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Existence of Three Phases for a $P(\varphi)_2$ Model of Quantum Field

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Abstract. In the two-dimensional model of the quantum field theory with lagrangean density $:\frac{1}{2}(\partial_\mu\varphi)^2-(\frac{1}{2}-\nu)\varphi^2+\lambda^{1/2}\varphi^4-\frac{1}{2}\lambda\varphi^6:$ there exist (at least) three different phases for small λ and some $\nu(\lambda)$.

1. Introduction

In recent years much of the work in constructive two-dimensional quantum field theory was devoted to the study of phase transitions [6, 7, 2, 3]. In most models considered so far phase transitions were accompanied by spontaneous symmetry breakdown. The exception is the Fröhlich's proof of existence of two phases for a $(\lambda(Q(\varphi)+\varepsilon P(\varphi))-\nu\varphi^2-\mu\varphi)_2$ model [3, Theorem 7.6].

We consider a model with the 6th order polynomial interaction $:\frac{1}{2}\lambda\varphi^6-\lambda^{1/2}\varphi^4-\nu\varphi^2-\mu\varphi:$ and show that for small λ, some $\nu(\lambda)$ ($\nu(\lambda)\to 0$ when $\lambda\to 0$) and $\mu=0$ there are at least three different states corresponding in the Euclidean framework to the formal expression

$$\frac{1}{Z}e^{-\int:(\frac{1}{2}\lambda\varphi^6-\lambda^{1/2}\varphi^4-\nu\varphi^2-\mu\varphi):}d\mu_1\,,$$

where $d\mu_1$ denotes the free, mass 1 measure. Appearance of only two of them is connected with spontaneous breakdown of the $\varphi\mapsto-\varphi$ symmetry.

Conventional wisdom based on the mean field approximation predicts existence of three phases for the considered model (see [7, 14]). Our result shows that the quantum corrections do not distroy the qualitative character of the picture based on the classical approximation to the effective potential. This does not seem to be obvious as the Wick ordering tends to fill up the middle minimum of $\frac{1}{2}\lambda x^6-\lambda^{1/2}x^4+(\frac{1}{2}-\nu)x^2$ relative to the two others. The problem of multiplicity of phases for $P(\varphi)_2$ models was also studied in [13], where existence of only two phases was predicted for a class of polynomials however not embracing the case studied here.

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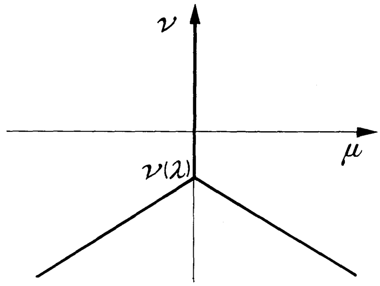

Fig. 1

Our result indicates that the phase diagram in the μ, ν plane should look somewhat as on Figure 1. However we are far from proving this even locally. Our method, which is patterned after the Fröhlich-Simon-Spencer's proof of existence of phase transition in a statistical-mechanical model with no single-spin symmetry [3, 5] and after the Fröhlich's proof of existence of phase transition for $(\lambda(Q(\varphi) + \varepsilon P(\varphi)) - \nu\varphi^2 - \mu\varphi)_2$ [3], seems not very well suited for deeper analysis of the phase diagram even in the statistical-mechanical lattice case with discrete spin where a powerful method of Pirogov and Sinai [11] is available. Giving limited information it works however for continuous spin and, as we demonstrate here, also in the continuum case.

2. The Strategy and the Main Result

Our strategy for proving existence of at least three phases in a $P(\varphi)_2$ model can be illustrated best in the simpler case of a $\mathbb{Z}^d$ lattice spin 1 system ($d \geqq 2$). As we mentioned before it follows closely the Fröhlich-Simon-Spencer's proof [5, Theorem 3.5] of existence of phase transition in the lattice model with no symmetry single-spin distribution for some value of external magnetic field (compare also [3, Section 8.2c]).

Denote a spin configuration by $\sigma = (\sigma_x)_{x \in \mathbb{Z}^d}$, $\sigma_x \in \mathbb{R}^1$. We consider a model with the measure on configurations formally given by

$$d\mu_{J,\nu} = \frac{1}{Z} \exp\left[-\frac{J}{2} \sum_{\{x,y\}} (\sigma_x - \sigma_y)^2\right] \prod_x (\delta(\sigma_x + 1) + \delta(\sigma_x) + \delta(\sigma_x - 1)) \exp(\nu\sigma_x^2), \quad (1)$$

where $\sum_{\{x,y\}}$ denotes the sum over pairs of nearest neighbor lattice sites, $J \geqq 0$, ν is real. Take the infinite volume pressure $\alpha_\infty^{J,\nu}$ connected with this model. This is a convex function of ν (and J). $\lim_{J \to \infty} \alpha_\infty^{J,\nu}$ is easily computable and equals 0 for $\nu \leqq 0$ and ν for $\nu \geqq 0$.

Now by the Gaussian domination bound of Fröhlich et al. [5] it can be easily shown that for each $\varepsilon > 0$ there exists J_0 such that for $J > J_0$ (in periodic states)

$$\langle \sigma_x^2 (1 - \sigma_y^2) \rangle_{J,\nu} \leqq \tfrac{1}{4} - \varepsilon \quad (2)$$

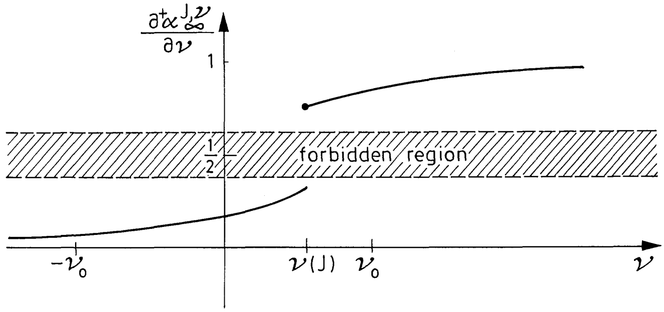

Fig. 2

for any ν, x, y. But whenever $\frac{\partial \alpha_\infty^{J,\nu}}{\partial \nu}$ exists $\langle \sigma_x^2(1-\sigma_y^2)\rangle_{J,\nu}$ clusters (argument of Guerra [8]). Thus

$$\langle \sigma_x^2 \rangle_{J,\nu}(1-\langle \sigma_x^2\rangle_{J,\nu}) \leqq \tfrac{1}{4} - \varepsilon$$

and there is a forbidden interval for $\langle \sigma_x^2\rangle_{J,\nu}$ around $\frac{1}{2}$.

Take now $\nu_0 > 0$.

$$\lim_{J\to\infty} \langle \sigma_x^2 \rangle_{J,-\nu_0} = \frac{\partial}{\partial \nu}\bigg|_{\nu=-\nu_0} \lim_{J\to\infty} \alpha_\infty^{J,\nu} = 0\,, \tag{3}$$

$$\lim_{J\to\infty} \langle \sigma_x^2 \rangle_{J,\nu_0} = \frac{\partial}{\partial \nu}\bigg|_{\nu=\nu_0} \lim_{J\to\infty} \alpha_\infty^{J,\nu} = 1\,. \tag{4}$$

When $J \geqq J_1(\varepsilon, \nu_0)$ we conclude that in the plot of the right-hand derivative $\frac{\partial^+ \alpha_\infty^{J,\nu}}{\partial \nu}$ (versus ν) there is a jump over the forbidden interval at some $\nu(J)$, $|\nu(J)| \leqq \nu_0$.

Three phases can be constructed as follows. We obtain the 1st state by taking limit of periodic ones $\langle\ \rangle_{J,\nu}$ when $\nu \uparrow \nu(J)$. Then we construct $\langle\ \rangle_{J,\nu}^{\pm}$ states by turning off external magnetic field in periodic states and take their limit when $\nu \downarrow \nu(J)$ in turn. This way we arrive at the 2nd and the 3rd state. Expectations of σ_x^2 in the 1st state and the two others differ since they lie below and above the forbidden interval, respectively. Then, using the Peierls argument, one shows that the 2nd and the 3rd states develope non-vanishing expectations of σ_x differing by sign.

Alternatively one can construct three different states using appropriate $+$, $-$, or 0 boundary conditions.

We choose

$$P_{\lambda,\nu}(x) = \tfrac{1}{2}\lambda x^6 - \lambda^{1/2} x^4 - \nu x^2\,, \qquad \lambda > 0\,. \tag{5}$$

ν will be always restricted to the interval $[-\frac{1}{20}, \frac{1}{20}]$. Let

$$U_{\lambda,\nu}^{\Lambda} := \int_{\Lambda} :P_{\lambda,\nu}(\varphi):\,. \tag{6}$$

Throughout the paper : : will denote the Wick ordering with respect to the free, mass 1 measure (Gaussian with covariance $-\Delta + 1$).

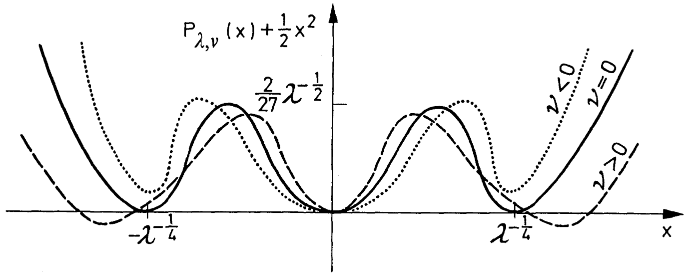

Fig. 3

We deal with the infinite volume half-Dirichlet states $d\mu_{\lambda,\nu}$ [12]. Thus

$$d\mu_{\lambda,\nu}=\lim_{\Lambda\to\infty}\frac{1}{Z}e^{-U_{\lambda,\nu}^{\Lambda}}d\mu_{1,\Lambda}^{D}=\lim_{\Lambda\to\infty}d\mu_{\lambda,\nu}^{\Lambda}, \tag{7}$$

where $d\mu_{1,\Lambda}^{D}$ is Gaussian with Dirichlet boundary condition covariance $-\Delta_{\Lambda}^{D}+1$ and the limit of measures is understood in terms of moments or characteristic functionals [12].

In tree approximation phases are determined by minima of $P_{\lambda,\nu}(x)+\frac{1}{2}x^2$. The latter is a polynomial with three local minima: one at zero (value zero) and two others at $\pm\xi_+$,

$$\xi_+\equiv\xi_+^{\lambda,\nu}=[\tfrac{2}{3}+(\tfrac{1}{9}+\tfrac{2}{3}\nu)^{1/2}]^{1/2}\lambda^{-1/4}$$

(value

$$E_0\equiv E_0^{\lambda,\nu}=\tfrac{1}{27}[1-18\nu-(1+6\nu)^{3/2}]\lambda^{-1/2},$$

$E_0<0$ when $\nu>0$, $E_0=0$ when $\nu=0$, $E_0>0$ when $\nu<0$). Thus from naive considerations it follows that three phases should occur for small λ at some $\nu(\lambda)$ close to zero.

However the Wick ordering tends to change this picture: it deepens the external wells with respect to the middle one. Nevertheless the behavior of pressures and expectations in λ is determined for small λ in the leading term by the naive classical picture. This makes possible application of the strategy described above.

Introduce in R^2 a lattice $\{\Delta_\alpha\}$ of squares of volume $|\Delta|=10^{-2}$. Our basic technical result is

Proposition 1. a) *For each* $\varepsilon>0$ *there exists* $0<\lambda_0(\varepsilon)$ *such that for each* λ, $0<\lambda\leqq\lambda_0(\varepsilon)$, *each* ν *and each* α, β

$$\left|\lambda\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right)\right\rangle_{\lambda,\nu}\right|\leqq\varepsilon. \tag{8}$$

b) *For* $0<\nu_0\leqq\frac{1}{20}$

$$\lim_{\lambda\to 0}\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,-\nu_0}=0, \tag{9}$$

$$\lim_{\lambda\to 0}\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu_0}=\tfrac{2}{3}+(\tfrac{1}{9}+\tfrac{2}{3}\nu_0)^{1/2}. \tag{10}$$

c) *There exists* $C>0$ *such that for* $0<\lambda\leqq\lambda_0$ *and all* ν

$$\left|\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}-\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu}\right|\leqq C. \tag{11}$$

Let $\alpha_\infty^{\lambda,\nu}$ denote the pressure connected with our model. Thus

$$\alpha_\infty^{\lambda,\nu}=\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-U_{\lambda,\nu}^{\Lambda}}d\mu_1. \tag{12}$$

$\alpha_\infty^{\lambda,\nu}$ is a convex function of ν. We shall also need the following

Proposition 2. *If* $\frac{\partial\alpha_\infty^{\lambda,\nu}}{\partial\nu}$ *exists then* $\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right\rangle_{\lambda,\nu}$ *clusters in mean.*

From Propositions 2 and 1.a with $\varepsilon\leqq\frac{1}{8}$ it follows that for $0<\lambda\leqq\lambda_0$ whenever $\frac{\partial\alpha_\infty^{\lambda,\nu}}{\partial\nu}$ exists $\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}$ cannot fall into $]\frac{1}{2}-\delta,\frac{1}{2}+\delta[$. Taking, if necessary, smaller λ_0 we can assume that

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,-\nu_0}\leqq\tfrac{1}{2}-\delta$$

and (13)

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu_0}\geqq\tfrac{1}{2}+\delta$$

(Proposition 1.b). Define

$$A_\lambda:=\left\{\nu\in[-\tfrac{1}{20},\tfrac{1}{20}]:\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}\leqq\tfrac{1}{2}-\delta\right\}. \tag{14}$$

A_λ is not empty since $-\nu_0\in A_\lambda$. Because $\langle:\varphi^2:(\Delta)\rangle_{\lambda,\nu}$ is non-decreasing in ν (2nd GKS), A_λ is convex. It is also bounded above by ν_0. Let

$$\nu(\lambda):=\sup A_\lambda. \tag{15}$$

$$\begin{aligned}\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu(\lambda)}&=\lambda^{1/2}\lim_{\Lambda\to\infty}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu(\lambda)}^{\Lambda}\\&=\lambda^{1/2}\lim_{\Lambda\to\infty}\lim_{\nu\uparrow\nu(\lambda)}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}^{\Lambda}=\lambda^{1/2}\lim_{\nu\uparrow\nu(\lambda)}\lim_{\Lambda\to\infty}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}^{\Lambda}\\&=\lim_{\nu\uparrow\nu(\lambda)}\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}\leqq\tfrac{1}{2}-\delta.\end{aligned} \tag{16}$$

(We could interchange the limits because of monotonicity of $\langle:\varphi^2:(\Delta)\rangle_{\lambda,\nu}^{\Lambda}$ both in Λ and ν.)

Hence, by Proposition 1.c,

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu(\lambda)}\leqq\tfrac{1}{2}(1-\delta) \tag{17}$$

for $0<\lambda\leqq\lambda_0$ and all ν $\left(\text{as we can take } \lambda_0 \text{ such that } \lambda_0^{1/2}C\leqq\frac{\delta}{2}\right)$.

Now introduce external field and define $d\mu_{\lambda,\nu,\mu}$ as

$$\lim_{\Lambda\to\infty}\frac{1}{Z}e^{-U^{\Lambda}_{\lambda,\nu}+\mu\varphi(\Lambda)}d\mu^{D}_{1,\Lambda}$$

and limit states $\langle\ \rangle^{\pm}_{\lambda,\nu}$ and $\langle\ \rangle^{\pm}_{\lambda,\nu+0}$ by

$$\left\langle\prod_i\varphi(f_i)\right\rangle^{\pm}_{\lambda,\nu}:=\lim_{\mu\downarrow 0}\left\langle\prod_i\varphi(f_i)\right\rangle_{\lambda,\nu,\pm\mu}, \tag{18}$$

$$\left\langle\prod_i\varphi(f_i)\right\rangle^{\pm}_{\lambda,\nu+0}:=\lim_{\nu'\downarrow\nu}\left\langle\prod_i\varphi(f_i)\right\rangle^{\pm}_{\lambda,\nu'},\qquad f_i\geqq 0. \tag{19}$$

$\left(\text{We use monotonocity of } \left\langle\prod_i\varphi(f_i)\right\rangle_{\lambda,\nu,\mu} \text{ in both } \nu \text{ and } \mu.\right)$

Choose a sequence $\nu_n\downarrow\nu(\lambda)$ of points such that $\frac{\partial\alpha^{\lambda,\nu_n}_{\infty}}{\partial\nu}$ exist.

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle^{\pm}_{\lambda,\nu(\lambda)+0}\geqq\lim_{\nu\downarrow\nu(\lambda)}\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu}=\lim_{n\to\infty}\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu_n}.$$

But

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu_n}\geqq\tfrac{1}{2}+\delta$$

since it cannot lie in $]\frac{1}{2}-\delta,\frac{1}{2}+\delta[$ and $\nu_n\notin A_\lambda$. Thus (Proposition 1.c)

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu_n}\geqq\tfrac{1}{2}(1+\delta) \tag{20}$$

and hence

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle^{\pm}_{\lambda,\nu(\lambda)+0}\geqq\tfrac{1}{2}(1+\delta)>\tfrac{1}{2}(1-\delta)\geqq\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu(\lambda)}.$$

States $\langle\ \rangle^{\pm}_{\lambda,\nu(\lambda)+0}$ differ from $\langle\ \rangle_{\lambda,\nu(\lambda)}$.

We are left with showing that $\langle\ \rangle^{+}_{\lambda,\nu(\lambda)+0}\neq\langle\ \rangle^{-}_{\lambda,\nu(\lambda)+0}$.

Proposition 3. *There exists $D>0$ such that for each λ, $0<\lambda\leqq\lambda_0$, each ν, $\nu>\nu(\lambda)$, each α,β*

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right\rangle_{\lambda,\nu}\geqq D. \tag{21}$$

From Proposition 3 it follows that

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right\rangle^{+}_{\lambda,\nu}\geqq D.$$

But, again in virtue of the argument of Guerra [8, 4], $\langle\varphi(\Delta_\alpha)\varphi(\Delta_\beta)\rangle^{+}$ clusters. Hence

$$\lambda^{1/4}\left\langle\frac{1}{|\Delta|}\varphi(\Delta)\right\rangle^{+}_{\lambda,\nu}\geqq D^{1/2}$$

and consequently

$$\lambda^{1/4}\left\langle\frac{1}{|\Delta|}\varphi(\Delta)\right\rangle^{+}_{\lambda,\nu(\lambda)+0}\geqq D^{1/2}. \tag{22}$$

Thus

$$\langle\varphi(\Delta)\rangle^{+}_{\lambda,\nu(\lambda)+0}=-\langle\varphi(\Delta)\rangle^{-}_{\lambda,\nu(\lambda)+0}>0. \tag{23}$$

We summarize the result in

Theorem. *Let $0<\nu_0\leqq\frac{1}{20}$. Let $0<\lambda_0=\lambda_0(\nu_0)$ be small enough. For each λ, $0<\lambda\leqq\lambda_0$, there exists $\nu(\lambda)$, $|\nu(\lambda)|\leqq\nu_0$, such that the states $\langle\ \rangle_{\lambda,\nu(\lambda)}$, $\langle\ \rangle^{-}_{\lambda,\nu(\lambda)+0}$ and $\langle\ \rangle^{+}_{\lambda,\nu(\lambda)+0}$ constructed above are different.*

The following sections are devoted to proofs of Propositions 1–3.

3. Proofs

The main tools in proving Propositions 1–3 are the chessboard estimate and the Gaussian domination bound.

1. Chessboard Estimate [4]

We shall need the chessboard estimate in the following form (it is not difficult to obtain it from the original one of [4]).

Let $F_\alpha\geqq 0$ be a function depending on field φ at points of a lattice square Δ_α. We shall consider functions of the following type

$$F_\alpha=f(:\varphi^2:(\Delta_\alpha),\varphi(\Delta_\alpha))\chi_I\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right), \tag{24}$$

where f is either a polynomial or $f(x,y)=\exp(ax+by)$ and χ_I is the characteristic function of an interval I.

Then

$$\left\langle\prod_\alpha F_\alpha\right\rangle_{\lambda,\nu}\leqq\exp\left[\sum_\alpha(\alpha^{\lambda,\nu}_\infty(F_\alpha)-\alpha^{\lambda,\nu}_\infty)|\Delta|\right], \tag{25}$$

where

$$\alpha^{\lambda,\nu}_\infty(F_\alpha)=\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int\prod_{\Delta_\beta\subset\Lambda}(F_\alpha)_\beta e^{-U^{\Lambda}_{\lambda,\nu}}d\mu_1. \tag{26}$$

Here, as throughout this paper, Λ runs through the set of squares built up from lattice squares. $(F_\alpha)_\beta$ denotes the function "living" in the lattice square Δ_β obtained from F_α by subsequent reflections in lines separating lattice squares.

2. Gaussian Domination Bound [2, 3, 5]

$$\left\langle\exp\left[\sum_{i=0,1}\varphi(\partial_i g^i)\right]\right\rangle_{\lambda,\nu}\leqq\exp\left[\sum_i\|g^i\|^2_{L^2}\right]. \tag{27}$$

This weak form of the Gaussian domination bound follows from the Glimm-Jaffe $\nabla\varphi$ bounds.

We start with the easiest (as standard).

Proof of Proposition 2. We use the argument of Guerra [8,4]. By 2nd GKS

$$\left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta_\alpha) \frac{1}{|\Delta|} :\varphi^2:(\Delta_\beta)\right\rangle_{\lambda,\nu} \geqq \left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta)\right\rangle^2_{\lambda,\nu}. \tag{28}$$

Put

$$\theta(\Lambda) := \sum_{\Delta_\alpha \subset \Lambda} (:\varphi^2: + B)(\Delta_\alpha) \quad \text{with} \quad B \in \mathbb{R}^1. \tag{29}$$

From the Hölder inequality and the chessboard estimate we get

$$\begin{aligned}
&\exp[\chi\langle\theta(\Lambda)^2\rangle^{1/2}_{\lambda,\nu}] \\
&\leqq 2\cosh[\chi\langle\theta(\Lambda)^2\rangle^{1/2}_{\lambda,\nu}] \leqq 2\langle\cosh(\chi\theta(\Lambda))\rangle_{\lambda,\nu} \\
&\leqq \exp[(\alpha^{\lambda,\nu+\chi}_\infty + \chi B - \alpha^{\lambda,\nu}_\infty)|\Lambda|] + \exp[(\alpha^{\lambda,\nu-\chi}_\infty - \chi B - \alpha^{\lambda,\nu}_\infty)|\Lambda|] \\
&\leqq 2\exp[(\alpha^{\lambda,\nu+\chi}_\infty + \chi B - \alpha^{\lambda,\nu}_\infty)|\Lambda|]
\end{aligned}$$

for each $0 < \chi \leqq \chi_0$ and each Λ if B is big enough. Hence

$$\frac{1}{|\Lambda|}\langle\theta(\Lambda)^2\rangle^{1/2}_{\lambda,\nu} \leqq \frac{\ln 2}{\chi|\Lambda|} + \frac{\alpha^{\lambda,\nu+\chi}_\infty - \alpha^{\lambda,\nu}_\infty}{\chi} + B$$

and

$$\limsup_{\Lambda\to\infty} \frac{1}{|\Lambda|}\langle\theta(\Lambda)^2\rangle^{1/2}_{\lambda,\nu} \leqq \frac{\partial^+\alpha^{\lambda,\nu}_\infty}{\partial\nu} + B.$$

On the other hand, as free and half-Dirichlet pressures coincide [12]

$$\begin{aligned}
\frac{\partial^-\alpha^{\lambda,\nu}_\infty}{\partial\nu} &\leqq \limsup_{\Lambda\to\infty} \frac{\partial\alpha^{\lambda,\nu}_{\Lambda,\mathrm{HD}}}{\partial\nu} \\
&= \limsup_{\Lambda\to\infty} \left\langle \frac{1}{|\Lambda|} :\varphi^2:(\Lambda)\right\rangle^{\Lambda}_{\lambda,\nu} \leqq \left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta)\right\rangle_{\lambda,\nu}
\end{aligned}$$

by monotonicity of the half-Dirichlet Schwinger functions in Λ. Hence if $\frac{\partial\alpha^{\lambda,\nu}_\infty}{\partial\nu}$ exists then

$$\limsup_{\Lambda\to\infty} \frac{1}{|\Lambda|^2}\langle\theta(\Lambda)^2\rangle_{\lambda,\nu} \leqq \left(\left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta)\right\rangle_{\lambda,\nu} + B\right)^2. \tag{30}$$

Consequently by (28) and (30)

$$\begin{aligned}
\left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta)\right\rangle^2_{\lambda,\nu} &\leqq \liminf_{\Lambda\to\infty} \frac{1}{|\Lambda|^2}\left\langle\left(\sum_{\Delta_\alpha\subset\Lambda} :\varphi^2:(\Delta_\alpha)\right)^2\right\rangle_{\lambda,\nu} \\
&\leqq \limsup_{\Lambda\to\infty} \frac{1}{|\Lambda|^2}\left\langle\left(\sum_{\Delta_\alpha\subset\Lambda} :\varphi^2:(\Delta_\alpha)\right)^2\right\rangle_{\lambda,\nu} \\
&\leqq \left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta)\right\rangle^2_{\lambda,\nu}
\end{aligned}$$

and $\left\langle \frac{1}{|\Delta|} :\varphi^2:(\Delta_\alpha) \frac{1}{|\Delta|} :\varphi^2:(\Delta_\beta)\right\rangle_{\lambda,\nu}$ clusters in mean. □

Proof of Proposition 1. We shall proceed in a series of lemmas. First five lemmas give bounds on various pressures. In proving them we follow the way paved by Glimm et al. in [7].

Denote

$$a_0 := \tfrac{1}{3}\lambda^{-1/4}, \quad a_1 := \tfrac{1}{2}\lambda^{-1/4},$$
$$a_2 := (\tfrac{2}{5})^{1/2}\lambda^{-1/4}, \quad a_3 := (\tfrac{1}{2})^{1/2}\lambda^{-1/4}. \tag{31}$$

$0 < a_0 < a_1 < a_2 < a_3$.

Lemma 1. *Let* $\varepsilon = \frac{1}{20}$.

$$\int e^{-(1+\varepsilon)U^{\Lambda}_{\lambda,v}} \prod_{\Delta_\alpha \subset \Lambda} \chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right) d\mu_1 \leqq e^{0(1)|\Lambda|}, \tag{32}$$

where $0(1)$ *does not depend on* λ, $0<\lambda \leqq \lambda_0$, v *and* Λ.

Proof of Lemma 1.

$$\int e^{-(1+\varepsilon)U^{\Lambda}_{\lambda,v}} \prod_{\Delta_\alpha \subset \Lambda} \chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right) d\mu_1$$
$$\leqq \Big\| \exp[-(1+\varepsilon)U^{\Lambda}_{\lambda,v} - (\tfrac{1}{2}-\eta):\varphi^2:(\Lambda)$$
$$-\zeta:(\delta\varphi)^2:(\Lambda)] \prod_{\Delta_\alpha \subset \Lambda} \chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\Big\|_{p'}$$
$$\cdot \|\exp[\zeta:(\delta\varphi)^2:(\Lambda) + (\tfrac{1}{2}-\eta):\varphi^2:(\Lambda)]\|_p, \tag{33}$$

where $\delta\varphi(x) := \varphi(x) - \frac{1}{|\Delta|}\varphi(\Delta_\alpha)$ if $x \in \Delta_\alpha$. For the rest of the paper we choose $\eta = \frac{1}{20}$, $\zeta = 300$, $p - 1 = \frac{1}{20}$.

Estimation of both terms is more or less standard [7]. The second term, by conditioning with respect to Neumann boundary condition Gaussian measure [7, 9], is dominated by

$$\left(\int e^{p[\zeta:(\delta\varphi)^2:(\Delta) + (\frac{1}{2}-\eta):\varphi^2:(\Delta)]} d\mu^N_{1,\Delta}\right)^{\frac{|\Lambda|}{p|\Delta|}}$$

which is easily computable and finite

$$\left(\text{equal}\left(\det{}_2\,[1 - p(2\zeta P + 1 - 2\eta)(-\Delta^N_\Delta + 1)^{-1}]\right)^{-\frac{|\Lambda|}{2p|\Delta|}}\right)$$

under the condition that in $L^2(\Delta)$ the operator

$$1 - p(2\zeta P + 1 - 2\eta)(-\Delta^N_\Delta + 1)^{-1} \quad \text{is strictly positive}. \tag{34}$$

Here P denotes the projection in $L^2(\Delta)$ onto the subspace of functions with vanishing integral and Δ^N_Δ is the Laplace operator in $L^2(\Delta)$ with Neumann boundary condition on $\partial\Delta$. Since the lowest eigenvalue of $-\Delta_\Delta^N P$ is $\frac{\pi^2}{|\Delta|}$, (34) holds if

$$p(2\zeta+1-2\eta)\left(\frac{\pi^2}{|\Delta|}+1\right)^{-1}<1$$

and

$$p(1-2\eta)<1\,.$$

This is the case for the chosen values of η, ζ, $|\Delta|$, and p. Thus the second term in the right hand side of (33) is bounded by $\exp(0(1)|\Lambda|)$.

In order to bound the first term it is sufficient to show that for each $q<\infty$

$$\left\|\exp\left[-(1+\varepsilon)U^{\Delta}_{\lambda,v}-(\tfrac{1}{2}-\eta):\varphi^2:(\Delta)-\zeta:(\delta\varphi)^2:(\Delta)\right]\right.$$
$$\left.\cdot\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\|_q\leqq 0(1) \tag{35}$$

since then we can use the checkerboard estimate of Guerra et al. [9].

To prove (35) we introduce the special cut-off field $\varphi_\varkappa$, the same Glimm et al. use in [7] except for adoption to the length scale of our lattice $\{\Delta_\alpha\}$. It has the nice property that $\varphi(\Delta)=\varphi_\varkappa(\Delta)$. (35) follows in a routine way [see e.g. 1, 12, 7] from two estimates uniform in λ, $0<\lambda\leqq\lambda_0$, and v.

1. $$\left\|(1+\varepsilon)\int_\Delta :P_{\lambda,v}(\varphi):+(\tfrac{1}{2}-\eta):\varphi^2:(\Delta)+\zeta:(\delta\varphi)^2:(\Delta)\right.$$
$$-(1+\varepsilon)\int_\Delta :P_{\lambda,v}(\varphi_\varkappa):-(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:(\Delta)$$
$$\left.-\zeta:(\delta\varphi_\varkappa)^2:(\Delta)\right\|_2\leqq 0(1)\varkappa^{-\delta} \tag{36}$$

for some $\delta>0$,

2. $$(1+\varepsilon)\int_\Delta :P_{\lambda,v}(\varphi_\varkappa):+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:(\Delta)+\zeta:(\delta\varphi_\varkappa)^2:(\Delta)$$
$$-\ln\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\geqq -0(1)(\ln\varkappa)^3 \tag{37}$$

(both for $\varkappa\geqq\varkappa_0$).

(36) is known (see [7], Lemma II.3.2.4). We shall prove (37) which is analogical to the "Wick ordering lower bound" of [7].

$$\begin{aligned}(1+\varepsilon)&:P_{\lambda,v}(\varphi_\varkappa):+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:=(1+\varepsilon)[\tfrac{1}{2}\lambda\varphi_\varkappa^6-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)\varphi_\varkappa^4\\&+(-v+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2)\varphi_\varkappa^2+vC_\varkappa-3\lambda^{1/2}C_\varkappa^2-\tfrac{15}{2}\lambda C_\varkappa^3]\\&+(\tfrac{1}{2}-\eta)\varphi_\varkappa^2-(\tfrac{1}{2}-\eta)C_\varkappa\geqq(1+\varepsilon)[\tfrac{1}{2}\lambda\varphi_\varkappa^6-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)\varphi_\varkappa^4\\&+\left(\frac{\frac{1}{2}-\eta}{1+\varepsilon}-v+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2\right)\varphi_\varkappa^2\Bigg]-0(1)(\ln\varkappa)^3\,,\end{aligned} \tag{38}$$

where $C_\varkappa=\int\varphi_\varkappa^2 d\mu_1=0(\ln\varkappa)$ for large $\varkappa$ (see (II.2.1.6) of [7]).

Polynomial $ax^6 - bx^4 + cx^2$ $(a, b, c, (b^2 - 3ac) > 0)$ has two local maxima at $x^2 = (b - (b^2 - 3ac)^{1/2})/3a$, a local minimum at zero and two local minima at $x^2 = (b + (b^2 - 3ac)^{1/2})/3a$. In our case we take

$$a = \tfrac{1}{2}\lambda, \quad b = \lambda^{1/2}(1 + \tfrac{15}{2}\lambda^{1/2}C_\varkappa), \quad c = \frac{\frac{1}{2} - \eta}{1 + \varepsilon} - v + 6\lambda^{1/2}C_\varkappa + \tfrac{45}{2}\lambda C_\varkappa^2, \tag{39}$$

$$b^2 - 3ac = \left(\frac{\frac{1}{4} + \varepsilon + \frac{3}{2}\eta}{1 + \varepsilon} + \tfrac{3}{2}v\right)\lambda + 6\lambda^{3/2}C_\varkappa + \tfrac{45}{2}\lambda^2 C_\varkappa^2 > 0. \tag{40}$$

The value of the polynomial at the external minima is equal

$$\frac{1}{27a^2}[-2(b^2 - 3ac)(b + (b^2 - 3ac)^{1/2}) + 3abc]$$

$$\geqq -\frac{2}{27a^2}[(b^2 - 3ac)(b + (b^2 - 3ac)^{1/2})] \geqq -\lambda^{-1/2}(1 + 8\lambda^{1/2}C_\varkappa)^3, \tag{41}$$

as

$$b^2 - 3ac \leqq \lambda(1 + 8\lambda^{1/2}C_\varkappa)^2 \quad \text{and} \quad b \leqq \lambda^{1/2}(1 + 8\lambda^{1/2}C_\varkappa).$$

In external minima

$$x^2 = (b + (b^2 - 3ac)^{1/2})/3a \geqq \frac{\lambda^{1/2} + \frac{1}{5}\lambda^{1/2}}{\frac{3}{2}\lambda} = \tfrac{4}{5}\lambda^{-1/2} \geqq a_3^2.$$

Hence if $\varphi_\varkappa \in [-a_3, a_3]$ then

$$\begin{aligned}
&\tfrac{1}{2}\lambda\varphi_\varkappa^6 - \lambda^{1/2}(1 + \tfrac{15}{2}\lambda^{1/2}C_\varkappa)\varphi_\varkappa^4 + \left(\frac{\frac{1}{2} - \eta}{1 + \varepsilon} - v + 6\lambda^{1/2}C_\varkappa + \tfrac{45}{2}\lambda C_\varkappa^2\right)\varphi_\varkappa^2 \\
&\quad \geqq \min\{0, \text{ value of the polynomial at } a_3\} \\
&\quad = \min\{0, \lambda^{-1/2}[(\tfrac{1}{16} - \tfrac{1}{2}\eta - \tfrac{1}{2}v - \tfrac{3}{16}\varepsilon - \tfrac{1}{2}v\varepsilon)(1 + \varepsilon)^{-1} \\
&\qquad + \tfrac{9}{8}\lambda^{1/2}C_\varkappa + \tfrac{45}{4}\lambda C_\varkappa^2]\} = 0.
\end{aligned} \tag{42}$$

Moreover

$$\zeta : (\delta\varphi_\varkappa)^2 : \geqq \zeta(\delta\varphi_\varkappa)^2 - 0(1)\ln\varkappa \geqq -0(1)(\ln\varkappa)^3. \tag{43}$$

From (38), (42), and (43) we conclude that if $\varphi_\varkappa \in [-a_3, a_3]$ then

$$(1 + \varepsilon) : P_{\lambda,v}(\varphi_\varkappa) : + (\tfrac{1}{2} - \eta) : \varphi_\varkappa^2 : + \zeta : (\delta\varphi_\varkappa)^2 : \geqq -0(1)(\ln\varkappa)^3. \tag{44}$$

Now if $\varphi_\varkappa \notin [-a_3, a_3]$ but $\frac{1}{|\Delta|}\varphi(\Delta) \in [-a_2, a_2]$ then by (38), (41), and (43)

$$\begin{aligned}
&(1 + \varepsilon) : P_{\lambda,v}(\varphi_\varkappa) : + (\tfrac{1}{2} - \eta) : \varphi_\varkappa^2 : + \zeta : (\delta\varphi_\varkappa)^2 : \\
&\quad \geqq -(1 + \varepsilon)\lambda^{-1/2}(1 + 8\lambda^{1/2}C_\varkappa)^3 + \zeta(\delta\varphi_\varkappa)^2 - 0(1)(\ln\varkappa)^3 \\
&\quad \geqq -(1 + \varepsilon)\lambda^{-1/2}(1 + 8\lambda^{1/2}C_\varkappa)^3 + \zeta(a_3 - a_2)^2 - 0(1)(\ln\varkappa)^3 \\
&\quad \geqq -(1 + \varepsilon)\lambda^{-1/2}(1 + 8\lambda^{1/2}C_\varkappa)^3 + \tfrac{4}{3}\lambda^{-1/2} - 0(1)(\ln\varkappa)^3 \geqq -0(1)(\ln\varkappa)^3.
\end{aligned} \tag{45}$$

Integrating out (42) and (43) over Δ we obtain (37). □

Lemma 2.

$$\int e^{-U^{\Lambda}_{\lambda,\nu}} \prod_{\Delta_\alpha \subset \Lambda} \chi_{[-a_2,-a_1]\cup[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right) d\mu_1$$
$$\leqq \exp[(0(1) - \tfrac{1}{100}\lambda^{-1/2})|\Lambda|]. \tag{46}$$

Proof of Lemma 2. We proceed the same way as when proving Lemma 1 and are left only with showing that for $q<\infty$

$$\left\| \exp[-U^{\Delta}_{\lambda,\nu} - (\tfrac{1}{2}-\eta):\varphi^2:(\Delta) - \zeta:(\delta\varphi)^2:(\Delta)]\chi_{[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\|_q$$
$$\leqq 0(1)\exp\left(-\frac{|\Delta|}{100}\lambda^{-1/2}\right),$$

and this follows in turn from the bound

$$\int_\Delta :P_{\lambda,\nu}(\varphi_\varkappa): + (\tfrac{1}{2}-\eta):\varphi^2_\varkappa:(\Delta) + \zeta:(\delta\varphi_\varkappa)^2:(\Delta)$$
$$-\ln\chi_{[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right) \geqq \frac{|\Delta|}{100}\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{47}$$

If $\varphi_\varkappa \in [a_0, a_3]$ then

$$:P_{\lambda,\nu}(\varphi_\varkappa): + (\tfrac{1}{2}-\eta):\varphi^2_\varkappa: + \zeta:(\delta\varphi_\varkappa)^2: \geqq \tfrac{1}{2}\lambda\varphi^6_\varkappa$$
$$-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)\varphi^4_\varkappa + (\tfrac{1}{2}-\eta+\nu+6\lambda^{1/2}C_\varkappa + \tfrac{45}{2}\lambda C^2_\varkappa)\varphi^2_\varkappa - 0(1)(\ln\varkappa)^3$$
$$\geqq \min\{\text{values of } \tfrac{1}{2}\lambda x^6 - \lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)x^4$$
$$+(\tfrac{1}{2}-\eta-\nu+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C^2_\varkappa)x^2 \text{ at } a_0 \text{ and } a_3\} - 0(1)(\ln\varkappa)^3$$
$$=\lambda^{-1/2}\min\left\{\frac{32}{(27)^2} - \tfrac{1}{9}(\eta+\nu) + \tfrac{31}{54}\lambda^{1/2}C_\varkappa + \tfrac{5}{2}\lambda C^2_\varkappa,\ \tfrac{1}{16} - \tfrac{1}{2}(\eta+\nu)\right.$$
$$\left. + \tfrac{9}{8}\lambda^{1/2}C_\varkappa + \tfrac{45}{4}\lambda C^2_\varkappa\right\} - 0(1)(\ln\varkappa)^3 \geqq \tfrac{1}{100}\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{48}$$

If $\varphi_\varkappa \notin [a_0, a_3]$ but $\dfrac{1}{|\Delta|}\varphi(\Delta) \in [a_1, a_2]$ then [compare (41)]

$$:P_{\lambda,\nu}(\varphi_\varkappa): + (\tfrac{1}{2}-\eta):\varphi^2_\varkappa: + \zeta:(\delta\varphi_\varkappa)^2: \geqq -\lambda^{-1/2}(1+8\lambda^{1/2}C_\varkappa)^3$$
$$+\zeta(\min\{a_3-a_2, a_1-a_0\})^2 \geqq -\lambda^{-1/2}(1+8\lambda^{1/2}C_\varkappa)^3 + \tfrac{4}{3}\lambda^{-1/2}$$
$$-0(1)(\ln\varkappa)^3 \geqq \tfrac{1}{3}\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{49}$$

Gathering (48) and (49) we get (47). □

Let

$$b_0 := -(1-(\tfrac{1}{3})^{1/2})\lambda^{-1/4}, \quad b_1 := \lambda^{-1/4}, \quad b_2 := 2\lambda^{-1/4}. \tag{50}$$

$b_0 < b_1 < b_2$.

Let $h \in C^\infty(\mathbb{R}^1)$, $0 \leqq h \leqq 1$, $h(x) = \begin{cases} 1 & \text{if } x \leqq 0 \\ 0 & \text{if } x \geqq 1 \end{cases}$. Let for $l>0$ $f_l(x) := h\left(|x| - \dfrac{l}{2}\right)$.

For $\Lambda = \left[-\frac{l}{2}, \frac{l}{2}\right] \times \left[-\frac{l}{2}, \frac{l}{2}\right]$ put

$$g(x) := \zeta_+ f_l(x^0) f_l(x^1). \tag{51}$$

$g \in C_0^\infty(R^2)$.

Next lemma will be used to estimate the input of minima of the action at $\varphi = \pm\zeta_+$ to expectations $\langle F \rangle_{\lambda,\nu}$.

Lemma 3. *Let* $\varepsilon = \frac{1}{20}$.

$$\int e^{-(1+\varepsilon)\left[\int_\Lambda (:P_{\lambda,\nu}(\varphi+g): - E_0) + \varphi((-\Delta+1)g) + \frac{1}{2}(g|(-\Delta+1)g)_{L^2}\right]}$$
$$\cdot \prod_{\Delta_\alpha \subset \Lambda} \chi_{[bo,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right) d\mu_1 \leqq \exp[0(1)|\Lambda| + C_{\lambda,\nu}|\partial\Lambda|] \tag{52}$$

[where, as always, 0(1) does not depend on λ, $0 < \lambda \leqq \lambda_0$, ν, and Λ]. We remind that $E_0 = \frac{1}{27}[1 - 18\nu + (1+6\nu)^{3/2}]\lambda^{-1/2}$.

Proof of Lemma 3. The left side of (52) equals

$$\int e^{-(1+\varepsilon)\int_\Lambda (:P_{\lambda,\nu}(\varphi+\xi_+): - E_0 + \xi_+\varphi + \frac{1}{2}\xi_+^2)}$$
$$\cdot \prod_{\Delta_\alpha \subset \Lambda} \chi_{[bo,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right) e^{-(1+\varepsilon)\varphi(\chi_{\sim\Lambda}(-\Delta+1)g)} e^{-\frac{1+\varepsilon}{2}\int_{\sim\Lambda}(\|\nabla g\|^2 + |g|^2)} d\mu_1$$
$$\leqq \Bigg\| \exp\Big[-(1+\varepsilon)\int_\Delta (:P_{\lambda,\nu}(\varphi+\xi_+): - E_0 + \xi_+\varphi + \tfrac{1}{2}\xi_+^2)$$
$$- (\tfrac{1}{2} - \eta) :\varphi^2:(\Delta) - \zeta :(\delta\varphi)^2:(\Delta)\Big] \chi_{[bo,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\Bigg\|_q^{\frac{|\Lambda|}{|\Delta|}}$$
$$\cdot \|\exp[-(1+\varepsilon)\varphi(\chi_{\sim\Lambda}(-\Delta+1)g) + (\tfrac{1}{2} - \eta) :\varphi^2:(\Lambda) + \zeta :(\delta\varphi)^2:(\Lambda)]\|_p \tag{53}$$

for q large enough (we have used the checkerboard estimate).

Conditioning with respect to the Neumann boundary condition measure we bound the second norm by $\exp[0(1)|\Lambda| + C_{\lambda,\nu}|\partial\Lambda|]$. As before we shall be finished if we show that

1. $$\Bigg\|(1+\varepsilon)\int_\Delta (:P_{\lambda,\nu}(\varphi+\xi_+): - E_0 + \xi_+\varphi + \tfrac{1}{2}\xi_+^2) + (\tfrac{1}{2} - \eta) :\varphi^2:(\Delta)$$
$$+ \zeta :(\delta\varphi)^2:(\Delta) - (1+\varepsilon)\int_\Delta (:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+): - E_0 + \xi_+\varphi_\varkappa + \tfrac{1}{2}\xi_+^2)$$
$$- (\tfrac{1}{2} - \eta) :\varphi_\varkappa^2:(\Delta) - \zeta :(\delta\varphi_\varkappa)^2:(\Delta)\Bigg\|_2 \leqq 0(1)\varkappa^{-\delta}, \tag{54}$$

2. $$(1+\varepsilon)\int_\Delta (:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+): - E_0 + \xi_+\varphi_\varkappa + \tfrac{1}{2}\xi_+^2) + (\tfrac{1}{2} - \eta) :\varphi_\varkappa^2:(\Delta)$$
$$+ \zeta :(\delta\varphi_\varkappa)^2:(\Delta) - \ln \chi_{[bo,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right) \geqq -0(1)(\ln \varkappa)^3. \tag{55}$$

To prove (54) we notice that

$$\begin{aligned}
&:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2\\
&=\tfrac{1}{2}\lambda:\varphi_\varkappa^6:+3\lambda\xi_+:\varphi_\varkappa^5:+(\tfrac{15}{2}\lambda\xi_+^2-\lambda^{1/2}):\varphi_\varkappa^4:\\
&\quad+(10\lambda\xi_+^3-4\lambda^{1/2}\xi_+):\varphi_\varkappa^3:+(\tfrac{15}{2}\lambda\xi_+^4-6\lambda^{1/2}\xi_+^2):\varphi_\varkappa^2:. \qquad (56)
\end{aligned}$$

Since all coefficients at Wick powers are bounded uniformly in λ, $0<\lambda\leqq\lambda_0$, and ν (54) follows similarly as (36) before.

We pass to proof of (55) which constitutes the most involved part of the paper. Denote $\varphi_\varkappa+\xi_+=:x$.

$$\begin{aligned}
&:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2+\frac{\frac{1}{2}-\eta}{1+\varepsilon}:\varphi_\varkappa^2:\\
&=\tfrac{1}{2}\lambda x^6-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)x^4+(-\nu+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2)x^2\\
&\quad+\xi_+(x-\xi_+)+\tfrac{1}{2}\xi_+^2+\frac{\frac{1}{2}-\eta}{1+\varepsilon}(x-\xi_+)^2-\left(\frac{\frac{1}{2}-\eta}{1+\varepsilon}-\nu\right)C_\varkappa\\
&\quad-3\lambda^{1/2}C_\varkappa^2-\tfrac{15}{2}\lambda C_\varkappa^3\\
&\geqq\tfrac{1}{2}\lambda x^6-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)x^4+(\tfrac{1}{2}-\nu+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2)x^2\\
&\quad-E_0-\frac{\eta+\frac{1}{2}\varepsilon}{1+\varepsilon}(x-\xi_+)^2-0(1)(\ln\varkappa)^3. \qquad (57)
\end{aligned}$$

Consider the polynomial

$$\begin{aligned}
w_\varkappa(x):=&\tfrac{1}{2}\lambda x^6-\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)x^4\\
&+(\tfrac{1}{2}-\nu+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2)x^2. \qquad (58)
\end{aligned}$$

$w_\varkappa$ takes its minimal value at $x=\pm\xi_{+,\varkappa}$, where

$$\begin{aligned}
\xi_{+,\varkappa}^2=&\tfrac{2}{3}\lambda^{-1}[\lambda^{1/2}(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)+(\lambda(1+\tfrac{15}{2}\lambda^{1/2}C_\varkappa)^2\\
&-\tfrac{3}{2}\lambda(\tfrac{1}{2}-\nu+6\lambda^{1/2}C_\varkappa+\tfrac{45}{2}\lambda C_\varkappa^2))^{1/2}]\\
=&\lambda^{-1/2}[\tfrac{2}{3}+5\lambda^{1/2}C_\varkappa+(\tfrac{1}{9}+\tfrac{8}{3}\lambda^{1/2}C_\varkappa+10\lambda C_\varkappa^2+\tfrac{2}{3}\nu)^{1/2}]\\
=&\lambda^{-1/2}[\tfrac{2}{3}+(\tfrac{1}{9}+\tfrac{2}{3}\nu)^{1/2}+\lambda^{1/2}C_\varkappa[5+(\tfrac{8}{3}+10\lambda^{1/2}C_\varkappa)\\
&\cdot((\tfrac{1}{9}+\tfrac{2}{3}\nu+\tfrac{8}{3}\lambda^{1/2}C_\varkappa+10\lambda C_\varkappa^2)^{1/2}+(\tfrac{1}{9}+\tfrac{2}{3}\nu)^{1/2})^{-1}]]\\
=&\xi_+^2+0(1)C_\varkappa. \qquad (59)
\end{aligned}$$

Straightforward computation gives for the minimal value $E_\varkappa$ of $w_\varkappa$

$$\begin{aligned}
E_\varkappa=&E_0-[(\tfrac{1}{6}+5\nu)C_\varkappa+5\lambda^{1/2}C_\varkappa^2+\tfrac{25}{2}\lambda C_\varkappa^3]\\
&-\tfrac{1}{27}\lambda^{-1/2}[(1+6\nu+24\lambda^{1/2}C_\varkappa+90\lambda C_\varkappa^2)^{3/2}-(1+6\nu)^{3/2}]\\
\geqq&E_0-0(1)(\ln\varkappa)^3. \qquad (60)
\end{aligned}$$

If $\varphi_\varkappa+\xi_+\in[a_1,+\infty[$ then

$$w_\varkappa(x)\geqq\tfrac{1}{12}(x-\xi_{+,\varkappa})^2+E_0-0(1)(\ln\varkappa)^3. \qquad (61)$$

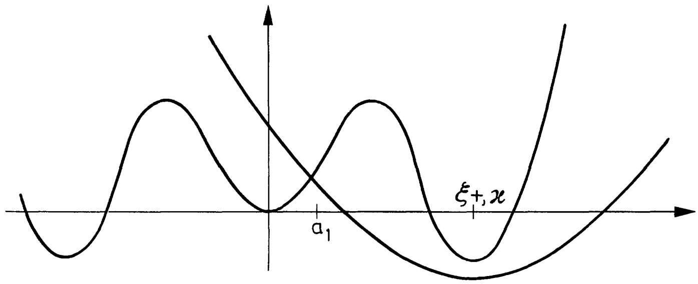

Fig. 4

Indeed, from a simple analysis based on Figure 4 it follows that (61) is a consequence of three facts:

a) $w_\varkappa(\xi_{+,\varkappa}) \geqq E_0 - 0(1)(\ln\varkappa)^3$ which is (60),
b) $w''_\varkappa(\xi_{+,\varkappa}) \geqq \frac{1}{6}$ which holds as the direct inspection shows,
c) $w_\varkappa(a_1) \geqq \frac{1}{12}(a_1 - \xi_{+,\varkappa})^2 + E_0 - 0(1)(\ln\varkappa)^3$. (62)

To show (62) we notice that

$$w_\varkappa(a_1) = \lambda^{-1/2}\left(\frac{9}{128} - \frac{v}{4}\right) + \frac{33}{32}C_\varkappa + \frac{45}{8}\lambda C_\varkappa^2. \tag{63}$$

$$\begin{aligned}
&\tfrac{1}{12}(\xi_{+,\varkappa} - a_1)^2 + E_0 \\
&= \tfrac{1}{12}\left(\frac{\xi_{+,\varkappa} - a_1^2}{\xi_{+,\varkappa} + a_1}\right)^2 + E_0 \leqq \tfrac{1}{12}\left(\frac{\xi_+^2 - a_1^2 + 0(1)C_\varkappa}{\xi_+ + a_1}\right)^2 + E_0 \\
&\leqq \tfrac{1}{12}\left(\frac{\xi_+^2 - a_1^2}{\xi_+ + a_1}\right)^2 + E_0 + 0(1)(\ln\varkappa)^2 \\
&\leqq \tfrac{1}{40}\lambda^{-1/2} + E_0 + 0(1)(\ln\varkappa)^2 \\
&\leqq \begin{cases} (\frac{1}{40} - v)\lambda^{-1/2} + 0(1)(\ln\varkappa)^2 & \text{if} \quad v \leqq 0, \\ \frac{1}{40}\lambda^{-1/2} + 0(1)(\ln\varkappa)^2 & \text{if} \quad v \geqq 0. \end{cases}
\end{aligned} \tag{64}$$

From (63) and (64) we get (62).

Now for $x \in [a_1, +\infty[$ (60) and (61) give

$$\begin{aligned}
&w_\varkappa(x) - E_0 - \frac{\eta + \frac{\varepsilon}{2}}{1+\varepsilon}(x - \xi_+)^2 - 0(1)(\ln\varkappa)^3 \\
&\geqq \tfrac{1}{12}(x - \xi_{+,\varkappa})^2 - \tfrac{1}{14}(x - \xi_+)^2 - 0(1)(\ln\varkappa)^3 \\
&= \tfrac{1}{12}((x - \xi_+) - (\xi_{+,\varkappa} - \xi_+))^2 - \tfrac{1}{14}(x - \xi_+)^2 - 0(1)(\ln\varkappa)^3 \\
&\geqq -\tfrac{1}{2}(\xi_{+,\varkappa} - \xi_+)^2 - 0(1)(\ln\varkappa)^3,
\end{aligned}$$

where we have used

$$(a-b)^2 \geqq \tfrac{6}{7}a^2 - 6b^2.$$

Hence by (59)

$$w_\varkappa(x)-E_0-\frac{\eta+\frac{\varepsilon}{2}}{1+\varepsilon}(x-\xi_+)^2-0(1)(\ln\varkappa)^3$$
$$\geqq -\frac{1}{2}\left(\frac{\xi_{+,\varkappa}^2-\xi_+^2}{\xi_{+,\varkappa}+\xi_+}\right)^2-0(1)(\ln\varkappa)^3\geqq -0(1)(\ln\varkappa)^3\,. \tag{65}$$

Gathering (57), (65), and (43) we get

$$(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2)+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:+\zeta:(\delta\varphi_\varkappa)^2:$$
$$\geqq -0(1)(\ln\varkappa)^3 \quad \text{if} \quad \varphi_\varkappa+\xi_+\in[a_1,+\infty[\,. \tag{66}$$

Now if $\varphi_\varkappa+\xi_+\notin[a_1,+\infty[$ but $\frac{1}{|\Delta|}\varphi(\Delta)\in[b_0,+\infty[$ then by (57) and (60)

$$(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2)+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:+\zeta:(\delta\varphi_\varkappa)^2:$$
$$\geqq(1+\varepsilon)(E_\varkappa-E_0)-\left(\eta+\frac{\varepsilon}{2}\right)\varphi_\varkappa^2+\zeta(\delta\varphi_\varkappa)^2-0(1)(\ln\varkappa)^3$$
$$\geqq -\left(\eta+\frac{\varepsilon}{2}\right)\varphi_\varkappa^2+\zeta(\varphi_\varkappa-b_0)^2-0(1)(\ln\varkappa)^3\,. \tag{67}$$

Now

$$\frac{\zeta(\varphi_\varkappa-b_0)^2}{\varphi_\varkappa^2}\geqq\frac{\zeta(a_1-\xi_+-b_0)^2}{(a_1-\xi_+)^2} \quad \text{for} \quad \varphi_\varkappa<a_1-\xi_+\,.$$

By direct computation

$$\frac{\zeta(\varphi_\varkappa-b_0)^2}{\varphi_\varkappa^2}\geqq\frac{1}{10}\geqq\eta+\frac{\varepsilon}{2}. \tag{68}$$

Thus from (67) and (68)

$$(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2)+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:+\zeta:(\delta\varphi_\varkappa)^2:$$
$$\geqq -0(1)(\ln\varkappa)^3 \quad \text{if} \quad \varphi_\varkappa\notin[a_1-\xi_+,+\infty[\quad \text{and} \quad \frac{1}{|\Delta|}\varphi(\Delta)\in[b_0,+\infty[\,. \tag{69}$$

(66) and (69) give (55). □

Lemma 4. *Let* $\varepsilon=\frac{1}{20}$.

$$\int e^{-(1+\varepsilon)\left[\int_\Lambda(:P_{\lambda,\nu}(\varphi+g):-E_0)+\varphi((-\Delta+1)g)+\frac{1}{2}(g|(-\Delta+1)g)_{L^2}\right]}$$
$$\cdot\prod_{\Delta_\alpha\subset\Lambda}\chi_{[b_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)d\mu_1\leqq\exp[(0(1)-9\lambda^{-1/2})|\Lambda|+C_{\lambda,\nu}|\partial\Lambda|]\,. \tag{70}$$

Proof of Lemma 4. Proceeding as in Proof of Lemma 3 one is left with showing an analog of (55):

$$(1+\varepsilon)\int_\Delta(:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2)+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2:(\Delta)+\zeta:(\delta\varphi_\varkappa)^2:(\Delta)$$
$$-\ln\chi_{[b_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\geqq 9|\Delta|\lambda^{-1/2}-0(1)(\ln\varkappa)^3\,. \tag{71}$$

We shall distinguish three cases.

1st Case. $\varphi_\varkappa \in [b_1, +\infty[$ i.e. $x \in [b_1 + \xi_+, +\infty[$ and

$$b_1 + \xi_+ \geqq \tfrac{4}{3}\xi_{+,\varkappa}. \tag{72}$$

Then (compare Fig. 4)

$$w_\varkappa(x) \geqq \tfrac{1}{12}(x - \xi_{+,\varkappa})^2 + E_0 + 9\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{73}$$

Indeed, we must only check that

$$w_\varkappa(b_1 + \xi_+) \geqq \tfrac{1}{12}(b_1 + \xi_+ - \xi_{+,\varkappa})^2 + E_0 + 9\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{74}$$

But

$$w_\varkappa(b_1 + \xi_+) \geqq w_\varkappa((\tfrac{7}{2})^{1/2}\lambda^{-1/4}) \tag{75}$$

as

$$b_1 + \xi_+ \geqq (\tfrac{7}{2})^{1/2}\lambda^{-1/4} \geqq \xi_{+,\varkappa}. \tag{76}$$

But

$$w_\varkappa((\tfrac{7}{2})^{1/2}\lambda^{-1/4}) \geqq 10\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \tag{77}$$

Furthermore by (59)

$$\begin{aligned} &\tfrac{1}{12}(b_1 + \xi_+ - \xi_{+,\varkappa})^2 + E_0 + 9\lambda^{-1/2} \leqq \tfrac{1}{108}\xi_{+,\varkappa}^2 + E_0 + 9\lambda^{-1/2} \\ &\quad \leqq \tfrac{1}{108}\xi_+^2 + E_0 + 9\lambda^{-1/2} + 0(1)\ln\varkappa \leqq 10\lambda^{-1/2} + 0(1)\ln\varkappa. \end{aligned} \tag{78}$$

Thus (73) is proven.

Proceeding further as when proving (66) we obtain from (57) and (73)

$$\begin{aligned} &(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa + \xi_+): - E_0 + \xi_+\varphi_\varkappa + \tfrac{1}{2}\xi_+^2) \\ &\quad + (\tfrac{1}{2} - \eta):\varphi_\varkappa^2: + \zeta:(\delta\varphi_\varkappa)^2: \geqq 9\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \end{aligned} \tag{79}$$

2nd Case.

$$\varphi_\varkappa \in [b_1, +\infty[, \qquad b_1 + \xi_+ \leqq \tfrac{4}{3}\xi_{+,\varkappa}.$$

By (59)

$$(b_1 + \xi_+)^2 \leqq \tfrac{16}{9}\xi_+^2 + 0(1)C_\varkappa.$$

Hence

$$\tfrac{3}{2}\lambda^{-1/2} \leqq 0(1)C_\varkappa. \tag{80}$$

From (66) and (80) we get

$$\begin{aligned} &(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa + \xi_+): - E_0 + \xi_+\varphi_\varkappa + \tfrac{1}{2}\xi_+^2) + (\tfrac{1}{2} - \eta):\varphi_\varkappa^2: \\ &\quad + \zeta:(\delta\varphi_\varkappa)^2: \geqq -0(1)(\ln\varkappa)^3 \geqq 9\lambda^{-1/2} - 0(1)(\ln\varkappa)^3. \end{aligned} \tag{81}$$

3rd Case.

$$\varphi_\varkappa \notin [b_1, +\infty[, \qquad \frac{1}{|\varDelta|}\varphi(\varDelta) \in [b_2, +\infty[.$$

An analog of (67) holds:

$$(1+\varepsilon)(:P_{\lambda,\nu}(\varphi_\varkappa+\xi_+):-E_0+\xi_+\varphi_\varkappa+\tfrac{1}{2}\xi_+^2)+(\tfrac{1}{2}-\eta):\varphi_\varkappa^2: \\ +\zeta:(\delta\varphi_\varkappa)^2:\geqq-\left(\eta+\frac{\varepsilon}{2}\right)\varphi_\varkappa^2+\zeta(\varphi_\varkappa-b_2)^2-0(1)(\ln\varkappa)^3. \tag{82}$$

Minimal value of $-\left(\eta+\frac{\varepsilon}{2}\right)\varphi_\varkappa^2+\zeta(\varphi_\varkappa-b_2)^2$ is attained at $\varphi_\varkappa=b_1$. Hence

$$-\left(\eta+\frac{\varepsilon}{2}\right)\varphi_\varkappa^2+\zeta(\varphi_\varkappa-b_2)^2\geqq-\left(\eta+\frac{\varepsilon}{2}\right)b_1^2+\zeta(b_1-b_2)^2\geqq 9\lambda^{-1/2}. \tag{83}$$

(71) follows from (79), (81) and (82) with (83).

Lemmas 1–4 provide upper bounds on pressures. We shall also need a lower bound on $\alpha_\infty^{\lambda,\nu}$.

Lemma 5.

$$\alpha_\infty^{\lambda,\nu}\geqq\max\{0,-E_0\}. \tag{84}$$

Proof of Lemma 5. 1. By the Jensen inequality

$$\alpha_\infty^{\lambda,\nu}=\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-U_{\lambda,\nu}^{\Lambda}}d\mu_1\geqq\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln e^{-\int U_{\lambda,\nu}^{\Lambda}d\mu_1}=0.$$

2. We translate $d\mu_1$ by the function g [see (51)] and use (compare I.4.10 of [7])

$$\int F(\varphi-g)d\mu_1=\int e^{-\varphi((-\Delta+1)g)-\frac{1}{2}(g|(-\Delta+1)g)_{L^2}}F(\varphi)d\mu_1. \tag{85}$$

Hence

$$\begin{aligned}&\int e^{-\int_\Lambda(:P_{\lambda,\nu}(\varphi):-E_0)}d\mu_1=\int e^{-\int_\Lambda(:P_{\lambda,\nu}(\varphi+g):-E_0)}\\&\quad\cdot e^{-\varphi((-\Delta+1)g)-\frac{1}{2}(g|(-\Delta+1)g)_{L^2}}d\mu_1\\&\quad=\int e^{-\int_\Lambda(:P_{\lambda,\nu}(\varphi+\xi_+):-E_0+\xi_+\varphi+\frac{1}{2}\xi_+^2)-\varphi(\chi_{\sim\Lambda}(-\Delta+1)g)}\\&\quad\cdot e^{-\frac{1}{2}\int_{\sim\Lambda}(\|\nabla g\|^2+|g|^2)}d\mu_1\\&\quad\geqq\exp\left[-\int\left(\int_\Lambda(:P_{\lambda,\nu}(\varphi+\xi_+):-E_0+\xi_+\varphi+\tfrac{1}{2}\xi_+^2)+\varphi(\chi_{\sim\Lambda}(-\Delta+1)g)\right.\right.\\&\qquad\left.\left.+\tfrac{1}{2}\int_{\sim\Lambda}(\|\nabla g\|^2+|g|^2)\right)d\mu_1\right]=\exp\left[-\tfrac{1}{2}\int_{\sim\Lambda}(\|\nabla g\|^2+|g|^2)\right]\\&\quad\geqq\exp[-C_{\lambda,\nu}|\partial\Lambda|],\end{aligned} \tag{86}$$

where we have used (56). From (86) we get

$$\alpha_\infty^{\lambda,\nu}\geqq-E_0-\lim_{\Lambda\to\infty}C_{\lambda,\nu}\frac{|\partial\Lambda|}{|\Lambda|}=-E_0. \quad \square$$

Having proven inequalities for pressures we can now employ the chessboard estimate in estimation process.

Lemma 6. *Let $F_{\lambda,\nu}$ be a positive polynomial in $:\varphi^2:(\Delta)$ and $\varphi(\Delta)$ such that for each $q<\infty$ there exists $C_q<\infty$ such that for each λ, $0<\lambda\leqq\lambda_0$, and each ν*

$$\|F_{\lambda,\nu}\|_q=(\int(F_{\lambda,\nu})^q d\mu_1)^{1/q}\leqq C_q. \tag{87}$$

Then

$$\left\langle F_{\lambda,\nu}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}\leqq\begin{cases}0(1) & \text{for } \nu\leqq 0\\ 0(1)\,e^{E_0|\Delta|} & \text{for } \nu\geqq 0\end{cases}. \tag{88}$$

Proof of Lemma 6. By the chessboard estimate (25)

$$\left\langle F_{\lambda,\nu}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}\leqq\exp\left[\left(\alpha_\infty^{\lambda,\nu}\left(F_{\lambda,\nu}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right)-\alpha_\infty^{\lambda,\nu}\right)|\Delta|\right]. \tag{89}$$

$$\begin{aligned}
&\alpha_\infty^{\lambda,\nu}\left(F_{\lambda,\nu}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right)\\
&=\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-U_{\lambda,\nu}^\Lambda}\prod_{\Delta_\beta\subset\Lambda}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)(F_{\lambda,\nu})_\beta d\mu_1\\
&\leqq\frac{1}{1+\varepsilon}\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-(1+\varepsilon)U_{\lambda,\nu}^\Lambda}\prod_{\Delta_\beta\subset\Lambda}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)d\mu_1\\
&\quad+\frac{1}{(1+\varepsilon)'}\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int\prod_{\Delta_\beta\subset\Lambda}(F_{\lambda,\nu})_\beta^{(1+\varepsilon)'}d\mu_1\leqq 0(1),
\end{aligned}\tag{90}$$

where we have used Lemma 1 to bound the first term and the checkerboard estimate [9] together with (87) to bound the 2nd one.

Inserting (90) and (84) to (89) we get (88). □

Lemma 7.

$$\left\langle\prod_{\alpha\in A}\chi_{[-a_2,-a_1]\cup[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\right\rangle_{\lambda,\nu}\leqq\exp[(-\tfrac{1}{100}|\Delta|\lambda^{-1/2}+0(1))|A|], \tag{91}$$

where $|A|$ denotes the number of elements of A.

Proof of Lemma 7. By the chessboard estimate (25)

$$\begin{aligned}
&\left\langle\prod_{\alpha\in A}\chi_{[-a_2,-a_1]\cup[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\right\rangle_{\lambda,\nu}\\
&\leqq\exp\left[\left(\alpha_\infty^{\lambda,\nu}\left(\chi_{[-a_2,-a_1]\cup[a_1,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right)-\alpha_\infty^{\lambda,\nu}\right)|\Delta||A|\right]\\
&\leqq\exp[(-\tfrac{1}{100}|\Delta|\lambda^{-1/2}+0(1))|A|]
\end{aligned}$$

in virtue of Lemmas 2 and 5. □

Lemma 8. *Let $F_{\lambda,\nu}$ be a positive polynomial in $:\varphi^2:(\Delta)$ and $\varphi(\Delta)$. Define $F'_{\lambda,\nu}$ by*

$$F'_{\lambda,\nu}(T) := F_{\lambda,\nu}(T+\xi_+). \tag{92}$$

Suppose that for each λ, $0<\lambda\leqq\lambda_0$, and each ν

$$\|F'_{\lambda,\nu}\|_q \leqq C_q. \tag{93}$$

Then

$$\left\langle F_{\lambda,\nu}\chi_{[b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu} \leqq \begin{cases} 0(1)e^{-E_0|\Delta|} & \text{for } \nu\leqq 0 \\ 0(1) & \text{for } \nu\geqq 0 \end{cases} \tag{94}$$

and

$$\left\langle F_{\lambda,\nu}\chi_{[b_2+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu} \leqq 0(1)e^{-8|\Delta|\lambda^{-1/2}}. \tag{95}$$

Proof of Lemma 8. We estimate

$$\alpha_\infty^{\lambda,\nu}\left(F_{\lambda,\nu}\chi_{[b_i+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right), \quad i=0,2,$$

translating $d\mu_1$ by g [see (51) and (85)].

$$\begin{aligned}
&\alpha_\infty^{\lambda,\nu}\left(F_{\lambda,\nu}\chi_{[b_i+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right) \\
&= \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-U^\Lambda_{\lambda,\nu}}\prod_{\Delta_\beta\subset\Lambda}\chi_{[b_i+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)(F_{\lambda,\nu})_\beta d\mu_1 \\
&= \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\ln\int e^{-\int_\Lambda :P_{\lambda,\nu}(\varphi+g):-\varphi((-\Delta+1)g)-\frac{1}{2}(g|(-\Delta+1)g)_{L^2}} \\
&\quad\cdot\prod_{\Delta_\beta\subset\Lambda}\chi_{[b_i,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)(F'_{\lambda,\nu})_\beta d\mu_1 \\
&\leqq \lim_{\Lambda\to\infty}\frac{1}{(1+\varepsilon)|\Lambda|}\ln\int e^{-(1+\varepsilon)\left[\int_\Lambda(:P_{\lambda,\nu}(\varphi+g):-E_0)+\varphi((-\Delta+1)g)+\frac{1}{2}(g|(-\Delta+1)g)_{L^2}\right]} \\
&\quad\cdot\prod_{\Delta_\beta\subset\Lambda}\chi_{[b_i,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)d\mu_1 + \lim_{\Lambda\to\infty}\frac{1}{(1+\varepsilon)'|\Lambda|}\ln\int\prod_{\Delta_\beta\subset\Lambda}(F'_{\lambda,\nu})_\beta^{(1+\varepsilon)'}d\mu_1 - E_0 \\
&\leqq \begin{cases} 0(1)-E_0 & \text{if } i=0 \\ -8\lambda^{-1/2}+0(1) & \text{if } i=2 \end{cases}.
\end{aligned} \tag{96}$$

We have used Lemmas 3 and 4 and (93).

(94) and (95) follow from (96) and Lemma 5 by the chessboard estimate. □

Denote

$$\chi_{-1} := \chi_{]-\infty,-a_2[}, \qquad \chi_0 := \chi_{[-a_2,a_2]}, \qquad \chi_1 := \chi_{]a_2,+\infty[}.$$

We shall also use shorthand χ_a^α for $\chi_a\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)$, $a=0,\pm 1$.

Lemma 9. *Let* $a, b = 0, \pm 1, a \neq b$.

$$\langle \chi_a^\alpha \chi_b^\beta \rangle_{\lambda, \nu} \xrightarrow[\lambda \to 0]{} 0 \quad \textit{uniformly in} \quad \nu, \alpha, \beta\,. \tag{97}$$

Proof of Lemma 9 (Following the Fröhlich's proof [2, 3] of existence of phase transition in $(\lambda Q(\varphi) - \nu\varphi^2)_2$).

By use of the Peierls argument in the form established in [6] and [3] (97) can be derived from

$$\left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma} \chi_{a_{\alpha\beta}}^\alpha \chi_{b_{\alpha\beta}}^\beta \right\rangle_{\lambda,\nu} \leqq \exp[(-\delta\lambda^{-1/2} + 0(1))|\gamma|]\,, \qquad \delta > 0\,. \tag{98}$$

Here γ is any set of $|\gamma|$ neighboring pairs of lattice squares and $a_{\alpha\beta} \neq b_{\alpha\beta}$. We can also assume that all pairs in γ are disjoint (using the Hölder inequality to separate them if this is not the case) and that $b_{\alpha\beta} \neq 0$.

Write

$$\chi_0 = \tilde{\chi}_0 + \tilde{\tilde{\chi}}_0\,, \tag{99}$$

where

$$\tilde{\chi}_0 := \chi_{[-a_1, a_1]} \tag{100}$$

and

$$\tilde{\tilde{\chi}}_0 := \chi_{[-a_2, -a_1[\cup]a_1, a_2]}\,. \tag{101}$$

Put also $\tilde{\chi}_{\pm 1} := \chi_{\pm 1}$.

$$\left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma} \chi_{a_{\alpha\beta}}^\alpha \chi_{b_{\alpha\beta}}^\beta \right\rangle_{\lambda,\nu} \leqq \sum_{\gamma'} \left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma \setminus \gamma'} \tilde{\chi}_{a_{\alpha\beta}}^\alpha \tilde{\chi}_{b_{\alpha\beta}}^\beta \right\rangle_{\lambda,\nu}^{1/2} \left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma'} \tilde{\tilde{\chi}}_0^\alpha \right\rangle_{\lambda,\nu}^{1/2}. \tag{102}$$

In $\sum_{\gamma'}$ γ' runs through the subsets of γ composed of pairs $(\Delta_\alpha, \Delta_\beta)$ such that $a_{\alpha\beta} = 0$.

$$\begin{aligned}&\left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma} \tilde{\chi}_{a_{\alpha\beta}}^\alpha \tilde{\chi}_{b_{\alpha\beta}}^\beta \right\rangle_{\lambda,\nu} \\ &\quad \leqq \left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma} e^{b_{\alpha\beta} \frac{1}{|\Delta|}(\varphi(\Delta_\beta) - \varphi(\Delta_\alpha)) - (a_2 - a_1)} \right\rangle_{\lambda,\nu}.\end{aligned} \tag{103}$$

It is easy to choose functions $g_{\alpha\beta}^i$ such that

$$b_{\alpha\beta} \frac{1}{|\Delta|} (\varphi(\Delta_\beta) - \varphi(\Delta_\alpha)) = \sum_{i=0}^{1} \varphi(\partial_i g_{\alpha\beta}^i)$$

(see [3, proof of Theorem 7.2]).

Using the Gaussian domination bound (27) we get

$$\begin{aligned}\left\langle \prod_{(\Delta_\alpha, \Delta_\beta) \in \gamma} \tilde{\chi}_{a_{\alpha\beta}}^\alpha \tilde{\chi}_{b_{\alpha\beta}}^\beta \right\rangle_{\lambda,\nu} &\leqq \exp\left[\sum_{(\Delta_\alpha, \Delta_\beta) \in \gamma} \sum_i \|g_{\alpha\beta}^i\|_{L^2}^2 - (a_2 - a_1)|\gamma| \right] \\ &\leqq \exp[(-\tfrac{1}{8}\lambda^{-1/2} + 0(1))|\gamma|]\,.\end{aligned} \tag{104}$$

Inserting (104) and (91) of Lemma 7 into (102) we obtain (98). □

Lemma 10.

$$\lambda^n\langle(:\varphi^2:(\Delta))^{2n}\rangle_{\lambda,v}\leqq 0(1),$$
$$\lambda^n\langle(\varphi(\Delta))^{4n}\rangle_{\lambda,v}\leqq 0(1). \tag{105}$$

Proof of Lemma 10.

$$\lambda^n\langle(:\varphi^2:(\Delta))^{2n}\rangle_{\lambda,v}\leqq\lambda^n\left\langle(:\varphi^2:(\Delta))^{2n}\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,v}$$
$$+2\lambda^n\left\langle(:\varphi^2:(\Delta))^{2n}\chi_{[b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,v}\quad\text{as}\quad b_0+\xi_+<a_2.$$

Now using Lemma 6 with $F_{\lambda,v}=(:\varphi^2:(\Delta))^{2n}$ and Lemma 8 with $F_{\lambda,v}=\lambda^n(:\varphi^2:(\Delta))^{2n}$ (i.e. $F'_{\lambda,v}=(\lambda^{1/2}:\varphi^2:(\Delta)+2\lambda^{1/2}\xi_+\varphi(\Delta)+\lambda^{1/2}\xi_+^2|\Delta|)^{2n}$) we obtain

$$\lambda^n\langle(:\varphi^2:(\Delta))^{2n}\rangle_{\lambda,v}\leqq\lambda^n 0(1)+0(1)\leqq 0(1).$$

Proof of the second inequality is identical. □

Now we are prepared to prove Proposition 1.

$$\lambda\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right)\right\rangle_{\lambda,v}$$
$$=\lambda\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right)\right\rangle_{\lambda,v}$$
$$+\lambda\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\chi_{]-\infty,-a_2[\cup]a_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\right.$$
$$\left.\cdot\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right)\chi_{]-\infty,-b_0-\xi_+[\cup]b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)\right\rangle_{\lambda,v}$$
$$+2\lambda\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta_\alpha)\chi_{]a_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta_\beta)\right)\right.$$
$$\left.\cdot\chi_{[-b_0-\xi_+,b_0+\xi_+]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)\right\rangle_{\lambda,v}. \tag{106}$$

We bound the terms of the right hand side of (106).

$$|\text{1st term}|\leqq\lambda^{1/2}\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,v}^{1/2}$$
$$\cdot\left[\lambda^{1/2}\xi_+^2+\left(\lambda\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\right\rangle_{\lambda,v}\right)^{1/2}\right]\leqq\lambda^{1/2}0(1) \tag{107}$$

in virtue of Lemmas 6 and 10.

$$|\text{2nd term}|\leqq 2\lambda^{1/4}\left(\lambda\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\right\rangle_{\lambda,v}\right)^{1/2}$$
$$\cdot\left(\lambda^{1/2}\left\langle\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{]b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,v}\right)^{1/2}$$
$$\leqq\lambda^{1/4}0(1), \tag{108}$$

where we have used Lemma 10 and Lemma 8 with

$$F_{\lambda,\nu}=\lambda^{1/2}\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2$$

i.e.

$$F'_{\lambda,\nu}=\left(\lambda^{1/4}\frac{1}{|\Delta|}:\varphi^2:(\Delta)+2\lambda^{1/2}\xi_+\frac{1}{|\Delta|}\varphi(\Delta)\right)^2$$

which is $d\mu_1$-integrable with any power uniformly in λ small and ν.

$$\begin{aligned}|\text{3rd term}|&\leqq 2\left(\lambda^2\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^4\right\rangle_{\lambda,\nu}\right)^{1/4}\\&\quad\cdot\left[\lambda^{1/2}\xi_+^2+\left(\lambda^2\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^4\right\rangle_{\lambda,\nu}\right)^{1/4}\right]\\&\quad\cdot\left\langle\chi_{]a_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right)\right\rangle_{\lambda,\nu}\\&\leqq 0(1)\varepsilon\quad\text{if}\quad 0<\lambda\leqq\lambda_0(\varepsilon)\end{aligned}\tag{109}$$

in virtue of Lemmas 9 and 10.

(106) together with (107)–(109) give (8) and Proposition 1.a) is proven. We pass to point b).

$$\begin{aligned}\lambda^{1/2}\left|\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,-\nu_0}\right|&\leqq\lambda^{1/2}\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,-\nu_0}^{1/2}\\&\quad+2\left\langle\lambda\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{[b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,-\nu_0}^{1/2}\\&\leqq 0(1)(\lambda^{1/2}+\exp(-E_0|\Delta|))\leqq 0(1)(\lambda^{1/2}+\exp(-\tfrac{1}{2}\nu_0|\Delta|\lambda^{-1/2}))\xrightarrow[\lambda\to 0]{}0.\end{aligned}\tag{110}$$

$$\begin{aligned}&\left|\frac{2}{3}+\left(\frac{1}{9}+\frac{2}{3}\nu_0\right)^{1/2}-\lambda^{1/2}\left\langle\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu_0}\right|\\&\quad=\lambda^{1/2}\left|\left\langle\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right\rangle_{\lambda,\nu_0}\right|\\&\quad\leqq 2\lambda^{1/4}\left\langle\lambda^{1/2}\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{[b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu_0}^{1/2}\\&\quad\quad+\left\langle\lambda\left(\xi_+^2-\frac{1}{|\Delta|}:\varphi^2:(\Delta)\right)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\right)\right\rangle_{\lambda,\nu_0}^{1/2}\\&\quad\leqq 0(1)(\lambda^{1/4}+\exp(E_0|\Delta|))\leqq 0(1)(\lambda^{1/4}+\exp(-\nu_0|\Delta|\lambda^{-1/2}))\xrightarrow[\lambda\to 0]{}0.\end{aligned}\tag{111}$$

We have used Lemmas 6 and 8. (110) and (111) prove Proposition 1.b).

Now

$$\left|\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)-\frac{1}{|\Delta|^2}\varphi(\Delta)\right)^2\right\rangle_{\lambda,\nu}\right|$$
$$\leqq\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)-\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}^{1/2}$$
$$+2\left\langle\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)-\frac{1}{|\Delta|^2}\varphi(\Delta)\right)^2\chi_{[b_0+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}^{1/2}\leqq 0(1) \tag{112}$$

in virtue of Lemmas 6 and 8 $\Big[$estimating the second term from Lemma 8 we put $F_{\lambda,\nu}=\left(\frac{1}{|\Delta|}:\varphi^2:(\Delta)-\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right)^2=F'_{\lambda,\nu}\Big]$. (112) proves (11) and completes the proof of Proposition 1. □

Proof of Proposition 3. First notice that for $\nu>\nu(\lambda)$ and λ small enough

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu}\geqq\frac{1}{2} \tag{113}$$

[see (20)]. Using Lemma 6 and (95) of Lemma 8 we obtain

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\right\rangle_{\lambda,\nu}=\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}$$
$$+2\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\chi_{]a_2,b_2+\xi+[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}$$
$$+2\lambda^{1/2}\left\langle\frac{1}{|\Delta|^2}\varphi(\Delta)^2\chi_{[b_2+\xi_+,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}$$
$$\leqq 2\lambda^{1/2}(b_2+\xi_+)^2\left\langle\chi_{]a_2,b_2+\xi+[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}$$
$$+(\lambda^{1/2}+\exp(-8|\Delta|\lambda^{-1/2}))0(1)\leqq 20\left\langle\chi_{[a_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}+\frac{1}{4} \tag{114}$$

for λ small enough.

(113) and (114) give

$$\frac{1}{80}\leqq\left\langle\chi_{[a_2,+\infty[}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}. \tag{115}$$

Now, with use of the notation of Lemma 9

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right\rangle_{\lambda,\nu}=2\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_1^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_1^\beta\right\rangle_{\lambda,\nu}$$
$$+\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_0^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_0^\beta\right\rangle_{\lambda,\nu}$$
$$+\lambda^{1/2}\sum_{a\neq b}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_a^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_b^\beta\right\rangle_{\lambda,\nu}. \tag{116}$$

By Lemma 6

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_0^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_0^\beta\right\rangle_{\lambda,\nu}$$
$$\leqq\lambda^{1/2}\left\langle\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)^2\chi_{[-a_2,a_2]}\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)\right\rangle_{\lambda,\nu}$$
$$\leqq\lambda^{1/2}0(1)\leqq\tfrac{1}{8}\cdot 10^{-2} \tag{117}$$

for λ small enough. Moreover

$$\lambda^{1/2}\sum_{a\neq b}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_a^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_b^\beta\right\rangle_{\lambda,\nu}$$
$$\leqq\sum_{a\neq b}\left(\lambda\left\langle\left(\frac{1}{|\Delta|}\varphi(\Delta)\right)^4\right\rangle_{\lambda,\nu}\right)^{1/2}\langle\chi_a^\alpha\chi_b^\beta\rangle_{\lambda,\nu}^{1/2}\leqq\tfrac{1}{8}\cdot 10^{-2} \tag{118}$$

for λ small enough (and all ν) by Lemmas 9 and 10. Using (117) and (118) we obtain from (116)

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right\rangle_{\lambda,\nu}\geqq 2\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\chi_1^\alpha\frac{1}{|\Delta|}\varphi(\Delta_\beta)\chi_1^\beta\right\rangle_{\lambda,\nu}$$
$$-\tfrac{1}{4}\cdot 10^{-2}\geqq 2\lambda^{1/2}a_2^2\langle\chi_1^\alpha\chi_1^\beta\rangle_{\lambda,\nu}-\tfrac{1}{4}\cdot 10^{-2}=\tfrac{4}{5}\langle\chi_1^\alpha\rangle_{\lambda,\nu}$$
$$-\tfrac{4}{5}\langle\chi_1^\alpha\chi_0^\beta+\chi_1^\alpha\chi_{-1}^\beta\rangle_{\lambda,\nu}-\tfrac{1}{4}\cdot 10^{-2}\geqq\tfrac{4}{5}\langle\chi_1^\alpha\rangle_{\lambda,\nu}-\tfrac{1}{2}\cdot 10^{-2} \tag{119}$$

again for λ small enough and all ν (Lemma 9).

From (115) and (119)

$$\lambda^{1/2}\left\langle\frac{1}{|\Delta|}\varphi(\Delta_\alpha)\frac{1}{|\Delta|}\varphi(\Delta_\beta)\right\rangle_{\lambda,\nu}\geqq\frac{1}{80}-\frac{1}{200}=\frac{3}{4}\cdot 10^{-2}.$$

Taking $D=\frac{3}{4}\cdot 10^{-2}$ we are done. □

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References

1. Dimock, J., Glimm, J.: Measures on Schwartz distribution space and applications to $P(\varphi)_2$ field theories. Advan. Math. **12**, 58—83 (1974)
2. Fröhlich, J.: Phase transitions in two dimensional quantum field models. ZIF, University of Bielefeld. Preprint
3. Fröhlich, J.: Phase transitions, Goldstone bosons and topological superselection rules. Acta Phys. Austr. Supl. **15**, 133—269 (1976)
4. Fröhlich, J., Simon, B.: Pure states for general $P(\varphi)_2$ theories: construction, regularity and variational equality. Princeton Preprint
5. Fröhlich, J., Simon, B., Spencer, T.: Infrared bounds, phase transitions and continuous symmetry breaking. Commun. math. Phys. **50**, 79—95 (1976)
6. Glimm, J., Jaffe, A., Spencer, T.: Phase transitions for φ_2^4 quantum fields. Commun. math. Phys. **45**, 203—216 (1975)

7. Glimm, J., Jaffe, A., Spencer, T.: Convergent expansion about mean field theory. I and II. Ann. Phys. **101**, 610—630 (1976); **101**, 631—669 (1976)
8. Guerra, F., Rosen, L., Simon, B.: Nelson's symmetry and the infinite behavior of the vacuum in $P(\varphi)_2$. Commun. math. Phys. **27**, 10—22 (1972)
9. Guerra, F., Rosen, L., Simon, B.: The $P(\varphi)_2$ Euclidean quantum field theory as classical statistical mechanics. I and II. Ann. Math. **101**, 111—189 (1975); **101**, 191—259 (1975)
10. Guerra, F., Rosen, L., Simon, B.: Boundary conditions for the $P(\varphi)_2$ Euclidean field theory. Ann. Inst. H. Poincaré **25**, 231—334 (1976)
11. Pirogov, S. A., Sinai, Ya. G.: Phase diagrams of the classical lattice systems. I and II. Teor. Mat. Fis. **25**, 358—369 (1975); **26**, 61—76 (1976)
12. Simon, B.: The $P(\varphi)_2$ Euclidean (quantum) field theory. Princeton, New Jersey: Princeton University Press 1974
13. Sinai, Ya. G.: 1977 lectures in IHES, Bures-sur-Yvette
14. Glimm, J., Jaffe, A.: On approach to the critical point. Ann. Inst. H. Poincaré **22**, 109—122 (1975)

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