Commun. math. Phys. 59, 167—196 (1978)

Communications in Mathematical Physics

Perturbations of Flows on Banach Spaces and Operator Algebras* **

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Abstract. For automorphism groups of operator algebras we show how properties of the difference $\|\alpha_t-\alpha'_t\|$ are reflected in relations between the generators δ_α, δ'_α. Indeed for a von Neumann algebra $\mathcal{M}$, with separable predual we show that if $\|\alpha_t-\alpha'_t\|\leqq 0.28$ for small t, then $\delta_\alpha=\gamma\circ(\delta'_\alpha+\delta)\circ\gamma^{-1}$ where γ is an inner automorphism of $\mathcal{M}$ and δ is a bounded derivation of $\mathcal{M}$. If the difference $\|\alpha_t-\alpha'_t\|=O(t)$ as $t\to 0$, then $\delta_\alpha=\delta'_\alpha+\delta$ and if $\|\alpha_t-\alpha'_t\|\leqq 0.28$ for all t then $\delta_\alpha=\gamma\circ\delta'_\alpha\circ\gamma^{-1}$. We prove analogous results for unitary groups on a Hilbert space and C_0, C_0^* groups on a Banach space.

§ 1. Introduction

Questions of perturbations of dynamics have received considerable attention from various points of view. In this paper we will consider one parameter groups of operators acting on dual Banach spaces and one parameter groups of *-automorphism of operator algebras. Recent considerations have centered on studying the behaviour of the difference of two dynamics as the parameter t goes to zero. Bucholz and Roberts [5] consider the situation where $\|\alpha_t-\alpha'_t\|\to 0$ as $t\to 0$, where α_t, α'_t are *-automorphism groups of a simple C^*-algebra or a von Neumann algebra. Among other things they show that the generator of α_t is related to that of α'_t by twisting and then adding a bounded perturbation. In [14] Robinson considers the question of the proximity of C_0 semi-groups of operators, U_t, V_t on a Banach space and characterizes $\|U_t-V_t\|=O(t^\alpha)$, $0<\alpha\leqq 1$. One also sees in [14] that the pointwise behaviour of U_t-V_t, i.e. the behaviour of $\|(U_t-V_t)(x)\|$ for x belonging to the domain of the generator of V_t is worth studying. Other aspects of perturbation of dynamics has been studied in [20]; and in [21] (concluding remarks).

For automorphism groups of von Neumann algebras our results fall into three categories. We consider the cases where $\|\alpha_t-\alpha'_t\|$ is, small for t small, small for all t,

* This paper subsumes an earlier work of the same title which appeared as a report from Z.I.F. der Universität Bielefeld

** With partial support of the U.S. National Science Foundation

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0010-3616/78/0059/0167/$06.00

or $O(t)$ as $t \to 0$. Results of Kadison and Ringrose suggest consideration of the first case. We deal with the first two for von Neumann algebras with separable pre-dual as we are in need of Borel cross-section theorems. The last two situations are in a sense part of the first. More precisely; if $\|\alpha_t - \alpha'_t\|$ is small for t small then (Theorem 3.6) $\delta_\alpha = \gamma \circ (\delta'_\alpha + \delta) \circ \gamma^{-1}$ where δ_α, δ'_α are the generators of α_t, α'_t respectively, δ is a bounded derivation and γ is an inner automorphism of the von Neumann algebra. Note that this is the same form as the main result of [5], where it is assumed that that the difference $\|\alpha_t - \alpha'_t\|$ goes to zero or $t \to 0$. The difference in these two cases is precisely the continuity of the orbit of the unitary giving γ, under α_t. The last two cases represent "parts" of the one just described. If the automorphisms are close (enough) for all t then $\delta_\alpha = \gamma \circ \delta_{\alpha'} \circ \gamma^{-1}$; (Theorem 3.5) if the difference $\|\alpha_t - \alpha'_t\|$ is $O(t)$ as $t \to 0$ then $\delta_\alpha = \delta_{\alpha'} + \delta$ (Theorem 3.1). Here γ is again an inner automorphism and δ a bounded derivation. Theorem 3.5 generalizes a result of Reynolds [26].

For von Neumann algebras in standard form $O(t)$ behaviour yields an explicit relation for the generators of the canonical unitaries. Indeed this shows that Araki's [3] perturbation of the modular automorphism group is characterized by $O(t)$ behaviour of the difference. One might incline to the view that $O(t)$ behaviour for automorphism groups of simple C^*-algebras would yield the same result as in the von Neumann algebra case. This however fails totally, as examples show. In some cases, e.g. quasi-free automorphisms of the Clifford algebra (Theorem 4.3) the result is the same as for von Neumann algebras.

The general results for automorphism groups have analogs for unitary groups in Hilbert space (indeed we need these for the former). These results are discussed in Section 2.

Finally we deal with examples which show that the above situations arise. One of the examples shows that it is possible to have $\|\alpha_t - \alpha'_t\| = a$ for all $t \in \mathbf{R} \setminus \{0\}$ where $0 \leqq a \leqq 2$.

§ 2. Banach and Hilbert Space Theory

In this section we deal with unitary groups and C_0 or C_0^* groups. The results we obtain for unitary groups will be used in our treatment of automorphism groups of von Neumann algebras in Section 3.

Recall [6] that a C_0-semigroup, U_t, of operators on a Banach space X is a homomorphism from $\mathbf{R}$ into the bounded operators on X such that $t \to U_t x$ is continuous and $U_0 = I$ (the identity operator). A C_0^*-group is a dual group of a C_0-group.

Theorem 2.1. *Let U, V be two C_0 or C_0^* groups, operating on a Banach space X, with generators S, T respectively and let I denote this identity operator.*

The following conditions are equivalent:

1. *There exists ε_1, $0 < \varepsilon_1 < 1$ and $\delta_1 > 0$ such that*

$$\|U_t V_{-t} - I\| \leqq 1 - \varepsilon_1$$

for $0 \leqq t \leqq \delta_1$.

2. *There exists ε_2, $0 < \varepsilon_2 < 1$, $\delta_2 > 0$, a bounded operator $P : \mathscr{D}(T) \to X$ and a bounded operator $\Omega : X \to X$ with bounded inverse, such that*

$$S = \Omega (T + P) \Omega^{-1}$$

and

$$\|U_t\Omega^{-1}U_{-t}\Omega-I\|\leqq 1-\varepsilon_2\,,\qquad 0\leqq t\leqq\delta_2\,.$$

Under these conditions Ω *may be defined by* $\Omega=\frac{1}{\delta_1}\int_0^{\delta_1}dsU_sV_{-s}$. *One has*

$$\|I-\Omega\|\leqq 1-\varepsilon_1\,,$$
$$\|U_t\Omega^{-1}U_{-t}\Omega-I\|\leqq\|U_tV_{-t}-I\|+O(t)\,,\quad as\quad t\to 0$$
$$\|U_t\Omega V_{-t}-\Omega\|=O(t)\quad as\quad t\to 0\,.$$

Proof. 1.$\Rightarrow$2.

Define

$$\Omega=\frac{1}{\delta_1}\int_0^{\delta_1}dsU_sV_{-s}\,.$$

It follows that $\|I-\Omega\|\leqq 1-\varepsilon_1$ and so Ω is a bounded operator with bounded inverse. Introduce χ_t by

$$\chi_t=\Omega^{-1}U_t\Omega V_t\,.$$

As in [14]

$$\frac{\chi_{t+h}-\chi_t}{h}=\frac{1}{\delta_1 h}\Omega^{-1}\int_{\delta_1}^{\delta_1+h}dsU_{s+t}V_{-s-t}-\frac{1}{\delta_1 h}\Omega^{-1}\int_0^h dsU_{s+s}V_{-s-t}\,.$$

This implies that χ_t is strongly differentiable in the C_0-case and weak*-differentiable in the C_0^*-case. The derivative is given by

$$\frac{d\chi_t}{dt}=\Omega^{-1}U_t(U_{\delta_1}V_{-\delta}-I)V_{-t}/\delta_1\,.$$

For $x\in X$ we have

$$\frac{(U_t-I)\Omega x}{t}=\frac{\Omega(V_t-I)x}{t}+\frac{\Omega(\chi_t-I)V_tx}{t}\,,$$

so that if $x\in\mathscr{D}(T)$ the right hand side converges, showing that $\Omega x\in\mathscr{D}(S)$ and

$$S\Omega x=\Omega Tx+\Omega Px\,.$$

Here

$$P=\left.\frac{d\chi_t}{dt}\right|_{t=0}=\Omega^{-1}(U_{\delta_1}V_{-\delta_1}-I)/\delta_1\,.$$

Similarly if $x\in\mathscr{D}(S)$ then $\Omega^{-1}x\in\mathscr{D}(T)$ and

$$\Omega^{-1}Sx=T\Omega^{-1}x+P\Omega^{-1}x\,.$$

We thus conclude that

$$\mathscr{D}(S)=\Omega\mathscr{D}(T)\quad\text{and}\quad S=\Omega(T+P)\Omega^{-1}\,.$$

Finally one computes that

$$U_t\Omega^{-1}U_{-t}\Omega-I=(U_tV_{-t}-I)(V_t\Omega^{-1}U_{-t}\Omega-I)$$
$$+\frac{1}{\delta_1}\Omega^{-1}\left[\int_0^t dsU_s(I-U_{\delta_1}V_{-\delta_1})V_{-s}\right]V_t\Omega^{-1}U_{-t}\Omega$$
$$+(U_tV_{-t}-I)\,.$$

The existence of ε_2, δ_2 is now clear. If one notes that $\Omega^{-1}U_t\Omega$ is the group of operators with generator $T+P$, perturbation theory immediately yields that

$$(*)\quad \|V_t\Omega^{-1}U_{-t}\Omega-I\|=O(t)\,,\qquad t\to 0$$

and thus

$$(**)\quad \|U_t\Omega^{-1}U_{-t}\Omega-I\|\leqq\|U_tV_{-t}-I\|+O(t)\,,$$

yielding the final statements of the theorem.

2.⇒1.

Define $Q=-\Omega P\Omega^{-1}$. Thus $T=\Omega^{-1}(S+Q)\Omega$. If $\hat{U}$ is the group generated by $S+Q$,

$$U_tV_{-t}-I=U_t\Omega^{-1}U_{-t}(U_t\hat{U}_{-t}-I)\Omega+(U_t\Omega^{-1}U_{-t}\Omega-I)\,.$$

Perturbation theory gives $\|U_t\hat{U}_{-t}-I\|=O(t)$, as $t\to 0$ and therefore

$$\|U_tV_{-t}-I\|\leqq\|U_t\Omega^{-1}U_{-t}\Omega-I\|+O(t)\,,\qquad t\to 0\,,$$

completing the proof.

For unitary groups on a Hilbert space Theorem 2.1 can be improved.

Theorem 2.2. *Let $U_t=\exp itH$, $V_t=\exp itK$ be strongly continuous unitary groups on a Hilbert space.*

The following conditions are equivalent:

1. *There exists ε_1, $0<\varepsilon_1<\sqrt{2}$ and $\delta_1>0$ such that*

$$\|U_t-V_t\|\leqq\sqrt{2}-\varepsilon\,,\qquad 0\leqq t\leqq\delta_1\,.$$

2. *There exists ε_2, $0<\varepsilon_2<\sqrt{2}$, $\delta_2>0$, a bounded self-adjoint operator P and a unitary operator W such that*

$$H=W(K+P)W^*$$
$$\|U_tW^*U_{-t}W-I\|\leqq\sqrt{2}-\varepsilon_2\,,\qquad 0\leqq t\leqq\delta_2\,.$$

If these conditions are satisfied, then W can be chosen as the unitary operator occuring in the polar decomposition of the invertible operator

$$\Omega=\frac{1}{\delta_1}\int_0^{\delta_1}dtU_t-V_{-t}\,.$$

Moreover $\|W-I\|\leqq\sqrt{2}-\varepsilon_1$.

Proof. The proof that 1.⇒2. relies on the following lemma.

Lemma 2.3 (U. Haggerup). *Let $\mathscr{U}$ be a collection of unitaries on a Hilbert space $\mathscr{H}$ and assume there exists an $\varepsilon>0$ such that if $U\in\mathscr{U}$ then $\|U-I\|<\sqrt{2}-\varepsilon$. If Ω is an operator in the σ-weakly closed convex hull of $\mathscr{U}$ then Ω is invertible with bounded inverse.*

Proof. For any bounded operator A on $\mathscr{H}$, let $W(A)=\{(A\psi,\psi);\ \|\psi\|=1\}$ be its numerical range. Then for any $U\in\mathscr{U}$, $W(U)$ is contained in the convex set

$$\left\{z\in C;|z|\leqq 1, \operatorname{Re} z\geqq \sqrt{2}\varepsilon-\frac{\varepsilon^2}{2}\right\}.$$

Thus $W(\Omega)$ is contained in this set. As $\operatorname{Sp}(\Omega)\subseteq W(\Omega)$, [32], it follows that Ω is invertible.

Now we return to the proof of the theorem. Consider the operator

$$\Omega=\frac{1}{\delta_1}\int_0^{\delta_1} ds U_s V_{-s}.$$

It follows from the assumptions on $\|U_tV_{-t}-I\|$, and the above lemma, that Ω is invertible. The calculations of Theorem 2.1 show that

$$\|U_t\Omega V_{-t}-\Omega\|=O(t)=\|V_t\Omega^*U_{-t}-\Omega^*\|.$$

Thus

$$\begin{aligned}\|V_t\Omega^*\Omega V_{-t}-\Omega^*\|&=\|(V_t\Omega^*U_{-t})(U_t\Omega V_t)-\Omega^*\Omega\|\\ &\leqq\|V_t\Omega^*U_{-t}-\Omega^*\|\,\|U_t\Omega V_{-t}\|\\ &\quad+\|U_t\Omega V_{-t}-\Omega\|\,\|\Omega^*\|=O(t)\quad\text{as}\quad t\to 0.\end{aligned}$$

This implies [12] that $|\Omega|^2$ lies in the domain of the derivation generating the automorphism group of $\mathscr{L}(\mathscr{H})$, implemented by V_t. From [21], we see that $|\Omega|^{-1}=(|\Omega|^2)^{-1/2}$ lies in the domain of that derivation and hence

$$\|V_t|\Omega|^{-1}V_{-t}-|\Omega|^{-1}\|=O(t),\quad\text{as}\quad t\to 0.$$

Taking W to be the unitary occuring in the polar decomposition of Ω we have

$$\begin{aligned}\|U_tWV_{-t}-W\|&=\|U_t\Omega V_{-t}V_t|\Omega|^{-1}V_{-t}-\Omega|\Omega|^{-1}\|\\ &\leqq\|U_t\Omega V_{-t}-\Omega\|\,\||\Omega|^{-1}\|\\ &\quad+\|\Omega\|\,\|V_t|\Omega|^{-1}V_{-t}-|\Omega|^{-1}\|\\ &=O(t).\end{aligned}$$

Introducing $\hat{V}_t=WV_tW^*$ and noting that $\|U_t-\hat{V}_t\|=O(t)$, we have that ([14, Corollary 3]) the generators of U_t and $\hat{V}_t$ differ by a bounded self-adjoint operator. This establishes the relation between H and K. Further we see that

$$\|U_tW^*U_{-t}-W^*\|=\|U_tWU_{-t}-W\|$$

and

$$\|U_tWU_{-t}-W\|\leqq\|(U_tW_{-t}-W)V_tU_{-t}\|+\|W(V_tU_{-t}-I)\|.$$

So by the above calculation the first term is $O(t)$ as $t \to 0$ and the second is smaller than $\sqrt{2} - \varepsilon_1$ for t small. Thus δ_2, and ε_2, exist.

2. $\Rightarrow$ 1. proceeds as in the proof of Theorem 2.1.

We may now obtain our estimate in $\|W - I\|$ by appealing to a result of Woronowicz [33]. As noted Ω has numerical range in

$$\left\{ z \in \mathbb{C},\ |z| \leqq 1,\ \operatorname{Re} z \geqq \sqrt{2}\varepsilon_1 - \frac{\varepsilon_1}{2} \right\}.$$

However Woronowicz's result shows that the unitary W has its spectrum there.

Hence $\|W - I\| \leqq \sqrt{2} - \varepsilon_1$.

The next result shows that the perturbation P in condition 2 of Theorem 2.2 can be eliminated if U_t and V_t are sufficiently close *for all t*.

After this paper was completed we found the paper [34], where a slightly weaker version of the following theorem is proved by the same method.

Theorem 2.4. *Let U_t, V_t be strongly continuous one-parameter unitary groups on a Hilbert space. Assume that $\|U_t - V_t\| \leqq k$ for all $t \in \mathbf{R}$, where $k < \sqrt{2}$. It follows that there exists a unitary W in the von Neumann algebra generated by $\{U_t V_{-t}, t \in \mathbf{R}\}$ such that*

$$U_t = W V_t W^*$$

for all $t \in \mathbf{R}$. Moreover $\|W - I\| \leqq k$.

Proof. Let $\mathcal{M}$ be the von Neumann algebra generated by U and V. Let m be an invariant mean on $\mathbf{R}$ and define $\Omega = m_s(U_s V_{-s})$. Since $\mathcal{M} = (\mathcal{M}_*)^*$ it follows that Ω is well defined and $\Omega \in \mathcal{M}$. But since

$$(U_s V_{-s}) V_t = U_t (U_{s-t} V_{-(s-t)})$$

it follows that $\Omega V_t = U_t \Omega$. An application of Lemma 2.3 shows that Ω is invertible and one then easily sees that $W V_t = U_t W$, where W is the unitary occuring in the polar decomposition of Ω.

The estimate on $\|W - I\|$ is obtained as in Theorem 2.3.

In Section 4 we will examine various examples which satisfy the conditions of Theorem 2.4.

We conclude this section by deriving a result on relatively bounded perturbations of C_0^* semigroups.

We need to recall some of the general theory and terminology of adjoint semigroups. We refer the reader to [6] for the details.

Let $t \to U_t$ be a strongly continuous (C_0) semigroup of operators on a Banach space X. The adjoint semigroup is $t \to U_t^*$ acting on the dual Banach space X^*. In general one no longer has strong continuity of the adjoint semigroup, but trivially it is weak *-continuous i.e. $t \to U_t^*(f)(x)$ is continuous for fixed f in X^* and x in X. The general theory tells us that if S is the generator of U then S^* is the weak

$*$-generator of U^*. It turns out that the set of elements, $f \in X^*$, for which $t \to U_t^* f$ is strongly continuous, is a strongly closed, invariant weak*-dense, subspace $X^*(U)$ of X^*. Further $\mathscr{D}(S^*) \subseteq X^*(U)$.

Let $U_{0,t}^*$ denote the restriction of U_t^* to $X^*(U)$. We write S_0^* for its generator.

Theorem 2.5. *Let U, V be strongly continuous semigroups on a Banach space X, with infinitesimal generators S, T respectively.*

The following six conditions are equivalent:

1. *(and 1_0)*

$$\|(U_t^* - V_t^*)f\| = O(t), \qquad t \to 0$$

for all $f \in \mathscr{D}(T^*)$ (for all $f \in \mathscr{D}(T_0^*)$).

2. *(and 2_0).*

One has $\mathscr{D}(S^*) \supseteq \mathscr{D}(T^*)$ (one has $\mathscr{D}(S^*) \supseteq \mathscr{D}(T_0^*)$) and there exist constants $a, b \geqq 0$ such that

$$\|(S^* - T^*)f\| \leqq a\|f\| + b\|T^* f\|$$

for all $f \in \mathscr{D}(T^*)$ (for all $f \in \mathscr{D}(T_0^*)$).

3. *The estimate*

$$\|(U_t^* - V_t^*)(1 + \varepsilon T^*)^{-1}\| = O(t)$$

is valid for all ε in an interval $0 < \varepsilon < \delta$.

4. *The estimate*

$$\|(1 + \varepsilon T)^{-1}(U_t - V_t)\| = O(t)$$

is valid for all ε in an interval $0 < \varepsilon < \delta$.

Proof. The proof is based on an extension of methods used in [14] and ultimately relies on de Leeuw's characterization [12] of generators of adjoint semi-groups; Viz

$$f \in \mathscr{D}(S^*) \quad \text{if and only if} \quad \|(U_t^* - 1)f\| = O(t) \quad \text{as} \quad t \to 0\,.$$

We show first that

1.$\Rightarrow$2.

If $f \in \mathscr{D}(T^*)$ then one has

$$\|(1 - U_t^*)f\| \leqq \|(1 - V_t^*)f\| + \|(V_t^* - U_t^*)f\| = O(t)\,.$$

Therefore $f \in \mathscr{D}(S^*)$, by our preliminary remarks, and so $\mathscr{D}(S^*) \supseteq \mathscr{D}(T^*)$.

We now make use of Hörmander's comparison theorem (Theorem II.6.2 of [18]) to obtain the estimate in 2.

Specifically we let

$$X_0 = X_1 = X_2 = X^*$$

$$T_1 = T^* \qquad T_2 = S^*\,.$$

The operators T^* and S^* are weak* closed, thus strongly closed, and so we may apply the quoted theorem to obtain constants $a', b' \geqq 0$ such that

$$\|S^* f\| \leqq a'\|f\| + b'\|T^* f\|$$

for all $f \in \mathscr{D}(T^*)$ or

2. $\|(T^* - S^*)f\| \leqq a\|f\| + b\|T^*f\|$

with $a = a', b = 1 + b'$.

Note that we have used a version of Hörmander's theorem where it is not assumed that T_1 and T_2 are densely defined. The proof of this version is the same as the original one.

2. $\Rightarrow$ 3.

If $f \in \mathscr{D}(T^*)$ then $V_t^* f \in \mathscr{D}(T^*) \subseteq \mathscr{D}(S^*)$ and

$$(U_t^* - V_t^*)f = \int_0^t ds U_s^*(T^* - S^*)V_{t-s}^* f .$$

Therefore,

$$\begin{aligned} \|(U_t^* - V_t^*)f\| &\leqq \int_0^t ds \|U_s^*\| \, \|(T^* - S^*)V_{t-s}^* f\| \\ &\leqq \int_0^t ds \|U_s\| (a\|V_{t-s}^* f\| + b\|T^* V_{t-s}^* f\| \\ &\leqq \int_0^t ds \|U_s\| \, \|V_{t-s}\| (a\|f\| + b\|T^* f\|) . \end{aligned}$$

We know there exist constants M, ω such that

$\|U_t\| \leqq M e^{\omega t}$; $\|V_t\| \leqq M e^{\omega t}$

and so we conclude that

$$\|(U_t^* - V_t^*)f\| \leqq t M^2 e^{\omega t} (a\|f\| + b\|T^* f\|) .$$

Now if $1 > \omega\varepsilon \geqq 0$ then

$$\|(1 + \varepsilon T)^{-1}\| = \left\| \int_0^\infty dt e^{-t} V_{-\varepsilon t} \right\| \leqq M(1 - \omega\varepsilon)^{-1}$$

and $\|T(1 + \varepsilon T)^{-1}\| \leqq \varepsilon^{-1}(1 + M(1 - \omega\varepsilon)^{-1})$.

If $g \in X^*$, we thus have the estimate,

$$\begin{aligned} &\|(U_t^* - V_t^*)(1 + \varepsilon T^*)^{-1} g\| \\ &\quad \leqq t M^2 e^{\omega t} \{a M(1 - \omega\varepsilon)^{-1} + b\varepsilon^{-1}(1 + M(1 - \omega\varepsilon)^{-1})\} \|g\| . \end{aligned}$$

Taking the supremum over g of norm 1 we obtain 3.

3. $\Leftrightarrow$ 4. Clear.

3. $\Rightarrow$ 1. This follows immediately from the fact that

$$\mathscr{D}(T^*) = \mathscr{R}((1 + \varepsilon T^*)^{-1}) .$$

$1_0. \Leftrightarrow 2_0. \Leftrightarrow 3.$

If we begin with 1_0. and repeat the reasoning of 1.⇒2. we obtain $1_0.\Rightarrow 2_0$. To apply Hörmander's theorem one need only verify that T_0^* is a closed operator. This is true since T_0^* is the generator of $V_{0,t}^*$.

A slight variation of the argument of 2.⇒3. (we take the Laplace transform directly on the adjoint semi-group) yields

$$\|(U_t^* - V_t^*)(1+\varepsilon T_0^*)^{-1}\| = O(t) .$$

Now let $f \in X^*(V)$ and $A \in X$. Then

$$(f, V_t A) = (V_{0,t}^* f, A) .$$

Taking Laplace transforms, one finds that

$$\begin{aligned}(f, (1+\varepsilon T)^{-1} A) &= \int_0^\infty dt\, e^{-t}(f, V_{\varepsilon t} A) \\ &= \int_0^t dt\, e^{-t}(V_{0,t}^* f, A) \\ &= ((1+\varepsilon T_0^*)^{-1} f, A) .\end{aligned}$$

Now $X^*(V) \geqq \mathcal{D}(T^*)$, [6], and so $X^*(V)$ contains all elements of the form

$$\int_{-\infty}^{\infty} dt E(t,s) V_t^* f ,$$

where $E(t,s) = (2\sqrt{\pi s})^{-1} \exp\left(-\frac{t^2}{4s}\right)$. Since we know $\|V_t^*\| \leqq M e^{\omega t}$, we can renormalize the $E(t,s)$, to conclude that the unit ball of $X^*(V)$ is weak* dense in the unit ball of X^*, we can then conclude from the above equality that

$$\|(1+\varepsilon T)^{-1}\| = \|(1+\varepsilon T_0^*)^{-1}\| .$$

Essentially the same argument yields

$$\begin{aligned}\|(1+\varepsilon T)^{-1}(U_t - V_t)\| &= \|(U_t^* - V_t^*)(1+\varepsilon T_0^*)^{-1}\| \\ &= O(t)\end{aligned}$$

and then $1_0 \Rightarrow 4$. However 4.⇔3., then establishing 1.⇒3.

Clearly 3.⇒1.⇒1_0 and so the proof of the theorem is complete.

The reader is referred to [14] for an estimate on the relative bound of $S^* - T^*$.

§ 3. Von Neumann Algebra Theory

In this section we obtain the principal results cited in the introduction.

We first digress to define a constant $C(\alpha, \beta) \in [0, \infty]$, for any two σ-weakly continuous, one-parameter *-automorphism groups, α_t, β_t of a von Neumann algebra

$$C(\alpha, \beta) = \overline{\lim_{t \to 0}} \frac{1}{|t|} \|\alpha_t - \beta_t\| .$$

We note that

$$\frac{1}{|s|}\|\alpha_s-\beta_s\| = \frac{1}{n\left|\frac{s}{n}\right|}\left\|\alpha_{n\cdot\frac{s}{n}}-\beta_{n\cdot\frac{s}{n}}\right\|$$

$$\leqq \frac{1}{n\left|\frac{s}{n}\right|}\sum_{k=1}^{n}\left\|\alpha_{\frac{s}{n}}^{n-k}\left(\alpha_{\frac{s}{n}}-\beta_{\frac{s}{n}}\right)\beta_{\frac{s}{n}}^{k-1}\right\|$$

$$\leqq \frac{1}{\left|\frac{s}{n}\right|}\left\|\alpha_{\frac{s}{n}}-\beta_{\frac{s}{n}}\right\|.$$

Thus $\|\alpha_t-\beta_t\|=O(t)$ if and only if $C(\alpha,\beta)<\infty$ and in that case

$$\|\alpha_t-\beta_t\| \leqq C(\alpha,\beta)|t|\,,$$

where $C(\alpha,\beta)$ is the best possible constant.

Theorem 3.1. *Let $\mathcal{M}$ be a von Neumann algebra; α_t, β_t σ-weakly continuous one parameter *-automorphism groups of $\mathcal{M}$ with generators δ_α, δ_β, respectively.*

The following conditions are equivalent:

1. $\|\alpha_t-\beta_t\|=O(t)$.
2. $\mathcal{D}(\delta_\alpha)=\mathcal{D}(\delta_\beta)$ *and there exists a bounded derivation δ, of $\mathcal{M}$, such that*

$$\delta_\alpha(x)-\delta_\beta(x)=\delta(x) \quad \textit{for all} \quad x\in\mathcal{D}(\delta_\alpha)=\mathcal{D}(\delta_\beta)\,.$$

In this case $\|\delta\|=C(\alpha,\beta)$.

If the above conditions are satisfied and α_t is implemented by a strongly continuous group of unitaries, U_t, there exists a strongly continuous unitary group V_t, implementing β_t such that

$$\|U_t-V_t\| \leqq \tfrac{1}{2}C(\alpha,\beta)|t|\,.$$

Proof. 1.⇒2.

Theorem 2 of [14] establishes that $\mathcal{D}(\delta_\alpha)=\mathcal{D}(\delta_\beta)$ and that $\delta_{\alpha\beta}\equiv\delta_\alpha-\delta_\beta$ is a bounded operator from $\mathcal{D}(\delta_\alpha)$ into $\mathcal{M}$ with bound $C(\alpha,\beta)$. However the norm closure of $\mathcal{D}(\delta_\alpha)$, say $\mathfrak{A}$, is a C^*-algebra which is σ-weakly dense in $\mathcal{M}$. By the norm continuity of $\delta_{\alpha\beta}$ it extends to a bounded derivation of this C^*-algebra into $\mathcal{M}$. It is then a result of Kadison, Lemma 3 of ([22] also see [11]), that $\delta_{\alpha\beta}$ has a unique ultraweakly continuous extension, δ, to $\mathcal{M}$. By the Kaplansky density theorem, the norm of this extension is also $C(\alpha,\beta)$. Note that although Kadison's lemma is stated for a derivation of a C^*-algebra into itself, the proof of the lemma does not rely in this fact.

2.⇒1.

For $x\in\mathcal{D}(\delta_\alpha)=\mathcal{D}(\delta_\beta)$ we have

$$(\alpha_t-\beta_t)(x)=\int_0^t ds\,\alpha_s(\delta_a-\delta_\beta)\beta_{t-s}(x)$$

and so

$$\|(\alpha_t-\beta_t)(x)\| \leqq |t| \, \|\delta\| \, \|x\| .$$

One then has $\|\alpha_t-\beta_t\|=O(t)$ by applying the Kaplansky density theorem.

To complete the proof of the theorem we note that by the derivation theorem, [15], there exists an $h=h^* \in \mathcal{M}$ with $\|h\| \leqq \frac{1}{2}\|\delta\| = \frac{1}{2}C(\alpha,\beta)$ and such that

$$\delta(x)=[ih,x], \qquad x \in \mathcal{M} .$$

If $U_t=e^{itH_\alpha}$, then, [4],

$$\delta_\alpha(x)=[iH_\alpha,x], \qquad x \in \mathcal{D}(\delta_\alpha) .$$

Thus

$$\delta_\beta(x)=[iH_\alpha-ih,x], \qquad x \in \mathcal{D}(\delta_\beta)=\mathcal{D}(\delta_\alpha) .$$

Defining $V_t \equiv e^{it(H_\alpha-h)}$, it follows by the Trotter product formula [18], that $V_t\mathcal{M}V_t^*=\mathcal{M}$. Thus, on defining $\beta_t'(x)=V_txV_t^*$ it is clear that $\delta_{\beta'}=\delta_\beta$, so that $\beta'=\beta$. Finally

$$\|U_t-V_t\| \leqq |t| \, \|ih\| = |t|\tfrac{1}{2}C(\alpha,\beta) .$$

We take a closer look at the situation of Theorem 3.1, when $\mathcal{M}$ is in standard form, [8, 1, 7, 6].

For a given faithful, normal, semifinite, weight on $\mathcal{M}$, let $\mathcal{P}^\natural$ be the natural cone, associated with $\mathcal{M}$ and J the corresponding modular conjugation. Every vector $\eta \in \mathcal{H}$ can be written uniquely as $\eta=\eta_1-\eta_2+i(\eta_3-\eta_4)$ where $\eta_i \in \mathcal{P}^\natural$ and $\eta_1 \perp \eta_2$, $\eta_3 \perp \eta_4$. Moreover there exists a bijection ζ from the normal positive functionals ϱ on $\mathcal{M}$ to vectors in $\mathcal{P}^\natural$ such that

$$\varrho(x)=\omega_{\zeta(\varrho)}(x) \quad \text{for} \quad x \in \mathcal{M} .$$

We have the inequality

$$\|\zeta(\varrho_1)-\zeta(\varrho_2)\|^2 \leqq \|\varrho_1-\varrho_2\| \leqq \|\zeta(\varrho_1)-\zeta(\varrho_2)\| \, \|\zeta(\varrho_1)+\zeta(\varrho_2)\| .$$

For any σ-weakly continuous representation of a topological group G, as *-automorphisms of $\mathcal{M}$, there exists a strongly continuous representation $g \to U_g$ of G, uniquely determined by the requirements:

$$\alpha_g(x)=U_gxU_g^* \qquad x \in \mathcal{M}, \quad g \in G$$
$$U_g\mathcal{P}^\natural=\mathcal{P}^\natural, \qquad g \in G .$$

This "canonical" representation is defined by

$$U_g\zeta(\varrho)=\zeta(\alpha_{g^{-1}}^*\varrho) .$$

Given two representations α, β of G, if U, V are the canonical unitaries described above, one easily sees that

$$\tfrac{1}{2}\|U_g-V_g\|^2 \leqq \|\alpha_{g^{-1}}-\beta_{g^{-1}}\| \leqq 2\|U_g-V_g\| .$$

In particular if φ is a faithful normal state on $\mathcal{M}$, then $\Delta_{\zeta(\varrho)}^{it}$ is the canonical unitary group associated to the modular automorphism group of φ.

When $G=R$ the next theorem states that the estimate $\frac{1}{2}\|U_g - V_g\|^2 \leqq \|\alpha_{g^{-1}} - \beta_{g^{-1}}\|$ can be improved.

Theorem 3.2[1]. *Let $\{\mathcal{M}, \mathcal{P}^\natural\}$ be a von Neumann algebra in standard form and J the modular conjugation associated to this pair. If α_t, β_t are σ-weakly continuous one-parameter *-automorphism groups of, let $U_t = e^{itH_\alpha}$, $V_t = e^{itH_\beta}$ be the canonical unitaries, described above, which implement α_t, β_t respectively. The following are equivalent:*

1. $\|\alpha_t - \beta_t\| = O(t)$.
2. $\|U_t - V_t\| = O(t)$.
3. *There exists an element $h = h^* \in \mathcal{M}$ such that*

$$H_\alpha = H_\beta + h - JhJ \ .$$

Furthermore $\|U_t - V_t\| \leqq C(\alpha, \beta)|t|$.

Proof. 3.⇒2.⇒1. are evident from Theorem 4 and its proof.

To show that 1.⇒3. we observe that Theorem 4 and the derivation theorem allow us to find $h = h^* \in \mathcal{M}$ with $\|h\| = \frac{1}{2}C(\alpha, \beta)$ and

$$\delta_\alpha(x) - \delta_\beta(x) = [ih, x] , \quad x \in \mathcal{D}(\delta_\alpha) = \mathcal{D}(\delta_\beta) \ .$$

Define a self-adjoint operator H on $\mathcal{D}(H_\beta)$ by

$$H = H_\beta + h - JhJ \ .$$

As $JhJ \in \mathcal{M}'$, the reasoning of Theorem 4 gives

$$\alpha_t(x) = e^{itH} x e^{-itH}, \quad x \in \mathcal{M} , \quad t \in \mathbf{R}$$

and

$$\|e^{itH} - e^{itH_\beta}\| \leqq |t| \, \|h - JhJ\| \leqq |t| C(\alpha, \beta) \ .$$

It remains to show that e^{itH} is the canonical unitary group for α_t (i.e. $H_\alpha = H$). For this we need to show that $e^{itH}\mathcal{P}^\natural \subseteqq \mathcal{P}^\natural$. However we know that $yJyJ\mathcal{P}^\natural \subseteqq \mathcal{P}^\natural$ for all $y \in \mathcal{M}$. Thus, by the Trotter product formula, if $\xi \in \mathcal{P}^\natural$, we have

$$e^{it(h - JhJ)}\xi = \lim_{n \to \infty} \left[e^{\frac{t}{n} ih} J e^{\frac{t}{n} ih} J \right]^n \xi \in \mathcal{P}^\natural \ .$$

Writing

$$e^{it(H_\beta + h - JhJ)} = \text{strong} \lim_{n \to \infty} \left[e^{\frac{it}{n} H_\beta} e^{i\frac{t}{n}(h - JhJ)} \right]^n$$

we see that $e^{itH}\mathcal{P}^\natural = \mathcal{P}^\natural$.

Remark. In particular we see from Theorem 5, that the Araki's perturbation [3] of the modular automorphism group [1] is characterized by $O(t)$ behaviour of the related modular automorphism groups.

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[1] This result has been improved by U. Haagerup who has shown that the spectrum of any automorphism is identical to the spectrum of the canonical implementing unitary $U(\alpha)$. Hence $\|U(\alpha) - U(\beta)\| = \|\alpha - \beta\|$ if $\|\alpha - \beta\| < 2$ and $\|U(\alpha) - U(\beta)\| \geqq \sqrt{3}$ if $\|\alpha - \beta\| = 2$

The rest of this section is devoted to the examination of groups of automorphisms which are close, but not necessarily of order t. First we need two results on cross-sections and cocycles.

Proposition 3.3. *Let α_t, β_t be weakly continuous automorphism groups of a von Neumann algebra with separable pre-dual. Suppose that $\|\alpha_t-\beta_t\|\leqq\varepsilon<2$ for $|t|<\delta$, $\delta>0$. There then exists a Borel mapping $t\to U_t$ from $\boldsymbol{R}$ into the unitary group, $\mathscr{U}(\mathscr{M})$, of $\mathscr{M}$ such that $\beta_t(x)=U_t\alpha_t(x)U_t^*$ for all $t\in\boldsymbol{R}$ and*

$$\|U_t-I\|\leqq\varepsilon' \quad \textit{for} \quad |t|\leqq\delta\,, \quad \textit{where} \quad (\varepsilon')^2=2\left(1-\sqrt{\frac{1-\varepsilon^2}{4}}\right).$$

Proof. An application of a result of Kadison and Ringrose [11] shows that $\gamma_t=\alpha_t\circ\beta_{-t}$ is inner for $|t|\leqq\delta$ and hence, by the cocycle identity for γ_t, for all $t\in\boldsymbol{R}$. Furthermore Lemma 5 of [11] shows that for each t, $|t|\leqq\delta$ we can choose the unitary so that $\|U_t-I\|^2\leqq2\left(1-\sqrt{1-\frac{\|\gamma_t-i\|^2}{4}}\right)=(\varepsilon')^2$, and this is the minimal possible norm of U_t-I.

We must now examine the Borel structure in the inner automorphism group, $\mathrm{Inn}(\mathscr{M})$. Indeed what we need is a variation on the proof of Theorem 4.13 of [29]. Recall that a topological space is called Polish if it is homeomorphic to a complete separable metric space. A subset of a Polish space (see [27]) is analytic if it is a continuous image of a Polish space and if X and Y are analytic Borel spaces and f is a $1-1$ Borel map of X onto Y, then f is a Borel isomorphism [27, p. 72]. Consider $\mathscr{U}(\mathscr{M})$, with the strong *-topology and $\mathscr{U}(\mathscr{Z})$, the closed subgroup of $\mathscr{U}(\mathscr{M})$, consisting of the unitaries in the center $\mathscr{Z}=\mathscr{M}\cap\mathscr{M}'$. Let $\mathscr{B}(\mathscr{M}_*)$ denote the bounded operators on $\mathscr{M}_*$ with the topology of pointwise norm convergence. Since $\mathscr{M}_*$ is separable, $\mathscr{U}(\mathscr{M})$ and $\mathscr{B}(\mathscr{M}_*)$ are Polish. Moreover, the subset $\mathrm{Inn}(\mathscr{M})\subseteq\mathscr{B}(\mathscr{M}_*)$ is analytic, since the canonical map $\mathscr{U}(\mathscr{M})\to\mathrm{Inn}(\mathscr{M})$ is continuous. The space $\mathrm{Inn}(\mathscr{M})$ will be the space Y in the result quoted above. The space $\mathscr{U}(\mathscr{M})/\mathscr{U}(\mathscr{Z})$ is to be X and f is the $1-1$ continuous map of an element in $\mathscr{U}(\mathscr{M})/\mathscr{U}(\mathscr{Z})$ to the automorphism it induces. We conclude that the Borel structure of $\mathrm{Inn}(\mathscr{M})$ is the same as that of $\mathscr{U}(\mathscr{M})/\mathscr{U}(\mathscr{Z})$.

Since the quotient map $\eta:\mathscr{U}(\mathscr{M})\to\mathscr{U}(\mathscr{M})/\mathscr{U}(\mathscr{Z})$ is continuous and open, it follows that the canonical map $f\circ\eta:\mathscr{U}(\mathscr{M})\to\mathrm{Inn}(\mathscr{M})$ is continuous and maps open sets into Borel sets. As $\mathscr{U}(\mathscr{M})$ is a Polish space it follows from Theorem 3.4.1 of [27] that $f\circ\eta$ admits a Borel cross section, i.e. there exists a Borel map $U:\mathrm{Inn}(\mathscr{M})\to\mathscr{U}(\mathscr{M})$ such that

$$\alpha(x)=U(\alpha)xU(\alpha)^*$$

for all $\alpha\in\mathrm{Inn}(\mathscr{M})$, $x\in\mathscr{M}$. The existence of $t\to U_t$ for $|t|>\delta$ follows immediately.

For $|t|\leqq\delta$ we proceed by defining

$$\mathscr{U}_{\varepsilon'}(\mathscr{M})=\{U\in\mathscr{U}(\mathscr{M});\|U-I\|\leqq\varepsilon'\}$$

$$\mathrm{Inn}_\varepsilon(\mathscr{M})=\{\alpha\in\mathrm{Inn}(\mathscr{M});\|\alpha-i\|\leqq\varepsilon\}\,.$$

The image of $\mathscr{U}_\varepsilon(\mathscr{M})$ in $\mathrm{Inn}(\mathscr{M})$ is just $\mathrm{Inn}_\varepsilon(\mathscr{M})$ by the Kadison-Ringrose theorem. $\mathscr{U}_{\varepsilon'}(\mathscr{M})$ is closed in $\mathscr{U}(\mathscr{M})$, hence Polish, and furthermore $\mathscr{U}_{\varepsilon'}(\mathscr{M})$ is a G_δ set in

$\mathscr{U}(\mathscr{M})$, thus the image of $\mathscr{U}_{\varepsilon'}(\mathscr{M})$ in $\mathscr{U}(\mathscr{M})/\mathscr{U}(\mathscr{Z})$ is Borel. As the open sets in $\mathscr{U}_{\varepsilon'}(\mathscr{M})$ are intersections of open sets in $\mathscr{U}(\mathscr{M})$ with $\mathscr{U}_{\varepsilon'}(\mathscr{M})$, it follows that $f\circ\eta$, restricted to $\mathscr{U}_{\varepsilon'}(\mathscr{M})$, maps open sets in $\mathscr{U}_{\varepsilon'}(\mathscr{M})$ into Borel sets in $\mathrm{Inn}(\mathscr{M})$. Another application of Theorem 3.4.1 of [27] on $f\circ\eta:\mathscr{U}_{\varepsilon'}(\mathscr{M})\to\mathrm{Inn}_{\varepsilon}(\mathscr{M})$ implies the existence of a Borel cross section for $|t|\leqq\delta$.

We next prove an implementabiliary theorem for inner cocycles on von Neumann algebras. Related results abound in the literature. Suppose $\mathscr{M}$ is a von Neumann algebra with separable pre-dual $\mathscr{M}_*$ and $t\to\sigma_t=\beta_t\cdot\alpha_{-t}$ is a σ-weakly continuous pointwise inner cocycle in the automorphism group of $\mathscr{M}$. It was shown in [24, Theorem 1.2.8] that σ_t is unitarily implementable if $\mathscr{M}$ is a factor, and in [28, Theorem 5.3] this result is extended to arbitrary $\mathscr{M}$ if α is centre-fixing. If $\lim_{t\to 0}\|\sigma_t-\iota\|=0$, then in [5, Proposition 4.1] it is shown that there exists a norm continuous unitary cocycle implementing σ. This last result holds without the separability of $\mathscr{M}_*$. Some features of these results are evident in what follows.

Theorem 3.4. *Let $\mathscr{M}$ be a von Neumann algebra with separable predual and let α, β be two σ-weakly continuous oneparameter groups of automorphisms of $\mathscr{M}$. Assume there exists $\delta,\varepsilon>0$ such that $\varepsilon<\sqrt{\frac{71}{18}}\cong 0.47$ and*

$$\|\alpha_t-\beta_t\|\leqq\varepsilon \quad \textit{when} \quad |t|\leqq\delta\,.$$

Then there exists a σ-weakly continuous cocycle $t\to\Gamma_t$ in $\mathscr{U}(\mathscr{M})$ such that

$$\Gamma_{t+s}=\Gamma_t\alpha_t(\Gamma_s) \quad \textit{for} \quad t,s\in\boldsymbol{R}$$

$$\beta_t(x)=\Gamma_t\alpha_t(x)\Gamma_t^* \quad \textit{for} \quad t\in\boldsymbol{R}, \quad A\in\mathscr{M}$$

$$\|\Gamma_t-I\|\leqq 10\sqrt{2\left(1-\sqrt{1-\frac{\varepsilon^2}{4}}\right)}=5\varepsilon+0(\varepsilon^2) \quad \textit{for} \quad |t|<\frac{\delta}{4}\,.$$

Proof. The proof goes by standard cohomological considerations as in [5]. First note by Proposition 3.3 that there exists a Borel map $t\in\boldsymbol{R}\to U_t\in\mathscr{U}(\mathscr{M})$ such that

$$\beta_t(x)=U_t\alpha_t(x)U_t^*\,,\qquad A\in\mathscr{M}\,,\quad t\in\boldsymbol{R}$$

and

$$\|U_t-I\|\leqq\varepsilon'$$

for $|t|\leqq\delta$, where $\varepsilon'^2=2\left(1-\sqrt{1-\frac{\varepsilon^2}{4}}\right)$. Now, define

$$z(s,t)=U_s\alpha_s(U_t)U_{s+t}^{-1}\,.$$

Then $z:\boldsymbol{R}^2\to\mathscr{U}(\mathscr{M}\cap\mathscr{M}')$ is a centre-valued two-cocycle, i.e. $z(s,0)=z(0,t)=I$ for all s and t and

$$z(s,t)z(s+t,u)=\alpha_s(z(t,u))z(s,t+u)$$

for all s,t,u. From the definition of z we immediately get the estimate

$$\|z(s,t)-I\|\leqq 3\varepsilon' \quad \text{for} \quad |s|+|t|<\delta\,.$$

Define $t\in\boldsymbol{R}\to\lambda_t\in\mathscr{U}(\mathscr{M}\cap\mathscr{M})$ inductively by $\lambda_0=I$ and

$$\lambda_{\frac{\delta}{2}(t+n)}=\lambda_{\frac{\delta}{2}n}\, z\left(\frac{\delta}{2}n, \frac{\delta}{2}t\right)$$

for $0\leqq t\leqq 1$, $n\in\boldsymbol{Z}$.

Next, define $z'(s,t)\equiv\lambda_s\alpha_s(\lambda_t)z(s,t)\lambda_{s+t}^{-1}$ for $s,t\in\boldsymbol{R}$. By [5], Lemma 3.3, $z'(s,t)$ is a 2-cocycle satisfying

$$z'(p,t)=I \quad \text{for} \quad 0\leqq t\leqq\frac{\delta}{2},\ p\in\frac{\delta}{2}\boldsymbol{Z}$$

and

$$\|z'(s,t)-I\|\leqq 6\varepsilon' \quad \text{for} \quad 0\leqq s,t\leqq\frac{\delta}{2}.$$

By [5], Lemma 3.2a, $s\to\alpha_{-s}(z'(s,t))$ is periodic with period $\frac{\delta}{2}$, and hence the estimate on $z'(s,t)-I$ just derived extends to all s. Now, let log denote the principal value of the logarithm on the complex plane with a cut along the negative real axis. Now, since $\varepsilon<\sqrt{\frac{71}{18}}$ it follows that $6\varepsilon'<2$, and hence by the estimate on z', we can consistently define $y(s,t)=\log(z'(s,t))$ for $0\leqq t<\frac{\delta}{2}$, and the cocycle property of $z'(s,t)$ gives

$$y(s,0)=y(0,t)=0$$
$$y(s,t)+y(s+t,u)=\alpha_s(y(t,u))+y(s,t+u)$$

for $0\leqq t,\ u,\ t+u\leqq\frac{\delta}{2}$. Also $s,t\to y(s,t)$ is a Borel map. Proceeding as in [5], proof of Proposition 3.5, we define a Borel map $c:t\in\left[0,\frac{\delta}{2}\right]\to c_t\in\mathscr{M}\cap\mathscr{M}'$ by

$$c_t=-\frac{2}{\delta}\int_0^{\frac{\delta}{2}} ds\,\alpha_{-s}(y(s,t))$$

and $\lambda':t\in\left[0,\frac{\delta}{2}\right]\to\lambda'_t\in\mathscr{U}(\mathscr{M}\cap\mathscr{M}')$ by

$$\lambda'_t=\exp(c_t)$$

and compute that

$$z'(s,t)=\lambda'_{s+t}\alpha_s(\lambda_t'^{-1})\lambda_s'^{-1}$$

for $0\leqq s,t,s+t\leqq\frac{\delta}{2}$. Using the spectral radius formula for unitaries and self-adjoints and the definition of λ'_t from $z'(s,t)$ by means of $y(s,t)$ and c_t we immediately derive the estimates

$$\|y(s,t)\| \leqq \arccos\left(1-\frac{(6\varepsilon')^2}{2}\right); \qquad 0 \leqq t \leqq \frac{\delta}{2}$$

thus

$$\|c_t\| \leqq \arccos\left(1-\frac{(6\varepsilon')^2}{2}\right); \qquad 0 \leqq t \leqq \frac{\delta}{2};$$

and finally

$$\|\lambda'_t - I\| \leqq 6\varepsilon'$$

for $0 \leqq t \leqq \frac{\delta}{2}$. We now extend the map $t \in \left[0, \frac{\delta}{2}\right] \to \lambda'_t \in \mathscr{U}(\mathscr{M} \cap \mathscr{M}')$ to a Borel map of all of $\boldsymbol{R}$ and define

$$z''(s,t) \equiv \lambda'^{-1}_{s+t} z'(s,t) \lambda'_s \alpha_s(\lambda'_t).$$

Then z'' is a 2-cocycle, and

$$z''(s,t) = I$$

for $0 \leqq s, t \leqq \frac{\delta}{4}$.

Now, replacing $\frac{\delta}{2}$ by $\frac{\delta}{4}$ and ε' by 0, we define z''' from z'' as z' was defined from z. First define

$$\lambda''_0 = I$$

$$\lambda''_{\frac{\delta}{4}(t+n)} = \lambda''_{\frac{\delta}{4}n} z''\left(\frac{\delta}{4}n, \frac{\delta}{4}t\right); \qquad 0 \leqq t \leqq 1, \quad n \in \boldsymbol{Z}$$

and then

$$z'''(s,t) = \lambda''_s \alpha_s(\lambda''_t) z''(s,t) \lambda''^{-1}_{s+t}.$$

By [5], Lemma 3.3, z''' is a 2-cocycle such that

$$z'''(p,t) = I \quad \text{for} \quad 0 \leqq t \leqq \frac{\delta}{4}, \qquad p \in \frac{\delta}{4}\boldsymbol{Z}$$

and

$$z'''(s,t) = I \quad \text{for} \quad 0 \leqq s, t \leqq \frac{\delta}{4}.$$

Hence [5], Lemma 3.2b implies that $z'''(s,t) = I$ for all s, t. Defining $\lambda'''_s = \lambda''_s \lambda'_s \lambda_s$ we thus have

$$\begin{aligned} z(s,t) &= \lambda'''_{s+t} z'''(s,t) \lambda'''^{-1}_s \alpha_s(\lambda'''^{-1}_t) \\ &= \lambda'''_{s+t} \lambda'''^{-1}_s \alpha_s(\lambda'''^{-1}_t). \end{aligned}$$

Define

$$\Gamma_s = \lambda_s''' U_s .$$

Then by the definition of z from U, Γ_s is a two-cocycle:

$$\Gamma_{s+t} = \Gamma_s \alpha_s(\Gamma_t)$$

and

$$\beta_t(x) = \Gamma_t \alpha_t(x) \Gamma_t^* .$$

$t \to \Gamma_t$ is Borel, hence strongly continuous by [24], proof of Lemma 1.2.5. [We note the following brief proof. If $\mathcal{M}$ is in standard form, let ζ be the canonical map of $\mathcal{M}_{*+} \to \mathcal{P}^\natural$ [10]. As $\mathcal{M}_{*+}$ contains a norm-dense subsequence it follows from the estimate $\|\zeta(\omega_1) - \zeta(\omega_2)\|^2 \leqq \|\omega_1 - \omega_2\|$ that $\mathcal{P}^\natural$ contains a norm dense subsequence. Then $\mathcal{H} = \mathcal{P}^\natural - \mathcal{P}^\natural + i(\mathcal{P}^\natural - \mathcal{P}^\natural)$ is separable. If $s \to V_s$ is the canonical unitary implementation of α on $\mathcal{H}$ put $U_s = \Gamma_s V_s$, and note that $s \to U_s$ is a Borel map and a unitary representation of $\boldsymbol{R}$ (implementing β). As $\mathcal{H}$ is separable it follows that $s \to U_s$ is strongly continuous so then $s \to \Gamma_s = U_s V_{-s}$ is strongly continuous.] We have the estimates

$$\|U_t - I\| \leqq \varepsilon' \quad \text{for} \quad |t| \leqq \delta$$

$$\|\lambda_t - I\| \leqq 3\varepsilon' \quad \text{for} \quad 0 \leqq t \leqq \frac{\delta}{2}$$

$$\|\lambda_t' - I\| \leqq 6\varepsilon' \quad \text{for} \quad 0 \leqq t \leqq \frac{\delta}{2}$$

$$\|\lambda_t'' - I\| = 0 \quad \text{for} \quad 0 \leqq t \leqq \frac{\delta}{4}$$

and hence

$$\|\Gamma_s - I\| \leqq \varepsilon' + 3\varepsilon' + 6\varepsilon' = 10\varepsilon'$$

for $|s| \leqq \frac{\delta}{4}$.

Now we apply these result to the analysis of automorphisms which are close for all t thus generalizing a result of Reynolds [26] and Theorem 2.4.

Theorem 3.5. *Let $\mathcal{M}$ be a von Neumann algebra with separable predual, and let α, β be σ-weakly continuous one parameter groups of *-automorphisms of $\mathcal{M}$. Assume there exists a $0 \leqq \varepsilon \leqq \sqrt{\frac{199}{50}} \cong 0.28$ such that*

$$\|\alpha_t - \beta_t\| \leqq \varepsilon$$

for all t.

There exists an inner automorphism γ of $\mathcal{M}$ such that

$$\alpha_t = \gamma \circ \beta_t \circ \gamma^{-1} .$$

We can choose a unitary $W \in \mathcal{M}$, *giving* γ, *so that*

$$\|W-I\| \leqq 10 \sqrt{2\left(1-\sqrt{1-\frac{\varepsilon^2}{4}}\right)}.$$

Thus $\|\gamma - \iota\| \leqq 10\varepsilon + 0(\varepsilon^2)$.

Proof. By Theorem 3.4 there exists a strongly continuous unitary cocycle $t \to \Gamma_t$ in $\mathcal{U}(\mathcal{M})$ such that

$$\Gamma_{t+s} = \Gamma_t \beta_t(\Gamma_s)$$
$$\alpha_t(x) = \Gamma_t \beta_t(x) \Gamma_t^*$$

for all $x \in \mathcal{M}$, $t, s \in \boldsymbol{R}$, and such that

$$\|\Gamma_t - I\| \leqq \varepsilon' = 10 \sqrt{2(1-\sqrt{1-\varepsilon^2/4})} < \sqrt{2} \quad \text{for all} \quad t\,.$$

We may assume that $\mathcal{M}$ is in a standard representation, hence there exists a strongly continuous unitary group $t \to V_t$ such that

$$\beta_t(x) = V_t x V_t^*$$

for $x \in \mathcal{M}$, $t \in \boldsymbol{R}$ [10]. Define $U_t = \Gamma_t V_t$. Then $U_s U_t = \Gamma_s V_s \Gamma_t V_t = \Gamma_s \beta_s(\Gamma_t) V_{s+t} = \Gamma_{s+t} V_{s+t} = U_{s+t}$, so $t \to U_t$ is a strongly continuous unitary group, and

$$\alpha_t(x) = U_t x U_t^*$$

for $x \in \mathcal{M}$, $t \in \boldsymbol{R}$. Since $U_t V_{-t} = \Gamma_t \in \mathcal{M}$, it follows from Theorem 2.4 that there exists a unitary $W \in \mathcal{M}$ such that $U_t = W V_t W^*$. Defining $\gamma(x) = W x W^*$ we thus have

$$\alpha_t = \gamma \circ \beta_t \circ \gamma^{-1}$$

for all $t \in \boldsymbol{R}$.

The estimate on $\|W - I\|$ is essentially that at the end of Theorem 2.2

Theorem 3.6. *Let* $\mathcal{M}$ *be a von Neumann algebra with separable predual and let* α, β *be two* σ*-weakly continuous one-parameter groups of *-automorphism of* $\mathcal{M}$ *with generators* δ_α, *and* δ_β, *respectively.*

The following conditions are equivalent:

1. *There exists* ε_1, $0 \leqq \varepsilon_1 < \sqrt{199}/50 \cong 0.28$ *and* $\delta_1 > 0$ *such that*

$$\|\alpha_t - \beta_t\| \leqq \varepsilon_1$$

for $0 \leqq t \leqq \delta_1$.

2. *There exists* ε_2, $0 \leqq \varepsilon_2 \leqq \sqrt{199}/50$ *and* $\delta_2 > 0$, *an inner automorphism* γ *of* $\mathcal{M}$, *and a bounded derivation* δ *of* $\mathcal{M}$ *such that*

$$\delta_\alpha = \gamma \circ (\delta_\beta + \delta) \circ \gamma^{-1}$$

and

$$\|\alpha_t \circ \gamma^{-1} \circ \alpha_{-t} \circ \gamma - \iota\| \leqq \varepsilon_2$$

for $0 \leqq t \leqq \delta_2$.

If these conditions are satisfied then

$$\|\alpha_t - \beta_t\| = \|\alpha_t \circ \gamma^{-1} \circ \alpha_{-t} \circ \gamma - \iota\| + O(t)\,.$$

Moreover, a $W \in \mathcal{M}$ can be chosen, giving γ, such that

$$\|W-I\| \leqq 10\sqrt{2\left(1-\sqrt{1-\frac{\varepsilon_1^2}{4}}\right)}.$$

Thus $\|\gamma-\iota\| \leqq 10\varepsilon_1 + 0(\varepsilon_1^2)$.

Proof. $1 \Rightarrow 2$. Proceeding as in the proof of the preceding theorem, we may assume that $\mathcal{M}$ is in a standard representation, and find strongly continuous unitary groups U, V such that

$$\alpha_t(x) = U_t x U_t^*, \quad \beta^t(x) = V_t x V_t^*$$
$$U_t V_{-t} \in \mathcal{M} \quad \text{for} \quad t \in \boldsymbol{R}$$
$$\|U_t - V_t\| \leqq \varepsilon' < \sqrt{2} \quad \text{for} \quad |t| \leqq \delta/4 .$$

In this case $\Omega \equiv \frac{4}{\delta}\int_0^{\delta/4} dt U_t V_{-t} \in \mathcal{M}$. Hence by Theorem 2.2 there exists a unitary $W \in \mathcal{M}$ such that

$$\|U_t - W V_t W^*\| = O(t) .$$

Defining $\gamma(x) = W x W^*$ and $\hat{\beta}_t^\prime = \gamma \circ \beta_t \circ \gamma^{-1}$ we then have

$$\|\alpha_t - \hat{\beta}_t\| \leqq 2\|U_t - W V_t W^*\| = O(t) .$$

Hence by Theorem 3.1 there exists a bounded derivation δ' of $\mathcal{M}$ such that

$$\delta_\alpha = \delta_{\hat{\beta}} + \delta' = \gamma \circ \delta_\beta \circ \gamma^{-1} + \delta'$$
$$= \gamma \circ (\delta_\beta + \delta) \circ \gamma^{-1}$$

where $\delta = \gamma^{-1} \circ \delta' \circ \gamma$.

It follows immediately that

$$\alpha_t \circ \gamma^{-1} \circ \alpha_t \circ \gamma = \exp(t\delta_\alpha) - \exp(t\delta_\beta)$$
$$+ \exp(t\delta_\beta) - \exp(t(\delta_\beta + \delta))$$

and hence

$$\|\alpha_t \circ \gamma^{-1} \circ \alpha_{-t} \circ \gamma - \iota\| = \|\alpha_t - \gamma^{-1}\alpha_t \gamma\| = \|\alpha_t - \beta_t\| + O(t) .$$

Hence the estimate in 2 follows from the estimate in 1, and conversely the estimate in 1 follows from 2.

The estimate of $\|W-I\|$ is by now clear.

§ 4. Special Cases and Examples

In Theorem 3.1 we saw an example where the twist occurring in Theorems 2.1, 2.2, 3.6 was not necessary. Other such cases are given in Propositions 4.1 and 4.3.

Proposition 4.1. *Let U_t, V_t be C_0 or C_0^* groups of isometries on a Banach space X, with generators S, T respectively. Assume that $U_t V_s = V_s U_t$, $s, t \in R$ and that there exists $\varepsilon > 0$ such that $\|U_t - V_t\| < 2$ for $|t| \leqq \varepsilon$. Then $\|U_t - V_t\| = O(t)$ and $\mathcal{D}(S) = \mathcal{D}(T)$.*

Moreover $T-S$ is bounded in norm and extends to a bounded operator on X which is $\sigma(X,X_*)$ closed in the C_0^*-case.

Proof. Define $W_t=U_tV_{-t}$. Since U and V commute, $t\to W_t$ is a one parameter group of isometries of X. We claim $t\to W_tx$ is continuous in the appropriate topology. The C_0 case is trivial. For the C_0^* case we observe that U_t is the dual of U_t^* acting in the predual of X, X_*. The same being true for V_t one sees that W_t is the dual group of the strongly continuous group $V_{-t}^*U_t^*$ and so is weak* continuous.

Thus W_t is a continuous group of isometries such that $\|W_t-I\|<2$, for $|t|\leqq\varepsilon$. Let $\mathrm{Sp}(W)$ denote the spectrum of W as defined in [23]. If $p\in\mathrm{Sp}(W)$ it follows [24, Lemma 2.36], that there exists a sequence $\{x_n\}\subseteq X$ such that $\|x_n\|=1$, for all n and

$$\lim_{n\to\infty}\|W_tx_n-e^{-ipt}x_n\|=0\ ,$$

uniformly for t in a compact set. We have that $\|W_tx_n-x_n\|\geqq\|e^{-ipt}x_n-x_n\|-\|W_tx_n-e^{-ipt}x_n\|$. Then $\|W_t-\iota\|\geqq|e^{-ipt}-1|$ for $p\in\mathrm{Sp}(W)$. However if $|t|\leqq\varepsilon$ and $p\in\mathrm{Sp}(W)$ we then have $|e^{-ipt}-1|<2$. Thus $|p|\leqq\pi/\varepsilon$, i.e. the spectrum of W is bounded. Now by [25, Proposition 2.2], $t\to W_t$ is norm continuous and so has a bounded generator, continuous in the appropriate topology. As $U_t=W_tV_t$, the rest of the proposition follow easily.

Note that if $\varepsilon=\infty$ in Proposition 4.1, the proof shows that $\mathrm{Sp}(W)=\{0\}$. But then $W_t=\iota$ for all t [23] and so we get the following strengthening of Theorems 2.4 and 3.5 in this special case.

Corollary 4.2. *With the same notation and assumptions as in Proposition 11, if $\|U_t-V_t\|<2$ for all $t\in\mathbf{R}$, then $U_t=V_t$.*

Another example where the twist of Theorem 3.6 is not necessary is for quasi-free automorphism groups of the CAR algebra [13].

Given a separable complex Hilbert space $\mathscr{H}$, the CAR-algebra, $\mathfrak{A}(\mathscr{H})$, over $\mathscr{H}$ is the C^*-algebra generated by the identity and elements $a(f)$, where $f\to a(f)$ is a linear map of $\mathscr{H}$ satisfying the anti-commutation relations.

$$a(f)^*a(g)+a(g)a(f)^*=(g,f)I$$
$$a(f)a(g)+a(g)a(f)\quad=0\ .$$

This C^*-algebra is a UHF C^*-algebra (more recent terminology is "uniformly matricial C^*-algebra) and is a fortiori simple.

The automorphism groups we study are quasi-free automorphism groups. Such a group arises as follows: Let $t\to U_t$ be a one parameter group of unitaries of the underlying Hilbert space.

Define

$$\alpha_t(a(f))=a(U_tf)\ .$$

One readily sees that α_t extends to an automorphism group of $\mathfrak{A}(\mathscr{H})$.

This algebra has been studied in great detail. We shall make use of the fact [13] that there exists a representation (the Fock representation) of $\mathfrak{A}(\mathscr{H})$ where every quasi-free automorphism group is unitarily implemented.

The Fock state of the CAR-algebra is the unique state ω_F with the property that $\omega(a(f)^*a(g))=0$. This is a pure state. As this defining relation is invariant under quasi-free automorphisms, so is the state and then applying the GNS procedure for the state ω_F we find a representation of $\mathfrak{A}, \pi_{\omega_F}$, a Hilbert space $\mathscr{H}_F$ and unitaries $U^\alpha(t)$ on $\mathscr{H}_F$ such that

$$\pi_{\omega_F}(\alpha_t(x)) = U^\alpha(t)\pi_{\omega_F}(x)U^\alpha(-t) .$$

Araki's characterization, [2], of bounded derivations of $\mathfrak{A}(\mathscr{H})$ will play in a key role in the following theorem.

Theorem 4.3. *Let $\mathfrak{A}=\mathfrak{A}(\mathscr{H})$ be the* CAR *algebra and suppose α_t, β_t are two quasi-free automorphism groups of $\mathfrak{A}$ corresponding to unitary groups U_t, V_t respectively. Let H, K be the self-adjoint generators of U_t, V_t respectively and write $\alpha_t=\exp(t\delta_H)$, $\beta_t=\exp(t\delta_K)$ for the appropriate closed derivations of $\mathfrak{A}$.*

The following conditions are equivalent:
1. *$\mathscr{D}(H)=\mathscr{D}(K)$ and $\|H-K\|_{\mathrm{Tr}}<\infty$($\| \ \|_{\mathrm{Tr}}$ denotes the trace class norm.*
2. *$\|U_t-V_t\|_{\mathrm{Tr}}=O(t)$, $t\to 0$.*
3. *$\mathscr{D}(\delta_H)=\mathscr{D}(\delta_K)$ and $\|\delta_H-\delta_K\|<\infty$.*
4. *$\|\alpha_t-\beta_t\|=O(t)$, $t\to 0$.*

Proof. 1.⇒2. We have the integral relation

$$\frac{(U_t-V_t)}{t}=\frac{i}{t}\int_0^t dsU_s(H-K)V_{t-s} .$$

Therefore,

$$\|(U_t-V_t)/t\|_{\mathrm{Tr}} \leqq \|(H-K)\|_{\mathrm{Tr}} .$$

2.⇒1.

First we note that $\|(U_t-V_t)\|=O(t)$ where $\|\cdot\|$ denotes the usual operator norm on $\mathscr{H}$. Applying Corollary 3 of [14], we conclude that $\mathscr{D}(H)=\mathscr{D}(K)$, and $H-K$ has a bounded extension to all of $\mathscr{H}$. We write this extension as $H-K$. Let $D_t=(U_t-V_t)/t$ and suppose $\psi\in\mathscr{D}(H)=\mathscr{D}(K)$. Then

$$(D_t-i(H-K))\psi=\left(\frac{U_t-1}{t}-iH\right)\psi+\left(\frac{V_t-1}{t}-iK\right)\psi$$

and so D_t converges strongly to $i(H-K)$ as $t\to 0$. Since $D_t^*=D_{-t}$ converges to $-i(H-K)$ one has (the D_t are uniformly bounded) that

$$|D_t|^2=D_t^*D_t\to |H-K|^2 \text{ strongly} .$$

Since the $|D_t|^2$ are uniformly bounded, we may apply the functional calculus to conclude that

$$|D_t|\to |H-K| \text{ strongly} .$$

Finally one has for some M and any orthonormal basis $\{\psi_i\}$

$$M \geqq \overline{\lim_{t\to 0}} \|(\mathrm{U}_t - \mathrm{V}_t)/\mathrm{t}\|_{\mathrm{Tr}}$$

$$= \overline{\lim_{t\to 0}} \sum_{i\geqq 1} (|\mathrm{D}_t|\psi_i|\psi_i)$$

$$\geqq \sum_{i=1}^{N} \lim_{t\to 0} (|D_t|\psi_i|\psi_i)$$

$$= \sum_{i=1}^{N} (|H-K|\psi_i|\psi_i) \text{ for any integer } N\,.$$

Then $\|H-K\|_{\mathrm{Tr}} < \infty$.

3. ⇒ 4.

We again have the integral relation

$$\left(\frac{\alpha_t - \beta_t}{t}\right)(A) = \frac{1}{t}\int_0^t ds\, \alpha_s(\delta_H - \delta_K)\beta_{t-s}(A)$$

and so $\left\|\frac{\alpha_t - \beta_t}{t}\right\| \leqq \|\delta_H - \delta_K\|$.

4. ⇒ 3.

Applying the remarks preceding this theorem we may lift the automorphism groups α_t, β_t to automorphism groups of the von Neumann algebra $\mathscr{M}$ associated with the Fock space representation [in this case $\mathscr{M} = \mathscr{L}(\mathscr{H}_F)$]. We write $\tilde{\alpha}_t, \tilde{\beta}_t$ for the extensions. These are adjoint semi-groups and the operator norm of $(\tilde{\alpha}_t - \tilde{\beta}_t)/t$ in $\mathscr{M}$ is by Kaplansky's density theorem, [15], equal to that of $(\alpha_t - \beta_t)/t$ on $\mathfrak{A}$.

Let $\tilde{\delta}_H$ and $\tilde{\delta}_K$ be the generators of the extended automorphism groups. Then Theorem 2 of [14] shows that $\mathscr{D}(\tilde{\delta}_H) = \mathscr{D}(\tilde{\delta}_K)$ and $\|\tilde{\delta}_H - \tilde{\delta}_K\| < \infty$ on this common domain. We claim that the proof will be finished once we show that $\mathscr{D}(\delta_H) \cap \mathscr{D}(\delta_K)$ is a joint core of δ_H and δ_K. Indeed we know that $\tilde{\delta}_H - \tilde{\delta}_K$ is a bounded derivation, say δ, on the joint strong continuity subspace of $\tilde{\alpha}$ and $\tilde{\beta}$. Then $\delta(\mathscr{D}(\delta_H) \cap \mathscr{D}(\delta_K)) \subseteqq \mathfrak{A}$ and so by closure $\delta(\mathfrak{A}) \subseteqq \mathfrak{A}$; that is δ extends to a bounded derivation of $\mathfrak{A}$. Since $\delta_H = \delta_K + \delta$ on $\mathscr{D}(\delta_H) \cap \mathscr{D}(\delta_K)$, it then follows by taking the closure on both sides that $\mathscr{D}(\delta_H) = \mathscr{D}(\delta_K)$ and $\|\delta_H - \delta_K\| < \infty$.

We now complete the argument by showing that $\mathscr{D}(\delta_H) \cap \mathscr{D}(\delta_K)$ is indeed a joint core of δ_H and δ_K.

By restricting α_t, β_t to any $a(f)$ one has, using $\|a(f)\| = \|f\|$, that $\|(U_t - V_t)/t\| \leqq M$. Here the norm is the usual operator norm on $\mathscr{H}$. One then has, using [14, Corollary 3], that $\mathscr{D}(H) = \mathscr{D}(K)$. Taking f's from the latter subspace and generating the corresponding monomials in the $a(f)$'s gives a subspace of $\mathscr{D}(\delta_H) \cap \mathscr{D}(\delta_K)$ which is dense in $\mathfrak{A}$. Since $e^{itH}\mathscr{D}(H) = \mathscr{D}(H)$, $e^{itH}\mathscr{D}(K) = \mathscr{D}(K)$, the subspace is furthermore invariant under the action of α_t and β_t. Hence it is a core for both δ_H and δ_K by [21], Theorem 3.

1.$\Leftrightarrow$3.

This equivalence is essentially a restatenent of Araki's result [2] since the derivation induced by $H-K$ coincides with $\delta_H - \delta_K$.

We now give an abstract version of part of the previous theorem. It is necessary to impose an additional restriction as we show in Examples 2.5 and 4.6. The Fock space argument of the previous theorem can be replaced by a more general argument.

Theorem 4.4. *Let $\mathfrak{A}$ be a simple C^*-algebra with identity and strongly continuous one-parameter $*$-automorphism groups α_t, β_t. Write $\alpha_t = \exp(t\delta_\alpha)$; $\beta_t = \exp(t\delta_\beta)$ for derivations $\delta_\alpha, \delta_\beta$ of $\mathfrak{A}$. If $\|\alpha_t - \beta_t\| = O(t)$ then there exists a (faithful) representation π of $\mathfrak{A}$ such that α_t, β_t extend to σ-weakly continuous $*$-automorphism groups $\tilde{\alpha}_t, \tilde{\beta}_t$ of $\pi(\mathfrak{A})''$. If $\delta_{\tilde{\alpha}}, \delta_{\tilde{\beta}}$ denote the corresponding generators of $\tilde{\alpha}_t, \tilde{\beta}_t$ then $\mathcal{D}(\delta_{\tilde{\alpha}}) = \mathcal{D}(\delta_{\tilde{\beta}})$ and $\|\delta_{\tilde{\alpha}} - \delta_{\tilde{\beta}}\| < \infty$. If $\mathcal{D}(\delta_\alpha) \cap \mathcal{D}(\delta_\beta)$ is in addition a joint core of δ_α and δ_β then $\|\alpha_t - \beta_t\| = O(t)$ is equivalent with $\mathcal{D}(\delta_\alpha) = \mathcal{D}(\delta_\beta)$ and $\|\delta_\alpha - \delta_\beta\| < \infty$.*

Proof. Consider the orbit of any state under the action of $t \to \alpha_t$. By applying an invariant mean we obtain a state ω which is invariant for the automorphism group α_t i.e. $\omega(\alpha_t(x)) = \omega(x)$ for all $t \in \boldsymbol{R}$.

Via the GNS procedure, one has a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathfrak{A}$; and a unitary representation of $\boldsymbol{R}$, $t \to V_t \in \mathcal{L}(\mathcal{H}_\omega)$, such that

$$\pi_\omega(\alpha_t(x)) = V_t \pi_\omega(x) V_{-t} = \tilde{\alpha}_t(\pi_\omega(x)) \ .$$

The representation is faithful since $\mathfrak{A}$ is simple. Next note that for t sufficiently small

$$\|\alpha_t \cdot \beta_{-t} - \iota\| < 2 \ .$$

Here ι is the identity automorphism. Results of Kadison and Ringrose [11], then give a unitary $U_t \in \mathcal{M} = \pi_\omega(\mathfrak{A})''$ such that

$$\pi_\omega(\alpha_t(\beta_{-t}(x))) = V_t \pi_\omega(\beta_{-t}(x)) V_{-t} = U_t \pi_\omega(x) U_t^* \ .$$

Unwinding one has that

$$\pi_\omega(\beta_{-t}(x)) = V_{-t} U_t \pi_\omega(x) U_t^* V_t, \quad \text{for small} \quad t \ .$$

Thus β_t extends to an automorphism $\tilde{\beta}_t$ of $\mathcal{M}$ for all sufficiently small t, and by iteration, for all t. The extension is moreover σ-weakly continuous in t, for given $x \in \mathcal{M}$ and $\phi \in \mathcal{M}_*$ one has

$$|\phi(\beta_t(x) - x)| \leqq |\phi(\alpha_t(x) - x)| + \|\alpha_t(x) - \beta_t(x)\| \ .$$

But by Kaplansky's density theorem [15],

$$\sup_{x\in\mathcal{M}}\left\|\frac{\alpha_t(x)-\beta_t(x)}{\|x\|}\right\| = \sup_{x\in\mathfrak{A}}\left\|\frac{\alpha_t(x)-\beta_t(x)}{\|x\|}\right\| \leqq M|t|$$

for t small. Thus continuity follows by the group property.

We now have dual semi-groups and so may conclude the first part of the theorem by applying [14, Theorem 2].

If in addition one has that $\mathscr{D}(\delta_\alpha)\cap\mathscr{D}(\delta_\beta)$ is a joint core of δ_α and δ_β we reach our conclusion by arguing as in 4)$\Rightarrow$3) of the previous theorem.

Conversely suppose that $\mathscr{D}(\delta_\alpha)=\mathscr{D}(\delta_\beta)$ and $\|\delta_\alpha-\delta_\beta\|<\infty$. Then choosing x in the common domain one has the integral relation.

$$\alpha_t(x)-\beta_t(x)=\int_0^t ds\,\alpha_s(\delta_\alpha-\delta_\beta)\beta_{t-s}(x)\,.$$

The $O(t)$ behaviour is then clear since such x are dense. The general setting for the first two examples will be a C^*-algebra, with unit acting on a Hilbert space $\mathscr{H}$ and two strongly continuous one parameter automorphism groups α_t, β_t of $\mathfrak{A}$ which extend to weakly continuous automorphism groups of $\mathfrak{A}''$. Let $\delta_\alpha, \delta_\beta$ be the generators of α_t, β_t respectively as automorphism groups of $\mathfrak{A}$, and $\hat{\delta}_\alpha, \hat{\delta}_\beta$ the corresponding generators as groups of automorphisms of $\mathfrak{A}''$. It is clear that $\delta_\alpha\subseteqq\hat{\delta}_\alpha, \delta_\beta\subseteqq\hat{\delta}_\beta$. Moreover $\mathscr{D}(\delta_\alpha)=(1+\lambda\hat{\delta}_\alpha)^{-1}(\mathfrak{A})$, $\mathscr{D}(\delta_\beta)=(1+\lambda\hat{\delta}_\beta)^{-1}(\mathfrak{A})$ for any real λ.

In the following two examples δ_α is obtained from δ_β in the following manner: We choose a unitary $v\in M$ such that $v\in\mathscr{D}(\hat{\delta}_\alpha)$, $v\notin\mathscr{D}(\delta_\alpha)$ and set

$$\delta_\beta(x)=v^*\delta_\alpha(vxv^*)v=\sigma^{-1}(\delta_\alpha(\sigma(x)))\,,$$

where $\sigma(x)=vxv^*$ is an inner automorphism. This relation persists for the extensions $\hat{\delta}_\alpha, \hat{\delta}_\beta$ so that one has

a) $\mathscr{D}(\delta_\beta)=v^*\mathscr{D}(\delta_\alpha)v$;

b) $\mathscr{D}(\hat{\delta}_\beta)=\mathscr{D}(\hat{\delta}_\alpha)$;

c) $$\begin{aligned}\hat{\delta}_\beta(x) &= v^*\hat{\delta}_\alpha(x)xv^*v+v^*v\hat{\delta}_\alpha(x)v^*v\\ &\quad+v^*vx\hat{\delta}_\alpha(v^*)v\\ &=\hat{\delta}_\alpha(x)+[v^*\hat{\delta}_\alpha(v),x]\,.\end{aligned}$$

The last two conditions show that $\|\alpha_t-\beta_t\|=O(t)$, $t\to 0$, as may be seen by applying Theorem 2 of [15] or computing directly. We shall show that v may be chosen so as to make

$$\mathscr{D}(\delta_\beta)\cap\mathscr{D}(\delta_\alpha)=v^*\mathscr{D}(\delta_\alpha)v\cap\mathscr{D}(\delta_\alpha)$$

a non-dense subspace of $\mathfrak{A}$.

Example 4.5. In this example $\mathfrak{A}$ is the common strong continuity subspace of α and β but $\mathscr{D}(\delta_\alpha)\cap\mathscr{D}(\delta_\beta)$ in not even σ-weakly dense in $\mathfrak{A}$.

Let M_2 be the complex 2×2 matrices and $C(\boldsymbol{T})$ the (complex valued) continuous functions on the circle group $\boldsymbol{T}$ with the usual supremum norm. For $\mathfrak{A}$

we take $M_2 \otimes C(\boldsymbol{T})$ and consider $\mathfrak{A}$ represented on $= \mathbb{C}^2 \otimes L^2(\boldsymbol{T}) = L^2(\boldsymbol{T}) \oplus L^2(\boldsymbol{T})$, using Haar measure on $\boldsymbol{T}$.

Then $\mathfrak{A}'' = M_2 \otimes L^\infty(\boldsymbol{T})$, i.e. the von Neumann algebra of all 2×2 matrices with entries in $L^\infty(\boldsymbol{T})$.

Define $\alpha_t = \iota \otimes \tau_t$, where ι is the identity automorphism and $(\tau_t f)(x) = f(x-t)$ in the group defined by rotation of the circle (translations mod 2π). Then α_t is σ-weakly continuous on $\mathfrak{A}''$ and $\mathfrak{A}$ is just the strong continuity subspace of α_t.

Next let $\theta: \boldsymbol{T} \to \boldsymbol{R}$ be a uniformly Hölder continuous function i.e. $|\theta(t_2) - \theta(t_1)| < M|t_2 - t_1|$ for some constant M, but θ is non-differentiable on a dense set of points of. One such function θ is

$$\theta(s) = \sum_{n-1}^{\infty} \frac{1}{n!} |s - r_n| ,$$

where r_n is an enumeration of the rationals in $\boldsymbol{T}$. Let δ_τ and $\hat{\delta}_\tau$ be the generators of τ_t viewed as an automorphism group of $C(\boldsymbol{T})$, $L^\infty(\boldsymbol{T})$ respectively. Then $\theta \in \mathscr{D}(\hat{\delta}_\tau)$ (it is absolutely continuous) and $\theta \notin \mathscr{D}(\delta_\tau)$. Further $\|\hat{\delta}_\tau(\theta)\| \leqq M$. We verify directly that $e^{i\theta}$ is also Hölder continuous and hence in $\mathscr{D}(\hat{\delta}_\tau)$. Clearly $e^{i\theta} \notin \mathscr{D}(\delta_\tau)$.

Next define the unitary operator v by

$$\boldsymbol{T} \ni s \to \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta(s)} \end{pmatrix} .$$

Then $v \in \mathscr{D}(\hat{\delta}_\alpha)$, $v \notin \mathscr{D}(\delta_\alpha)$ and v is a unitary in $\mathfrak{A}$.

Given

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathfrak{A} ,$$

we have

$$v^* \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} v = \begin{pmatrix} f_{11} & e^{i\theta} f_{12} \\ e^{-i\theta} f_{21} & f_{22} \end{pmatrix} .$$

Now $\mathscr{D}(\delta_\alpha)$ consists of just the matrices (f_{ij}) where each f_{ij} is continuously differentiable. Since these functions are continuous f and $e^{i\theta} f$ can both be continuously differentiable only when $f = 0$. This shows that

$$\mathscr{D}(\delta_\alpha) \cap \mathscr{D}(\delta_\beta) = \left\{ \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \middle| \, f_{ii} \in \mathscr{D}(\delta_\tau) \right\}$$

so that $\mathscr{D}(\delta_\alpha) \cap \mathscr{D}(\delta_\beta)$ is not dense in $\mathfrak{A}$.

Example 4.6. This example is a variation of the first and provides us with a *simple* C^*-algebra with unit where $\mathscr{D}(\delta_\alpha) \cap \mathscr{D}(\delta_\beta)$ is not dense in $\mathfrak{A}$ and $\|\alpha_t - \beta_t\| = O(t)$.

We refer to $\mathfrak{A}, \alpha$, and v of the previous example as $\mathfrak{A}^0, \alpha^0, v^0$. Let t_0 be an irrational rotation of the circle and define

$$\tau = \alpha^0_{t_0} .$$

Then τ is an automorphism of $\mathfrak{A}^0$, which acts freely and ergodically on its spectrum. It follows from [9, 16] that the C^*-crossed product $\mathfrak{A}=C^*(\mathfrak{A}^0,\tau) =M_2\otimes C^*(C(\boldsymbol{T}),\tau)$ is simple. Since this crossed product is discrete we have a canonical imbedding of $\mathfrak{A}^0$ into $\mathfrak{A}$ and since α_t^0 commutes with τ,α_t^0 extends canonically to a strongly continuous group α_t of automorphisms of $\mathfrak{A}$. Using Zeller-Meier's approach [19] we define the von Neumann crossed product of $\mathfrak{A}^{0\prime\prime}=M_2\otimes L^\infty(\boldsymbol{T})$ by τ as $VN(\mathfrak{A}^{0\prime\prime},\tau)$. The simplicity of $\mathfrak{A}$ allows us to represent $\mathfrak{A}$ faithfully in $VN(\mathfrak{A}^{0\prime\prime},\tau)$. Keeping the same letter we see that $\mathfrak{A}''=VN(\mathfrak{A}^{0\prime\prime},\tau)$ and α_t extends to a σ-weakly continuous group of automorphisms of $VN(\mathfrak{A}^{0\prime\prime},\tau)$. Now $v=v^0\in\mathfrak{A}^0\subseteqq\mathfrak{A}$ and so $v\in\mathscr{D}(\hat{\delta}_\alpha)$, $v\notin\mathscr{D}(\delta_\alpha)$. One can then show, as in Example 1, that

$$\mathscr{D}(\delta_\alpha)\cap\mathscr{D}(\delta_\beta)\subseteqq C^*(\mathfrak{A}^{00},\tau)\ ,$$

viewed as a subalgebra of $VN(\mathfrak{A}^{0\prime\prime},\tau)$, where $\mathfrak{A}^{00}=\left\{\begin{pmatrix} f_{11} & 0\\ 0 & f_{22}\end{pmatrix}\middle|\, f_{ij}\in\mathscr{D}(\delta_\tau)\right\}$.

The techniques used in the first two examples can be used to show that the saturation property of de Leeuw [13] does not serve to characterize the domain of the generator of a one parameter automorphism group of a simple C^*-algebra with unit. This property says that given a one parameter group of automorphism τ_t the domain of its generator is identical with those x such that $\|\tau_t(x)-x\|=O(t)$.

Example 4.7. There exists a simple C^*-algebra $\mathfrak{A}$ with unit, a one parameter group. τ_t, of automorphisms of $\mathfrak{A}$ with generator δ and an element $x\in\mathfrak{A}$ such that $\|\tau_t(x)-x\|=O(t)$, $t\to 0$, but $x\notin\mathscr{D}(\delta)$. The simple C^*-algebra $\mathfrak{A}$ is the C^*-crossed product of $\mathfrak{A}_0=C(\boldsymbol{T})$ by an irrational rotation. The group of automorphisms τ_t is that gotten by lifting to the crossed product the automorphism of $C(\boldsymbol{T})$ given by rotation through an angle t. Call this automorphism τ_t^0. Since $\mathfrak{A}_0$ is embedded in $\mathfrak{A}$ we have $\delta_0\subseteqq\delta$ and $\mathscr{D}(\delta_0)=\mathscr{D}(\delta)\cap\mathfrak{A}_0$ where δ_0,δ are the generators of τ_t^0,τ_t respectively. If we take for $x\in\mathfrak{A}_0$ a Hölder continuous function f, which is non-differentiable, one has that $x\notin\mathscr{D}(\delta_0)$ and by the above remark $x\notin\mathscr{D}(\delta)$.

We show in the next example that commutativity is required for Proposition 4.1. The following two examples delineate the distinction between our results and those obtained in [5].

Example 4.8. For any $0\leqq\varepsilon\leqq 2$ there exists two strongly continuous one-parameter unitary groups U_t, V_t on a separable Hilbert space $\mathscr{H}$ such that

$$\|U_t-V_t\|=\varepsilon \qquad t\in\boldsymbol{R}\backslash\{0\}\ .$$

Proof. Let $\mathscr{H}=L^2(R)$ and define U_t by

$$(U_tf)(s)=f(s-t)\ .$$

For ε given, choose $\delta=\arcsin(\varepsilon/2)$ and define a function θ by

$$\theta(s)=\delta\sin(s^2)\ .$$

Next define a unitary operator W on by

$$(Wf)(s)=e^{i\theta(s)}f(s)$$

and define

$$V_t = WU_tW^* .$$

Then

$$\|V_t - U_t\| = \|WU_tW^*U_{-t} - 1\| .$$

By an easy calculation:

$$(WU_tW^*U_{-t}f)(s) = e^{i(\theta(s)-\theta(s-t))}f(s)$$

so

$$\|V_t - U_t\| = \sup_s |e^{i(\theta(s)-\theta(s-t))} - 1| .$$

It is then clear that

$$\|U_t - V_t\| \leqq |e^{i(\delta+\delta)} - 1| = \varepsilon$$

for all t. Next fix $t \neq 0$ and $\varepsilon' > 0$. We will exhibit an s such that

$$\theta(s) = \delta$$

$$|\theta(s-t) + \delta| < \varepsilon'$$

i.e.

$$|\theta(s) - \theta(s-t) - 2\delta| < \varepsilon' .$$

This will end the proof. It amounts to showing that there exists integers n, m such that simultaneously:

$$s^2 - (s-t)^2 \cong \pi + 2n\pi$$

$$s^2 = \frac{\pi}{2} + 2m\pi .$$

Since $s^2 - (s-t)^2 = 2st - t^2$, this amounts to finding n, m such that

$$2\sqrt{\frac{\pi}{2} + 2m}\, t - t^2 \cong \pi + 2n\pi .$$

But since the derivative of the function $x \to \sqrt{x}$ tends to 0 as $x \to +\infty$, while the function itself tends to ∞, this is clearly possible for any ε'. This shows that $\|U_t - V_t\| \geqq \varepsilon$ for $t \neq 0$ i.e.

$$\|U_t - V_t\| = \varepsilon \qquad t \neq 0 .$$

One easily refines the technique of the proof given here to show that $\mathrm{Sp}(WU_tW^*U_{-t}) = e^{i[-2\delta, 2\delta]}$. This implies the following:

Example 4.9. If $\mathscr{M} = \mathscr{L}(\mathscr{H})$, $\mathscr{H}$ a separable Hilbert space, then for any δ such that $0 \leqq \delta \leqq 2$ there exists two σ-weakly continuous one-parameter groups α_t, β_t of *-automorphisms of $\mathscr{L}(\mathscr{H})$ such that

$$\|\alpha_t - \beta_t\| = \delta \qquad t \in \boldsymbol{R} \backslash \{0\} .$$

Proof. Use U_t, V_t from Example 4.8 with $\varepsilon = \sqrt{2(1-\sqrt{1-\varepsilon^2/4})}$ and set $\alpha_t = U_t \circ U_t^*$, $\beta_t = V_t \circ V_t^*$. Then

$$\|\beta_t - \alpha_t\| = \|\beta_t \alpha_{-t} - i\|$$
$$= \|\mathrm{ad}(WU_t W^* U_{-t}) - \mathrm{ad}(1)\| = \delta .$$

Next we give a physically more interesting example of unitary groups such that $\|U_t - V_t\| < \varepsilon$ for all t.

Example 4.10 (Smooth Interactions). Let U_t, V_t be two unitary groups on the Hilbert space $\mathscr{H}$ with generators H_0 and $H = H_0 + V$ where V is a self-adjoint operator such that

$$\| \, |V|^{1/2}(H_0 - z)^{-1} |V|^{1/2} \| \leqq N < 1$$

for all z with $\mathrm{Im}\, z \neq 0$. It follows by a perturbation calculation that

$$\| \, |V|^{1/2}(H - z)^{-1} |V|^{1/2} \| \leqq N(1-N)^{-1}, \qquad \mathrm{Im}\, z \neq 0 .$$

Moreover if $\|A\|_H$ is defined by

$$\|A_H\|^2 = \sup_{\psi \in \mathscr{H}} \int_{-\infty}^{\infty} dt \|A e^{iHt} \psi\|^2 / \|\psi\|^2$$

then one may deduce [30] that

$$\| \, |V|^{1/2} \|_{H_0}^2 < 2N , \qquad \| \, |V|^{1/2} \|_H^2 < 2N(1-N)^{-1} .$$

But one then has

$$|(\varphi, (U_t - V_t)\psi)| = \left| \int_0^t ds (\varphi, U_s V V_{t-s} \psi) \right|$$
$$\leqq \|\varphi\| \, \|\psi\| \, \| \, |V|^{1/2} \|_{H_0} \| \, |V|^{1/2} \|_H .$$

Therefore

$$\|U_t - V_t\| \leqq 2N(1-N)^{1/2} .$$

[In particular if $N < (\sqrt{5} - 1)/2$ then $\|U_t - V_t\| < 2$ for all $t \in \boldsymbol{R}$.] It is also possible to show that

$$\|(H_0 - z)^{1/2} (U_t - V_t)(H - z)^{1/2}\| \leqq 2|t| N(1-N)^{1/2} .$$

A specific situation in which this example applies is given by $= L^2(\boldsymbol{R}^3)$, $H_0 = -\nabla^2$ and V a multiplication operator $(V\psi)(x) = V(x)\psi(x)$ such that $V(x)$ is real and

$$(1/4\pi) \left[\int dx dy |V(x)| \, |V(y)| / (x-y)^2 \right]^{1/2} < 1 .$$

This example occurs in scattering theory and one can show that

$$W = \underset{t \to \infty}{\text{strong limit}} \, U_t V_{-t}$$

exists, is unitary, and $U_t W = W V_t$ (see for example, [31]).

Acknowledgement. We are pleased to acknowledge useful conversations with H. Araki. G. Elliot, U. Haagerup, R. V. Kadison, C. Moore, J. Roberts, and M. Takesaki. For invitations which made this collaboration possible we are all grateful to Ludwig Streit at the Universität Bielefeld, the first two authors to the Universite de Toulon and C.N.R.S. Marseille, the first and third named authors to the Mathematics Department at The Pennsylvania State University. The first named author is grateful for support from the Norwegian Research Council for Science and Humanities and the second named for the support of the Alexander von Humboldt Stiftung.

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Communicated by J. Glimm

Received September 27, 1977; in revised form December 5, 1977