Commun. math. Phys. 58, 113—129 (1978)

Communications in
Mathematical
Physics

Selfadjointness of the Liouville Operator for Infinite Classical Systems*

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Abstract. We study some properties of the time evolution of an infinite one dimensional hard cores system with singular two body interaction. We show that the Liouville operator is essentially antiselfadjoint on the algebra of local observables. Some consequences of this result are also discussed.

1. Introduction

In the last years the time evolution of infinite classical particle systems has been studied by various authors [1–8]. The main problem to solve was to give an existence theorem for the infinite equations of motion which formally read as:

$$\left.\begin{aligned} &\frac{d}{dt}p_i(t)=F_i(q(t))\\ &\frac{d}{dt}q_i(t)=p_i(t)\\ &p_i(0)=p_i \qquad q_i(0)=q_i\,, \qquad i\in\mathbb{Z}\end{aligned}\right\} \tag{1.1}$$

where $q(t)\equiv\{q_i(t)\}_{i=-\infty}^{i=+\infty}$ and $p(t)\equiv\{p_i(t)\}_{i=-\infty}^{i=+\infty}$ are respectively the positions and momenta of the particles and $F_i(q(t))$ is the force on the "i-th" particle induced by the others, and p_i, q_i are the initial data. A trajectory $(p,q)\to(p(t),q(t))$ satisfying Equations (1.1) for all $t\in\mathbb{R}$ may be found if we make suitable hypothesis on the interactions and on the regularity of the initial conditions (p,q). The set $\hat{\mathfrak{X}}$ of these couples (p,q) (phase points) is large enough to have full measure with respect to the equilibrium measure ν, and this allows to construct a triple $(\mathfrak{X}, S_t, \nu)$ where $\mathfrak{X}$ is the phase space and S_t is a ν-almost everywhere defined one parameter group of ν-invariant transformations satisfying Equations (1.1). Nevertheless till now very little is known about the physical properties of the dynamics; for example it is possible to exhibit explicitely the initial conditions $x\in\hat{\mathfrak{X}}$ for which S_tX is defined only in the one dimensional case [1, 7] and, by the choice of particular interactions,

* Research partially supported by the Consiglio Nazionale delle Ricerche, G.N.F.M.

0010-3616/78/0058/0113/$03.40

at most in two dimensions [8]. In the other cases [2–6] one may show that a ν-full measure set of initial conditions can evolve, but it is not possible to characterize explicitely the initial phase points.

A Cauchy problem which is related to the dynamical problem (1.1) is the following:

$$\left.\begin{aligned} &\frac{d}{dt} f_t = \mathscr{L} f_t \\ &f_{t=0} = f \qquad f \in L_2(\mathfrak{X}, \nu) \end{aligned}\right\} \tag{1.2}$$

where $\mathscr{L}$, the Liouville operator, is defined as:

$$\mathscr{L} f = \sum_i \left\{ \frac{\partial f}{\partial q_i} p_i + \frac{\partial f}{\partial p_i} F_i(q) \right\} \tag{1.3}$$

and makes sense as an antisymmetric operator in $L_2(\mathfrak{X}, \nu)$ on a suitable algebra of functions $\mathfrak{A} \subset L_2(\mathfrak{X}, \nu)$. Each function in $\mathfrak{A}$ depends only on the coordinates of particles that fall in a fixed bounded region (see Section 2 for a precise definition) so that the series in (1.3) contains only a finite number of elements different from zero.

It is possible to show [9] that if $\mathscr{L}$ is essentially antiselfadjoint (e.a.s.) on $\mathfrak{A}$, there exist a dynamical flow $(\mathfrak{X}, T_t, \nu)$ such that:

$$(U_t f)(x) = f(T_t x), \qquad t \in \mathbb{R}, \qquad x \in \mathfrak{X}, \qquad f \in L_2(\mathfrak{X}, \nu) \tag{1.4}$$

where $U_t = e^{\bar{\mathscr{L}} t}$ is the unitary group generated by the closure of $\mathscr{L}$ and $T_t = S_t$ if S_t is a solution of the problem (1.1). In this paper we study the essential antiselfadjointness of $\mathscr{L}$ on $\mathfrak{A}$, starting from a solution of Equations (1.1) for a particular interacting system. Such problem was discussed in [10] and solved for the free gas case.

The most relevant consequence of being $\mathscr{L}$ e.a.s. on $\mathfrak{A}$ is the essential locality of the motion.

For finite systems in open bounded regions, the knowledge of the vector field (forces and momenta) at any point of the phase space is not sufficient to determinate the motion, since we have also to specify the boundary conditions; in fact each of them give a different generator for the motion, which reduces to the usual Liouville operator on the algebra of the differentiable functions with compact support.

In analogy with the finite case we may conclude that the antiselfadjointness property means that the "boundary" effects which arise from the infinity will occur with probability zero. An example of these effects may be found in [4, p. 50].

Finally we remark that the e.a.s. property implies also a particular kind of unicity for the motion. In fact, if $\mathscr{L}$ has different antiselfadjoint extentions, it could be possible to have other dynamical flows different from S_t. A result in this direction has been obtained by Lanford [4, Proposition 3, p. 60], who proved the unicity of S_t in the class of the ν-preserving a.e. defined flows that satisfy the equations of motion.

In this paper, we study a one dimensional infinite hard cores system interacting via a singular two body potential with finite range.

Using a recent result of Dobrushin and Fritz [7] we prove the e.a.s. of $\mathscr{L}$ on an algebra of local functions. The crucial point is the knowledge of the time growth of the mean energy in order to obtain an estimate on the derivatives of the position and momentum of the particles w.r.t. the initial data.

We shall consider a class of invariant states, containing the equilibrium ones, on which we impose some technical requirements. In Section [5] we use the results of this paper, together with those of [11], and we prove that the only stable (w.r.t. local perturbations) state in the class under consideration, which also satisfies some ergodic properties, must be an equilibrium state.

In Section [2] notation, definitions and results are stated, in Section [3] we derive the estimates on the derivatives in order to prove in Section [4] the main result of this paper. The Appendix is devoted to the short proofs of substantially already known results that we report for sake of completeness.

2. Notation, Definitions and Results

Definition 2.1 (Phase spaces). Let $\Delta \subset \mathbb{R}$ be a Borel set; we will denote by $\mathfrak{X}(\Delta)$ the phase space of finite or countably infinite hard cores in Δ:

$$\mathfrak{X}(\Delta)=\{x\equiv(q(x),p(x))|q(x)\equiv\{q_i\},\,p(x)\equiv\{p_i\},\,i\in\mathbb{Z}$$
$$\cdot q_i<q_{i+1},q_i\in\Delta,|q_i-q_j|>\delta \text{ if } i\neq j,\,p_i\in\mathbb{R}\}\,.$$

Here q_i denotes the position of the "i-th" particle (the particles are ordered denoting by q_0 the first one on the right of the origin) p_i its momentum and δ the hard core length.

In what follows we will put $\mathfrak{X}=\mathfrak{X}(\mathbb{R})$. If Δ is bounded, then $\mathfrak{X}(\Delta)$ may be thought as $\bigcup\limits_{n\geqq 0}\tilde{\mathfrak{X}}_{\Delta,n}$, where $\tilde{\mathfrak{X}}_{\Delta,n}$ is the subset of the symmetrization of $(\Delta\times\mathbb{R})^n$ satisfying the hard core condition, and $\tilde{\mathfrak{X}}_{\Delta,0}$ consists of only one element which describes the vacuum in Δ. A mapping $\Pi_\Delta:\mathfrak{X}\to\mathfrak{X}(\Delta)$ is defined, where $\Pi_\Delta(x)\equiv x_\Delta=\{(p_i,q_i)|q_i\in q(x),\ q_i\in\Delta\}$. Furthermore $\mathfrak{X}$ may be canonically identified with the subset of $\mathfrak{X}(\Delta)\times\mathfrak{X}(\Delta^c)$ $(\Delta^c=\mathbb{R}-\Delta)$ consisting of all the couples $(x,y)\equiv x\cup y$, $x\in\mathfrak{X}(\Delta)$, $y\in\mathfrak{X}(\Delta^c)$, compatible with the hard core condition.

Definition 2.2 (Topologies and σ-algebras). A metrizable topology is given on each $\mathfrak{X}(\Delta)$, specified by means of the following convergence: a sequence $\{x_n\}\subset\mathfrak{X}(\Delta)$ is said to converge to $x\in X(\Delta)$ iff for all open bounded sets $\Lambda\subset\Delta$ such that $q(x)\cap\partial\Lambda=\emptyset$ then:

i) $q(x_n)\cap\Lambda\to q(x)\cap\Lambda$ point by point in $\mathbb{R}$ and

ii) $p(x_n)|_{q(x_n)\cap\Lambda}\to p(x)|_{q(x)\cap\Lambda}$ pointwise.

In the sequel we will consider measures defined on the σ-algebras Σ_Δ of all Borel sets of $\mathfrak{X}(\Delta)$. We put $\Sigma\equiv\Sigma_{\mathbb{R}}$.

For any Borel bounded set $\Delta\subset\mathbb{R}$ let us denote by $\Sigma_{\Delta,n}$ the σ-algebra of the Borel sets in $\tilde{\mathfrak{X}}_{\Delta,n}$; then Σ_Δ may be defined alternatively as:

$$\Sigma_\Delta\equiv\{A\subset\mathfrak{X}(\Delta)|A\cap\tilde{\mathfrak{X}}_{\Delta,n}\in\Sigma_{\Delta,n}\forall n\}\,.$$

It is not hard to see that the Σ_Δ's generate Σ.

Definition 2.3 (Interaction). The interaction is described by a two body potential ϕ, $\phi:(\delta, +\infty)\to\mathbb{R}$ satisfying:

i) $\lim_{r\to\delta^+} \phi(r) = +\infty$;

ii) $\phi \in C^\infty((\delta, +\infty))$;

iii) $\phi(r)=0$ if $r \geqq R$;

iv) there exist constants A, A_1, A_2, B_1, B_2, λ, σ, $\lambda>0$, $2>\sigma>1$, such that:

$$\phi(r) \geqq (r-\delta)^{-\lambda} + A$$

$$|\phi'(r)(r-\delta)| \leqq A_1\phi(x) + B_1$$

$$|\phi''(r)| \leqq (A_2\phi(r) + B_2)^\sigma$$

we note that the above conditions are fulfilled if $\phi(r) \sim (r-\delta)^{-\delta}$ near $r=\delta$, with $\lambda>2$.

Definition 2.4 (States). A state μ is a probability Borel measure on $\mathfrak{X}$. $\mathfrak{X}$ is regular enough so that there exist the conditional probabilities $\mu_\Delta(dx|y)$ of μ w.r.t. any Σ_{Δ^c} $\mu_\Delta(dx|y)$ may be thought for μ_{Δ^c}-almost all $y \in \mathfrak{X}(\Delta^c)$ as a probability measure on $\mathfrak{X}(\Delta)$. μ_{Δ^c} is the probability measure induced by μ on $\mathfrak{X}(\Delta^c)$ via the projection Π_{Δ^c}.

Furthermore $\mu_\Delta(dx|y)$ is concentrated on the configurations $x \in \mathfrak{X}(\Delta)$ compatible with y, and it results, for any bounded measurable function f on $\mathfrak{X}$:

$$\int_{\mathfrak{X}} f(x)\mu(dx) = \int_{\mathfrak{X}(\Delta^c)} \mu_{\Delta^c}(dy) \int_{\mathfrak{X}(\Delta)} \mu(dx|y) f(x\cup y).$$

Definition 2.5 (Equilibrium states). A state ν is called an Equilibrium State for the potential ϕ with inverse temperature β and chemical potential $\bar{\mu}$ [12], if its conditional probabilities are: (Δ Borel bounded set)

$$\nu(dx_\Delta|y) = \frac{dx_\Delta}{Z_{\Delta^c}(y)} \exp\{-\beta H(x_\Delta|y)\}$$

where: (m is the mass of the particles)

$$H(x_\Delta|y) = \sum_{q_i\in\Delta} p_i^2/2m + U(x_\Delta) + W(x_\Delta|y)$$

$$U(x_\Delta) = \frac{1}{2} \sum_{\substack{i,j \\ q_i, q_j\in\Delta \\ i\neq j}} \phi(|q_i - q_j|)$$

$$W(x_\Delta|y) = \sum_{\substack{i,j \\ q_i\in\Delta \\ q_j\in\Delta^c}} \phi(|q_i - q_j|).$$

dx_Δ is the measure on $\mathfrak{X}(\Delta)$ given by:

$$dx_\Delta = 1 + \sum_{n=1}^{\infty} dq_1 \ldots dq_n dp_1 \ldots dp_n \frac{\exp(\beta\bar{\mu}n)}{n!}$$

and $Z_{\Delta^c}(y)$ is a normalization factor.

Definition 2.6 (The reduced phase space $\hat{\mathfrak{X}}$). We define:

$$Q(x;\mu,\bar{\sigma})=\sum_{|q_i-\mu|<\bar{\sigma}}\frac{p_i^2}{2m}+\frac{1}{2}\sum_{\substack{i\neq j\\ |q_i-\mu|<\bar{\sigma}\\ |q_j-\mu|<\bar{\sigma}}}\phi(|q_i-q_j)$$
$$+BN(x;\mu,\bar{\sigma})$$
$$N(x;\mu,\bar{\sigma})=\mathrm{Card}\{q_i\in x|\,|q_i-\mu|<\bar{\sigma}\}.$$

B is a constant such that $Q(x;\mu,\bar{\sigma})\geqq 0$, for technical reasons we fix $B=B_1R/A_1\delta$ (see [7])

$$Q(x)=\left[\left(\sup_{\mu}\sup_{\bar{\sigma}>\log_+\mu}(Q(x;\mu,\bar{\sigma})/2\bar{\sigma})\right)\vee 1\right]$$
$$\cdot\log_+\mu=\log(|\mu|\vee e).$$

$Q(x)$ is a Σ-measurable function and hence the following set

$$\hat{\mathfrak{X}}=\{x\in\mathfrak{X}|Q(x)<+\infty\}$$

is measurable.

Definition 2.7 (Regular states). A state μ is called a regular state if:

i) μ is locally absolutely continuous with respect to the Lebesgue measure i.e. its conditional probability $\mu(dx|y)$ as measure on $\mathfrak{X}(\Delta)$ is absolutely continuous with respect to the measure dx_Δ (see Definitions 2.4 and 2.5).

ii) There exist positive constants b_1 sufficiently small b_2 sufficiently large, and b_3, such that:

$$\int\mu(dx)\exp[b_1Q(x;\eta,\bar{\sigma})]\leqq\exp b_2\bar{\sigma},$$
$$\mu(\{x\in\mathfrak{X}|q(x)\cap[\eta-\bar{\sigma},\eta+\bar{\sigma}]=\emptyset\})\leqq\exp-b_3\bar{\sigma}.$$

It is an easy consequence of Definition 2.5 that every equilibrium state is regular [7].

Lemma 2.1. *Let μ be a regular state. Then there exists a constant a (depending only on μ) such that:*

$$\mu(\{x|Q(x)\geqq s\})\leqq e^{-as}$$

and hence $\mu(\hat{\mathfrak{X}})=1$.

Proof. See the Appendix.

Definition 2.8 (Partial dynamics). A function: $\mathfrak{X}\ni x=(q_i,p_i)_{i=-\infty}^{i=+\infty}\to x^\alpha(t)=(q_i^\alpha(t),p_i^\alpha(t))_{i=-\infty}^{i=+\infty}$, $\alpha\in\mathbb{R}$ is called a partial dynamics if it is the solution of the following integral equations:

$$q_i^\alpha(t)=q_i+\vartheta_{i,\alpha}(x)\int_0^t ds\,p_i^\alpha(s)$$

$$p_i^\alpha(t)=p_i-\vartheta_{i,\alpha}(x)\int_0^t ds\sum_{j\neq i}\frac{\partial\phi(|q_j^\alpha(s)-q_i^\alpha(s)|)}{\partial q_i^\alpha(s)}$$

where

$$\vartheta_{i,\alpha}(x)=0 \quad \text{if either} \quad q_{i+1}\notin[-\alpha,\alpha] \quad \text{or} \quad q_{i-1}\notin[-\alpha,\alpha]$$
$$=1 \quad \text{otherwise}.$$

Definition 2.9 (Dynamics). A function: $\mathfrak{X}\ni x=(q_i,p_i)_{i=-\infty}^{+\infty}\to x(t)=(q_i(t),p_i(t))_{i=-\infty}^{+\infty}$ with values on $\mathfrak{X}$ is called a solution of the motion equations if:

$$q_i(t)=q_i+\int_0^t p_i(s)ds$$

$$p_i(t)=p_i+\int_0^t ds\sum_{j\neq i}\frac{\partial\phi(|q_j(s)-q_i(s)|)}{\partial q_i(s)}.$$

We shall use the following version of a theorem due to Dobrushin and Fritz [7].

Theorem 2.1 (Dobrushin, Fritz). *There exists a one parameter group of continuous transformations:*

$$S(t):\hat{\mathfrak{X}}\to\hat{\mathfrak{X}}, \qquad t\in\mathbb{R} \quad \textit{such that} \quad x(t)\equiv S(t)x$$

is a solution of the motion equations. Furthermore:

i) *For all* $t\in\mathbb{R}$

$$\lim_{\alpha\to\infty}x^\alpha(t)=\lim_{\alpha\to\infty}S^\alpha(t)x=x(t);$$

ii) $Q(x^\alpha(t))\leqq\mathscr{K}(t)Q(x)^\xi$

where $\xi=3\left(\frac{\lambda-1}{\lambda}\right)$ *and* $\mathscr{K}(t)$ *is a continuous function of* t.

Proof. See the Appendix.

Definition 2.10 (Koopman unitaries). Let μ be a regular, invariant state [i.e. $\mu(S_tA)=\mu(A)$) for all Borel sets A, and $t\in\mathbb{R}$]. For all $f\in L_2(\mathfrak{X},\mu)$, we define:

$$(U_tf)(x)=f(S_tx) \qquad t\in\mathbb{R} \qquad x\in\hat{\mathfrak{X}}.$$

It is easy to verify that U_t is a strongly continuous one parameter group of unitaries in $L_2(\mathfrak{X},\mu)$ [13].

Definition 2.11 (The class $\mathscr{R}$). We will be interested in the following to a particular set of regular invariant states with an additional regularity property with respect to the partial dynamics.

We define:

$\mathscr{R}=\{\mu|\mu$ is a regular time invariant state for which there esists a real bounded function $t\to H_\mu(t)$ not depending on α, such that $\mu(S^\alpha(t)A)\leqq H_\mu(t)\mu(A)$ for all $A\in\Sigma\}$.

If $\mu\in\mathscr{R}$, the operators:

$$(U_\alpha(t)f)(x)=f(S^\alpha(t)x) \qquad f\in L_2(\mathfrak{X},\mu)$$

are uniformly bounded operators in $L_2(\mathfrak{X}, \mu)$ and furthermore:

$$L_2 - \lim_{\alpha \to \infty} U_\alpha(t) f = U(t) f \qquad f \in L_2(\mathfrak{X}, \mu), \qquad t \in \mathbb{R}$$

as consequence of Theorem 2.1 and dominated convergence theorem. We observe that $\mathscr{R}$ is non void because it contains at least the equilibrium states that are $S^\alpha(t)$-invariant as a consequence of Definition 2.5 and the Liouville theorem.

Definition 2.12 (The algebras $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$). We define the algebra $\mathfrak{A}(\Lambda)$ of the functions $f: \mathfrak{X} \to \mathbb{C}$ described in terms of potentials $\{F^{(m)}\}$ via the formula:

$$f(x) = \sum_{n \geqq 0} \sum_{X^n \subset x} F^{(n)}(X^n), \qquad x \in \mathfrak{X}, \qquad X^n \in (\Lambda \times \mathbb{R})^n$$

where $F^{(n)}$ are C^∞, complex functions symmetric and with compact support in $(\Lambda \times \mathbb{R})^n$. We call $\mathfrak{A} \equiv \bigcup_\Lambda \mathfrak{A}(\Lambda)$ the algebra of local observables. Note that, $\mathfrak{A}$ is a complex algebra of bounded functions because of the hard core condition. We consider also the algebra $\tilde{\mathfrak{A}}$ of the functions generated by potentials $F^{(n)}$ that are C^∞, complex symmetric functions in $(\mathbb{R} \times \mathbb{R})^n$ with compact support in the region in which $|q_i - q_j| \geqq \delta$ and possibly diverging, together with all the derivatives, at $|q_i - q_j| = \delta$ at most as a power. If μ is a regular state, then $\mathfrak{A} \subset \tilde{\mathfrak{A}} \subset L_p(\mathfrak{X}, \mu)$ $(p \geqq 1)$. We finally remark that $\mathfrak{A}$ is $L_2(\mathfrak{X}, \mu)$-dense. It follows by observing that the Σ_Λ's generate Σ and that any bounded measurable function on $\mathfrak{X}(\Lambda)$ (thought of as function on $\mathfrak{X}$) may be pointwise approximated from below by a sequence in $\mathfrak{A}(\Lambda)$.

Definition 2.13 (The Liouville operator). We introduce an operator $\mathscr{L}: \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ defined by

$$(\mathscr{L} f)(x) = \sum_i \left\{ \frac{\partial f}{\partial q_i} p_i - \frac{\partial f}{\partial p_i} \sum_{j \neq i} \frac{\partial \phi}{\partial q_i} (|q_j - q_i|) \right\} (x).$$

The above sum is finite since f depends explicitely only on a finite number of coordinates.

For any $\mu \in \mathscr{R}$ we consider the antiselfadjoint operator $\tilde{\mathscr{L}}$ that generates $U(t)$ in $L_2(\mathfrak{X}, \mu)$. Then

$$\mathscr{L} = \tilde{\mathscr{L}}|_{\tilde{\mathfrak{A}}}.$$

Definition 2.14 (Poisson bracket). A bilinear form $\{\cdot, \cdot\}: \tilde{\mathfrak{A}} \times \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ is defined

$$\{f, g\}(x) = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)(x).$$

As in Definition 2.13 one realizes that the above sum is finite. The main theorem of this paper is the following

Theorem 2.2. *If $\mu \in \mathscr{R}$ then $\mathfrak{A}$ (and hence $\tilde{\mathfrak{A}}$) is a core for $\tilde{\mathscr{L}}$ in $L_2(\mathfrak{X}, \mu)$. That is $\mathscr{L}|_{\mathfrak{A}}^{**} = \overline{\mathscr{L}|_{\mathfrak{A}}} = \tilde{\mathscr{L}}$. (Here $\overline{\mathscr{L}|_{\mathfrak{A}}}$ denotes the closure of the restriction to $\mathfrak{A}$ of $\mathscr{L}$.)*

3. Estimates on the Derivatives

In this section we derive some estimates crucial for the proof of Theorem 2.2. Let us first define the quantities we want to estimate. We shall consider derivatives of $q_i^\alpha(x,t)$, $p_i^\alpha(x,t)$ with respect to some coordinate q_j. We note that $\frac{\partial q_i^\alpha(x,t)}{\partial q_j}$, $\frac{\partial p_i^\alpha(x,t)}{\partial q_j}$ exist only for those configurations $x\in\hat{\mathfrak{X}}$ such that $q_j(x)\neq\pm\alpha$. Otherwise $q_i^\alpha(x,t)$ as function of q_j will not be continuous. Let us fix $\eta\in\mathbb{R}$ and $\gamma\in[-\eta,\eta]$, and put $\alpha(n,\gamma)=n+\gamma$, $n\in\mathbb{Z}^+$. In the sequel we will put $\alpha=\alpha(n,\gamma)$. The set $\hat{\mathfrak{X}}_\gamma=\{x\in\hat{\mathfrak{X}}\,|\,q(x)\cap\pm\alpha=\emptyset\}$ has μ measure 1, if μ is a regular state, as follows by Definition 2.7, i), and furthermore for all $x\in\hat{\mathfrak{X}}_\gamma$, $\frac{\partial q^\alpha(x,t)}{\partial q_j}$, $\frac{\partial p^\alpha(x,t)}{\partial q_j}$ make sense. For any $t>0$, $x\in\hat{\mathfrak{X}}_\gamma$ let us put

$$u_{i,j}^\alpha(x,t)=\frac{\partial q_i^\alpha(x,t)}{\partial q_j}\qquad v_{i,j}^\alpha(x,t)=\frac{\partial p_i^\alpha(x,t)}{\partial q_j}\tag{3.1}$$

$$\varphi_{i,j}^\alpha(x,t)=(|u_{i,j}^\alpha(x,t)|\vee|v_{i,j}^\alpha(x,t)|)\tag{3.2}$$

$$\varphi_\lambda^\alpha(x,t)=\sup_i\sup_j\varphi_{i,j}^\alpha(x,t)\tag{3.3}$$

where the suprema are taken on the set of indices i for which $q_i^\alpha(x,t)\in\Lambda\equiv[-\lambda,\lambda]$ and on the set of indices j of the frozen particles.

Finally we define

$$\varphi^\alpha(x,t)=\sup_i\sup_j\varphi_{i,j}^\alpha(x,t)\tag{3.4}$$

where the suprema are taken without any condition

Proposition 3.1. *Let $\mu\in\mathscr{R}$ for all t, λ, η, $b>0$ and $p\geqq1$ we have*

$$\lim_{n\to\infty}(\log n)^b\sup_{\gamma\in[-\eta,\eta]}\|\varphi_\lambda^{n+\gamma}(\cdot,t)\|_p=0$$

where $\|\ \|_p$ denotes the L_p-norm.

Proof. By Definition 2.8, 3.1, and 3.2

$$\begin{aligned}(x\in\hat{\mathfrak{X}}_\gamma)&=u_{i,j}^\alpha(x,t)=\delta_{i,j}+\vartheta_{i,\alpha}(x)\int_0^t ds\,v_{i,j}^\alpha(x,s)\\ v_{i,j}^\alpha(x,t)&=\vartheta_{i,\alpha}(x)\int_0^t ds\frac{\partial}{\partial q_j}F_i(x^\alpha(s))\end{aligned}\tag{3.5}$$

where $F_i(x^\alpha(s))$ is the force on the i-particle induced by $x^\alpha(s)$. Then

$$\begin{aligned}|u_{i,j}^\alpha(x,t)|&\leqq\delta_{i,j}+\int_0^t ds\,|v_{i,j}^\alpha(x,s)|\\ |v_{i,j}^\alpha(x,t)|&\leqq\int_0^t ds\sum_k A_{i,k}^\alpha(x,s)|u_{k,j}^\alpha(x,s)|\end{aligned}\tag{3.6}$$

where

$$A^\alpha_{i,k}(x,s)=\left|\frac{\partial^2\phi(|q^\alpha_i(s)-q^\alpha_k(s)|)}{\partial q^\alpha_k(s)^2}\right|,\qquad i\neq k$$

$$A^\alpha_{i,i}(x,s)=\sum_{k\neq i}A^\alpha_{i,k}(x,s)\,. \tag{3.7}$$

Then by Definitions 2.3 and 2.6

$$\begin{aligned}A^\alpha_{i,i}(x,s)&\leqq\left[\sum_{k\neq i}(B_2+A_2)\phi(q^\alpha_i(s)-q^\alpha_k(s))\right]^\sigma\\&\leqq\left[\frac{2B_2}{\delta}+c_0Q(x^\alpha(s);q^\alpha_i(s),R)\right]^\sigma\end{aligned}\tag{3.8}$$

and hence here exists a constant c_1, not depending on x and s such that:

$$A^\alpha_{i,i}(x,s)\leqq c_1[Q(x^\alpha(s))\log_+\alpha]^\sigma \tag{3.9}$$

and therefore there exists c_2 such that:

$$\begin{aligned}u^\alpha(x,t)&\leqq 1+\int_0^t ds\,v^\alpha(x,s)\\ v^\alpha(x,t)&\leqq c_2\int_0^t ds[Q(x^\alpha(s))\log_+\alpha]^\sigma u^\alpha(x,s)\end{aligned}\tag{3.10}$$

where:

$$\begin{aligned}u^\alpha(x,t)&=\sup_i\sup_j|u^\alpha_{i,j}(x,t)|\\ v^\alpha(x,t)&=\sup_i\sup_j|v^\alpha_{i,j}(x,t)|\,.\end{aligned}\tag{3.11}$$

Therefore:

$$\varphi^\alpha(x,t)\leqq\sqrt{c_2}\,(Q(x^\alpha(t))\log_+\alpha)^{\sigma/2}\exp\sqrt{c_2}\int_0^t ds(Q(x^\alpha(s))\log_+\alpha)^{\sigma/2} \tag{3.12}$$

and hence by Theorem 2.1

$$\varphi^\alpha(x,t)\leqq\exp\sqrt{c_2}\,\{(1+t\mathscr{K}(t))Q(x)^{\xi\sigma/2}(\log_+\alpha)^{\sigma/2}\}\,. \tag{3.13}$$

On the other hand by (3.6) and (3.8) we obtain:

$$\varphi^\alpha_{i,j}(x,t)\leqq\delta_{i,j}+\int_0^t ds\,c_1(Q(x^\alpha(s))\log_+\alpha)^\sigma\,\tilde{\sum_k}\varphi^\alpha_{k,j}(x,s) \tag{3.14}$$

where $\tilde{\sum_k}$ means we sum on the indices of those particles that interact with the "i" particle at the instant s. Now we consider those configurations belonging to the set

$$\tilde{\mathfrak{X}}_n\equiv\left\{x\in\mathfrak{X}\,\middle|\,q(x)\cap\left[\frac{n}{2},n\right]\neq\emptyset\quad\text{and}\quad q(x)\cap\left[-n,-\frac{n}{2}\right]\neq\emptyset\right\}.$$

Since we are interested in those $\varphi^\alpha_{i,j}$ in which the "i" particle is in $[-\lambda,\lambda]$ at the instant t and q_j is frozen, $\delta_{i,j}=0$ in (3.14) if α is large enough. We may iterate the inequality (3.14) with $\delta_{k,j}=0$ until the index k may be equal to j. Successive iterations will not give any substantially better estimate than the a priori one (3.13).

With a fixed energy density $Q(x)$, any particle may not move too much because of Theorem 2.1, and so the number of admissible iterations is greater than $\frac{\alpha-2\lambda-\eta}{2(Vt+R)}$ where V is the maximum of the velocities. Since:

$$V \leqq \sup_{0\leqq s\leqq t} \sqrt{\frac{2}{m} Q(x^\alpha(s)) \log_+ \alpha} \tag{3.15}$$

we may find a constant c_3 depending only on m, t, λ, such that the number k of the iterations may be chosen:

$$k = \text{integer part of} \left[c_3 \frac{\alpha}{Q(x)^{\xi/2} \sqrt{\log \alpha}}\right] \vee 1 . \tag{3.16}$$

So we have, by (3.13) and (3.14):

$$\begin{aligned} \varphi_\lambda^\alpha(x,t) \leqq & \frac{t^k}{k!} c_1^k (Q(x)^{\xi/2} \log_+ \alpha)^{k\sigma} \mathscr{K}(t)^k (2R/\delta)^k \\ & \cdot \exp \sqrt{c_2}\, \{(t\mathscr{K}(t)+1) Q(x)^{\xi\sigma/2} (\log_+ \alpha)^{\sigma/2}\} . \end{aligned} \tag{3.17}$$

And hence there exist constants c_4 and c_5 not depending on x such that:

$$\begin{aligned} \varphi_\lambda^\alpha(x,t) \leqq & \frac{c_4^k}{k!} Q(x)^{\frac{k\sigma\xi}{2}} (\log_+ \alpha)^{k\sigma} \\ & \cdot \exp\{c_5 Q(x)^{\sigma\xi/2} (\log_+ \alpha)^{\sigma/2}\} . \end{aligned} \tag{3.18}$$

We shall use this inequality to estimate φ_λ^α for $Q(x)$ not too large. Otherwise, as we shall show below, we use the inequality (3.12).

Combining (3.15) and (3.18) one has:

$$\begin{aligned} \varphi_\lambda^\alpha(x,t) \leqq \exp\Big\{ & c_3 \alpha \log c_4 / Q(x)^{\xi/2} (\log_+ \alpha)^{1/2} \\ & + \frac{\sigma\xi}{2} c_3 \alpha \log_+ Q(x) / Q(x)^{\xi/2} (\log_+ \alpha)^{1/2} \\ & + c_3 \alpha \sigma \log_+ \log_+ \alpha / Q(x)^{\xi/2} (\log_+ \alpha)^{1/2} + c_5 Q(x)^{\sigma\xi/2} (\log_+ \alpha)^{\sigma/2} \\ & - (c_3 \alpha / Q(x)^{\xi/2} (\log_+ \alpha)^{1/2}) \log \frac{c_3 \alpha}{Q(x)^{\xi/2} (\log_+ \alpha)^{1/2}} \\ & + c_3 \alpha / Q(x)^{\xi/2} (\log_+ \alpha)^{1/2} \Big\} \\ \leqq \exp\Big\{ & \frac{c_3 \alpha}{Q(x)^{\xi/2} (\log \alpha)^{1/2}} (c_6 \log_+ Q(x) + c_7 \log_+ \log_+ \alpha \\ & + c_5 Q(x)^{\xi\left(\frac{\sigma+1}{2}\right)} (\log_+ \alpha)^{\frac{\sigma+1}{2}} / c_3 \alpha - \log c_3 \alpha) \Big\} \end{aligned} \tag{3.19}$$

with c_6, c_7 not depending on x.

Putting $Q(x) \leqq (\log\alpha)^l$, where l will be fixed later, we obtain for n large enough and suitable constant $c_8(l) < 1$:

$$\varphi_\lambda^\alpha(x,t) \leqq \exp\left\{-\frac{c_8(l)c_3\alpha}{(\log_+\alpha)^{(l\xi+1)/2}}\log c_3\alpha\right\}, \ x \in \tilde{\mathfrak{X}}_n. \tag{3.20}$$

On the other hand by (3.12)

$$\int_{\{x|Q(x)>(\log\alpha)^l\}} |\varphi_\lambda^\alpha(x,t)|^p \mu(dx) \leqq \sqrt{c_2} \int \mu(dx)\chi(x)$$

$$\cdot (Q(x^\alpha(t))\log_+\alpha)^{p\sigma/2} \cdot \exp p\sqrt{c_2}\int_0^t ds[Q(x^\alpha(s))\log_+\alpha]^{\sigma/2} \tag{3.21}$$

where χ is the characteristic function of the set $\{x|Q(x) > (\log\alpha)^l\}$. By the Jensen Inequality and Fubini Theorem the above quantity is bounded by:

$$\sqrt{c_2}\int_0^t \frac{ds}{t}\int \mu(dx)\chi(x)(Q(x^\alpha(t))\log_+\alpha)^{p\sigma/2} \exp p\sqrt{c_2}\,t(Q(x^\alpha(s)\log_+\alpha)^{\sigma/2}. \tag{3.22}$$

Finally using the Schwarz inequality and Definition 2.11:

$$\int |\varphi_\lambda^\alpha(x,t)|^p \chi(x)\mu(dx) \leqq \sqrt{c_2}\int_0^t \frac{ds}{t}\|\chi\|_2$$

$$\cdot [\int \mu(dx)(Q(x^2(s))\log_+\alpha)^{p\sigma} \exp 2p\sqrt{c_2}\,t[Q(x^\alpha(s))\log_+\alpha]^{\sigma/2}]^{1/2}$$

$$\leqq \sqrt{c_2}\|\chi\|_2 \left[\sup_{0\leqq s\leqq t} H_\mu(s)\right]^{1/2} [\int \mu(dx)(Q(x)\log_+\alpha)^{2p\sigma}$$

$$\cdot \int \mu(dx)\exp 4p\sqrt{c_2}\,t(Q(x)\log_+\alpha)^{\sigma/2}]^{1/4}$$

$$\leqq c_9(\log_+\alpha)^{2p\sigma} \sum_{r>(\log_+\alpha)^l} \exp -ar$$

$$\cdot \left[\sum_{r\geqq 0} \exp(4p\sqrt{c_2}\,t[r\log_+\alpha]^{\sigma/2})\exp -ar\right]^{1/4} \tag{3.23}$$

where $c_9 = \left[\sup_{0\leqq s\leqq t} H_\mu(s)\right]^{1/2} \sqrt{c_2}\int \mu(dx)Q(x)$. Hence there exist two positive numbers c_{10} and c_{11} not depending on α and x such that

$$\int |\varphi_\lambda^\alpha(x,t)|^p \chi(x)\mu(dx) \leqq \exp\{-c_{10}(\log n)^l + c_{11}(\log n)^{\sigma/(2-\sigma)}\}. \tag{3.24}$$

We now fix $l > \sigma/(2-\sigma)$ so the above quantity goes to zero for $n \to \infty$ more rapidly than any power of $\log n$ and hence:

$$(\log n)^{bp}\|\varphi_\lambda^\alpha(\cdot,t)\|_p^p \leqq (\log n)^{pb}\int |\varphi_\lambda^\alpha(x,t)|^p\chi(x)\mu(dx)$$

$$+\sup\{\varphi_\lambda^\alpha(x,t)^p | Q(x) \leqq (\log\alpha)^l \quad x \in \tilde{\mathfrak{X}}_n\}$$

$$+\sup\{\varphi_\lambda^\alpha(x,t)^p | Q(x) \leqq (\log\alpha)^l, \quad x \in \mathfrak{X}/\tilde{\mathfrak{X}}_n\} \cdot \mu(\mathfrak{X}/\tilde{\mathfrak{X}}_n)]$$

$$\cdot \alpha = h + \gamma \quad \gamma \in [-\eta, \eta]. \tag{3.25}$$

So the thesis in Proposition 3.1 is easily obtained by the use of the estimates (3.20), (3.24), (3.13), and Definition 2.7.

4. Proof of Theorem 2.2

We shall use the following Criterion:

Let $\mathfrak{H}$ be a complex Hilbert space, $U(t)=\exp(iHt)$ a strongly continuous group, and $\mathscr{D}\subset\mathscr{D}(H)$ a dense set.

Then $\mathscr{D}$ is an essentially selfadjointness domain for H if $U(t)\Psi\in\mathscr{D}(\overline{H\lceil_{\mathscr{D}}})$ for all $t\in\mathbb{R}$, $\Psi\in\mathscr{D}$.

The criterion may be proven by adapting Theorems VIII.11 or X.49 of [13].

In virtue of the above criterion, the proof of Theorem 2.2 will be achieved if we show that for all $f\in\mathfrak{A}$, $t\in\mathbb{R}$ it is possible to find a sequence $\{1_n\}_{n=1}^{\infty}\subset\mathfrak{A}$ such that $1_n\to U(t)f$, $\mathscr{L}1_n\to\bar{\mathscr{L}}U(t)f$ in $L_2(\mathfrak{X},\mu)$ for $n\to\infty$.

A natural candidate for $\{1_n\}$ would be $\{U_n(t)f\}$, but $U_n(t)f$, $n\in\mathbb{Z}^+$ does not belong to $\mathfrak{A}$ because is not continuous, so we must regularize the sequence $\{U_n(t)f\}$. Let us fix $t\in\mathbb{R}$, $f\in\mathfrak{A}([-\lambda,\lambda])$ and define for $n>\lambda$:

$$(V_n(t)f)(x)=\int(U_{n+\gamma}(t)f)(x)g(\gamma)d\gamma\,,\qquad x\in\hat{\mathfrak{X}} \tag{4.1}$$

where $0\leqq g\in c_0^\infty[-\eta,\eta]$, $\int g d\gamma=1$, and $2\eta<\delta$. Then for all $x\in\hat{\mathfrak{X}}$, $n\in\mathbb{Z}^+$, there exists at most two values of γ, denoted by $\gamma_n^+(x)$, $\gamma_n^-(x)\subset[-\eta,\eta]$ (one for side), for which $q_i(x)=\gamma_n^\pm(x)\pm n$ for some i, and hence $\left(\frac{\partial U_{n+\gamma}}{\partial q_i}f\right)(x)$ does not exist. So we have:

$$\begin{aligned}\frac{\partial}{\partial q_i}(V_n(t)f)(x)&=\frac{\partial}{\partial q_i}\left[\int_{-\eta}^{\gamma_n^\pm(x)}\int_{\gamma_n^\pm(x)}^{\eta}d\gamma g(\gamma)(U_{n+\gamma}(t)f)(x)\right]\\&=\int d\gamma g(\gamma)\frac{\partial}{\partial q_i}(U_{n+\gamma}(t)f)(x)+S_t^\pm(n,x)\end{aligned} \tag{4.2}$$

where:

$$\begin{aligned}S_t^\pm(n,x)&=g(\gamma_n^\pm(x))\lim_{\varepsilon\to0^+}[U_{n+\gamma_n^\pm(x)+\varepsilon}(t)f-U_{n+\gamma_n^\pm(x)-\varepsilon}(t)f](x)\\&=0\quad\text{if}\quad q(x)\cap[\pm n-\eta,\,\pm n+\eta\Big]=\phi\,.\end{aligned} \tag{4.3}$$

It is easily seen that $V_n(t)f\in\mathfrak{A}$.

Defining

$$K_{n,\gamma}^t(x)=\sum_j\left\{\frac{\partial U_{n+\gamma}(t)f}{\partial q_j}p_j-\frac{\partial U_{n+\gamma}(t)f}{\partial p_j}\sum_{r\neq j}\frac{\partial\phi}{\partial q_j}(|q_j-q_n|)\right\}(x) \tag{4.4}$$

we see that $K_{n,\gamma}^t(x)$ exists for almost all γ and it results

$$[\mathscr{L}(V_{n+\gamma}(t)f)](x)=\int K_{n,\gamma}^t(x)g(\gamma)d\gamma+H_t(x,n) \tag{4.5}$$

where we have posed:

$$H_t(x,n)=S_t^+(n,x)+S_t^-(n,x)\,. \tag{4.6}$$

For all γ for which $\frac{\partial}{\partial q_j}(U_{n+\gamma}(t)f)(x)$ exists we have:

$$\frac{\partial}{\partial q_j} U_{n+\gamma}(t)f(x)={\sum_i}^*\left\{\frac{\partial f}{\partial q_i}\bigg|_{X^{n+\gamma}(t)} \frac{\partial q_i^{n+\gamma}(t)}{\partial q_j}+\frac{\partial f}{\partial p_i}\bigg|_{X^{n+\gamma}(t)} \frac{\partial p_i^{n+\gamma}(t)}{\partial q_j}\right\} \tag{4.7}$$

where $\sum^*$ means that we sum on the indices of the moving particles. So:

$$K_{n+\gamma}^{+}(x)={\sum_i}^*\sum_j\left\{\frac{\partial f}{\partial q}\bigg|_{X^{n+\gamma}(t)}\left(\frac{\partial q_i^{n+\gamma}(t)}{\partial q_j}p_j-\frac{\partial q_i^{n+\gamma}}{\partial p_j}\sum_{i\neq r}\frac{\partial\phi}{\partial q_j}(|q_j-q_r|)\right)\right.$$

$$\left.+\frac{\partial f}{\partial p_i}\bigg|_{X^{n+\gamma}(t)}\left(\frac{\partial p_i^{n+\gamma}(t)}{\partial q_j}p_j-\frac{\partial p_i^{n+\gamma}}{\partial p_j}\sum_{i\neq r}\frac{\partial\phi}{\partial q_j}(|q_j-q_r|)\right)\right\}$$

$$={\sum_i}^*{\sum_j}^*\{\ \}+{\sum_i}^*{\sum_j}'$$

$$\left\{\frac{\partial f}{\partial q_i}\bigg|_{X^{n+\gamma}(t)}\frac{\partial q_i^{n+\gamma}(t)}{\partial q_j}p_j+\frac{\partial f}{\partial p_i}\bigg|_{X^{n+\gamma}(t)}\frac{\partial p_i^{n+\gamma}(t)}{\partial q_j}p_j\right\} \tag{4.8}$$

where ${\sum_j}'$ means that we sum on the frozen particles.

The first term in r.h.s. of (4.8) is just $U_{n+\gamma}(t)\mathscr{L}f$ so, calling $F_{n+\gamma}^t(x)\equiv{\sum_i}^*{\sum_j}'\{\ \}$ the second term in r.h.s. of (4.8), we have:

$$(4.5)=\int(U_{n+\gamma}(t)\mathscr{L}f)(x)g(\gamma)d\gamma+\int F_{n+\gamma}^t(x)g(\gamma)d\gamma+H_t(x,n). \tag{4.9}$$

If $h\in\tilde{\mathrm{B}}$, then:

$$\begin{aligned}\|U(t)h-V_n(t)h\|_2&=\int\mu(dx)|\int d\gamma g(\gamma)(U(t)h)(x)-(U_{n+\gamma}(t)h)(x)|^2\\&\leqq\int\mu(dx)\int d\gamma g(\gamma)|(U(t)h)(x)-(U_{n+\gamma}h)(x)|^2\\&\leqq\int d\gamma g(\gamma)\|U(t)h-U_{n+\gamma}(t)h\|_2^2\xrightarrow[n\to\infty]{}0\end{aligned} \tag{4.10}$$

in virtue of Theorem 2.1, so the proof of Theorem 2.2 will be completed if we show that the last two terms in r.h.s. of (4.9) go to zero as $n\to\infty$ in $L_2(\mathfrak{X},\mu)$ norm. On the basis of Theorem 2.1 $H_t(x,n)\xrightarrow[n\to\infty]{}0$ for a.e. $x\in\mathfrak{X}$ with the following bound:

$$|H_t(x,n)|\leqq\sup_{\gamma\in[-\eta,\eta]}4|(U_{n+\gamma}(t)f)(x)|\leqq4\|f\|_\infty \tag{4.11}$$

so by dominated convergence theorem $H_t(\cdot,n)\to0$ in $L_2(\mathfrak{X},\mu)$. Finally, because $\sum^*$ and $\sum'$ contain respectively no more than $2\lambda/\delta$ and R/δ terms, if $\sqrt{k_1}$ is a constant greater than any derivative of f; we obtain: [$\forall\gamma$ such that $q(x)\cap n+\gamma=\emptyset$]

$$\begin{aligned}|F_{n+\gamma}^t(x)|^2&\leqq4k_1\left(\frac{\lambda R}{\delta^2}\right)^2|\varphi_\lambda^{n+\gamma}(x,t)|^2Q(x)\log_+(n+R+\eta)\\&\leqq k_2|\varphi_\lambda^{n+\gamma}(x,t)|^2Q(x)\log_+n\end{aligned} \tag{4.12}$$

with k_2 a suitable constant.

But, with similar arguments leading to (4.10) one obtains:

$$\|\int F^t_{n+\gamma}(\cdot)g(\gamma)d\gamma\|_2^2 \leqq \int d\gamma g(\gamma)\|F_{n+\gamma}\|_2^2$$
$$\leqq 2\eta\|g\|_\infty \sup_{-\eta\leqq\gamma\leqq\eta} \|\mathrm{F}_{n+\gamma}\|_2^2$$
$$\leqq 2\eta k_2\|g\|_\infty \log_+ n\|Q\|_2 \sup_{-\eta\leqq\gamma\leqq\eta} \|\varphi_\lambda^{n+\gamma}(\cdot,t)\|_4^2 . \tag{4.13}$$

Therefore, using Proposition 3.1, we conclude:

$$L_2 - \lim_{n\to\infty} \mathscr{L}V_n(t)f = \tilde{\mathscr{L}}U(t)f. \tag{4.14}$$

5. Concluding Remarks

In this section, we discuss a consequence of Theorem 2.2. Before this, we make some comments about the result obtained. The same techniques used in this work may be applied to the dynamics of some anharmonic systems whose existence and unicity was discussed in [14]. The crucial point of our proof seems to be the good estimate (see Theorem 2.1) of the growth of the particles velocity in time and besides some technical difficulties, our approach seems applicable when one has a behavior not faster than a polynomial.

We conclude by applying Theorem 2.2 to the problem of characterizing the Equilibrium States in the class of all the stationary ones by means of a notion of stability [15, 16, 11]. We call a regular state $\mu\in\mathscr{R}$ stable if:

i) For all $f\in\tilde{\mathfrak{A}}$ there exists a state $\mu_{\lambda f}$ formally invariant for the time evolution generated by $\mathscr{L}+\lambda\{\cdot,f\}$ i.e. $\mu_{\lambda f}(\mathscr{L}g+\lambda\{g,f\})=0$ for all $g\in\tilde{\mathfrak{A}}$.

ii) $\dfrac{d\mu_{\lambda f}}{d\mu}=\varphi_{\lambda f}\in L_2(\mu)$.

iii) $\lim\limits_{\lambda\to 0}\dfrac{\varrho_{\lambda f}-1}{\lambda}$ exists in $L_2(\mathfrak{X},\mu)$.

iv) The derivatives iii) is continuous in f in the $L_2(\mathfrak{X},\mu)$ sense.

Then:

Theorem 5.1. *Let μ be a stable state in $\mathscr{R}$. Let us suppose $\mathscr{L}$ such that* 0 *is a simple eigenvalue of $\mathscr{L}$ (ergodicity), and* $\mathrm{Sp}\,\mathscr{L}=\mathbb{R}$. Then μ is an equilibrium state.

Proof. One easily verifies that all the hypothesis in Theorem 2.1 in [11] are satisfied so μ is K.M.S. The equivalence between K.M.S. and the equilibrium condition has been recently proven fully [17].

Acknowledgements. It is a pleasure to thank G. Gallavotti for useful discussions and B. Tirozzo for having informed us about the research of Dobrushin and Fritz. We thank also Prof. N. H. Kuiper for his Kind hospitality at I.H.E.S. where part of this work was done.

Appendix

Proof of Lemma 2.1. By Definition 2.7 and Tchebyshev inequality we have:

$$\mu(\{x|Q(x,\eta,\bar{\sigma})\geqq 2\bar{\sigma}s\})\leqq e^{-2b_1\bar{\sigma}s}e^{b_2\bar{\sigma}} \tag{A.1}$$

But:

$$\{x|Q(x)\geqq s\}\subset\bigcup_{\eta}\bigcup_{\bar\sigma>\log_+\eta}\{x|Q(x;\eta,\bar\sigma)\geqq 2\bar\sigma(s-1)\}\,. \tag{A.2}$$

And so

$$\sum_{k=-\infty}^{\infty}\sum_{n>\log_+k}e^{n(b_4-b_5s)}\geqq\mu(\{x|Q(x)\geqq s\}) \tag{A.3}$$

where b_5 and b_4 are constants, b_5 taking in account that we sum over integer numbers. Hence there exists a constant a for which the statement in Lemma 2.1 holds.

Proof of Theorem 2.1. The proof of Theorem 2.1 is essentially contained in [7]. Here we sketch only the proof of ii) and iii) that may be obtained by a slight modification of the ideas contained in that work. Let us define:

$$W(x)=\sup_{\bar m\in\mathbb{Z}}\ \sup_{\bar\sigma\geqq\log_+(x_{\bar m})}\frac{W_{\bar m}(x,\bar\sigma)}{2\bar\sigma}\vee 1 \tag{A.4}$$

where:

$$W_{\bar m}(x,\bar\sigma)=\sum_{i\in\mathbb{Z}}f(y_i-y_{\bar m},\bar\sigma)\cdot\left[2B+mv_i^2+\sum_{j\neq i}\phi(|q_i-q_j|)\right] \tag{A.5}$$

with $f(y,\bar\sigma)\in c^{\infty}(\mathbb{R}^2)$, $y\in\mathbb{R}$, $\bar\sigma>0$ satisfying the conditions:

$$\begin{aligned}&\text{i) } f(y,\bar\sigma)=1 \quad\text{if}\quad |y|\leqq\bar\sigma, f(y,\bar\sigma)=0 \quad\text{if}\quad |y|\geqq\bar\sigma+2R,\\ &\text{ii) } \left|\frac{\partial}{\partial y}f(y,\bar\sigma)\right|=\frac{\partial}{\partial\bar\sigma}f(y,\bar\sigma)\quad\forall y,\bar\sigma,\\ &\text{iii) } \left|\frac{\partial}{\partial u}f(u,\bar\sigma)\right|\leqq\frac{\partial}{\partial\bar\sigma}f(y,\bar\sigma)+\frac{\partial}{\partial\bar\sigma}f(z,\bar\sigma),\end{aligned} \tag{A.6}$$

if $\bar\sigma>R$ and $y\leqq u\leqq z\leqq y+R$, and

$$y_i=q_i'-i\delta \tag{A.7}$$

q_i being the coordinate of the "i" particle.

W is a smooth version of Q which takes in account the presence of the excluded volume

The comparison between Q and W is given by the following Lemma:

Lemma A.1. *There exist two constant a_1, a_2 such that $Q(x)\leqq q\leqq\infty$ implies $W(x)\leqq(a_1+a_2q^{\frac{1}{\lambda}+1})$.*

Proof. It is sufficient to explicit the proof of Lemma 2 in [7]. Now we are able to prove Theorem 2.1. It is shown in [7] that W_m does not increase to much in time uniformly in partial dynamics (Definition 2.8):

$$W_{\bar m}(x^{\alpha}(t),\bar\sigma)\leqq W_{\bar m}(x,r(0)),\qquad x=x^{\alpha}(0) \tag{A.8}$$

where $r(t)$ is defined by the following integral equation:

$$r(t)=\bar{\sigma}+2(A_1+1)\int_t^T V_{\bar{m}}(x^\alpha(s),r(s))ds \tag{A.9}$$

where:

$$V_{\bar{m}}(x^\alpha(t),\bar{\sigma})=[2W(x^\alpha(t))\log_+(|q_{\bar{m}}(t)|+\bar{\sigma}+3R)]^{1/2} \qquad 0\leqq t\leqq T. \tag{A.10}$$

Because $r(0)\geqq \log_+ q_{\bar{m}}$ if $\bar{\sigma}\geqq \log_+(q_{\bar{m}}(t))$, it results:

$$W(x^\alpha(T))\leqq W(x) \sup_{\bar{\sigma}\geqq \log_+(q_{\bar{m}}(T))}\left\{\frac{r(0)}{\bar{\sigma}}\right\}. \tag{A.11}$$

Using the subadditivity of $\log_+$, Equation (A.7) and the fact that:

$$|q_{\bar{m}}(t)|\leqq |q_{\bar{m}}(T)|+r(0)-\bar{\sigma} \tag{A.12}$$

we have (with a_3, a_4 constants large enough):

$$\begin{aligned} r(0)&\leqq \bar{\sigma}+2^{3/2}(A_1+1)\int_0^T W^{1/2}(x^\alpha(s))\{\log_+ r(0)+a_3\bar{\sigma}\}^{1/2}ds \\ &\leqq \left[1+\left\{\frac{a_4}{\bar{\sigma}^{1/2}}\log_+ r(0)+a_3\right\}\int_0^T W^{1/2}(x^\alpha(s))ds\right]\bar{\sigma}. \end{aligned} \tag{A.13}$$

Putting

$$L=\sup_{\bar{\sigma}>\log_+ q_m(T)}\left\{\frac{r(0)}{\bar{\sigma}}\right\} \tag{A.14}$$

then for all $\varepsilon:\frac{1}{2}>\varepsilon>0$, there exists a_5 such that:

$$\begin{aligned} L&\leqq a_5 L^\varepsilon\left[\int_0^T W^{1/2}(x^\alpha(s))ds+1\right] \\ &\leqq (a_5)^{1/(1-\varepsilon)}\left[\int_0^T W^{1/2}(x^\alpha(s))ds+1\right]^{1/(1-\varepsilon)} \end{aligned} \tag{A.15}$$

Using (A.8) one obtain:

$$W(x^\alpha(T))\leqq (a_5)^{1/(1-\varepsilon)}\left[\int_0^T W^{1/2}(x^\alpha(s))ds+1\right]^{1/(1-\varepsilon)} W(x). \tag{A.16}$$

This is an inequality, which bounds the grow of $W(T)$

$$W(x^\alpha(t))\leqq \tilde{\mathscr{K}}(t)W(x)^{(2-2\varepsilon)/(1-2\varepsilon)} \tag{A.17}$$

with $\tilde{\mathscr{K}}(t)$ continuous function on t. By Lemma A.1, ii) proven. Then we have found an a priori bound on the grow of $Q(x^\alpha(t))$ (uniformly on α) and so the possible displacement of every particle is bounded. The whole Theorem 2.1 is then proven following [7] or using an iterative method like [3, 5].

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Communicated by J. L. Lebowitz

Received May 31, 1977