Commun. math. Phys. 57, 31—50 (1977)

Communications in
Mathematical
Physics

The Energy-Momentum Spectrum in the Yukawa$_2$ Quantum Field Theory

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Abstract. We prove that the Yukawa$_2$ quantum field theory with periodic boundary conditions satisfies the spectral condition, i.e., the joint spectrum of the energy operator H and the momentum operator P is contained in the forward cone. In addition, the φ-bound is obtained.

1. Introduction

In the present paper the Yukawa$_2$ ($=Y_2$) interaction in two-dimensional space time is considered. This theory provides an example of the theory which satisfies all the Haag-Kastler axioms and many of the Wightman and Osterwalder-Schrader axioms. In the present paper we prove the spectral condition for the Y_2 theory with the periodic boundary conditions. In addition, we obtain the uniform bounds on the boson field operators and on their derivatives with respect to coordinates.

The proof of the spectrum condition is divided into the following parts.

In Section 2 we prove the boundedness below and the $N_{\tau,V}$ estimate for the Lorentz rotated Hamiltonian in the Y_2 theory,

$$\beta_0 H_{V,\sigma}+\beta P_V+c_1(\beta,\tau)\geqq c_2(\beta,\tau)N_{\tau,V}\,,\quad \tau<1\,,\quad \beta_0^2-\beta^2=1\,,\quad \beta_0>0$$

uniformly in the ultraviolet cut-off σ.

In Section 3 we prove the Osterwalder-Schrader positivity condition in the theory with $H_{0,V}(\beta)\,(=\beta_0 H_{0,V}+\beta P_V)$ as the Hamiltonian and P_V as the momentum operator.

In Section 4 we prove that the free vacuum $\Omega_{0,V}$ overlaps (see [1]) the vacuum for $H_V(\beta)=\beta_0 H_V+\beta P_V$.

In Section 5, using the Osterwalder-Schrader positivity in the spatial direction, we prove the main result, that the energy-momentum spectrum for the Y_2 interaction in a periodic box lies in the forward light cone. As a consequence, the uniform estimates on the derivatives of the boson field operators and the spectrum condition for the Y_2 interaction with the periodic boundary conditions follow (Section 6). In Section 6 the φ-bounds are also obtained.

The same results have been obtained for the $P(\varphi)_2$ theory with the periodic boundary conditions [14].

2. $\beta_0 H_{V,\sigma}+\beta P_V+c_1(\beta,\tau)\geqq c_2(\beta,\tau)N_{\tau,V}$

Let $H_{V,\sigma}$ be the renormalized Hamiltonian of the scalar or pseudoscalar Y_2 interaction with an ultraviolet cut-off σ in a periodic box V and P_V be the operator of momentum. We suppose that our ultraviolet cut-off σ is a sharp position cut-off.

Theorem 2.1.

$$\beta_0 H_{V,\sigma}+\beta P_V+c_1(\beta,\tau)\geqq c_2(\beta,\tau)N_{\tau,V}$$

uniformly in the ultraviolet cut-off σ. *Here* $\beta_0^2-\beta^2=1$, $\beta_0>0$, $\tau<1$, $c_2(\beta,\tau)>0$.

Proof of Theorem 2.1. The proof is completely the same as the one given by Glimm [2] (see also [3]). Though Glimm had used in his proof a sharp momentum cut-off, the same proof holds also in the case of a sharp position ultraviolet cut-off. (It holds also in the case of a general ultraviolet cut-off [3], but we restrict ourselves to sharp-position cut-offs.)

In Glimm's proof it is necessary to make the following replacements (we follow in our notations to the paper [2]):

$$\bar{\mu}(k)\to\bar{\mu}(k,\gamma):=\mu(k)+\gamma k-(3/4)(1-|\gamma|)\mu(k)^{\tau}\,,$$
$$\bar{\omega}(p)\to\bar{\omega}(p,\gamma):=\omega(p)+\gamma p-(3/4)(1-|\gamma|)\omega(p)^{\tau}\,,$$

Here $\gamma:=\beta/\beta_0$.

The operation $\bar{\Gamma}(\gamma)$ replaces the operation $\bar{\Gamma}$ and is an approximate inverse to

$$\bar{H}_{0,V}(\gamma)\quad(\bar{H}_{0,V}(\gamma):=H_{0,V}+\gamma P_V-(3/4)(1-|\gamma|)N_{\tau,V})\,.$$

We remark that in [2] Glimm denotes the "number" operator $N_{\tau,V}$ by F_τ.

The essential point is the translation invariance of the interaction Hamiltonian. The translation invariance implies that the renormalization constants of $H_{V,\sigma}$ and $H_{V,\sigma}+\gamma P_V$ coincide. Thus, Lemma 3.4 by Glimm [2] is valid and has the following form:

Lemma 2.2. *There is a constant* K *depending on* ϱ *but not on* σ, *such that*

$$d_\sigma=\left|\sum_{i=1}^{3} c^{(\gamma)}_{i\varrho\sigma}(\sim\Xi)-c_\sigma(\sim\Xi)\right|<K\,.$$

Here $c^{(\gamma)}_{i\varrho\sigma}(\sim\Xi)$ *denote the expression which are similar to* $c_{i\varrho\sigma}(\sim\Xi)$ *(see* [2]*), and which correspond to the Hamiltonian* $H_{V,\sigma}+\gamma P_V$.

The proof of this lemma coincides with that given in [2], because the momentum conservation gives

$$c^{(\gamma)}_{\sigma}(\sim\Xi)=c_\sigma(\sim\Xi)\,,\qquad c^{(\gamma)}_{\varrho\sigma}(\sim\Xi)=c_{\varrho\sigma}(\sim\Xi)\,.$$

In addition we need the slight modification of Lemma 4.2 [2], which, in our case, has the following form:

Lemma 2.3.

$$-\frac{2\pi}{V}\sum_{p\in\Gamma_V}\bar{\omega}(p,\gamma)\frac{\delta\bar{\Gamma}(\gamma)W_{1\varrho\sigma}}{\delta b'(p)}\frac{\delta\bar{\Gamma}(\gamma)W_{1\varrho\sigma}}{\delta b'^*(p)}$$

$$=\frac{1-\gamma}{2}\frac{2\pi}{V}\sum_{p\in\Gamma_V}Z_{\varrho\sigma}(p)^*Z_{\varrho\sigma}(p)+P_1(\sigma)\,,$$

$$-\frac{2\pi}{V}\sum_{p\in\Gamma_V}\bar{\omega}(p,\gamma)\frac{\delta\bar{\Gamma}(\gamma)W_{1\varrho\sigma}}{\delta b(p)}\frac{\delta\bar{\Gamma}(\gamma)W_{1\varrho\sigma}}{\delta b^*(p)}$$

$$=\frac{1+\gamma}{2}\frac{2\pi}{V}\sum_{p\in\Gamma_V}Z'_{\varrho\sigma}(p)^*Z'_{\varrho\sigma}(p)+P_2(\sigma)\,.$$

Here $\Gamma_V=\left\{\ldots-\frac{2\pi}{V},0,\frac{2\pi}{V},\ldots\right\}$ *is the discrete lattice of allowed momenta.*

The operators $P_1(\sigma)$, $P_2(\sigma)$ *are bounded by* $o(1)(N_{\tau,V}+1)$, $o(1)\to 0$ *as* $\varrho\to\infty$ *and* $o(1)$ *does not depend on* σ.

The proof of this lemma is the same as the one of Lemma 4.2 in [2]. The difference in the coefficients is due to the different asymptotics of the functions $\bar{\omega}(p_1,\gamma)$ and $\bar{\omega}(p_2,\gamma)$ for large $|\xi|$ in the region Ξ

$$\bar{\omega}(p_1,\gamma)\propto|\xi|(1-\gamma)/2$$

$$\bar{\omega}(p_2,\gamma)\propto|\xi|(1+\gamma)/2\,.$$

For $\gamma=0$ these asymptotics coincide.

The required inequality

$$\Delta_{\varrho\sigma}(\Xi)\geqq\frac{1-\gamma}{2}\frac{2\pi}{V}\sum_{p\in\Gamma_V}Z_{\varrho\sigma}(p)^*Z_{\varrho\sigma}(p)+\frac{1+\gamma}{2}\frac{2\pi}{V}\sum_{p\in\Gamma_V}Z'_{\varrho\sigma}(p)^*Z'_{\varrho\sigma}(p)$$

follows, as in [2], from Proposition 5.1 [2].

3. Osterwalder-Schrader Positivity in the Spatial Direction

Let $H_{0,V}$ and P_V be the Hamiltonian and the momentum of the free theory in a periodic box V.

In this section we show that the theory with the Hamiltonian $H_{0,V}(\beta)=\beta_0H_{0,V}+\beta P_V$ and the momentum P_V satisfies the Osterwalder-Schrader positivity condition in the spatial direction. Since boson operators are unbounded we also need the Osterwalder-Schrader positivity condition for bounded functions of the boson field.

Let us now define and construct the corresponding notions.

Let $\mathcal{M}_{\text{coh}}$, $\mathcal{M}_{\text{coh},+}$ be the algebras of coherent functions, $\sum \alpha_i \exp(i\varphi_V(h_i))$, where real $h_i \in C_0^\infty([-V/2, V/2])$ for the case $\mathcal{M}_{\text{coh}}$ and real $h_i \in C_0^\infty([0, V/2])$ for the case $\mathcal{M}_{\text{coh},+}$.

For the sake of convenience, we introduce a four-component space of test functions for the fermion field, and we set for such a four-component function $f(x,\alpha)$

$$\Psi(f) := \sum_{\alpha=1}^{2} \int dx \psi_{V,\alpha}(x) f(x,\alpha) + \sum_{\alpha=3}^{4} \int dx \bar{\psi}_{V,\alpha-2} f(x,\alpha)\,.$$

Let $\mathcal{B}(\mathcal{B}_+)$ be the operator algebra generated by the fermion operators $\Psi(f)$, where $f \in C_0^\infty([-V/2, V/2]) \otimes \mathbb{C}^4 (f \in C_0^\infty([0, V/2]) \otimes \mathbb{C}^4)$.

Let $\mathcal{P}(\mathcal{P}_+)$ be the operator algebra generated by the algebras $\mathcal{M}_{\text{coh}}$ and $\mathcal{B}(\mathcal{M}_{\text{coh},+}$ and $\mathcal{B}_+$, respectively).

Let $\mathfrak{A}(\mathfrak{A}_+)$ be the free associative algebra over the complex field, the generating set of which is the set $\mathbb{R} \times \mathcal{P}$ $(\mathbb{R} \times \mathcal{P}_+)$ where $\mathbb{R}$ is the set of reals (see [4]).

Let $a \in \mathfrak{A}$. Then a may be represented in the following form:

$$a = \sum_{k \in A} \alpha_k \prod_{j_k \in A_k} (t_{j_k} F_{j_k})\,, \tag{3.1}$$

where A is some finite *unordered* set and A_k are some finite *ordered* sets, $t_{j_k} \in \mathbb{R}$ and $F_{j_k} \in \mathcal{P}$. Moreover, the factors in the product $\prod_{j_k \in A_k}$ are ordered from left to right in correspondence with the order in A_k (we note that the algebra $\mathfrak{A}$ is non-commutative!). We define the linear functional S on the algebra $\mathfrak{A}$ in the following way. If $a \in \mathfrak{A}$, then a is of the form (3.1), and we set

$$S(a) := \sum_{k \in A} \alpha_k \left(\Omega_{0,V}, \bar{T} \prod_{j_k \in A_k} (\hat{F}_{j_k}(t_{j_k}) \Omega_{0,V}\right) \tag{3.2}$$

where

$$\hat{F}(t) = \exp(-tH_{0,V}(\beta)) F \exp(tH_{0,V}(\beta))$$

and $\bar{T}$ is the anti-time ordering over the time variables t, and if some of the t's coincide, then the corresponding F stand in (3.2) in the same order as they do in a.

Lemma 3.1. *The expression* (3.2) *defines the linear functional on the algebra* $\mathfrak{A}$.

Proof of Lemma 3.1. It is evident that the expression (3.2) defines the mapping S' from the set of words to the complex field. If a and b are two congruent words (with respect to the identities defining the algebra $\mathfrak{A}$ (see [4])), then, since $\mathcal{P}$ is an associative algebra, $S'(a) = S'(b)$, and, thus, S' defines the mapping S from $\mathfrak{A}$ into $\mathbb{C}$, and, as it may be easily seen, this mapping is the linear one. Lemma 3.1 is proved.

Now we want to build the operator Θ corresponding to the Euclidean operator of the space reflection, $x \to -x$, such that $S(\Theta(a)b)$ would satisfy the Osterwalder-Schrader positivity condition in the spatial direction for $a, b \in \mathfrak{A}_+$.

For this purpose we define the (antilinear) operator Θ on the algebra $\mathfrak{A}$. Let $a \in \mathfrak{A}$, then a is given by (3.1), where each F_{j_k} is

$$\mathrm{F}_{j_k} = \sum_{r \in R(j_k)} F(j_k, r) \prod_{l \in A(j_k, r)} \Psi(f(l, j_k, r)) \tag{3.3}$$

where

$$F(j_k, r) \in \mathscr{M}_{\text{coh}}, \quad f(l, j_k, r) \in C_0^\infty([-V/2, V/2]) \otimes \mathbb{C}^4$$

and $R(j_k)$ is some *unordered* finite set, $A(j_k, r)$ are *ordered* finite sets and the product is taken in correspondence with the order in $A(j_k, r)$.

Let

$$\Theta(a) = \sum_{k \in A} \alpha_k^* \prod_{j_k \in \bar{A}_k} (t_{j_k}, \bar{F}_{j_k}), \tag{3.4}$$

where $\bar{A}_k$ is the set *antiordered* with respect to A_k (i.e., if $j_1, j_2 \in A_k$ and $j_1 < j_2$ in A_k, then $j_1, j_2 \in \bar{A}_k$ and $j_1 > j_2$ in $\bar{A}_k$) and where

$$\bar{F}_{j_k} = \sum_{r \in R(j_k)} \vartheta_b F(j_k, r)^* \vartheta_b^{-1} \prod_{l \in \bar{A}(j_k, r)} \Psi(\vartheta_f(f(l, j_k, r))). \tag{3.5}$$

Here * denotes the complex conjugate and ϑ_b is the unitary operator in the Fock space $\mathscr{F}_V$ of the space reflection $x \to -x$ for the boson field $\vartheta_b \varphi_V(x) \vartheta_b^{-1} = \varphi_V(-x)$, and the mapping of the fermion test functions space is defined in the following way

$$\vartheta_f(f(x, \alpha)) = (-if^*(-x, 4), if^*(-x, 3), if^*(-x, 2), -if^*(-x, 1)) \tag{3.6}$$

ϑ_f acts on the fermion field in the following way. If g and h are a pair of two-component functions, then

$$\vartheta_f(g, h) = (h^* \gamma_1^E, g^* \gamma_1^{E*}).$$

We note that $\vartheta_f^2 = 1$.

We use the following representation for the γ-matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma_1^E = i\gamma_1, \quad \gamma_5 = \gamma_1 \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 3.2. *The expressions (3.4)–(3.6) define the antilinear operator Θ on the algebra $\mathfrak{A}$.*

Proof of Lemma 3.2. These expressions define the operator Θ' on the set of words. Then if a and b are two congruent words [4], then the words $\Theta'(a)$ and $\Theta'(b)$ are congruent and thus, Θ' maps congruent words into congruent words and thus, Θ' defines the mapping Θ of the classes of congruent words and so is the operator on the algebra $\mathfrak{A}$. It is clear that this operator is antilinear. Lemma 3.2 is proved.

Now we can formulate the main theorem of this section.

Theorem 3.3. *$S(\Theta(a)b)$ is an hermitian positive bilinear form on the algebra $\mathfrak{A}_+$.*

Proof of Theorem 3.3. The bilinearity follows from the linearity of S, antilinearity of $\Theta(\cdot)$ and from the fact that $\mathfrak{A}$ is an algebra (i.e. that the product ab is a bilinear operation in $\mathfrak{A}$).

To prove positivity we prove first of all that the two-point functions

$$S\left(\Theta\left(\sum_I \alpha_i(t_i,\varphi_V(h_i))\right)\sum_{I'}\alpha_{i'}(t_{i'},\varphi_V(h_{i'}))\right)$$

$$S\left(\Theta\left(\sum_I \alpha_i(t_i,\Psi(h_i))\right)\sum_{I'}\alpha_{i'}(t_{i'},\Psi(h_{i'}))\right) \tag{3.7}$$

are positive semidefinite for $h_i, h_{i'}\in C_0^\infty([0,V/2])$ or $h_i. h_{i'}\in C_0^\infty([0,V/2])\otimes\mathbb{C}^4$.

For this purpose we introduce the following two-point functions

$$G_{\beta V}(t_1,x_1;t_2,x_2)=\begin{cases}(\Omega_{0,V},\varphi_V(x_1)e^{-(t_2-t_1)H_{0,V}(\beta)}\varphi_V(x_2)\Omega_{0,V}) & \text{for } t_1\leqq t_2\\ (\Omega_{0,V},\varphi_V(x_2)e^{-(t_1-t_2)H_{0,V}(\beta)}\varphi_V(x_1)\Omega_{0,V}) & \text{for } t_1> t_2\end{cases}$$

and

$$S_{\beta V}(+;t_1,x_1;t_2,x_2)_{\alpha\alpha'}=\begin{cases}(\Omega_{0,V},\psi_{V,\alpha}(x_1)e^{-(t_2-t_1)H_{0,V}(\beta)}\bar\psi_{V,\alpha'}(x_2)\Omega_{0,V}) & \text{for } t_1\leqq t_2\\ -(\Omega_{0,V},\bar\psi_{V,\alpha'}(x_2)e^{-(t_1-t_2)H_{0,V}(\beta)}\psi_{V,\alpha}(x_1)\Omega_{0,V}) & \text{for } t_1> t_2\end{cases}$$

and (3.8)

$$S_{\beta V}(-;t_1,x_1;t_2,x_2)_{\alpha\alpha'}=\begin{cases}(\Omega_{0,V},\bar\psi_{V,\alpha}(x_1)e^{-(t_2-t_1)H_{0,V}(\beta)}\psi_{V,\alpha'}(x_2)\Omega_{0,V})\\ \qquad\text{for } t_1\leqq t_2\\ -(\Omega_{0,V},\psi_{V,\alpha'}(x_2)e^{-(t_1-t_2)H_{0,V}(\beta)}\bar\psi_{V,\alpha}(x_1)\Omega_{0,V})\\ \qquad\text{for } t_1> t_2\,.\end{cases}$$

We start with the consideration of the fermion two-point functions.

In accordance with the polarization principle to prove the Osterwalder-Schrader positivity it is sufficient to prove that the expressions

$$\sum_{\alpha,\alpha',\alpha''=1}^{2}\int dt_1dx_1dt_2dx_2S_{\beta V}(+;t_1,x_1;t_2,x_2)_{\alpha\alpha''}f_{\alpha'}^*(t_1,-x_1)\cdot(\gamma_1^E)_{\alpha'\alpha}f_{\alpha''}(t_2,x_2) \tag{3.9}$$

and

$$\sum_{\alpha,\alpha',\alpha''=1}^{2}\int dt_1dx_1dt_2dx_2S_{\beta V}(-;t_1,x_1;t_2,x_2)_{\alpha\alpha''}f_{\alpha'}^*(t_1,-x_1)(\gamma_1^{E*})_{\alpha'\alpha}f_{\alpha''}(t_2,x_2)$$

are positive for $f_\alpha(t,x)\in C_0^\infty(\mathbb{R}\times[0,V/2])$ and $\alpha=1,2$.

First we consider the function $S_{\beta V}(+)$. We introduce the function

$$S_\beta(+;t_1,x_1;t_2,x_2)_{\alpha\alpha'}=\begin{cases}(\Omega_0,\psi_\alpha(x_1)e^{-(t_2-t_1)H_0(\beta)}\bar\psi_{\alpha'}(x_2)\Omega_0) & \text{for } t_1\leqq t_2\\ -(\Omega_0,\bar\psi_{\alpha'}(x_2)e^{-(t_1-t_2)H_0(\beta)}\psi_\alpha(x_1)\Omega_0) & \text{for } t_1> t_2\end{cases}$$

Here $\psi_\alpha(x), H_0(\beta), \Omega_0$ are the free fermion field, free Lorentz rotated Hamiltonian and the free vacuum in the full Fock space, i.e., in the Fock space corresponding to the free boundary conditions.

The functions $S_{\beta V}(+)$ and $S_\beta(+)$ are translation invariant and

$$S_{\beta V}(+;t_1-t_2,x_1-x_2)=\sum_{n=-\infty}^{+\infty}S_\beta(+;t_1-t_2,x_1-x_2+nV)$$

where the series converges in the sense of distributions.

We use the Lorentz rotation to replace $S_\beta(+)$ by $S(+)(:=S_\beta(+)|_{\beta=0})$

$$U(\beta)^{-1}\bar{\psi}(0,x)\Omega_0=\bar{\psi}(-x\beta,x\beta_0)\Lambda(\beta)\Omega_0$$
$$U(\beta)^{-1}\psi(0,x)\Omega_0=\Lambda^{-1}(\beta)\psi(-x\beta,x\beta_0)\Omega_0\,.$$

Besides,

$$\psi(t,x)=\int dx' S(t,x-x')(-i\gamma_0)\psi(0,x')$$
$$\bar{\psi}(t,x)=\int dx'\bar{\psi}(0,x')(i\gamma_0)\bar{S}(t,x-x')$$

where $S:=(i\partial\!\!\!/+m)D$ and D is the Pauli-Jordan function, $\bar{S}=\gamma_0 S^+\gamma_0$ and S^+ is the hermitian conjugate to S (see [5]).

Let $\eta_k(t)$ be a smooth function, $\eta_k(t)=1$ for $|t|\geqq 2/k$ and $\eta_k(t)=0$ for $|t|\leqq 1/k$. Then (3.9) is equal to

$$\begin{aligned}&\sum_n \lim_k \int dt_1dx_1dt_2dx_2 \int dx'_1dx'_2\eta_k(t_1-t_2)\\ &\quad\cdot f^*(t_1,-x'_1)\gamma_1^E\Lambda^{-1}(\beta)S(-\beta(x'_1+nV/2),\beta_0(x'_1+nV/2)-x_1)(-i\gamma_0)\\ &S^E(t_1-t_2,x_1-x_2)(i\gamma_0)\bar{S}(-\beta(x'_2-nV/2),\beta_0(x'_2-nV/2)-x_2)\\ &\Lambda(\beta)f(t_2,x'_2)\,.\end{aligned}\tag{3.10}$$

We use here the fact that for $t\neq 0$

$$S(+\,;t,x)=S^E(t,x)=(2\pi)^{-2}\int d^2p\,\frac{m-ip\!\!\!/^E}{m^2+p^2}\,e^{-ip_1t-ip_2x}\,.$$

Note also that

$$\int dx' S(-\beta(x'+nV/2),\ \beta_0(x'+nV/2)-x)f(t,x')\in\mathcal{S}(\mathbb{R}^2)$$

if $f\in\mathcal{S}(\mathbb{R}^2)$, etc.

Then let

$$g_n(t,x)=\int dx' i\gamma_0\bar{S}(-\beta(x'-nV/2),\ \beta_0(x'-nV/2)-x)\Lambda(\beta)f(t,x')\,.$$

Using the relations (see [5])

$$\gamma_0\Lambda(\beta)\gamma_0=\Lambda(\beta)^{-1}$$
$$\gamma_1^E\Lambda(\beta)\gamma_1^E=-\gamma_1\Lambda(\beta)\gamma_1=\Lambda(\beta)^{-1}$$
$$\gamma_0S(t,x)\gamma_0=S(t,-x)$$
$$\gamma_1^ES(t,x)\gamma_1^E=-S(-t,x)$$

(3.10) may be written in the following form

$$\sum_n \lim_k \int dt_1dt_2dx_1dx_2\eta_k(t_1-t_2)\vartheta(g_n(t_1,x_1)S^E(t_1-t_2,x_1-x_2)g_n(t_2,x_2)\tag{3.11}$$

where $\vartheta(g(t,x))=g^*(t,-x)\gamma_1^E$.

The supports of $S(=(i\partial\!\!\!/+m)D)$ lie in the light cone, [5], so, in the region where the integrand is non-zero $(i=1,2)$

$$(x'_i+nV/2)(\beta_0-\beta)\leqq x_i\leqq(x'_i+nV/2)(\beta_0+\beta),\ \text{if}\ (x'_i+nV/2)\beta>0\,;$$
$$(x'_i+nV/2)(\beta_0+\beta)\leqq x_i\leqq(x'_i+nV/2)(\beta_0-\beta),\ \text{if}\ (x'_i+nV/2)\beta<0\,.$$

Thus, the signs of x_i and $x_i'+nV/2$ coincide. Since $f_\alpha \in C_0^\infty(\mathbb{R}\times[0,V/2])$, so the signs of x_1 and x_2 coincide, and $|x_1+x_2|=|x_1|+|x_2|\geqq\varepsilon(f)>0$.

Since for $|x_1+x_2|\geqq\varepsilon(f)>0$ $S^E(t_1-t_2,x_1+x_2)$ is an infinitely differentiable function, so as a result we obtain that (3.9) is equal to

$$\sum_n \int dt_1 dt_2 dx_1 dx_2 g_n^*(t_1,x_1)\gamma_1^E S^E(t_1-t_2,|x_1|+|x_2|)g_n(t_2,x_2)\geqq 0 .$$

The sum is positive, since the n-th term of the sum is the norm of the vector

$$\int dx g_n(t,x)\exp(-|x|H_0)\bar{\psi}(t)\Omega_0 .$$

This proves that the two-point function $S_{\beta V}(+;t_1,x_1;t_2,x_2)$ satisfies Osterwalder-Schrader positivity condition in the spatial direction. In the same way we prove that $S_{\beta V}(-;t_1,x_1;t_2,x_2)$ and $G_{\beta V}(t_1,x_1;t_2,x_2)$ satisfy the Osterwalder-Schrader positivity condition. This implies that the two-point functions (3.7) and

$$\int dt_1 dt_2 S(\Theta(t_1,\varphi_V(f_1(t_1,\cdot)))(t_2,\varphi_V(f_2(t_2,\cdot))))$$
$$\int dt_1 dt_2 S(\Theta(t_1,\Psi(f_1(t_1.\cdot)))(t_2,\Psi(f_2(t_2,\cdot))))$$

are positive semidefinite for $f_1,f_2\in C_0^\infty(\mathbb{R}\times[0,V/2])$ or

$$f_1,f_2\in C_0^\infty(\mathbb{R}\times[0,V/2])\otimes\mathbb{C}^4 .$$

Now there are two positive semidefinite bilinear forms which are defined on $C_0^\infty(\mathbb{R}\times[0,V/2])$ and $C_0^\infty(\mathbb{R}\times[0,V/2])\otimes\mathbb{C}^4$, respectively. We form two Hilbert spaces $\mathscr{F}_{\beta b1}$ and $\mathscr{F}_{\beta f1}$, respectively, by dividing out by the vectors of norm 0 and completing. Let $\mathscr{F}_{\beta b}$ be the symmetric and $\mathscr{F}_{\beta f}$ the anti-symmetric Fock spaces over the Hilbert spaces $\mathscr{F}_{\beta b1}$ and $\mathscr{F}_{\beta f1}$, respectively.

Let

$$\mathscr{F}_\beta=\mathscr{F}_{\beta b}\otimes\mathscr{F}_{\beta f} .$$

Now let us proceed to the proof of the positive (semi) definiteness of the bilinear form $S(\Theta(a)b)$.

We show that

$$S(\Theta(a)b)=(N(a),N(b))_{\mathscr{F}_\beta} \tag{3.12}$$

where $N(a)$ is a linear mapping from the algebra $\mathfrak{A}_+$ into the Hilbert space $\mathscr{F}_\beta$. In fact, $N(\cdot)$ is the normal ordering.

First we define the mappings N_b and N_f (the normal ordering of bosons and fermions, respectively).

Let $\mathfrak{A}_{+,b}$ and $\mathfrak{A}_{+,f}$ be the subalgebras of $\mathfrak{A}_+$ with the generating sets $\mathbb{R}\times\mathscr{M}_{\mathrm{coh},+}$ and $\mathbb{R}\times\mathscr{B}_+$, respectively.

The mapping $N_b:\mathfrak{A}_{+,b}\to\mathscr{F}_{\beta,b}$ we define in the following way.

If $a\in\mathfrak{A}_{+,b}$, then

$$a=\sum_{k\in A}\alpha_k\prod_{j_k\in A_k}\left(t_{j_k},\sum_{r\in R(j_k)}\alpha(j_k,r)\exp(i\varphi_V(h(j_k,r)))\right) .$$

We define $N_b(\cdot)$ by

$$N_b(a)=\sum_{k\in A}\alpha_k \sum_{r(\cdot)\in\mathscr{R}_k}\Bigg\{\Bigg(\prod_{j_k\in A_k}\alpha(j_k,r(j_k))\Bigg)$$
$$\cdot\exp\Bigg(-\tfrac{1}{2}\int dx'dx''dt'dt''\Bigg(\sum_{j_k\in A_k}h_{j_k,r(j_k)}(x')\delta(t_{j_k}-t')\Bigg)$$
$$\cdot G_{\beta V}(t'-t'',x'-x'')\Bigg(\sum_{j_k\in A_k}h_{j_k,r(j_k)}(x'')\delta(t_{j_k}-t'')\Bigg)\Bigg)$$
$$\cdot \operatorname{Exp}\Bigg(i\sum_{j_k\in A_k}h_{j_k,r(j_k)}\otimes\delta t_{j_k}\Bigg)\Bigg\}$$

where $\mathscr{R}_k$ is the set of all functions from A_k into $\bigcup_{j_k\in A_k} R(j_k)$ with the following properties $r(j_k)\in R(j_k)$. By δ_t we denote the translated δ-function $\delta_t:=\delta(\cdot-t)$ and

$$\operatorname{Exp}(if)=1\oplus\bigotimes_{n=1}^{\infty}\frac{i^n}{n!}f(x_1,t_1)\otimes_s f(x_2,t_2)\otimes_s\ldots\otimes_s f(x_n,t_n)$$

is the coherent vector in the space $\mathscr{F}_{\beta b}$.

Now we define the fermion normal ordering $N_f(\cdot)$. If $a\in\mathfrak{A}_{+,f}$, then

$$a=\sum_{k\in A}\alpha_k\prod_{j_k\in A_k}\Bigg(t_{j_k},\sum_{r\in R(j_k)}\prod_{l\in L(j_k,r)}\Psi(f_{j_k,r,l})\Bigg)\,.$$

Let

$$N_f(a)=\sum_{k\in A}\alpha_k\sum_{r(\cdot)\in\mathscr{R}_k}N_f\Bigg(\prod_{j_k\in A_k}\Bigg(\prod_{l\in L(j_k,r)}(t_{j_k},\Psi(f_{j_k,r(j_k),l}))\Bigg)\Bigg)$$

and on monomials we define $N_f(\cdot)$ in the following way. We define $N_f(\cdot)$ on the monomial as the formal normal ordering, i.e., as the sum over all contractions (pairings) of the fermion operators. Then in each term of this sum the contracted and uncontracted fermion factors appear. We replace each contracted pair

$$\ldots\underbrace{(t,\Psi(f))\ldots(t',\Psi(f'))}\ldots$$

by its $\bar{T}$ product

$$S((t,\Psi(f))(t',\Psi(f')))$$

and in correspondence with the parity of the permutation the sign (± 1) appears. Each term in the sum over contractions gives a vector in the fermion Hilbert space.

If uncontracted fermion operators have test functions and times, say, $\ldots(t_1,f_1)\ldots(t_2,f_2)\ldots(t_n,f_n\ldots$, then this term of the sum over contractions gives the vector with the wave function $f_1\delta_{1_1}\otimes_a f_2\delta_{t_2}\otimes_a\ldots\otimes_a f_n\delta_{t_n}$ multiplied by the product of all contractions of this term and each contraction enters with its sign (± 1). The full contracted terms give the vacuum vector in $\mathscr{F}_{\beta f}$ [i.e., the vector $(1,0,0\ldots)$] multiplied by the sum of the products of all corresponding contractions with their signs.

Now let $a\in\mathfrak{A}_+$, then

$$a=\sum_{k\in A}\alpha_k\prod_{j_k\in A_k}\Bigg(t_{j_k},\sum_{r\in R(j_k)}F(j_k,r)Q(j_k,r)\Bigg)$$

where $F(j_k, r) \in \mathcal{M}_{\mathrm{coh},+}$ and $Q(j_k, r) \in \mathcal{B}_+$.

We set

$$N(a) = \sum_{k \in A} \alpha_k \sum_{r(\cdot) \in \mathcal{R}_k} \left\{ N_b \left(\prod_{j_k \in A_k} (t_{j_k}, F(j_k, r(j_k))) \right) \otimes N_f \left(\prod_{j_k \in A_k} (t_{j_k}, Q(j_k, r(j_k))) \right) \right\}.$$

It is easy to see that $N(\cdot)$ is a linear mapping from the algebra $\mathfrak{A}_+$ into the Hilbert space $\mathcal{F}_\beta$.

Now let us prove the equality (3.12). For this purpose we shall calculate $(N(a), N(b))_{\mathcal{F}_\beta}$ and $S(\Theta(a)b)$.

The relation

$$\begin{aligned}
&\langle \bar{T} \exp(i\widehat{\varphi_V(h_1)})(t_1) \dots \exp(i\widehat{\varphi_V(h_n)})(t_n) \rangle \\
&\quad = \exp \left(- \sum_{i<j} \int dx' dx'' h_i(x') G_{\beta V}(t_i - t_j, x' - x'') h_j(x'') \right. \\
&\qquad \left. - \tfrac{1}{2} \sum_{i=j} \int dx' dx'' h_i(x') G_{\beta V}(0, x' - x'') h_i(x'') \right),
\end{aligned}$$

the definition of $N(\cdot), S(\cdot)$ and Wick's theorem imply that to prove the equality (3.12) it is sufficient to prove that

$$\begin{aligned}
&G_{\beta V}(t_1, x_1; t_2, x_2)^* = G_{\beta V}(t_1, -x_1; t_2, -x_2) \\
&G_{\beta V}(t_1, x_1; t_2, x_2) = G_{\beta V}(-t_1, -x_1; -t_2, -x_2) \\
&\langle \bar{T} \Psi(\widehat{f_1})(t_1) \Psi(\widehat{f_2})(t_2) \rangle^* = \langle \bar{T} \Psi(\widehat{\vartheta_f(f_2)})(t_2) \Psi(\widehat{\vartheta_f(f_1)})(t_1) \rangle .
\end{aligned}$$

But these relations follow from the definitions of $G_{\beta V}$, Ψ, ϑ_f by the direct calculation.

Thus,

$$S(\Theta(a)b) = (N(a), N(b))_{\mathcal{F}_\beta} .$$

Since $(\,,\,)_{\mathcal{F}_\beta}$ is the scalar product in the Hilbert space $\mathcal{F}_\beta$ and N is a linear operator, so $S(\Theta(a)b)$ is a positive bilinear form. Theorem 3.3 is proved.

4. The Vacuum Overlap

To prove the vacuum overlap results we use the arguments of Seiler and Simon [1].

We start with the convergence of $H_{V,\sigma}$ as $\sigma \to \infty$.

Lemma 4.1. *As $\sigma \to \infty$ $H_{V,\sigma}$ converge in the sense of strong resolvent convergence to the self-adjoint operator H_V.*

Proof of Lemma 4.1. The proof of this lemma is similar to the proof of Theorem 3.4 [1], reformulated for the case of the periodic boundary conditions and it follows from the fact that all the estimates and bounds of [6] can also be obtained in the periodic case. Lemma 4.1 is proved.

The resolvent convergence and the commutativity of $H_{V,\sigma}$ and P_V imply that H_V commute strongly with P_V.

Lemma 4.2. *The operator $H_V + \gamma P_V$ is bounded below, self-adjoint and has the compact resolvent for $-1 < \gamma < 1$.*

Proof of Lemma 4.2. The proof follows from the uniform bound of Theorem 2.1 and from the compactness of the resolvent of $N_{\tau,V}$ for $\tau > 0$. Lemma 4.2 is proved.

Now we define Jost states [1]. Since $H_{0,V}^2 \geqq P_V^2$, the vector valued distribution $\phi_1(x_1) \dots \phi_n(x_n)\Omega_{0,V}$ [where ϕ is either $\varphi_V(x)$ or $\Psi(x)$] is the boundary value of a vector-valued function (= Jost state), analytic in the region $\operatorname{Im} z_1$, $\operatorname{Im}(z_2 - z_1), \dots, \operatorname{Im}(z_n - z_{n-1}) \in V_+$, where V_+ is the forward light cone. By cyclicity of the vacuum, the set of linear combinations of Jost states is dense in $\mathscr{F}_V$. We call a Jost state β Euclidean if and only if each z_k is $z_k = (x_k + i\beta t_k, i\beta_0 t_k)$ with x_k, t_k real, $\beta_0 = (\beta^2 + 1)^{1/2}$ and, moreover, the t_k's are non-coincident.

We call a vector a β "good" Jost state if it is an integral over space variables x and a sum over fermion indices of β Euclidean Jost states with a function $\bigotimes_{k=1}^{n} f_k$, $f_k \in C_0^\infty([-V/2, V/2])$ or $f_k \in C_0^\infty([-V/2, V/2]) \otimes \mathbb{C}^4$ and $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$ are all positive. We say the state is supported in $(a, b) \times (c, d)$ if $\operatorname{supp} f_k \subset (a, b)$ [that is, $\operatorname{supp} f_k(x, \alpha) \subset (a, b)$ in the case of a fermion test function] and $c < t_k < d$.

We call a vector a bounded β Jost state supported in $(a, b) \times (c, d)$ if it has the form

$$\exp(-t_1 H_{0,V}(\beta)) A_1 \exp(-(t_2 - t_1) H_{0,V}(\beta)) \cdot A_2 \dots \exp(-(t_n - t_{n-1}) H_{0,V}(\beta)) A_n \Omega_{0,V}$$

where each A_k is either $\exp(i\varphi_V(h_k))$ with real $h_k \in C_0^\infty([-V/2, V/2])$, $\operatorname{supp} h_k \subset (a, b)$, or $\Psi(f_k)$ with $f_k \in C_0^\infty([-V/2, V/2]) \otimes \mathbb{C}^4$, $\operatorname{supp} f_k(\cdot, \alpha) \subset (a, b)$ and, in addition, $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$ are all positive and $c < t_k < d$.

Lemma 4.3. *Fix a, b, c, d with $-V/2 < a < b < V/2$, $0 < c < d$. The linear combinations of bounded β Jost states with support in $(a, b) \times (c, d)$ are dense in Fock space $\mathscr{F}_V$.*

Proof of Lemma 4.3. The proof is similar to the one of Lemma 5.2 [1]. Suppose, η is orthogonal to all bounded β Jost states with the support property. Then η is orthogonal to all β good Jost states. By taking the smearing functions to delta functions, η is orthogonal to all β Euclidean Jost states with the support property. By analyticity, it is then orthogonal to all Jost states and hence is zero. The lemma is proved.

Theorem 4.4. *The Fock vacuum $\Omega_{0,V}$ overlaps the vacuum for $\beta_0 H_V + \beta P_V$.*

Proof of Theorem 4.4. To prove the theorem we prove the estimate

$$(\eta, \exp(-tH_V(\beta)\eta)) \leqq (\Omega_{0,V}, \exp(-tH_V(\beta))\Omega_{0,V})^{1/2} (\eta', \exp(-tH_V(\beta))\eta')^{1/2} \tag{4.1}$$

for appropriate η and use Lemma 5.1 [1].

We shall prove the inequality (4.1) for all η which are bounded β Jost states supported in $[V/8, V/4] \times [1, 2]$.

We note that though in the formulation of Lemma 5.1 [1] an estimate of the type (4.1) is needed for a dense set of vectors, in fact, as can easily be seen, it is sufficient to prove such an estimate for a set of vectors the linear combinations of which are dense in the Hilbert space.

In principle, the slightly modified estimate can also be proved for linear combinations of bounded β Jost states.

After this remark we proceed to the proof of (4.1). To prove the estimate (4.1) we shall approximate the expression

$$(\eta, \exp(-tH_V(\beta))\eta)$$

by the ones for which we apply the Osterwalder-Schrader positivity condition of Theorem 3.3.

Let an ultraviolet cut-off be given by an even function $\chi_\sigma(x) = \sigma\chi(\sigma x)$, where $\chi(x)$ is a positive $C_0^\infty([-1,1])$ function with the total integral one.

Let

$$H_I(A(\sigma)) = \lambda \int_{A(\sigma)} dx :\bar{\psi}_{V,\sigma}(x)\Gamma\psi_{V,\sigma}(x):\varphi_{V,\sigma}(x) ,$$

where $\Gamma = 1$ or $i\gamma_5$ and the set $A(\sigma)$ is

$$A(\sigma) = [-V/2, -V/2 + 2\sigma^{-1}] \cup [-2\sigma^{-1}, 2\sigma^{-1}] \cup [V/2 - 2\sigma^{-1}, V/2] .$$

Corollary 2.1.2 [7], which is also valid in the periodic boundary conditions case, allows to obtain the following simple bound

$$\pm H_I(A(\sigma)) \leqq \int_{A(\sigma)} dx M_\sigma(N_{\tau,V} + 1) \leqq c\sigma^{-\tau/3}(N_{\tau,V} + 1)$$

uniformly in σ.

The operator $H_I(A(\sigma))$ is bounded with respect to $H_{V,\sigma}$ with the relative bound zero and $\beta\beta_0^{-1}P_V$ is bounded with respect to $H_{V,\sigma}$ with the bound less than 1. Thus, [8, p. 287–288]

$$H'_{V,\sigma}(\beta) := \beta_0 H_{V,\sigma} + \beta P_V - \beta_0 H_I(A(\sigma))$$

is self-adjoint operator on $\mathscr{D}(H_{V,\sigma}) = \mathscr{D}(H_{0,V})$.

Then

$$\begin{aligned} &\|(H_{V,\sigma}(\beta) + \zeta)^{-1} - (H'_{V,\sigma}(\beta) + \zeta)^{-1}\| \\ &\quad \leqq \|(H'_{V,\sigma}(\beta) + \zeta)^{-1}(N_{\tau,V} + 1)^{1/2}\| \\ &\qquad \cdot \|(N_{\tau,V} + 1)^{-1/2} H_I(A(\sigma))(N_{\tau,V} + 1)^{-1/2}\| \\ &\qquad \cdot \|(N_{\tau,V} + 1)^{1/2}(H_{V,\sigma}(\beta) + \zeta)^{-1}\| \\ &\quad \leqq c(\beta)\sigma^{-\tau/3} \end{aligned}$$

uniformly in σ for sufficiently large σ and for sufficiently large positive ζ not depending on σ.

But

$$\begin{aligned} \operatorname*{s-lim}_{\sigma\to\infty} \exp(isH_{V,\sigma}(\beta)) &= \operatorname*{s-lim}_{\sigma\to\infty} \exp(is\beta_0 H_{V,\sigma}) \exp(is\beta P_V) \\ &= \exp(is\beta_0 H_V) \exp(is\beta P_V) = \exp(isH_V(\beta)) \end{aligned}$$

thus,

$$\underset{\sigma\to\infty}{s\text{-}\lim}(H'_{V,\sigma}(\beta)+\zeta)^{-1}=(H_V(\beta)+\zeta)^{-1}$$

for sufficiently large ζ.

Since $H'_{V,\sigma}(\beta)$ is bounded below uniformly in σ, so [8, p. 502]

$$\underset{\sigma\to\infty}{s\text{-}\lim}\exp(-tH'_{V,\sigma}(\beta))=\exp(-tH_V(\beta)).$$

Now we make the next approximation.

Let

$$W_+(\sigma)=-\tfrac{1}{2}\delta m^2(V,\sigma)\int_0^{V/2}dx:\varphi_V^2(x):$$

and

$$W_-(\sigma)=-\tfrac{1}{2}\delta m^2(V,\sigma)\int_{-V/2}^{0}dx:\varphi_V^2(x):$$

where $\delta m^2(V,\sigma)$ is the (divergent) boson mass renormalization. $W_\pm(\sigma)\in L_p$ for some $p>2$ [9].

Let

$$W_+(n,\sigma)=\begin{cases}W_+(\sigma) & \text{for } |W_+(\sigma)|\leqq n\\ n & \text{for } |W_+(\sigma)|>n\end{cases}.$$

We define

$$W_-(n,\sigma)=\exp(-iP_VV/2)W_+(n,\sigma)\exp(iP_VV/2)$$

and

$$W(n,\sigma)=W_+(n,\sigma)+W_-(n,\sigma).$$

Let

$$H_{I,\pm}(\varrho,\sigma)=\lambda\int_{\substack{[-V/2,V/2]\backslash A(\sigma)\\ \pm x>0}}dx:\bar{\psi}_{V,\sigma}(x)\Gamma\psi_{V,\sigma}(x):\varrho^{-1}\sin(\varrho\varphi_{V,\sigma}(x))$$

and

$$H_I(\varrho,\sigma)=H_{I,+}(\varrho,\sigma)+H_{I,-}(\varrho,\sigma)$$

and let

$$H'_I(\sigma)=\lambda\int_{[-V/2,V/2]\backslash A(\sigma)}dx:\bar{\psi}_{V,\sigma}(x)\Gamma\psi_{V,\sigma}(x):\varphi_{V,\sigma}(x).$$

Since $W_\pm(\sigma)\in L_p$ for some $p>2$ [9], so $W(n,\sigma)\to W(\sigma)$ in any L_q norm with $q<p$ (Lemma 3.5 [1]). On the domain $F\cap\mathscr{D}(H_{0,V})$, where F is the set of vectors with a finite number of particles,

$$\begin{aligned}&\underset{n\to\infty}{s\text{-}\lim}\lim_{\varrho\to0}(H_{0,V}+W(n,\sigma)+H_I(\varrho,\sigma)+\gamma P_V)\\ &\quad=\underset{n\to\infty}{s\text{-}\lim}(H_{0,V}+W(n,\sigma)+H'_I(\sigma)+\gamma P_V)\\ &\quad=H_{0,V}(\delta m^2)+H'_I(\sigma)+\gamma P_V\end{aligned}\tag{4.2}$$

where $\gamma=\beta\beta_0^{-1}$ and

$$H_{0,V}(\delta m^2)=H_{0,V}-\tfrac{1}{2}\delta m^2(V,\sigma)\int dx:\varphi_V^2(x):.$$

Since $F\cap\mathscr{D}(H_{0,V})$ is a core for $H_{0,V}(\delta m^2)$ [9, Theorem 4.2d] and γP_V is the operator bounded with respect to $H_{0,V}(\delta m^2)$ with the bound <1, and $H_I'(\sigma)$ is bounded with respect to $H_{0,V}(\delta m^2)$ with the bound zero, so [8, p. 287–288, 429] the resolvents of the operators in (4.2) converge strongly as $\varrho\to 0$, $n\to\infty$.

Since $\|\exp(-tW(n))\|_1$ is bounded uniformly in n for each t, so Corollary 2.14 [9, p. 133] implies that

$$\beta_0(H_{0,V}+W(n,\sigma))+\beta P_V\geqq -c$$

uniformly in n.

Since $H_I(\varrho,\sigma)$ and $H_I'(\sigma)$ are infinitely small perturbation, we have [8, p. 502]

$$\begin{aligned}&\operatorname*{s\text{-}lim}_{n\to\infty}\lim_{\varrho\to 0}\exp(-t(\beta_0(H_{0,V}+W(n,\sigma)+H_I(\varrho,\sigma))+\beta P_V))\\&\quad=\exp(-t(\beta_0(H_{0,V}(\delta m^2)+H_I'(\sigma))+\beta P_V)).\end{aligned}$$

Now let us consider the operator

$$\exp(-t(\beta_0(H_{0,V}+W(n,\sigma)+H_I(\varrho,\sigma))+\beta P_V)).$$

The operators $W(n,\sigma)$ and $H_I(\varrho,\sigma)$ are bounded operators and thus

$$\begin{aligned}&\exp(-t(\beta_0(H_{0,V}+W(n,\sigma)+H_I(\varrho,\sigma)+\beta P_V))\\
&=\operatorname*{s\text{-}lim}_{m\to\infty}\Bigg[\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)\\
&\quad\cdot\exp\left(-\frac{t}{m}\beta_0(W_-(n,\sigma)+H_{I,-}(\varrho,\sigma))\right)\\
&\quad\cdot\exp\left(-\frac{t}{m}\beta_0(W_+(n,\sigma)+H_{I,+}(\varrho,\sigma))\right)\Bigg]^m\\
&=\operatorname*{s\text{-}lim}_{m\to\infty}\lim_{R\to\infty}\Bigg\{\Bigg[\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)\\
&\quad\cdot\sum_r^R(-1)^r\frac{t^r\beta_0^r}{m^r r!}(W_-(n,\sigma)+H_{I,-}(\varrho,\sigma))^r\\
&\quad\cdot\sum_r^R(-1)^r\frac{t^r\beta_0^r}{m^r r!}(W_+(n,\sigma)+H_{I,+}(\varrho,\sigma))^r\Bigg]^m\Bigg\}.\end{aligned}\tag{4.3}$$

And finally, we make the last approximation. We approximate $W_\pm$ by the coherent functions and replace the integration over the box in $H_{I,\pm}(\varrho,\sigma)$ by the summation.

Kaplansky's density theorem [10, p. 46] implies that the bounded real function $W_+(n,\sigma)$ can be approximated by the real functions $W_+(N,n,\sigma)\in\mathscr{M}_{\mathrm{coh},+}$ with $\|W_+(N,n,\sigma)\|\leqq\|W_+(n,\sigma)\|$. We define

$$W_-(N,n,\sigma)=\exp(-iP_VV/2)W_+(N,n,\sigma)\exp(iP_VV/2).$$

Let

$$H_{I,\pm}(N,\varrho,\sigma):=N^{-1}\sum_{n\in N_\pm}:\bar{\psi}_{V,\sigma}(n)\Gamma\psi_{V,\sigma}(n):\varrho^{-1}\sin(\varrho\varphi_{V,\sigma}(n))$$

where

$$N_+:=\{[0,V/2]\backslash A(\sigma)\}\cap Z_N\,,$$
$$N_-:=\{[-V/2,0]\backslash A(\sigma)\}\cap Z_N$$

and $Z_N=\{x\in Z_N|Nx \text{ is integer}\}$.

Let

$$B_\pm=\sum_r^R(-1)^r\frac{t^r\beta_0^r}{m^r r!}(W_\pm(N,n,\sigma)+H_{I,\pm}(N,\varrho,\sigma))^r$$

then the expression in the curly brackets in the right hand side of (4.3) is equal to

$$\underset{N\to\infty}{s\text{-lim}}\left(\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)B_-B_+\right)^m\,.$$

As the result of all the approximations we obtain the expression for which Theorem 3.3 is applicable.

Let η be a bounded β Jost state supported in the region $[V/8,V/4]\times[1,2]$. Then

$$\begin{aligned}&\left(\eta,\left(\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)B_-B_+\right)^m\eta\right)\\&\quad=S\left(\eta^\sim\prod_{k=1}^m(kt/m,B_-)(kt/m,B_+)\eta_t\right)\end{aligned}\tag{4.4}$$

where we have used the same notation both for η as the element of the Fock space $\mathscr{F}_V$, and for the element of the algebra $\mathfrak{A}$, and where $(\cdot)_t$, $(\cdot)^\sim$ are the following operations on the algebra $\mathfrak{A}$. If a is given by (3.1), (3.3), then

$$a_t=\sum_{k\in A}\alpha_k\prod_{j_k\in A_k}(t_{j_k}+t,F_{j_k})$$

$$a^\sim=\sum_{k\in A}\alpha_k^*\prod_{j_k\in\bar{A}_k}(-t_{j_k},F_{j_k}^\sim)$$

where

$$F_{j_k}^\sim=\sum_{r\in R(j_k)}F(j_k,z)^*\prod_{l\in\bar{A}(j_k,r)}\Psi(f(l,j_k,r))^+\,.$$

Here $\Psi(f)^+$ is the operator which is the hermitian adjoint to the operator $\Psi(f)$. We note that $\Psi(f)^+=\Psi(f^+)$, where f^+ is the following mapping of fermion test functions. If (g,h) is a pair of two components functions, then $(g,h)^+=(h^*\gamma_0,g^*\gamma_0)$. We note that

$$\vartheta_f(f^+)=-(\vartheta_f(f))^+\,.\tag{4.5}$$

Since B_+ contain an even number of fermion operators and since fermion operators anticommute in S for separated points, so (4.4) is equal to

$$S\left(\prod_{k=1}^m(kt/m,B_-)\prod_{k=1}^m(kt/m,B_+)\eta^\sim\eta_t\right).$$

Then, from the definition of Θ and $B_\pm$ it follows that

$$\prod_{k=1}^{m}(kt/m, B_-)=\Theta\left(\prod_{k=m}^{1}(kt/m, B_+)\right).$$

This equality is connected with the Euclidean invariance of the Y_2 interaction. Using Theorem 3.3 and Schwarz's inequality we obtain that (4.4) is less than

$$S\left(\Theta\left(\prod_{k=m}^{1}(kt/m, B_+)\right)\prod_{k=m}^{1}(kt/m, B_+)\right)^{1/2}\cdot$$
$$S\left(\Theta\left(\prod_{k=1}^{m}(kt/m, B_+)\eta^{\sim}\eta_t\right)\prod_{k=1}^{m}(kt/m, B_+)\eta^{\sim}\eta_t\right)^{1/2}. \tag{4.6}$$

Taking into account (4.5) and using that $B_\pm$ contain an even number of fermion operators once more to reordered the factors in (4.6), we obtain

$$\begin{aligned}&\left(\Omega_{0,V}, \left(\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)B_-B_+\right)^m\Omega_{0,V}\right)^{1/2}\\&\left(\eta', \left(\exp\left(-\frac{t}{m}H_{0,V}(\beta)\right)B_-B_+\right)^m\eta'\right)^{1/2}\end{aligned} \tag{4.7}$$

where η' is some vector in the Fock space $\mathscr{F}_V$ $(\eta'=\bar{T}(\eta\Theta(\eta))\Omega_{0,V})$.

It should be noted that just to express the second factor in (4.6) in the Hamiltonian form it is essential that η is a Jost state, i.e., a monomial of fermion operators rather than a linear combination of Jost states. For a linear combination of Jost states we would obtain a sum of expressions like $\pm(\eta', (\ )\eta'')$, where η', η'' are Jost states, but, nevertheless, this sum may easily be estimated and gives the slightly modified estimate (4.1).

Taking $N, R, m, \varrho, n, \sigma\to\infty$ in (4.7) we obtain

$$\begin{aligned}&(\eta, \exp(-tH_V(\beta))\eta)\\&\quad\leqq(\eta', \exp(-tH_V(\beta))\eta')^{1/2}(\Omega_{0,V}, \exp(-tH_V(\beta))\Omega_{0,V})^{1/2}.\end{aligned} \tag{4.8}$$

By Lemma 5.1 [1], Theorem 4.4 is proved.

5. $H_V^2 \geqq P_V^2$

In this section we proceed to the proof of the main theorem. We normalize the vacuum energy so that inf spectrum $H_V=0$.

Theorem 5.1. *The joint spectrum of H_V and P_V lies in the forward light cone, that is,*

$$\beta_0 H_V+\beta P_V\geqq 0, \quad H_V^2-P_V^2\geqq 0,$$
$$E(\beta)=\inf\operatorname{spectrum} H_V(\beta)=0, \quad \pi(\beta)=\pi(\beta=0).$$

Here $\pi(\beta)$ denotes the projection onto the vacuum subspace of $H_V(\beta)$.

Proof of Theorem 5.1. By Theorem 4.4 [or by the inequality (4.8)]

$$\begin{aligned}\inf\operatorname{spectrum} H_V(\beta)&=-\lim_{t\to\infty}t^{-1}\ln(\Omega_{0,V}, \exp(-tH_V(\beta))\Omega_{0,V})\\&=-\lim_{t\to\infty}t^{-1}\ln(\Omega_{0,V}, \exp(-tH_V)\Omega_{0,V})=0.\end{aligned}$$

Hence

$$\beta_0 H_V + \beta P_V \geqq 0 .$$

The commutativity of H_V and P_V implies the commutativity of $\pi(\beta)$ and P_V. Thus

$$\pm P_V \pi(\beta) \leqq c(\beta) H_V(\beta) \pi(\beta) = 0 ,$$

hence, $H_V \pi(\beta) = 0$, $H_V(\beta)\pi(\beta=0) = 0$, and so $\pi(\beta) = \pi(\beta = 0)$.

Furthermore, the commutativity of H_V and P_V implies that $H_V^2 - \beta_0^{-2}\beta^2 P_V^2 \geqq 0$, and so, by limits as $\beta \to \infty$

$$H_V^2 - P_V^2 \geqq 0 .$$

This completes the proof of the theorem.

Corollary 5.2. *Let* $f \in C_0^\infty([-V/2, V/2])$, *then*

$$0 \leqq H_V + \pi_V(f) + 1/2\|f\|_2^2$$
$$0 \leqq H_V + \gamma P_V \pm (\pi_V(f) - \gamma \nabla \varphi_V(f)) + 1/2\|f\|_2^2$$

where $\gamma = \beta\beta_0^{-1}$.

Proof of Corollary 5.2. We use, as in [11], the unitary operator $U = \exp(i\varphi_V(f))$ to transform the Hamiltonian $H_V(\beta)$. We show that on $\mathscr{D}(H_V)$

$$U(H_V + \gamma P_V)U^{-1} = H_V + \gamma P_V - (\pi_V(f) - \gamma \nabla \varphi_V(f)) + 1/2\|f\|_2^2 . \tag{5.1}$$

To prove this statement we use Theorem 3 by McBryan [12].

First we show that

$$i[\varphi_V(f), H_V + \gamma P_V] = -\pi_V(f) + \gamma \nabla \varphi_V(f)$$

as bilinear forms on $\mathscr{D}(H_V^2) \times \mathscr{D}(H_V^2)$.

Indeed, by the $N_{\tau,V}$ estimates the left and right hand sides are defined correctly on $\mathscr{D}(H_V^2) \times \mathscr{D}(H_V^2)$. Let $R = (H_V + \gamma P_V + \zeta)^{-1}$ and $R_\sigma = (H_{V,\sigma} + \gamma P_V + \zeta)^{-1}$, where ζ is positive sufficiently large and $H_{V,\sigma}$ is the ultraviolet cut-off Hamiltonian (more precisely, $H_{V,\sigma}$ is $H_{V,\sigma} - E_V$, where E_V is the vacuum energy of the Hamiltonian without the ultraviolet cut-off. Remember that H_V is normalized so that $\inf \text{spectrum } H_V = 0$).

By Theorem 2.1, by the commutativity and by Lemma 4.1 R_σ is defined correctly for sufficiently large ζ and $s\text{-}\lim R_\sigma = R$.

Each vector $x \in \mathscr{D}(H_V^2)$ is of the form $x = R^2 y$ and the $N_{\tau,V}$ estimates imply that $\mathscr{D}(\varphi_V(\mathrm{f})) \supset \mathscr{D}(H_V)$, so we obtain

$$\begin{aligned}
& i\{((H_V + \gamma P_V)x, \varphi_V(f)x) - (\varphi_V(f)x, (H_V + \gamma P_V)x)\} \\
&\quad = i\{(Ry, \varphi_V(f)x) - (\varphi_V(f)x, Ry)\} \\
&\quad = i\{(\varphi_V(f)Ry, x) - (x, \varphi_V(f)Ry)\} \\
&\quad = \lim_\sigma i\{(\varphi_V(f)Ry, x_\sigma) - (x_\sigma, \varphi_V(f)Ry)\}
\end{aligned}$$

where $x_\sigma = R_\sigma R y$ and then we have

$$= \lim_\sigma i\{((H_{V,\sigma} + \gamma P_V)x_\sigma, \varphi_V(f)x_\sigma) - (\varphi_V(f)x_\sigma, (H_{V,\sigma} + \gamma P_V)x_\sigma)\} .$$

The vectors $x_\sigma \in \mathscr{D}(H_{V,\sigma}) = \mathscr{D}(H_{0,V})$.

As an operator equality on $\mathscr{D}(H_{0,V}) \cap F$, where F are vectors with a finite number of particles,

$$i[\varphi_V(f), H_{V,\sigma} + \gamma P_V] = -\pi_V(f) + \gamma \nabla \varphi_V(f) .$$

Since the domain $\mathscr{D}(H_{0,V}) \cap F$ is a core for $H_{V,\sigma} + \gamma P_V$, $\varphi_V(f)$, $\nabla\varphi_V(f)$, $\pi_V(f)$, then for $x_\sigma \in \mathscr{D}(H_{0,V})$

$$i\{((H_{V,\sigma} + \gamma P_V)x_\sigma, \varphi_V(f)x_\sigma) - (\varphi_V(f)x_\sigma, (H_{V,\sigma} + \gamma P_V)x_\sigma)\}$$
$$= (x_\sigma, (\pi_V(f) - \gamma \nabla \varphi_V(f))x_\sigma)$$

and by the $N_{\tau,V}$ estimates we obtain in the limit as $\sigma \to \infty$

$$i\{((H_V + \gamma P_V)x, \varphi_V(f)x) - (\varphi_V(f)x, (H_V + \beta P_V)x)\}$$
$$= (x, (\pi_V(f) - \gamma \nabla \varphi_V(f))x) .$$

Since $\mathscr{D}(H_V^2)$ is a core for $H_V + \gamma P_V$ and by the $N_{\tau,V}$ estimates this bilinear form extends on $\mathscr{D}(H_V) \times \mathscr{D}(H_V)$. Since $\|(\pi_V(f) - \gamma \nabla \varphi_V(f))R\| < \infty$ and $\|R^{1/2} \varphi_V(f) R^{1/2}\| < \infty$, so Theorem 3 by McBryan [12] implies that $U(s)\mathscr{D}(H_V) \subset \mathscr{D}(H_V)$ [here $U(s) = \exp(is\varphi_V(f))$], $(H_V + \gamma P_V)U(s)$ is strongly continuous on $\mathscr{D}(H_V)$ and thus for $x \in \mathscr{D}(H_V)$

$$\frac{d}{ds}(x, U(s)(H_V + \gamma P_V)U(-s)x)$$
$$= i\{(\varphi_V(f)U(-s)x, (H_V + \gamma P_V)U(-s)x)$$
$$- ((H_V + \gamma P_V)U(-s)x, \varphi_V(f)U(-s)x)\}$$
$$= -(U(-s)x, (\pi_V(f) - \gamma \nabla \varphi_V(f))U(-s)x)$$
$$= (x, (-\pi_V(f) + \gamma \nabla \varphi_V(f) + \tfrac{1}{2}\|f\|_2^2)x) .$$

Integrating the last equality over $s \in [0, 1]$ we obtain

$$(x, U(H_V + \gamma P_V)Ux)$$
$$= (x, (H_V + \gamma P_V - (\pi_V(f) - \gamma \nabla \varphi_V(f)) + \tfrac{1}{2}\|f\|_2^2)x)$$

i.e., the equality 5.1. This implies the statements of Corollary 5.2.

6. φ-Bounds and the Energy-Momentum Spectrum in the Infinite Volume

To extend the results of the pervious section to the case of the infinite volume we need some sort of vacuum expectation values in the infinite volume (Schwinger or Wightman functions, or a state on the quasilocal algebra of observables or on the quasilocal field algebra). The simplest way, we think, is to obtain uniform φ-bounds and then, using the compactness, to construct Wightman functions. These Wightman functions, as the limit of the finite volume ones, satisfy the spectral condition.

Theorem 6.1. *Let $f \in C_0^\infty([-\frac{1}{2},\frac{1}{2}])$ and $\|f\|_\alpha^2 = \int |\hat{f}(k)|^2(k^2+1)^\alpha dk$ then for $V \geqq 1$*

$$\pm \varphi_V(f) \leqq H_V + c_1\|f\|_{-1}^2 + c_2$$
$$\varphi_V(f)^2 \leqq c_3\|f\|_{-1/2}^2(H_V+1)$$

for suitable constants c_1, c_2, c_3 (not depending on V).

Proof of Theorem 6.1. Let us prove the first bound. Let $E_V(f)$ be the ground state energy for $H_V + \varphi_V(f)$. To prove φ-bounds we need only to prove [13,1].

$$-E_V(f) \leqq -E_V + c_1\|f\|_{-1}^2 + c_2 .$$

Let integer n be such that $2^n \leqq V < 2^{n+1}$. Let F be the function obtained by translating f by 1/2 unit and taking the sum of the translation and its reflection about $x=0$. Then, as in [1], we first claim that

$$-E_V(f) \leqq -1/2E_V - 1/2E_V(F) . \tag{6.1}$$

This follows as in the proof of the vacuum overlap (Theorem 4.4) and as in the proof of Theorem 5.1 (cf. with the proof of Theorem 7.1 [1]). By iterating (6.1) and using the translation invariance we obtain

$$-E_V(f) \leqq -E_V + 2^{-n}E_V - 2^{-n}E_V(F_n) ,$$

where F_n is obtained by iterating the passage from $f \to F$ n times. Now by the vacuum overlap [which holds for $H_V + \varphi_V(F_n)$ if $n \geqq 1$ since F_n is symmetric]

$$-E_V(F_n) = \lim_{t\to\infty} t^{-1} \ln \int \exp(\varphi_V(F_n \otimes \chi(0,t))) \det_{\text{ren}}(1+K_{V,t}) d\mu_{0,V} .$$

As in [6] one can easily obtain the bound

$$\begin{aligned} &\leqq \overline{\lim_{t\to\infty}}\, t^{-1} \ln \left(\int \det_{\text{ren}}(1+K_{V,t})^2 d\mu_{0,V}\right)^{1/2} \\ &\quad \cdot \left(\int \exp(2\varphi_V(F_n \otimes \chi(0,t))) d\mu_{0,V}\right)^{1/2} \\ &\leqq c_1' V + 2^n c_2' \|f\|_{-1}^2 \end{aligned}$$

and

$$-E_V(f) \leqq -E_V + 2^{-n}V\left(\frac{E_V}{V} - \frac{E_V(F_n)}{V}\right) \leqq -E_V + c_1\|f\|_{-1}^2 + c_2 .$$

In the same way we prove the second bound. Theorem 6.1 is proved.

Theorem 6.2. *The Wightman functions for the Y_2 interaction with the periodic boundary conditions exist as distributions.*

Proof of Theorem 6.2. The φ-bounds and the arguments of Glimm and Jaffe [13] imply the uniform estimates on the Wightman functions for a finite volume and, thus, the compactness of the Wightman functions with box cut-offs. Theorem 6.2 is proved.

Theorem 6.3. *The infinite volume limit Wightman functions for the Y_2 theory with the periodic boundary conditions satisfy the spectral condition.*

Proof of Theorem 6.3. Since the Wightman functions are translation invariant, so by the Stone theorem the unitary representation of the group of translations is of the form

$$U(t,x)=\int \exp(itp_0+ixp_1)E(d^2p)$$

and on the Wightman domain the measure $E(d^2p)$ is the weak limit of measures $E_V(d^2p)$, corresponding to the Wightman functions in a finite volume. But these measures satisfy the spectral condition and so $E(d^2p)$ also satisfies the spectrum condition. Theorem 6.3 is proved.

Acknowledgements. One of the authors (E.H.) is grateful to M.I.Shirokov, and the other—to A. M. Vasilev for their friendly encouragement. We are also indebted to E. Seiler and B. Simon for having sent the preprint of [1] prior to publication.

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Communicated by A. Jaffe

Received April 8, 1977