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Properties of Certain Matrices Related to the Equilibrium Configuration of the One-Dimensional Many-Body Problems with the Pair Potentials $V_1(x) = -\log|\sin x|$ and $V_2(x) = 1/\sin^2 x$

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Abstract. It is shown that at equilibrium certain matrices associated to the one-dimensional many-body problems with the pair potentials $V_1(x) = -\log|\sin x|$ and $V_2(x) = 1/\sin^2 x$ have a very simple structure. These matrices are those that characterize the small oscillations of these systems around their equilibrium configurations, and, for the second system, the Lax matrices that demonstrate its integrability.

1. Results

Consider the two classical one-dimensional many-body systems characterized by the hamiltonians

$$H_s = \tfrac{1}{2}p^2 + V_s(x), \qquad s = 1, 2, \tag{1.1}$$

with

$$V_1(x) = -\sum_{j>k=1}^{n} \log|\sin(x_j - x_k)|, \tag{1.2a}$$

$$V_2(x) = \sum_{j>k=1}^{n} \sin^{-2}(x_j - x_k), \tag{1.2b}$$

and by the corresponding equations of motion

$$\ddot{x}_i = \sum_{k=1}^{n}{}' \operatorname{cotg}(x_j - x_k), \qquad j = 1, 2, \ldots, n, \tag{1.3a}$$

$$\ddot{x}_j = 2\sum_{k=1}^{n}{}' \operatorname{cotg}(x_j - x_k)\sin^{-2}(x_j - x_k), \qquad j = 1, 2, \ldots, n. \tag{1.3b}$$

Here, and always in the following, x resp. p indicate the n-dimensional vector of coordinates x_j resp. p_j, $j = 1, 2, \ldots, n$, while a prime appended to a sum indicates that the singular term must be omitted.

Let us, for definiteness, assume that the particles are labeled so that

$$x_j < x_{j+1}, \quad j = 1, 2, \ldots, n-1, \tag{1.4}$$

and assume moreover that there exist a constant x_0 such that

$$|x_j - x_0| < \pi, \quad j = 1, 2, \ldots, n. \tag{1.5}$$

Clearly these assumptions imply no loss of generality, in view of the singular and periodic nature of the forces.

The system characterized by the hamiltonian H_1 was discussed by Dyson [1], who noted that its equilibrium configuration is given by the formula

$$x_j = \bar{x}_j \equiv x_0 + \pi j/n, \quad j = 1, 2, \ldots, n, \tag{1.6}$$

with x_0 an arbitrary constant. This corresponds to the formula

$$\sum_{k=1}^{n}{}' \operatorname{cotg}(\bar{x}_j - \bar{x}_k) = 0, \quad j = 1, 2, \ldots, n. \tag{1.7}$$

It is not known whether this "Dyson system" is integrable. But we prove in the next section that the circular frequencies of the normal modes of the small oscillations of this system around its equilibrium configuration (1.6) are given by the simple formula

$$\omega_s^2 = 2s(n-s), \quad s = 1, 2, \ldots, n. \tag{1.8}$$

As implied by the standard theory of small oscillations [applied to the equilibrium configuration (1.6) of the Dyson system with equations of motion (1.3a)], these numbers coincide with the n eigenvalues of the hermitian matrix of rank n

$$\bar{M}_{jk} = \delta_{jk} \sum_{l=1}^{n}{}' \sin^{-2}(\bar{x}_j - \bar{x}_l) - (1 - \delta_{ik}) \sin^{-2}(\bar{x}_j - \bar{x}_k), \tag{1.9}$$

where of course the $\bar{x}_j$'s are given by (1.6).

The system characterized by the hamiltonian H_2 was introduced by Sutherland [2] in the quantal context. Its complete integrability in the classical case was proved by Moser [3], who introduced the two Lax matrices.

$$L_{ik} = \delta_{ik} p_j + i(1 - \delta_{jk}) \operatorname{cotg}(x_j - x_k), \tag{1.10}$$

$$M_{jk} = \delta_{jk} \sum_{l=1}^{n}{}' \sin^{-2}(x_j - x_l) - (1 - \delta_{jk}) \sin^{-2}(x_j - x_k), \tag{1.11}$$

and noted the equivalence of the Lax equation

$$L_t = i[L, M] \tag{1.12}$$

to the equations of motion (1.3b). There follows of course that the eigenvalues λ_s of the matrix L are constants of the motion. Moser showed that these n quantities are independent [3], and it was subsequently easily shown that they are in involution [4].

We prove in the following Section that the equilibrium configuration for this "Sutherland-Moser system" coincides with that, Equation (1.6), of the Dyson system; this of course implies that

$$\sum_{k=1}^{n}{}' \operatorname{cotg}(\bar{x}_j - \bar{x}_k) \sin^{-2}(\bar{x}_j - \bar{x}_k) = 0, \quad j = 1, 2, \ldots, n. \tag{1.13}$$

We prove moreover that at equilibrium, i.e. for $p=0$ and $x=\bar{x}$ [with the components $\bar{x}_j$ of $\bar{x}$ given by (1.6)], the Lax matrix $\bar{L}\equiv L(p=0, x=\bar{x})$ has the eigenvalues[1]

$$\bar{\lambda}_s=2s-n, \quad s=1,2,\ldots,n-1\,; \quad \bar{\lambda}_n=0\,. \tag{1.14}$$

Note that also the eigenvalues of the second Lax matrix M of Equation (1.11) are, at equilibrium, given by a very simple formula, namely

$$\bar{\mu}_s=2s(n-s)\,, \qquad s=1,2,\ldots,n\,, \tag{1.15}$$

as implied by the statement reported above after Equation (1.8), and by the identity at equilibrium of M, Equation (1.11), with $\bar{M}$, Equation (1.9). Note moreover that the two matrices $\bar{L}$ and $\bar{M}$ commute [as implied by (1.12); of course at equilibrium $L_t=0$]; and it is shown below that to the eigenvalues $\bar{\lambda}_s$ and $\bar{\mu}_s$ there corresponds the same eigenvector. Since it is apparent by inspection [using (1.7)] that the eigenvector corresponding to the eigenvalues $\bar{\lambda}_n=\bar{\mu}_n=0$ is the vector $u=(1,1,\ldots,1)$, a comparison of the spectra (1.14) and (1.15) implies that these two matrices are related by the simple formula

$$\bar{M}=\tfrac{1}{2}[n^2 I-(\bar{L}')^2]\,, \tag{1.16}$$

where I is the unit matrix of rank n and L' is related to L by the formula

$$L'=L+J\,, \tag{1.17}$$

J being the idempotent matrix having all elements equal to unity,

$$J_{jk}=1\,, \quad J^2=nJ\,. \tag{1.18}$$

Indeed it is well known that the matrix J has the eigenvalue 0 with multiplicity $n-1$, and the (nondegenerate) eigenvalue n corresponding to the eigenvector u. Note that the matrix L' could be used in place of L in the Lax formula (1.12), since addition of the constant matrix J to L changes neither the l.h.s. nor the r.h.s. of (1.12), since clearly (1.11) and (1.18) imply

$$MJ=JM=0\,. \tag{1.19}$$

Of course the matrix $\bar{L}'$ appearing in the r.h.s. of (1.16) is the matrix L' of (1.17) evaluated at equilibrium, i.e. for $p=0$ and $x=\bar{x}$, with the components of $\bar{x}$ given by (1.6); and its eigenvalues are given by the simple formula $\bar{\lambda}'_s=2s-n$, $s=1,2,\ldots,n$ [5].

Squaring explicitly the matrix $\bar{L}$ and using the trigonometric identity

$$\operatorname{cotg}\alpha\operatorname{cotg}\beta=-1-[\operatorname{cotg}\alpha-\operatorname{cotg}\beta]\operatorname{cotg}(\alpha-\beta) \tag{1.20}$$

together with (1.7) one easily shows that

$$(\bar{L})^2=-2\bar{M}-nJ+I+3\operatorname{diag}\left[\sum_{k=1}^{n}{}' \sin^{-2}(\bar{x}_j-\bar{x}_k)\right]; \tag{1.21}$$

1 Note that the eigenvalue 0 is always present, and it has multiplicity 2 for even n

and a comparison of this formula with (1.16) implies [using (1.18) and the formula $J\bar{L}=\bar{L}J=0$ that follows from (1.7)] the formula

$$\sum_{k=1}^{n}{}' \sin^{-2}(\bar{x}_j-\bar{x}_k)=\tfrac{1}{3}(n^2-1), \quad j=1,2,\ldots,n. \tag{1.22}$$

Note that this formula implies that all the diagonal elements of $\bar{M}$ are equal and that

$$\operatorname{tr}\bar{M}=\tfrac{1}{3}n(n^2-1), \tag{1.23}$$

a formula that is easily seen to be compatible with (1.15). Indeed (1.22) could also have been obtained noting by inspection [using (1.6)] that all the diagonal elements of $\bar{M}$ are equal, and then evaluating the trace of $\bar{M}$ from (1.15).

Analysis of the small oscillations of the Sutherland-Moser system around its equilibrium configuration implies in addition that the matrix

$$\begin{aligned}\bar{N}_{jk}=4\delta_{jk}\sum_{l=1}^{n}{}' [\tfrac{3}{2}-\sin^2(\bar{x}_j-\bar{x}_l)]\sin^{-4}(\bar{x}_j-\bar{x}_l)\\ -4(1-\delta_{jk})[\tfrac{3}{2}-\sin^2(\bar{x}_j-\bar{x}_k)]\sin^{-4}(\bar{x}_j-\bar{x}_k)\end{aligned} \tag{1.24}$$

[with $\bar{x}_j$ given by (1.6)], whose eigenvalues provide the squares of the circular frequencies of the normal modes, satisfies the matrix equality

$$\bar{N}=\bar{M}^2, \tag{1.25}$$

with $\bar{M}$ given by (1.9). This result, proved in the following section, implies that also the eigenvectors of $\bar{M}$ and $\bar{N}$ coincide, and that the eigenvalues $\bar{\nu}_s$ of $\bar{N}$ are given by the simple formula

$$\bar{\nu}_s=4s^2(n-s)^2, \quad s=1,2,\ldots,n. \tag{1.26}$$

The fact that the square roots of all these numbers are integers imply the complete periodicity of the small oscillations of the Sutherland-Moser system around its equilibrium configuration; hardly a surprising result, since the full motion of the Sutherland-Moser system is completely periodic [6].

From (1.26) there easily follows that

$$\operatorname{tr}\bar{N}=\tfrac{2}{15}n(n^4-1). \tag{1.27}$$

Noting that the diagonal elements of $\bar{N}$ are all equal, one easily obtains from this formula [also using (1.22)] the sum rule

$$\sum_{k=1}^{n}{}' \sin^{-4}(\bar{x}_j-\bar{x}_k)=\tfrac{1}{45}(n^2-1)(n^2+11). \tag{1.28}$$

Let us emphasize that the results reported above, and proved below, not only display some remarkably simple properties of the Dyson and Sutherland-Moser systems, but also imply a number of nontrivial relations for trigonometric functions of rational angles, including some remarkable diophantine equations. In view of the possible interest of these formulae, they will be also reported elsewhere.

Let us finally mention that analogous results can be obtained starting from other "solvable" many-body models [7]. Some of the results for the zeros of classical polynomials and other special functions that can be obtained in this manner go beyond those recently discovered [8]; they also will be published elsewhere.

2. Proofs

Let us begin by proving the coincidence of the equilibrium configurations for the Dyson and the Sutherland-Moser systems, and also the validity of Equation (1.25). These results are immediate consequences of the following elementary theorem: [9].

Let the two hamiltonians

$$H^{(s)}(x,p)=\tfrac{1}{2}p^2+V^{(s)}(x), \qquad s=1,2, \tag{2.1}$$

be related by the formula

$$V^{(2)}(x)=\alpha\sum_{l=1}^{n}[\partial V^{(1)}(x)/\partial x_l]^2+\text{const}; \tag{2.2}$$

then they have the same equilibrium configuration, and the matrices $\bar{M}^{(s)}$ that characterize their behavior in the neighborhood of an equilibrium configuration are related by the formula

$$\bar{M}^{(2)}=2\alpha[\bar{M}^{(1)}]^2; \tag{2.3}$$

so that their normal modes coincide and the circular frequencies of their small oscillations around the equilibrium configuration are related by the formula

$$[\omega_s^{(2)}]^2=2\alpha[\omega_s^{(1)}]^4, \qquad s=1,2,\ldots,n. \tag{2.4}$$

For indeed this theorem is applicable in our case, since it is easy to prove (see Appendix) that

$$V_2(x)=\tfrac{1}{2}\sum_{l=1}^{n}[\partial V_1(x)/\partial x_l]^2+\tfrac{1}{6}\,n(n^2-1), \tag{2.5}$$

with $V_1(x)$ and $V_2(x)$ defined by Equations (1.2).

Let us proceed now to prove (1.8) [or equivalently (1.15)] and (1.14). To this end we introduce the one-dimensional many-body system characterized by the equations of motion

$$\ddot{x}_j=-a\dot{x}_j+\sum_{k=1}^{n}{}'\,[1-b(\dot{x}_j+\dot{x}_k)+2\dot{x}_j\dot{x}_k]\operatorname{cotg}(x_j-x_k), \qquad j=1,2,\ldots,n. \tag{2.6}$$

Clearly the equilibrium configuration of this system coincides with that of the Dyson system (and therefore also of the Sutherland-Moser system), since the r.h.s. of (1.3a) and (2.6) coincide at equilibrium. Note incidentally that for $a=b=0$ the behavior of the system (2.6) in the neighborhood of its equilibrium configuration coincides, in the linear approximation, with that of the Dyson model of Equation (1.3a).

The equations of motion (2.6) are exactly solvable [7]. Let us restrict attention to motions with fixed center-of-mass, a condition clearly consistent with (2.6). Then the exact solution [7][2] of (2.6) is a nonlinear combination of the functions

$$c_m(t) = c_m^{(+)} \exp(\alpha_m^{(+)} t) + c_m^{(-)} \exp(\alpha_m^{(-)} t), \qquad m = 0, \pm 1, \pm 2, \ldots, \pm n, \tag{2.7}$$

where the quantities $c_m^{(\pm)}$ are arbitrary constants (to be fixed by the initial conditions) and the quantities $\alpha_m^{(\pm)}$ are the two roots of the second degree equation[3]

$$\alpha^2 + (a + imb)\alpha + \tfrac{1}{2}(n^2 - m^2) = 0 . \tag{2.8}$$

Note that, for $m = n$, only the single solution $c_n(t) = \text{const}$ must be considered, corresponding to the single root $\alpha = 0$ of (2.8) for $m = n$ (indeed this contribution is associated to the center-of-mass degree of freedom[4]); moreover the quantities $c_m(t)$ enter in the solutions of (2.6) only for those values of m whose parity coincides with the parity of n [there are thus exactly n different $c_m(t)$'s that contribute][5].

Consider now the behavior of (2.6) in the neighborhood of its equilibrium configuration (1.6). Then setting $x = \bar{x} + \varepsilon$ in (2.6) and keeping only terms linear in ε one obtains [using (1.7)] the equation

$$\ddot{\varepsilon} + (a - ib\bar{L})\dot{\varepsilon} + \bar{M}\varepsilon = 0 . \tag{2.9}$$

Here we are using a vector notation: ε is the n-vector of components ε_j, while $\bar{L}$ and $\bar{M}$ are the matrices defined in the previous Section. But this formula must yield the same (not necessarily real) circular frequencies that appear in the exact solution of (2.6). A comparison of (2.8) and (2.9) (note the arbitrariness of a and b) implies that this is possible only if the matrices $\bar{L}$ and $\bar{M}$ commute[6], have common eigenvectors and have eigenvalues $-m$ and $\frac{1}{2}(n^2 - m^2)$, with the additional specification on the permitted values of m reported above after (2.8). This immediately implies (setting $m = n - 2s$) the results (1.8) [or equivalently (1.15)] and (1.14)[7].}

Appendix

From (1.2a) there follows that

$$\partial V_1(x)/\partial x_l = - \sum_{k=1}^{n}{}' \operatorname{cotg}(x_l - x_k) , \tag{A.1}$$

[2] Set $C = 1$, $E = a$, $B = 0$, $\beta = 1$, $A = \frac{1}{2}$, $D = b$ in Equation (3.6.9) of C

[3] See Equation (3.6.19) of C

[4] See Equation (3.6.23) of C

[5] See the statement after Equation (3.6.22) of C

[6] A fact already implied by the results proved above, as noted in the previous section

[7] The eigenvalue $\bar{\lambda}_n$, that is not determined by this argument, is given by the condition $\operatorname{tr} \bar{L} = 0$, implied by the very definition of $\bar{L}$

so that

$$\sum_{l=1}^{n} [\partial V_1(x)/\partial x_l]^2 = A + B \tag{A.2}$$

with

$$A = \sum_{l=1}^{n} \sum_{k=1}^{n}{}' \operatorname{cotg}^2(x_l - x_k) \tag{A.3}$$

$$= 2V_2(x) - n(n-1), \tag{A.4}$$

$$B = \sum_{l,k,j=1}^{n}{}'' \operatorname{cotg}(x_l - x_k) \operatorname{cotg}(x_l - x_j) \tag{A.5}$$

$$= - \sum_{l,j,k=1}^{n}{}'' \{[\operatorname{cotg}(x_l - x_k) - \operatorname{cotg}(x_l - x_j)] \operatorname{cotg}(x_j - x_k) + 1\} \tag{A.6}$$

$$= -2B - n(n-1)(n-2). \tag{A.7}$$

The double apex on the sum in (A.5) and (A.6) indicates that in the sum all terms in which two indices coincide must be omitted. The step from (A.5) to (A.6) uses the trigonometric identity (1.20). The step from (A.6) to (A.7) takes advantage of the fact that the 3 indices l, j, k are dummy. Clearly (A.7) implies

$$B = -\tfrac{1}{3} n(n-1)(n-2), \tag{A.8}$$

and this, together with (A.2) and (A.4), yields (2.5), q.e.d.

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