operators in $\mathscr{H}$.

Finally, we define the "time zero field". Consider σ_i with $i_1 = 0$ as a multiplication operator in $L^2(\mathscr{X}, \Sigma_+, d\mu)$. It is easy to show that σ_i leaves $\mathscr{N}$ invariant so we define $\hat{\sigma}_i$ on $L^2(\mathscr{X}, \Sigma_+, d\mu)/\mathscr{N}$ by

$$\hat{\sigma}_i \pi f = \pi \sigma_i f \quad \forall f \in L^2(\mathscr{X}, \Sigma_+, d\mu)$$

and extend to $\mathscr{H}$ as a self adjoint operator with $\hat{\sigma}_i^2 = 1$. The Gellman Low formula

$$\langle \sigma_j \sigma_k\rangle = (\Omega_0, \hat{\sigma}_0 T_1^{(j_1 - k_1)} \prod_{\alpha=2}^{\nu} T_\alpha^{(j_\alpha - k_\alpha} \hat{\sigma}_0 \Omega_0)$$

follows immediately from the definitions. $T_\alpha(2 \leqq \alpha \leqq \nu)$ can be written as $T_\alpha = e^{iP_\alpha}$ with spec $P_\alpha \subset [-\pi, \pi)$, and since the T_α's are commuting,

$$\prod_{\alpha=2}^{\nu} T_\alpha^{(j_\alpha - k_\alpha)} = e^{i(\boldsymbol{j}-\boldsymbol{k})\cdot \boldsymbol{P}}$$

and $\boldsymbol{P}$ is identified with the momentum operator.

The energy operator H would be defined by $-\log T_1$ after one proves that the null space of T_1 is trivial. As far as the author knows, this is still an open question, and therefore we shall place emphasis on T_1, instead of H.

b) Spectral Properties of the Two-Point Function

In this section, we establish the existence of one-particle states in the spectrum of the two-point function. The basic issue considered here is the analytic structure of the function

$$\tilde{G}^{(2)}(\underset{\sim}{p}) = \frac{1}{(2\pi)^{\nu/2}} \sum_{x \in Z^\nu} G^{(2)}(\underset{\sim}{x}) e^{-i\underset{\sim}{p}\underset{\sim}{x}} \tag{2.1}$$

where $G^{(2)}(\underset{\sim}{x}) \equiv \langle \sigma_0 \sigma_{\underset{\sim}{x}} \rangle - \langle \sigma_0 \rangle \langle \sigma_{\underset{\sim}{x}} \rangle$ is the truncated two-point function, $\underset{\sim}{p} \in \mathbb{T}^\nu$ (ν-dimensional torus) and $\underset{\sim}{p}\underset{\sim}{x} = p_1 x_1 + \ldots + p_\nu x_\nu$. We will show that $\tilde{G}^{(2)}(p_1, \bar{p})$ $[\underset{\sim}{p} \equiv (p_1, \bar{p})$ with $p_1 \in [-\pi, \pi)$ and $\bar{p} \in T^{\nu-1}]$ is meromorphic in a strip in the p_1 plane for each $\bar{p} \in T^{\nu-1}$, having a pole at a purely imaginary number $p_1 = iw(\bar{p})$. By general principles, this implies the existence of one-particle states, and in fact $w(\bar{p})$ can be interpreted as the energy of a particle with momentum $\bar{p}$. The meromorphic character of $\tilde{G}^{(2)}(p_1, \bar{p})$ follows from an exponential decay of $G^{(2)}(\underset{\sim}{x})$ and a stronger exponential decay of its convolution inverse.

We start by defining the two-point function measure. From the Gellman-Low formula, we know that

$$\langle \sigma_0 \sigma_{(x_1, \bar{x})} \rangle = \left(\Omega_0, \hat{\sigma}_0 T_1^{|x_1|} \prod_{\alpha=2}^{\nu} T_\alpha^{x_\alpha} \hat{\sigma}_0 \Omega_0 \right)_{\mathscr{H}},$$

where we write $\underset{\sim}{x} \in Z^\nu$ as $\underset{\sim}{x} = (x_1, \bar{x})$ with $x_1 \in \mathbb{Z}$ and $\bar{x} \in Z^{\nu-1}$. Let $\{E_\lambda\}$ and $\{F_\lambda^{(\alpha)}\}_{\alpha=2}^\nu$ be the spectral families of projection operators defined by

$$T_1 = \int_0^1 \lambda dE_\lambda \quad \text{and} \quad T_\alpha = \int_{(-\pi, \pi]} e^{i\lambda} dF_\lambda^{(\alpha)} (2 \leqq \alpha \leqq \nu).$$

Since $T_1, \{T_\alpha\}_{\alpha=2}^\nu$ are commuting operators, $dE_{\lambda_1} \prod_{\alpha=2}^{\nu} dF_{\lambda_\alpha}^{(\alpha)}$ is a projection valued measure on $(0,1) \times T^{\nu-1}$, where $T^{\nu-1}$ is the $(\nu-1)$ dimensional torus. Notice that

$$\langle \sigma_0 \sigma_{(x_1, x)} \rangle = \int_0^1 \int_{T^{\nu-1}} \lambda_1^{|x_1|} e^{i\vec{\lambda}\cdot\vec{x}} d\left(\hat{\sigma}_0 \Omega_0, E_{\lambda_1} \prod_{\alpha=2}^{\nu} F_{\lambda_\alpha}^{(\alpha)} \hat{\sigma}_0 \Omega_0 \right)_{\mathscr{H}}.$$

Two-point function measure is defined to be

$$d\mu(\lambda_1, \vec{\lambda}) = d\left(\hat{\sigma}_0 \Omega_0, E_{\lambda_1} \prod_{\alpha=2}^{\nu} F_\alpha^{(\alpha)} \hat{\sigma}_0 \Omega_0 \right)_{\mathscr{H}}.$$

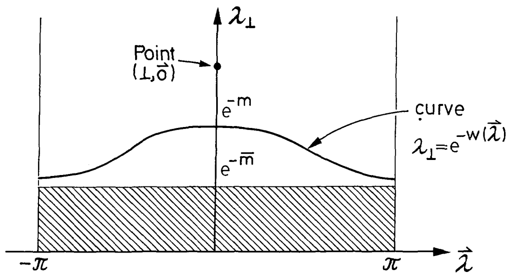

Fig. 1

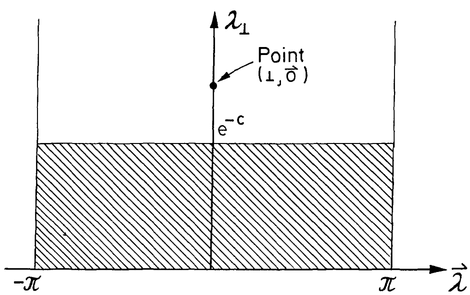

Fig. 2

Our goal is to show that when $v \geqq 3$ and the inverse temperature β is large, the support of $d\mu$ is contained in the set (see Fig. 1).

$$(1,\vec{0}) \cup \{(\lambda_1, \vec{\lambda}) : \lambda_1 = e^{-w(\vec{\lambda})}\} \cup \{(\lambda_1, \vec{\lambda}) : 0 \leqq \lambda_1 \leqq e^{-\bar{m}}\}$$

for suitable β-dependent constants $m, \bar{m}$ and function $w(\vec{\lambda})$. The structure of the support of $d\mu$ in a neighborhood of $\lambda = 1$ is obtained from the following result

Theorem 2.1. *For large β, $0 \leqq G^{(2)}(x_1, \vec{x} = \vec{0}) \leqq \text{const}\, e^{-c|x_1|}$.*

The first inequality is one of Griffiths.

The proof of the second inequality will be given in the next chapter. A simple consequence of this theorem is the

Corollary 2.1. *The support of $d\mu$ is contained in*

$$(1,\vec{0}) \cup \{\lambda_1, \vec{\lambda}) : 0 \leqq \lambda_1 \leqq e^{-c}\}$$

and

$$\mu\{(1,\vec{0})\} = \langle \sigma_0 \rangle^2 > 0$$

(see Fig. 2).

Proof. Write

$$\langle \sigma_0 \sigma_{(x_1, 0)} \rangle = \|E_{\{1\}} \hat{\sigma}_0 \Omega_0\|^2 + \int_{[0,1)} \lambda_1^{|x_1|} d\|E_{\lambda_1} \hat{\sigma}_0 \Omega_0\|^2 .$$

Letting $|x_1| \uparrow \infty$ and using Theorem 2.1 we conslude that $E_{\{1\}} \hat{\sigma}_0 \Omega_0 = \langle \sigma_0 \rangle \Omega_0$. From this it follows that $\mu\{(1,\vec{0})\} = \langle \sigma_0 \rangle^2$ and that μ has no mass on $\{(\lambda_1, \vec{\lambda}) : \lambda_1$

$=1, \vec{\lambda} \neq 0\}$. The exponential decay given by Theorem 2.1 also implies that $E_{(e^{-c},1)}\hat{\sigma}_0\Omega_0 = 0$ and the proof is complete.

The constant m in Figure 1 (which is interpreted as the rest mass of the "particle" described by the theory) is now defined to be

$$m = -\lim_{|x_1| \uparrow \infty} \frac{1}{|x_1|} \log G^{(2)}(x_1, 0).$$

We will show in the next chapter that $m \approx 4(\nu-1)\beta J$ for large β. Of course, c can be replaced by m in corollary.

We now study the support of $d\mu$ in the region $0 \leqq \lambda_1 \leqq e^{-m}$. The general strategy will be the following. First, it will be shown that $d\mu(\lambda_1, \vec{\lambda})$ is absolutely continuous in $\vec{\lambda}$ (w.r.t. Lebesgue measure) when $\lambda_1 \in [0, e^{-m}]$, and then we will obtain detailed information about the Radon-Nikodym derivative $d\mu(\lambda_1, \vec{\lambda})/d\vec{\lambda}$ for λ_1 in a neighborhood of e^{-m}. Combining these results, we will be able to justify the picture drawn in Figure 1.

We start with the following theorem.

Theorem 2.2. *For each* $\vec{p} \in T^{\nu-1}$, *there exists a positive measure* $d\varrho(\lambda_1; \vec{p})$ *supported in* $\lambda_1 \in [0, e^{-m}]$ *such that*

$$\tilde{G}^{(2)}(p_1, \vec{p}) = \int_{[0, e^{-m}]} \frac{1-\lambda_1^2}{1 - 2\lambda_1 \cos p_1 + \lambda_1^2} d\varrho(\lambda_1; \vec{p}). \tag{2.2}$$

Moreover, $d\varrho(\cdot\,; \mathbf{p})$ *is weakly continuous in* $\mathbf{p}$ *and for any continuous* $f(\vec{p})$ *and* $g(\lambda_1)$,

$$\int_{T^{\nu-1}} d\vec{p} f(\vec{p}) \int_{[0, e^{-m}]} g(\lambda_1) d\varrho(\lambda_1; \vec{p}) = (2\pi)^{\left(\frac{\nu}{2}-1\right)} \int_{[0, e^{-m}]} \int_{T^{\nu-1}} f(\vec{\lambda}) g(\lambda_1) d\mu(\lambda_1, \vec{\lambda}). \tag{2.3}$$

In (2.2) above, $\tilde{G}^{(2)}(p_1, \vec{p})$ is given by (2.1). This theorem can be heuristically derived as follows. Starting from the representation

$$\tilde{G}^{(2)}(x_1, \vec{x}) = \int_{[0, e^{-m}]} \int_{T^{\nu-1}} \lambda_1^{|x_1|} e^{i\vec{\lambda}\cdot\vec{x}} d\mu(\lambda_1, \vec{\lambda})$$

we obtain, taking Fourier transforms

$$\tilde{G}^{(2)}(p_1, \vec{p}) = \int_{[0, e^{-m}]} \int_{T^{\nu-1}} \frac{1-\lambda_1^2}{1 - 2\lambda_1 \cos p_1 + \lambda_1^2} (2\pi)^{\left(\frac{\nu}{2}-1\right)} \delta(\vec{\lambda} - \vec{p}) d\mu(\lambda_1, \vec{\lambda}). \tag{2.4}$$

Letting

$$d\varrho(\lambda_1; \vec{p}) = (2\pi)^{\left(\frac{\nu}{2}-1\right)} \int_{T^{\nu-1}} \delta(\vec{\lambda} - \vec{p}) d\mu(\lambda_1, \vec{\lambda})$$

we obtain Equation (2.2).

To prove the theorem, note that (2.4) is true after smearing with a smooth function of $\mathbf{p}$. For each $n > 1$, let

$$h_m(\mathbf{p}; \mathbf{q}) = \prod_{\alpha=2}^{\nu} \frac{1 - r_n^2}{1 - 2r_n \cos(p_\alpha - q_\alpha) + r_n^2}; \qquad r_n = \frac{n-1}{n}$$

and let

$$d\varrho_m(\lambda_1;\boldsymbol{q})=(2\pi)^{\left(\frac{\nu}{2}-1\right)}\int_{T^{\nu-1}} h_m(\boldsymbol{\lambda};\boldsymbol{q})d\mu(\lambda_1,\boldsymbol{\lambda}).$$

Thus, $d\varrho_m(\lambda_1;\boldsymbol{q})$ is a positive measure in λ_1 (for each fixed $\boldsymbol{q}\in T^{\nu-1}$) supported on $(0,e^{-m})$ and clearly

$$\int_{T^{\nu-1}} \tilde{G}^{(2)}(p_1,\boldsymbol{p})h_m(\boldsymbol{p};\boldsymbol{q})d\boldsymbol{p}=\int_{[0,e^{-m}]}\frac{1-\lambda_1^2}{1-2\lambda_1\cos p_1+\lambda_1^2}d\varrho_m(\lambda_1;\boldsymbol{q}).$$

Notice that $\|d\varrho_m(\cdot\,;\boldsymbol{q})\|\leqq \max_{\boldsymbol{p}\in T^{\nu-1}}|\tilde{G}^{(2)}(0,\boldsymbol{p})|<\infty$ [from Theorem 2.1 and the symmetry of $G^{(2)}(x)$, it follows that $\tilde{G}^{(2)}(p_1,\boldsymbol{p})$ is real analytic in $\boldsymbol{p}$ and analytic in p_1 on $|\mathrm{Im}\,p_1|<m$]. By compactness, $d\varrho_{n_j}(\cdot\,;\boldsymbol{q})\to d\varrho(\cdot\,;\boldsymbol{q})$ for some subsequence $\{n_j\}$, and this implies

$$\tilde{G}^{(2)}(p_1,\boldsymbol{q})=\int_{[0,e^{-m}]}\frac{1-\lambda_1^2}{1-2\lambda_1\cos p_1+\lambda_1^2}d\varrho(\lambda_1;\boldsymbol{q}).$$

Taking Fourier transform in the p_1 variable,

$$\begin{aligned}\tilde{G}^{(2)}(x_1,\boldsymbol{q})&\equiv\frac{1}{(2\pi)^{1/2}}\int_{-\pi}^{\pi}\tilde{G}^{(2)}(p_1,\boldsymbol{q})e^{ip_1x_1}dp_1\\&=(2\pi)^{1/2}\int_0^{e^{-m}}\lambda_1^{|x_1|}d\varrho(\lambda_1;\boldsymbol{q})\end{aligned}\tag{2.5}$$

showing that $\int_0^{e^{-m}}P(\lambda_1)d\varrho(\lambda_1;\boldsymbol{q})$ is continuous in $\boldsymbol{q}$ for any polynomial $P(\lambda_1)$. Since $\|d\varrho(\cdot\,;\boldsymbol{q})\|$ is bounded uniformly in $\boldsymbol{q}$, we see that $\int_0^{e^{-m}}g(\lambda_1)d\varrho(\lambda_1;\boldsymbol{q})$ is continuous in $\boldsymbol{q}$ for any continuous function $g(\lambda_1)$. From the distribution form of (2.4), and the considerations above, we have

$$\int_{T^{\nu-1}}d\boldsymbol{q}f(\boldsymbol{q})\int_0^{e^{-m}}P(\lambda_1)d\varrho(\lambda_1;\boldsymbol{q})=(2\pi)^{\left(\frac{\nu}{2}-1\right)}\int_0^{e^{-m}}\int_{T^{\nu-1}}P(\lambda_1)f(\boldsymbol{\lambda})d\mu(\lambda_1,\boldsymbol{\lambda})$$

for any polynomial $P(\lambda_1)$ and continuous f, and the proof of the theorem is complete.

An immediate consequence of Equation (2.3) and the dominated convergence theorem is the

Corollary 2.2. *For any intervals $\Delta_1\subset[0,e^{-m}]$ and $\vec{\Delta}\subset T^{\nu-1}$ ($\vec{\Delta}$ is just a product of intervals)*

$$\mu(\Delta_1\times\vec{\Delta})=\frac{1}{(2\pi)^{\left(\frac{\nu}{2}-1\right)}}\int_{\Delta}\varrho(\Delta_1;\vec{p})d\vec{p}\tag{2.6}$$

where $\varrho(\Delta_1;\boldsymbol{p})=\int_{\Delta_1}d\varrho(\lambda_1;\boldsymbol{p})$.

Our next task is to study properties of the measure $d\varrho(\lambda_1;\vec{p})$. We will show that for large β, this measure has the form

$$d\varrho(\lambda_1;\vec{p}) = Z(\vec{p})\delta(\lambda_1 - e^{-w(\vec{p})})d\lambda_1 + d\varrho'(\lambda_1;\vec{p}) \tag{2.7}$$

where $d\varrho'(\lambda_1;\vec{p})$ has support in $\lambda_1 \in [0, e^{-\bar{m}}]$ with $\bar{m} > m$, $Z(\vec{p})$ is positive and $w(\vec{p})$ is smooth. Combining now Corollary 2.1 and Equations (2.6) and (2.7) gives us the picture and Figure 1.

The function $w(\boldsymbol{p})$ is defined by

$$-\lim_{|x_1|\uparrow\infty} \frac{1}{|x_1|} \log \tilde{G}^{(2)}(x_1, \boldsymbol{p}) \equiv w(\boldsymbol{p}).$$

The limit exists because of the representation (2.5) and clearly $w(\boldsymbol{p}) \geqq m$. Also, a simple computation shows that $w(\vec{0}) = m$. Additional properties of $w(\vec{p})$ are summarized in the

Theorem 2.3. *$w(\vec{p})$ is real analytic in $\vec{p}$ and* $\lim_{\beta\uparrow\infty} \frac{w(\vec{p})}{m} = 1$ *uniformly in $\vec{p}$.*

The analyticity will be verified in the proof of Theorem 2.5. The other assertion will be proved in Appendix I of the next chapter.

The only undefined quantity in Figure 1 now is the constant $\bar{m}$. This constant is characterized in Theorem 2.4 below. Let

$$\tilde{\Gamma}^{(2)}(p_1, \vec{p}) = \frac{-1}{\tilde{G}^{(2)}(p_1, \vec{p})}.$$

Theorem 2.4. *For large β and $v \geqq 3$, there exists $\bar{m} > m$ such that $\tilde{\Gamma}^{(2)}(p_1, \vec{p})$ is real analytic in $\vec{p}$ and analytic in p_1 on $|\mathrm{Im}\, p_1| < \bar{m}$. Moreover, $(\bar{m}/m)$ can be taken close to $(4v-3)/(4v-4)$ by letting $\beta\uparrow\infty$.*

This theorem is the main result of this work. The interesting feature is that it is dimension dependent. In fact, the result is known to be false when $v = 2$. The proof of an equivalent version will be given in an appendix to the next section.

We are now ready to prove (2.7). We assume β so large that $m \leqq w(\vec{p}) < \bar{m} \forall \vec{p} \in T^{v-1}$.

Theorem 2.5. *$d\varrho(\lambda_1;\vec{p}) = Z(\vec{p})\delta(\lambda_1 - e^{-w(\vec{p})})d\lambda_1 + d\varrho'(\lambda_1;\vec{p})$ where $d\varrho'$ has support in $\lambda_1 \in [0, e^{-\bar{m}}]$ and $Z(\vec{p})$ is a positive smooth function.*

Proof. From Theorem 2.2

$$\tilde{G}^{(2)}(p_1, \vec{p}) = \varrho(\{0\};\vec{p}) + \int_{(0, e^{-m}]} \frac{1-\lambda_1^2}{1 - 2\lambda_1 \cos p_1 + \lambda_1^2} d\varrho(\lambda_1;\vec{p}).$$

Making the successive change of variables $\lambda_1 = e^{-\sigma}$ and then $a = -1 + \cosh\sigma$ we obtain

$$\tilde{G}^{(2)}(p_1, \vec{p}) = \varrho(\{0\};\vec{p}) + \int_{-1+\cosh\bar{m}}^{\infty} \frac{dv(a;\vec{p})}{1 - \cosh p_1 + a}$$

with a certain positive measure $d\nu(a;\vec{p})$. Hence, $\tilde{G}^{(2)}(p_1,\vec{p})$ is an Herglotz function of $(\cos p_1 - 1)$ (see [5] for properties of Herglotz functions which will be used below) and therefore so is $\tilde{\Gamma}^{(2)}(p_1,\vec{p})$. As such, it has the following general representation

$$\tilde{\Gamma}^{(2)}(p_1,\vec{p})=\alpha(\vec{p})(\cos p_1-1)+\beta(\vec{p})+\int_{-1+\cosh m}^{\infty}\left[\frac{1}{1-\cos p_1+a}-\frac{a}{1+a^2}\right]d\eta(a;\vec{p})$$

where $\alpha(\vec{p})\geqq 0$, $\beta(\vec{p})$ is real and $(1+a^2)^{-1}d\eta$ is a finite, positive measure. The support of $d\eta$ is $[-1+\cos m,\infty)$ because $\tilde{G}^{(2)}(p_1,\vec{p})$ does not vanish when $(\cos p_1-1)$ runs through $(-\infty,-1+\cosh m)$. The analyticity of $\tilde{\Gamma}^{(2)}(p_1,\vec{p})$ in the region $|\mathrm{Im}\,p_1|<\bar{m}$ implies that $d\eta$ is in fact supported in $[-1+\cosh\bar{m},\infty)$, so that $\tilde{G}^{(2)}(p_1,\vec{p})$ is meromorphic in the region $-1+\cosh m\leqq -1+\cos p_1<-1+\cosh\bar{m}$. Now, notice that

$$\frac{\partial}{\partial(\cos p_1-1)}\tilde{\Gamma}^{(2)}(p_1,\vec{p})=\alpha(\vec{p})+\int_{-1+\cosh\bar{m}}^{\infty}\frac{d\eta(a,\vec{p})}{(1-\cos p_1+a)^2}>0 \tag{2.8}$$

in that region, and therefore $\tilde{\Gamma}^{(2)}(p_1,\vec{p})$ can have at most one simple zero in the region. It is easy to see that in fact,

$$\tilde{\Gamma}^{(2)}(\pm iw(\vec{p}),\vec{p})=0 \tag{2.9}$$

and if we let

$$\frac{\partial}{\partial(\cos p_1-1)}\tilde{\Gamma}^{(2)}(p_1,\vec{p})|_{p_1=\pm iw(\vec{p})}=\frac{1}{Z_1(\vec{p})} \tag{2.10}$$

then, $\tilde{G}^{(2)}(p_1,\vec{p})-\dfrac{Z_1(\vec{p})}{\cosh w(\vec{p})-\cos p_1}$ is an Herglotz function in $(\cos p_1-1)$ which is analytic on $\cos p_1-1<\cosh\bar{m}-1$. By the uniqueness theorem on representations of such functions [5], it follows that $d\nu(a;\vec{p})$ is of the form

$$d\nu(a;\vec{p})=Z_1(\vec{p})\delta(a+1-\cosh w(\vec{p}))da+d\nu'(a;\vec{p})$$

where $\mathrm{supp}\,d\nu'(a;\vec{p})\subset[-1+\cosh\bar{m},\infty)$. Going back from $d\nu$ to $d\varrho$, we conclude that

$$d\varrho(\lambda_1;\vec{p})=Z(\vec{p})\delta(\lambda_1-e^{-w(\vec{p})})d\lambda_1+d\varrho'(\lambda_1;\vec{p})$$

where $Z(\vec{p})=Z_1(\vec{p})/\sinh w(\vec{p})$ and $\mathrm{supp}\,d\varrho'\subset[0,e^{-\bar{m}}]$. This proves (2.7). The positivity and smoothness of $Z(\vec{p})$ follows from Equations (2.8) and (2.10). The real analyticity of $w(\vec{p})$ in $\vec{p}$ is a consequence of Equations (2.8) and (2.9) and the implicit function theorem.

III. Exponential Decay of $\Gamma^{(2)}(\underset{\sim}{x})$

In this chapter, Theorems 2.3 and 2.4 will be proven. We will work in position instead of momentum space. Thus, Theorem 2.4 will follow from a strong [as compared to $G^{(2)}(\underset{\sim}{x})$] exponential decay of $\Gamma^{(2)}(\underset{\sim}{x})$ as $|x|\uparrow\infty$. To prove these results, we use a generalized Gibbs measure, depending on several complex variables and study detailed properties of the corresponding finite volume two-point function and its matrix inverse. The estimates we obtain are independent of the volume, and therefore can be carried to the infinite volume limit. Similar methods have been used in Field Theory [13].

We now introduce the generalized Gibbs measure. Let $\Lambda \subset Z^\nu$ be a hypercube (sides with length $2n$) containing the origin. If $A \subset \Lambda$, define

$$\langle \sigma_A \rangle_{\Lambda,\{w,z\}} = \frac{1}{Z_{\Lambda,\{w,z\}}} \sum_{\vec{\sigma}} \sigma_A \prod_{p=-n}^{n} \prod_{\substack{|i-j|=1 \\ i_1=j_1}} w_p^{(1-\sigma_i\sigma_j)} \prod_{\substack{|i-j|=1 \\ i_1 \neq j_1}} z^{(1-\sigma_i\sigma_j)}$$

where $Z_{\Lambda,\{w,z\}}$ is a normalization factor (such that $\langle 1 \rangle_{\Lambda,\{w,z\}} = 1$) and $\sigma_A = \prod_{i \in A} \sigma_i$. The summation $\sum_{\vec{\sigma}}$ is over all configurations of spins in Λ and we adopt the convention that $\sigma_i = +1$ if $i \notin \Lambda$. Note that if we set $w_p = e^{-\beta J}$, we recover the original Gibbs measure in Λ with $+$ boundary conditions.

Let $G_\Lambda^{(2)}(i,j;\{w,z\}) = \langle \sigma_i \sigma_j \rangle_{\Lambda,\{w,z\}} - \langle \sigma_i \rangle_{\Lambda,\{w,z\}} \langle \sigma_j \rangle_{\Lambda,\{w,z\}}$ be the corresponding truncated two-point function ($i,j \in \Lambda$). The basic result about this function is

Theorem 3.1. *There are positive constants r, C (independent of Λ, i, j) such that* $G_\Lambda^{(2)}(i,j;\{w,z\})$ is analytic on $|w_p|$, $|z| < r$, *and* $|G_\Lambda^{(2)}| \leqq C$ *there.*

The proof of this theorem is lengthy but standard, and is based on a low temperature contour expansion of the Minlos-Sinai type [9]. We will not present it here. For a complete discussion see [14] or [2] for related results. Before proceeding with the specific theorems, let us make a general outline of our approach.

Let $\Gamma_\Lambda^{(2)}(i,j;\{w,z\})$ be (minus) the matrix inverse of $G_\Lambda^{(2)}$. As will turn out, $\Gamma_\Lambda^{(2)}$ is singular at $w = z = 0$. We isolate the singular part by writing

$$\Gamma_\Lambda^{(2)}(i,j;\{w,z\}) = M_\Lambda(i,j;\{w,z\}) S_\Lambda(j,j;\{w,z\})$$

where M_Λ is analytic on $|w_p|$, $|z| < r'$ (r' independent of Λ) and uniformly bounded; S_Λ is analytic on $0 < |w_p|$, $|z| < r'$ having poles in z and w_j. By controlling the behavior of M_Λ and S_Λ near the origin we will be able to obtain bounds on $\Gamma_\Lambda^{(2)}$ which can be carried to the infinite volume limit. Next, we obtain lower bounds for $G_\Lambda^{(2)}(i,j)$ and the final result follows by comparing these two bounds.

We consider now the more specific results, starting with the structure of $G_\Lambda^{(2)}$.

Theorem 3.2. $G_\Lambda^{(2)}(i,j;\{w,z\}) = z^4 \prod_{p=i_1}^{j_1} w_p^{4(\nu-1)} F_\Lambda(i,j;\{w,z\})$ *where F_Λ is analytic on* $|z|, |w_p| < r$.

Proof. We express $G_\Lambda^{(2)}$ in terms of duplicate variables

$$\begin{aligned} G_\Lambda^{(2)}(i,j;\{w,z\}) &= \tfrac{1}{2} \langle\!\langle (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') \rangle\!\rangle_{\Lambda,\{w,z\}} \\ &\equiv \frac{1}{2} \frac{1}{Z^2_{\Lambda,\{w,z\}}} \sum_{\vec{\sigma},\vec{\sigma}'} (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') \\ &\quad \prod_{p=-n}^{n} \prod_{\substack{|k-l|=1 \\ k_1=l_1=p}} w_p^{(2-\sigma_k\sigma_l - \sigma_k'\sigma_l')} \prod_{\substack{|k-l|=1 \\ k \neq l_1}} z^{(2-\sigma_k\sigma_l - \sigma_k'\sigma_l')} \end{aligned} \tag{3.1}$$

Suppose $\vec{\sigma}=\vec{\sigma}'=+1$ on the hyperplane $k_1=p$ $(i_1\leqq p\leqq j_1)$. (We then say that the configurations are trivial on that hyperplane.) Consider the configuration $(\vec{s},\vec{s}')$ obtained by interchanging $\vec{\sigma},\vec{\sigma}'$ only to the right of $k_1=p$. It is clear that the contributions of $(\vec{\sigma},\vec{\sigma}')$ and $(\vec{s},\vec{s}')$ cancel each other. Thus, we have only to consider configurations which are not both trivial on $k_1=p$. Each such configuration has $w_p^{4(\nu-1)}$ as a factor. This can be seen as follows. Assume $\vec{\sigma}$ is not trivial on $k_1=p$. Consider the Peierls contours on $k_1=p$ corresponding to $\vec{\sigma}$. These are the contours separating the plus from the minus spins. They are constructed by drawing an elementary hypersquare bisecting the bond $\{i,j\}$ for all nearest neighbor pairs such that $\sigma_i\sigma_j=-1$. These contours are closed because of the boundary condition. The exponent of w_p is $2\cdot$(total hyperarea (surface area) of the contours associated with $\vec{\sigma}$)$+2\cdot$(total hyperarea associated with $\vec{\sigma}'$). Since $\vec{\sigma}$ is not trivial and the smallest hyperarea is $2(\nu-1)$ (corresponding to an elementary hypercube), we obtain the factor $w_p^{4(\nu-1)}$. Although the picture above was made for $\nu\geqq 3$, the result is still valid when $\nu=2$. Since any non-trivial Peierls contours in Λ must be closed, it is also clear that each non-vanishing term in Equation (3.1) has z^4 as a factor. The theorem follows from these observations and Theorem 3.1.

From Theorems 3.1 and 3.2 and the maximum modulus theorem

$$|G_\Lambda^{(2)}(i,j;\{w,z\})|\leqq C\left(\frac{|z|}{r}\right)^4\prod_{p=i_1}^{j_1}\left(\frac{|w_p|}{r}\right)^{4(\nu-1)} \tag{3.2}$$

and this result leads directly to the exponential decay of the (infinite volume) $G^{(2)}$, Theorem 2.1. We consider now the question of invertibility of the matrix $G_\Lambda^{(2)}$ when $|z|$, $|w_p|$ are small. The problem is somewhat delicate because $G_\Lambda^{(2)}=0$ when $w_p=z=0$, but, as we shall see, $G_\Lambda^{(2)}$ is invertible if w_p, $z\neq 0$. The idea is to separate explicitly the diagonal part of $G_\Lambda^{(2)}: G_\Lambda^{(2)}=P_\Lambda^{(2)}+R_\Lambda^{(2)}$ where

$$P_\Lambda^{(2)}(i,j;\{w,z\})=G_\Lambda^{(2)}(i,j;\{w,z\})\delta_{ij}.$$

Then, one shows that $P_\Lambda^{(2)}$ is invertible if w_p, $z\neq 0$ so that $G_\Lambda^{(2)}=P_\Lambda(1+P_\Lambda^{-1}R_\Lambda)$ and finally that $P_\Lambda^{-1}R_\Lambda$ has a small norm when $|w_p|$, $|z|$ are small. The result then follows.

Theorem 3.3. *There exist* $0<r'\leqq r$ *(independent of* Λ*) such that*

$$|G_\Lambda^{(2)}(i,i;\{w,z\})|\geqq |z|^4|w_{i_1}|^{4(\nu-1)} \quad \text{whenever } |z|,\quad |w_p|<r'. \tag{3.3}$$

Proof. Consider the function $F_\Lambda(i,i;\{w,z\})=G_\Lambda^{(2)}(i,i;\{w,z\})/z^4w_{i_1}^{4(\nu-1)}$. By Theorems 3.1 and 3.2 F_Λ is analytic and uniformly bounded (by $C/r^{4\nu}$) on $|w_p|$, $|z|<r$, and it is not difficult to see that $F_\Lambda(i,i;\{w=0,z=0\})=4$. Therefore, from Schwarz's lemma (see, e.g., [8] we have

$$|F_\Lambda(i,i;\{w,z\})-F_\Lambda(i,i;\{w=0,z=0\})|<3$$

if $|w_p|$, $|z|<r'$ for a certain $r'\leqq r$ independent of Λ and the theorem follows easily from this inequality.

An immediate consequence of this theorem is that P_Λ^{-1} exists if z, $w_p\neq 0$. Let Q_Λ be the matrix $P_\Lambda^{-1}R_\Lambda$. Combining Equations (3.2) and (3.3) we see that $Q_\Lambda(i,j;\{w,z\})$ is analytic on $|w_p|$, $|z|<r'$ and

$$|Q_\Lambda(i,j;\{w,z\})|\leqq\frac{C}{r^{4\nu}}\prod_{\substack{p=i_1\\ p\neq i_1}}^{j_1}\left(\frac{|w_p|}{r}\right)^{4(\nu-1)}(1-\delta_{ij}). \tag{3.4}$$

From this, it follows that the matrix norm $\|Q_\Lambda\| < 1/2$ when $|z|, |w_p| < r'' \leqq r'$ (independent of Λ) so that $(1+Q_\Lambda)^{-1}$ exists. In conclusion, $G_\Lambda^{(2)^{-1}} = (1+Q_\Lambda)^{-1} P_\Lambda^{-1}$ exists when $0 < |z|, |w_p| < r''$. We set $\Gamma_\Lambda^{(2)} \equiv -G_\Lambda^{(2)^{-1}}$ and $M_\Lambda \equiv (1+Q_\Lambda)^{-1}$. Note that M_Λ is analytic on $|z|, |w_p| < r''$ and $\|M_\Lambda\| \leqq 2$. Also, from Equation (3.4) we have

$$\frac{\partial^m}{\partial w_p^m} Q_\Lambda(i,j;\{w,z\})\bigg|_{w_p=0} = 0 \quad \text{if} \quad 0 \leqq m \leqq (4\nu-5) \quad \text{and} \quad i_1 < p \leqq j_1 . \tag{3.5}$$

Writing $l^2(\Lambda) = l^2(\Lambda'_p) + l^2(\Lambda''_p)$ where $\Lambda'_p = \{k \in \Lambda : k_1 < p\}$ and $\Lambda''_p = \{k \in \Lambda : k_1 \geqq p\}$ and looking at Q_Λ as an operator on $l^2(\Lambda)$, the above relations say that $\frac{\partial^m}{\partial m_p^m} Q_\Lambda\Big|_{w_p=0}$ $(0 \leqq m \leqq 4\nu-5)$ reduce the subspace $l^2(\Lambda''_p)$. Thus, the same is true for $M_\Lambda(\{w,z\})|_{w_p=0}$ and since by Leibnitz formula $(m \geqq 1)$

$$\frac{\partial^m}{\partial w_p^m} M_\Lambda = -\sum_{s=0}^{m-1} \binom{m}{s} \frac{\partial^s M_\Lambda \partial^{m-s} Q_\Lambda}{\partial w_p^s \partial w_p^{m-s}} M_\Lambda \tag{3.6}$$

it follows that

$$\frac{\partial^m}{\partial_{w_p^m}} M_\Lambda(i,j;\{w,z\})\bigg|_{w_p=0} = 0 \quad \text{if} \quad 0 \leqq m \leqq (4\nu-5) \quad \text{and} \quad i_1 < p \leqq j_1 . \tag{3.7}$$

The next theorem extends the above equation to one more derivative and is the main result of this work.

Theorem 3.4. *If $\nu \geqq 3$ and $|w_p|, |z| < r''$, then*

$$\frac{\partial^m}{\partial w_p^m} M_\Lambda(i,j;\{w,z\})\bigg|_{w_p=0} = 0 \quad \textit{if} \quad 0 \leqq m \leqq (4\nu-4) \quad \textit{and} \quad i_1+1 < p \leqq j_1 .$$

Notice that the statement above is dimension dependent. The proof (which is given in Appendix II) depends in a crucial way on the characterization of configurations in $(\nu-1)$ dimensional hyperplanes whose associated Peierls contours have minimal (non-trivial) hyperarea. When $\nu > 2$, these are precisely those having exactly one spin -1 and all the others equal to $+1$, Figure 3. When $\nu = 2$, a hyperplane is just a one-dimensional line and we can have many more possibilities. For example, all configurations in Figure 4 on following page have minimal "hyperarea".

Corollary 3.1. *If $\nu \geqq 3$, $|w_p|, |z| < r''$ and $i_1+2 \leqq j_1$, then*

$$|M_\Lambda(i,j;\{w,z\})| \leqq 2 \prod_{p=i_1+2}^{j_1} \left(\frac{|w_p|}{r''}\right)^{(4\nu-3)} .$$

Proof. Follows from Theorem 3.4 the maximum modulus theorem and the bound $|M_\Lambda(i,j;\{w,z\})| \leqq \|M_\Lambda\| \leqq 2$.

We now let all the variables $\{w,z\}$ be equal to $z > 0$. Combining Theorem 3.3 with Corollary 3.1, we get

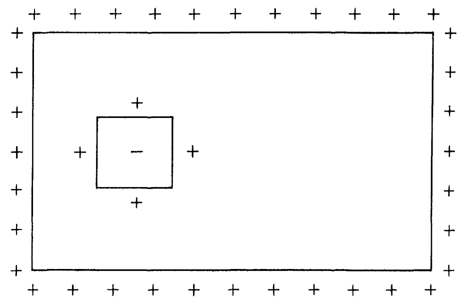

Fig. 3

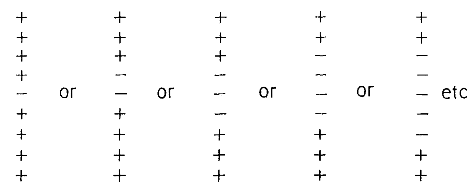

Fig. 4

Corollary 3.2. *If* $v \geqq 3$, $0 < z < r''$ *and* $i_1 + 2 \leqq j_1$, *then*

$$|\Gamma_{\Lambda}^{(2)}(i,j;z)| \leqq \frac{2}{(r'')^{(4v-3)[|i_1-j_1|-1]}} z^{(4v-3)|i_1-j_1|-(8v-3)}.$$

This is the required upper bound for $\Gamma_{\Lambda}^{(2)}$. It is not difficult to prove that $\lim_{|\Lambda| \uparrow \infty} \Gamma_{\Lambda}^{(2)}(i,j;z)$ exists and are the matrix elements of the operator inverse of $-G^{(2)}$ in $l^2(Z^v)$. Since the bound above is independent of Λ, it carries to the infinite volume limit. Using the notation of Section II namely a point in Z^v is written as $(x_1, \vec{x})$, then from Corollary 3.2 given $\varepsilon > 0$,

$$|\Gamma^{(2)}(x_1, \vec{x})| \leqq z^{[(4v-3)-\varepsilon]|x_1|} \tag{3.8}$$

for sufficiently large $|x_1|$ and small z (depending on ε). We next obtain a lower bound for $G^{(2)}(x_1, \vec{x} = \vec{0})$.

Lemma 3.1. *For any integer* $N \geqq 1$, *there exists* $r(N) < r''$ *(independent of* Λ*) such that*

$$G_{\Lambda}^{(2)}((i_1, 0, \ldots, 0), (j_1, 0, \ldots, 0); z) \geqq z^{4(v-1)N+(8v-4)}$$

whenever $0 < z < r(N)$ *and* $|i_1 - j_1| = N$.

Proof. Similar to Theorem 3.3. Consider

$$F_{\Lambda}(i,j;\{w,z\}) = G_{\Lambda}^{(2)}(i,j;\{w,z\}) / z^4 \prod_{p=i_1}^{j_1} w_p^{4(v-1)}$$

which is analytic and bounded by $C/r^{4(v-1)N+(8v-4)}$ on $|w_p|, |z| < r$. Since $F_{\Lambda}(i,j;\{w=0, z=0\}) = 4$, the result follows from Schwarz's lemma, as before.

Theorem 3.5. *For any integer* $N \geqq 1$,

$$G^{(2)}(x_1, \vec{x}=\vec{0}) \geqq z^{\left[4(v-1)+\frac{(8v-4)}{N}\right]|x_1|}$$

whenever $0<z<r(N)$ *and* $|x_1| \geqq N$.

Proof. Let $p=|x_1|/N$. From Section II, we have $G^{(2)}(N, \vec{x}=\vec{0}) = \int_0^{e^{-m}} \lambda^N d\sigma(\lambda)$ for a certain positive measure $d\sigma$ with total mass <1. Thus,

$$G^{(2)}(N, \vec{x}=\vec{0}) \leqq \left(\int_0^{e^{-m}} \lambda^{Np} d\sigma \right)^{\frac{1}{p}} \left(\int_0^{e^{-m}} d\sigma \right)^{\frac{1}{p'}} \quad (1/p + 1/p' = 1)$$

or

$$G^{(2)}(N, \vec{x}=\vec{0}) \leqq \left(\int_0^{e^{-m}} \lambda^{|x_1|} d\sigma \right)^{\frac{1}{p}} = G^{(2)}(x_1, \vec{0})^{\frac{1}{p}}.$$

Since the estimate in Theorem 3.5 is also true in the infinite volume limit, we have for $0<z<r(N)$

$$G^{(2)}(x_1, \vec{x}=\vec{0}) \geqq z^{[4(v-1)N+(8v-4)]p} = z^{\left[4(v-1)+\frac{(8v-4)}{N}\right]|x_1|}.$$

Corollary 3.3. *Assume* $v \geqq 3$ *and let* $\varepsilon > 0$ *and* $N > 0$ *be given. There exists* $r_0(\varepsilon, N)$ *and* $\alpha(\varepsilon, N)$ *such that if* $0<z<r_0(\varepsilon, N)$ *and* $|x_1| \geqq \alpha(\varepsilon, N)$, *then*

$$|\Gamma^{(2)}(x_1, \vec{x})| \leqq G^{(2)}(x_1, \vec{x}=0)^{\gamma(\varepsilon, N)}$$

where

$$\gamma(\varepsilon, N) = \frac{(4v-3)-\varepsilon}{(4v-4)+\dfrac{(8v-4)}{N}}.$$

Proof. Follows from Corollary 3.2 and Theorem 3.5. Theorem 2.4 is a simple consequence of this corollary.

Appendix I. Proof of Theorem 2.3

In this appendix, we shall give a proof of the following result:

$$\lim_{z \downarrow 0} \frac{w(\vec{p})}{m} = 1 \quad (\text{uniformly in } \vec{p}).$$

We start with a strengthening of Theorem 3.2 using the same notation

Theorem A1.1.

$$G_\Lambda^{(2)}(i,j;\{w,z\}) = z^{4[\|i-j\|+1]} \prod_{p=i_1}^{j_1} w_p^{4(v-1)} H_\Lambda(i,j;\{w,z\})$$

where H_Λ *is analytic on* $\{w_p\}$, $|z|<r$ *and*

$$\|i-j\| = \max_{2 \leqq \alpha \leqq v} \{|i_\alpha - j_\alpha|\}.$$

Proof. The argument goes as in Theorem 3.2 but now we consider also hyperplanes perpendicular to the direction given by $|i_\alpha - j_\alpha| = \|i-j\|$, which gives the contribution $z^{4[\|i-j\|+1]}$.

We let from now on $w_p = z$. By using Vitali's theorem (and Theorem 3.1) we can take the $\Lambda \uparrow \infty$ to obtain

$$G^{(2)}(x_1, \vec{x}; z) = z^{4(\nu-1)|x_1| + 4\|\vec{x}\| + 4\nu} H(x_1, \vec{x}; z)$$

with analytic $G^{(2)}$ and H. As was pointed out in Lemma 3.1 $H(x_1, \vec{x} = \vec{0}; z=0) = 4$. Let

$$\tilde{G}^{(2)}(x_1, \vec{p}; z) = \frac{1}{(2\pi)^{(\nu-1)/2}} \sum_{\vec{x}} G^{(2)}(x_1, \vec{x}; z) e^{-i\vec{p}\cdot\vec{x}}.$$

It is easy to see that this definition agrees with the one given in Section II when $z>0$. Note that $\tilde{G}^{(2)}(x_1, \vec{p}; z)$ is analytic for small $|z|$ and $|p_i|$. Writing

$$\tilde{G}^{(2)}(x_1, \vec{p}; z) = z^{4(\nu-1)|x_1| + 4\nu} H(x_1, \vec{x} = 0; z) + z^{4(\nu-1)|x_1| + 4\nu} \sum_{\vec{x} \neq 0} z^{4\|\vec{x}\|} H(x_1, \vec{x}; z) e^{-i\vec{p}\cdot\vec{x}}$$

it is clear that $\tilde{G}^{(2)}(x_1, \vec{p}; z)/z^{4(\nu-1)|x_1|+4\nu}$ is analytic for small $|z|$ (say $|z|<\eta$). Since also $|\tilde{G}^{(2)}(x_1, \vec{p}; z)| \leqq D$ in that region (for some $D>0$), *we have by the maximum modulus theorem*

$$|\tilde{G}^{(2)}(x_1, \vec{p}; z)| \leqq D \frac{|z|^{4(\nu-1)|x_1|+4\nu}}{\eta}$$

and therefore, $|\tilde{G}^{(2)}(x_1, \vec{p}; z)| \leqq |z|^{[4(\nu-1)-\varepsilon]|x_1|}$ for small $|z|$ and large $|x_1|$ (depending on ε).

We now obtain a lower bound for $\tilde{G}^{(2)}(x_1, \vec{p}; z)$. Since

$$\tilde{G}^{(2)}(x_1, \vec{p}; z)/z^{4(\nu-1)|x_1|+4\nu}|_{z=0} = 4,$$

there exists $\xi(x_1)$ (independent of $\vec{p}$) such that

$$|\tilde{G}^{(2)}(x_1, \vec{p}; z)| \geqq |z|^{4(\nu-1)|x_1|+4\nu} \quad \text{if} \quad |z| < \xi(x_1).$$

Now, when $z>0$, it was shown in Section II that $\tilde{G}^{(2)}$ has the integral representation

$$\tilde{G}^{(2)}(x_1, \vec{p}; z) = \int_0^{e^{-m}} \lambda^{|x_1|} d\varrho(\lambda; \vec{p})$$

where $d\varrho(\lambda; \vec{p})$ is a positive measure in λ for each fixed $\vec{p} \in T^{\nu-1}$, with a bounded (independent of $\vec{p}$) total mass. Using an argument similar to the one in Theorem 3.5, we conclude that given $N \geqq 1$

$$\tilde{G}^{(2)}(x_1, \vec{p}; z) \geqq z^{\left[4(\nu-1) + \frac{4\nu}{N}\right]|x_1|}$$

when z is small and $|x_1|$ is large (depending on N).

Therefore,

$$\frac{\tilde{G}^{(2)}(x_1, \vec{p}; z)}{\tilde{G}^{(2)}(x_1, \vec{0}; z)} \geqq z^{\left[\frac{4\nu}{N} + \varepsilon\right]|x_1|}$$

and

$$w(\vec{p})-m=-\lim_{|x_1|\uparrow\infty}\frac{1}{|x_1|}\log\frac{\tilde{G}^{(2)}(x_1,\vec{p};z)}{\tilde{G}^{(2)}(x_1,\vec{0};z)}\leqq-\left(\frac{4v}{N}+\varepsilon\right)\log z.$$

On the other hand, since

$$\log\tilde{G}^{(2)}(x_1,\vec{p}=\vec{0},z)\leqq[4(v-1)-\varepsilon]|x_1|\log z$$

we see that

$$m\geqq-[4(v-1)-\varepsilon]\log z$$

and the conclusion is

$$\frac{w(\vec{p})}{m}-1\leqq\frac{\left(\frac{4v}{N}+\varepsilon\right)}{4(v-1)-\varepsilon}$$

which completes the proof of the theorem. Notice that from the bounds above, it also follows that

$$\lim_{z\downarrow 0}\frac{m}{-4(v-1)\log z}=1.$$

Appendix II. Proof of Theorem 3.4

In this appendix, we prove Theorem 3.4 for the case $m=(4v-4)$; the other cases are covered by Equation (3.7). The proof will follow from the

Lemma. *Assume* $|z|,|w_p|<r''$, $z\neq 0$ *and* $v\geqq 3$. *Then* (a) *If* $i_1<p<j_1$,

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}}G_A^{(2)}(i,j;\{w,z\})\Big|_{w_p=0}=(4v-4)!\frac{1}{4}(z^2-z^{-2})^2$$
$$\cdot\sum_{l_1=p}G_A^{(2)}(i,l-\hat{e}_1;\{w,z\})\Big|_{w_p=0}\;G_A^{(2)}(l+\hat{e}_1,j;\{w,z\})\Big|_{w_p=0}$$

where the summation is over all $l\in A$ *such that* $l_1=p$ *and* $l-\hat{e}_1=(l_1-1,l_2,\dots,l_v)$, $l+\hat{e}_1=(l_1+1,l_2,\dots,l_v)$.

(b) *If* $i_1<p=j_1$,

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}}G_A^{(2)}(i,j;\{w,z\})\Big|_{w_p=0}=-(4v-4)!(z^2-z^{-2})^2$$
$$\cdot G_A^{(2)}(i,j-\hat{e}_1;\{w,z\})\Big|_{w_p=0}\;\langle z^{2\sigma_{j+\hat{e}}}\rangle_{A,\{w,z\}}\Big|_{w_p=0}.$$

Assuming the lemma, let us prove Theorem 3.4. Thus, suppose $i_1+1<p\leqq j_1$, $v\geqq 3$ and for the moment, that $z\neq 0$. From Equations (3.5)–(3.7),

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}}M_A(i,j;\{w,z\})\Big|_{w_p=0}=-\sum_{\substack{k_1<p\\ l_1\geqq p}}M_A(i,k;\{w,z\})\Big|_{w_p=0}$$
$$\cdot\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}}Q_A(k,l;\{w,z\})\Big|_{w_p=0}\;M_A(l,j;\{w,z\})\Big|_{w_p=0}. \tag{A.1}$$

From now on, we simplify the notation by writing $M_A(i,j)$ instead of $M_A(i,j;\{w,z\})$ and $M_A(i,j;w_p=0)$ instead of $M_A(i,j;\{w,z\})|w_p=0$. Let V_A be the matrix $V_A=I+Q_A$. Thus, $M_A=V_A^{-1}$ and since $G_A^{(2)}=P_A V_A$, $G_A^{(2)}(k,l)=G_A^{(2)}(k,k)V_A(k,l)$. From Equation (3.5) and Leibnitz formula,

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} G_A^{(2)}(k,l;w_p=0)=G_A^{(2)}(k,k;w_p=0)\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} V_A(k,l;w_p=0)$$

if $k_1<p\leqq l_1$. Applying the lemma, we have

(a) If $k_1<p<l_1$:

$$G_A^{(2)}(k,k;w_p=0)\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} V_A(k,l;w_p=0)=(4v-4)!\frac{1}{4}(z^2-z^{-2})^2$$
$$\cdot\sum_{m_1=p} G_A^{(2)}(k,m-\hat{e}_1;w_p=0)G_A^{(2)}(m+\hat{e}_1,l;w_p=0).$$

Now, assume for the meoment that $w_{k_1}\neq 0$. Then, by Theorem 3.3 $G_A^{(2)}(k,k;w_p=0)\neq 0$ and we can divide both members of the above expression by this term:

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} V_A(k,l,w_p=0)=(4v-4)!\frac{1}{4}(z^2-z^{-2})^2$$
$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} V_A(k,l,w_p=0)=(4v-4)!\frac{1}{4}(z^2-z^{-2})^2$$
$$\cdot\sum_{m_1=p} V_A(k,m-\hat{e}_1;w_p=0)G_A^{(2)}(m+\hat{e}_1,l;w_p=0). \tag{A.2}$$

Since both members are analytic in $|w_{k_1}|<r''$, this formula holds even when $w_{k_1}=0$. In a similar way, we have

(b) If $k_1<p=l_1$

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} V_A(k,l;w_p=0)=-(4v-4)!(z^2-z^{-2})^2$$
$$\cdot V_A(k,l-\hat{e}_1;w_p=0)\langle z^{2\sigma_{l+\hat{e}}}\rangle_{A;w_p=0}. \tag{A.3}$$

Noting that in Equation (A.1) the derivative of $Q_A(k,l)$ is the same as the derivative of $V_A(k,l)$ we have using Equations (A.2) and (A.3)

$$\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} M_A(i,j;w_p=0)=-(4v-4)!\frac{1}{4}(z^2-z^{-2})^2\cdot\sum_{\substack{k_1<p\\ l_1>p}}\sum_{m_1=p}$$
$$\cdot M_A(i,k;w_p=0)V_A(k,m-\hat{e}_1;w_p=0)G_A^{(2)}(m+\hat{e}_1,l;w_p=0)$$
$$\cdot M_A(l,j;w_p=0)+(4v-4)!(z^2-z^{-2})^2\sum_{\substack{k_1<p\\ l_1=p}} M_A(i,k;w_p=0)$$
$$\cdot V_A(k,l-\hat{e}_1;w_p=0)M_A(l,j;w_p=0)\langle z^{2\sigma_{l+\hat{e}}}\rangle_{A;w_p=0}. \tag{A.4}$$

Now, in the first sum,

$$\sum_{k_1<p} M_A(i,k;w_p=0)V_A(k,m-\hat{e}_1;w_p=0)$$
$$=\sum_{k\in A} M_A(i,k;w_p=0)V_A(k,m-\hat{e}_1;w_p=0)=\delta(i,m-\hat{e}_1)=0$$

because $i_1<m_1-1$ $(=p-1)$. The first equality above follows from Equation (3.7). Thus, the first sum in (A.4) is zero. The second sum is also zero because if $l_1=p$,

$$\sum_{k_1<p} M_A(i,k;w_p=0)V_A(k,l-\hat{e}_1;w_p=0)=\delta(i,l-\hat{e}_1)=0$$

and we conclude that $\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}} M_A(i,j;w_p=0)=0$. Since M_A is analytic at $z=0$, the above result (which was derived assuming $z\neq 0$) is also valid for $z=0$. The proof of Theorem 3.4 is complete.

We turn to the

Proof of the Lemma. As in Theorem 3.2, we express $G_A^{(2)}$ in terms of duplicate variables:

$$G_A^{(2)}(i,j)=\frac{1}{2Z^2_{A,\{w,z\}}}\sum_{\vec{\sigma},\vec{\sigma}'}(\sigma_i-\sigma_i')(\sigma_j-\sigma_j')\prod_{q=-n}^{n}\prod_{\substack{|k-l|=1\\ k_1=l_1=q}} \cdot w_q^{(2-\sigma_k\sigma_l-\sigma_k'\sigma_l')}\prod_{\substack{|k-l|=1\\ k_1\neq l_1}} z^{(2-\sigma_k\sigma_l-\sigma_k'\sigma_l')} \tag{A.5}$$

and look at this as a function of w_p, $i_1<p\leqq j_1$, all the other variables being held fixed. Thus, $G_A^{(2)}(i,j)$ is a ratio of two polynomials in w_p, and from Theorem 3.2, it has the form

$$G_A^{(2)}(i,j)=\frac{1}{2}\frac{Aw_p^{4(v-1)}+(\text{higher powers of } w_p)}{B+(\text{higher powers of } w_p)}$$

and therefore,

$$\frac{1}{(4v-4)!}\frac{4(v-1)}{w_p^{4(v-1)}}G_A^{(2)}(i,j;w_p=0)=\frac{1}{2}\frac{A}{B}. \tag{A.6}$$

B is easily calculated: $B=Z^2_{A,w_p=0}$. To calculate A, we have to isolate all terms in the numerator of (A.5) having $w_p^{4(v-1)}$ as a factor. If $v\geqq 3$, this is easily done. Indeed, in this case, one of the configurations (say $\vec{\sigma}'$) must be trivial on the hyperplane $k_1=p$ and the other configuration must have exactly one spin in $k_1=p$ pointing down, all other spins in $k_1=p$ pointing up. For then the exponent of w_p—which is $2\cdot$(total hyperarea of the contours in $k_1=p$ associated with $\vec{\sigma}$)$+2\cdot$(total hyperarea in $k_1=p$ of $\vec{\sigma}'$)—is equal to $2\cdot 2(v-1)+2\cdot 0=4(v-1)$. And it is clear that any $(\vec{\sigma},\vec{\sigma}')$ contributing $4(v-1)$ to the exponent of w_p must be of the above form. Note that this last remark is false when $v=2$. In this case, again one of the configurations must be trivial on $k_1=p$ but the character of the other configuration on $k_1=p$ need not be restricted as before. In fact, when $v=2$, the hyperplane $k_1=p$ is one dimensional and each of the configurations below contributes the same amount to the exponent of w_p.

$$\begin{array}{cccc} + & + & + & + \\ + & + & + & + \\ + & - & - & - \\ - & - & - & - \\ + & + & - & + \\ + & + & + & + \\ + & + & + & + \end{array} \quad \ldots \text{ etc.}$$

From now on, we assume $v \geqq 3$. Let

$$F^{(p)}(\vec{\sigma}, \vec{\sigma}') = \prod_{\substack{q=-n \\ q \neq p}}^{n} \prod_{\substack{|k-l|=1 \\ k_1=l_1=q}} w_q^{(2-\sigma_k\sigma_l-\sigma_k'\sigma_l')} \prod_{\substack{|k-l|=1 \\ k_1 \neq l_1}} z^{(2-\sigma_k\sigma_l-\sigma_k'\sigma_l')}.$$

Let $\Phi(p)$ be the set of all configurations $\vec{\sigma}$ in Λ which are trivial on the hyperplane $k_1=p$. With this notation

$$B = Z^2_{\Lambda, w_p=0} = \sum_{\vec{\sigma}, \vec{\sigma}' \in \Phi(p)} F^{(p)}(\vec{\sigma}, \vec{\sigma}').$$

Let $\Theta_l(p)$ be the set of configurations $\vec{\sigma}$ in Λ such that $\sigma_k = +1 \forall k \neq l$ with $k_1 = p$, and $\sigma_l = -1$ (the point l is assumed to be in the same hyperplane: $l_1 = p$). From the observations above,

$$A = 2 \sum_{l_1=p} \sum_{\substack{\vec{\sigma} \in \Theta_l(p) \\ \vec{\sigma}' \in \Phi(p)}} (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') F^{(p)}(\vec{\sigma}, \vec{\sigma}'). \tag{A.7}$$

The factor of 2 comes from the interchange $(\vec{\sigma}, \vec{\sigma}') \leftrightarrow (\vec{\sigma}', \vec{\sigma})$. The sets $\Theta_l(p)$ and $\Phi(p)$ are in one-to-one correspondence: to each $\vec{\sigma} \in \Phi(p)$ we associate $\vec{s}^l \in \Theta_l(p)$ obtained by flipping the spin $\vec{\sigma}_l$. If $i_1 < p < j_1$, then we can write Equation (A.7) as

$$A = 2 \sum_{l_1=p} \sum_{\vec{\sigma}, \vec{\sigma}' \in \Phi(p)} (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') F^{(p)}(\vec{s}^l, \vec{\sigma}'). \tag{A.8}$$

But clearly,

$$F^{(p)}(\vec{s}^l, \vec{\sigma}') = F^{(p)}(\vec{\sigma}, \vec{\sigma}') z^{2(\sigma_{l+\hat{e}_1} + \sigma_{l-\hat{e}_1})}; \; (\sigma_{l=+1})$$

and therefore we see that

$$\frac{1}{2} \frac{A}{B} = \sum_{l_1=p} \langle\!\langle (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') z^{2(\sigma_{l+\hat{e}_1} + \sigma_{l-\hat{e}_1})} \rangle\!\rangle_{\Lambda, w_p=0}. \tag{A.9}$$

The effect of setting $w_p=0$ is to decouple the region to the left of $k_1=p$ from the region to the right of that hyperplane. Thus,

$$\begin{aligned} &\langle\!\langle (\sigma_i - \sigma_i')(\sigma_j - \sigma_j') z^{2(\sigma_{l+\hat{e}_1} + \sigma_{l-\hat{e}_1})} \rangle\!\rangle_{\Lambda, w_p=0} \\ &= \langle\!\langle (\sigma_i - \sigma_i') z^{2\sigma_{l-\hat{e}_1}} \rangle\!\rangle_{\Lambda, w_p=0} \langle\!\langle (\sigma_j - \sigma_j') z^{2\sigma_{l+\hat{e}_1}} \rangle\!\rangle_{\Lambda, w_p=0}. \end{aligned}$$

Now, clearly

$$z^{2\sigma_{l-\hat{e}_1}}=\frac{1}{2}(z^2+z^{-2})+\frac{\sigma_{l-e_1}}{2}(z^2-z^{-2})$$

so that

$$\langle\langle(\sigma_i-\sigma_i')z^{2\sigma_{l-\hat{e}_1}}\rangle\rangle_{\Lambda,w_p=0}=\frac{1}{2}(z^2-z^{-2})\langle\langle(\sigma_i-\sigma_i')\sigma_{l-e_1}\rangle\rangle_{\Lambda,w_p=0}$$

$$=\frac{1}{2}(z^2-z^{-2})[\langle\sigma_i\sigma_{l-\hat{e}_1}\rangle_{\Lambda,w_p=0}-\langle\sigma_i\rangle_{\Lambda,w_p=0}\langle\sigma_{l-\hat{e}_1}\rangle_{\Lambda,w_p=0}]$$

$$=\frac{1}{2}(z^2-z^{-2})G_\Lambda^{(2)}(i,l-\hat{e}_1;w_p=0).$$

We have used the facts $\langle\langle(\sigma_i-\sigma_i')\rangle\rangle_{\Lambda,w_p=0}=\langle\sigma_i\rangle_{\Lambda,w_p=0}-\langle\sigma_i\rangle_{\Lambda,w_p=0}$ and $\langle\langle\sigma_i'\sigma_{l-\hat{e}_1}\rangle\rangle_{\Lambda,w_p=0}=\langle\sigma_i\rangle_{\Lambda,w_p=0}\langle\sigma_{l-\hat{e}_1}\rangle_{\Lambda,w_p=0}$. Similarly

$$\langle\langle(\sigma_j-\sigma_j')z^{2\sigma_{l+e}}\rangle\rangle_{\Lambda,w_p=0}=\frac{1}{2}(z^2-z^{-2})G_\Lambda^{(2)}(l+\hat{e}_1,j;w_p=0).$$

Taking these results into Equation (A.9), we therefore conclude that when $i_1<p<j_1$,

$$\frac{1}{(4v-4)!}\frac{\partial^{4(v-1)}}{\partial w_p^{4(v-1)}}G^{(2)}(i,j;w_p=0)=\frac{1}{2}(z^2-z^{-2})^2$$

$$\cdot\sum_{l_1=p}G_\Lambda^{(2)}(i,l-\hat{e}_1;w_p=0)G_\Lambda^{(2)}(l+\hat{e}_1,j;w_p=0)$$

which proves Part (a) of the lemma. To prove Part (b), assume $i_1<p=j_1$. Then from Equation (A.7)

$$A=-4\sum_{\substack{\vec{\sigma}\in\Theta_j(p)\\ \vec{\sigma}'\in\Phi(p)}}(\sigma_i-\sigma_i')F^{(p)}(\vec{\sigma},\vec{\sigma}')=-4\sum_{\vec{\sigma},\vec{\sigma}'\in\Phi(p)}(\sigma_i-\sigma_i')F^{(p)}(\vec{s}^j,\vec{\sigma}')$$

or

$$A=-4\sum_{\vec{\sigma},\vec{\sigma}'\in\Phi(p)}(\sigma_i-\sigma_i')z^{2(\sigma_{j-\hat{e}_1}+\sigma_{j+\hat{e}_1})}F^{(p)}(\vec{\sigma},\vec{\sigma}').$$

Thus,

$$\frac{1}{2}\frac{A}{B}=-2\langle\langle(\sigma_i-\sigma_i')z^{2(\sigma_{j-\hat{e}_1}+\sigma_{j+\hat{e}_1})}\rangle\rangle_{\Lambda,w_p=0}$$

$$=-2\langle\langle(\sigma_i-\sigma_i')z^{2\sigma_{j-\hat{e}_1}}\rangle\rangle_{\Lambda,w_p=0}\langle\langle z^{2\sigma_{j+\hat{e}_1}}\rangle\rangle_{\Lambda,w_p=0}$$

$$=-(z^2-z^{-2})G_\Lambda^{(2)}(i,j-\hat{e}_1;w_p=0)\langle z^{2\sigma_{j+\hat{e}_1}}\rangle_{\Lambda,w_p=0}$$

and the proof is complete.

Acknowledgements. I would like to thank Thomas Spencer for his patience, insights and constant encouragement during the course of this work. I would also like to thank James Glimm for several useful conversations.

References

1. Burnap, C.A.: Thesis, Harvard University (1976)
2. Brascamp, H.J., Kunz, H.: Commun. math. Phys. **32**, 93 (1973)
3. Camp, W.J.: Phys. Rev. B **6**, 960 (1972); **7**, 3187 (1973)
4. Camp, W.J., Fisher, M.E.: Phys. Rev. B **6**, 946 (1972)
5. Donoghue, Jr., W.: Monotone matrix functions and analytic continuation. Berlin-Heidelberg-New York: Springer 1974
6. Glimm, J., Jaffe, A.: Functional integral methods in quantum field theory. To be published in the *1976 Cargese lectures volume*
7. Glimm, J., Jaffe, A., Spencer, T.: The particle structure of the weakly coupled $P(\phi)_2$ model. In: Constructive quantum field theory. Lecture notes in physics, Vol. 25. Berlin-Heidelberg-New York: Springer 1973
8. Gunning, R.C., Rossi, H.: Analytic functions of several complex variables. New Jersey: Prentice-Hall, Inc. 1965
9. Minlos, R.A., Sinai, Ja.G.: Trans. Moscow Math. Soc. **17**, 237 (1967); **19**, 121 (1968)
10. Osterwalder, K., Schrader, R.: Commun. math. Phys. **31**, 83 (1973); **42**, 281 (1975)
11. Paes-Leme, P.J.: Thesis, New York University (1977)
12. Ruelle, D.: Statistical mechanics. New York: Benjamin 1969
13. Spencer, T.: Commun. math. Phys. **44**, 143 (1975)
14. Schor, R.: Thesis, New York University (1977)
15. Spencer, T., Zirilli, F.: Commun. math. Phys. **49**, 1 (1976)

Communicated by A. Jaffe

Received October 31, 1977; in revised form November 28, 1977