Commun. math. Phys. 57, 1—30 (1977)

Communications in
Mathematical
Physics

The Pole-Factorization Theorem in S-Matrix Theory*

Daniel Iagolnitzer

Service de Physique Théorique, Centre d'Études Nucléaires de Saclay, F-Gif-sur-Yvette, France

Henry P. Stapp

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA

Abstract. Previous derivations of physical-region discontinuity formulas in S-matrix theory make use of an ad hoc assumption according to which certain sets of singularities associated with mixed-α Landau diagrams cancel among themselves. The aim of the present work is to prove the simplest of these discontinuity formulas, namely, the pole-factorization theorem for a $3\to3$ equal-mass process below the 4-particle threshold, without using this mixed-α cancellation assumption. The result is derived from macro-causality, unitarity and two weak regularity assumptions on scattering functions and bubble diagram functions.

Contents

page
Abstract ... 1
1. Introduction ... 1
2. Two Assumptions on scattering and bubble diagram functions ... 5
(a) Assumption on $u=0$ points ... 5
(b) Analyticity assumption (no sprout assumption) ... 6
3. Pole-factorization theorem (proof using the mixed-α concellation assumption) ... 8
4. Analysis of mixed-α Landau diagrams ... 10
5. New proof of the pole-factorization theorem ... 17
(a) Main steps of the proof ... 17
(b) Preliminary mathematical results ... 18
(c) Details on the proof ... 19
Appendix I: Treatment of mixed-α diagrams ... 23
Appendix II: Comparison to treatment of [12] ... 28
Appendix III: Three equal-mass particles can collide only thrice ... 29
References ... 30

1. Introduction

The basic quantities of interest in the study of systems of massive particles with short-range interactions are the scattering functionals S_{IJ} between sets I and J of

* This work was supported by the United States Energy Research and Development Administration, and by Centre d'Études Nucleaires de Saclay

initial and final particles. The collection of these functionals is the S-matrix. From general principles, the S matrix is known to be unitary ($S^{-1}=S^{\dagger}$), from which it follows that each S_{IJ}, and also its connected part S_{IJ}^c, is a well-defined tempered distribution on the space of all real on-mass-shell initial and final energy-momentum 4-vectors $p_k(p_k^2=\boldsymbol{p}_{ko}^2-p_k^2=m^2, p_{ko}>0, \forall k)$. This distribution can be written (if we exclude the exceptional points P_{exc} where all the 4-vectors p_k are parallel) in the form:

$$S_{IJ}^c=f_{IJ}\times\delta^4\left(\sum_{i\in I}p_i-\sum_{j\in J}p_j\right), \tag{1}$$

where f_{IJ} is a well-defined distribution on the physical region $\mathscr{M}_{IJ}$ of the process $I\to J$, i.e. on the space of all vectors $p=\{p_k\}$ subject, for each k, to the above-mentioned mass-shell constraints and satisfying energy-momentum conservation $\left(\sum_{i\in I}p_i=\sum_{j\in J}p_j\right)$.

The distribution f_{IJ} is called the scattering function of the process $I\to J$. Important advances were made at the end of the sixties in the derivation and physical understanding of the physical-region analytic structure of the scattering functions in the multiparticle case. On the one hand, a macroscopic causality property was formulated and shown to be equivalent to certain basic physical-region analyticity properties [1]. These properties ensure, in particular, that for each process $I\to J$, there is a unique analytic function $\boldsymbol{f}_{IJ}$, defined in a domain of the complexified mass-shell manifold $\boldsymbol{\mathscr{M}}_{IJ}$, such that f_{IJ} is equal to $\boldsymbol{f}_{IJ}$ at all points of $\mathscr{M}_{IJ}$ that do not lie on the $+\alpha$-Landau surfaces of connected graphs. Moreover, f_{IJ} is a "plus $i\varepsilon$" boundary value of $\boldsymbol{f}_{IJ}$ at almost all other points. A general discontinuity formula around the $+\alpha$-Landau surfaces was then derived from these analyticity properties and unitarity, plus a certain mixed-α cancellation assumption [2, 3, 4, 5].

The aim of the present work is to examine this latter assumption, as it applies to the simplest case, and replace it by more satisfactory ones.

Let us first briefly recall the general procedure of [2]. Consider for simplicity a point $P\varepsilon\mathscr{M}_{IJ}$ that lies on the $+\alpha$-Landau surface $L_1(D_+)$ associated with a given connected graph D, but does not lie in $L_1(D'_+)$ for any other graph $D'\neq D$. [The surface $L_1(D_+)$ is the full Landau surface $L(D_+)$ minus point that are solutions of the corresponding equations with some of the Landau constants α_l equal to zero, and minus also the points p where any two initial lines of D have parallel momenta, or any two final lines of D have parallel momenta.] This surface $L_1(D_+)$ is known [6] to be a codimension 1 analytic submanifold of $\mathscr{M}_{IJ}$. It therefore divides $\mathscr{M}_{IJ}$ locally into two parts, called respectively the physical and nonphysical sides of $L_1(D_+)$. (The physical side is always well-characterized by convexity properties of the surface [7]). From macrocausality (see above), f_{IJ} is known to be locally the plus $i\varepsilon$ boundary value of the analytic function $\boldsymbol{f}_{IJ}$. Moreover, f_{IJ} is analytic on both sides of $L_1(D_+)$.

Unitary is then used to write S_{IJ}^c in the form:

$$S_{IJ}^c=D_{IJ}+R_{IJ} \tag{2}$$

where $D_{IJ}=d_{IJ}\times\delta^4(\Sigma p_i-\Sigma p_j)$ and $R_{IJ}=r_{IJ}\times\delta^4(\Sigma p_i-\Sigma p_j)$, and d_{IJ} and r_{IJ} have the following properties:

(i) d_{IJ} is explicitly known to vanish on the nonphysical side of $L_1(D_+)$.

(ii) r_{IJ} is a certain sum of "bubble diagram functions" which is shown to be locally the boundary value of an analytic function $\boldsymbol{r}_{IJ}$ from "minus $i\varepsilon$" directions, provided the assumption of mixed-α cancellation is used (see below).

Since $d_{IJ}=0$ on the non-physical side of $L_1(D_+)$, one has $f_{IJ}=r_{IJ}$ in that region. Thus $d_{IJ}=f_{IJ}-r_{IJ}$ is, in some neighborhood of P, the discontinuity of f_{IJ} (i.e. the difference between the boundary values of the plus $i\varepsilon$ and minus $i\varepsilon$ analytic continuations of f_{IJ} around $L(D_+)$.

If P is a point of $L_1(D_+)$ that lies on other $+\alpha$-Landau surfaces $L_1(D'_+)$, then one cannot expect in general to derive a discontinuity formula in the sense previously mentioned. However, the results of [2] and the assumption of mixed-α cancellation can still lead to certain (essential support) properties of f_{IJ}, as will be shown in Section 3.

Recent mathematical developments in the study of the analytic structure of distributions [8] have led to sharper formulations of these various results [5, 9] and to the possibility of the derivation of the discontinuity formulas from weaker assumptions.

Let us briefly summarize the main facts. (For details, see [9].) The analytic structure of a general distribution f defined on a real analytic manifold $\mathscr{M}$ can be characterized at each real point p of $\mathscr{M}$ by a certain set of "singular directions" (in the cotangent space $T_p^*\mathscr{M}$ at p to $\mathscr{M}$). These directions are those along which the generalized Fourier transform of f at p does *not* decrease exponentially (in a well-defined sense). The set of singular directions at a real point p is called the essential support of f at p. The distribution f is analytic at p if and only if its essential support at p is empty. And f, near p, is a boundary value of an analytic function from the directions of an open cone Γ if and only if the essential support of f at p is contained in the closed dual cone C of Γ. (For more general results, see [8].)

If the manifold $\mathscr{M}$ is the physical region of a process $I\to J$ (minus the exception points P_{exc}), then a direction u in $T_p^*\mathscr{M}$ is uniquely characterized by a configuration (defined modulo global space-time translations and space-time dilations) of initial and final trajectories that do not all pass through a common point. There is one trajectory for each particle in I or J, and each trajectory is a full line in space-time parallel to the 4-momentum p_k of particle k.

The bubble diagram functions are functions F_B that arise in equations derived from unitarity and the decomposition of S into its "connected" components. The function F_B is associated with a connected bubble diagram B, which is a connected graph consisting of $+$ and $-$ bubbles connected by directed lines. Each line runs always from left to right and is associated with a certain physical particle. The function F_B is the integral, over all possible on-mass-shell internal 4-momenta, of the product of the momentum-space kernels $S^c_{I_bJ_b}$, resp. $(S^c_{I_bJ_b})^-$, associated with each $+$ bubble, resp. $-$ bubble, of B. $((S^c_{I_bJ_b})^-$ is the connected kernel of $S^{-1}=S^\dagger$). Each F_B, like each S_{IJ} or S^c_{IJ}, is a well-defined distribution [9d], which can be written (if the external 4-momenta are not all parallel) in the same form as (1):

$$F_B = f_B \times \delta^4(\Sigma p_i - \Sigma p_j) \tag{3}$$

where I and J denote the sets of external initial and final particles of B, and f_B is a distribution defined on the physical region $\mathscr{M}_{IJ}$.

The *structure theorem* [10] follows from macrocausality and unitarity. It says that the only possible singular directions u of f_B at any non $u=0$ point p (see definition below) are those corresponding to the configurations of external trajectories of at least one connected multiple-scattering space-time diagram $\mathscr{D}_B$.

A diagram $\mathscr{D}_B$ is a space-time network of vertices, directed external lines (which begin or end at a vertex, but not both), and directed internal lines (which join two vertices). It can be constructed by replacing each bubble b of B by a connected subdiagram $\mathscr{D}_b$ (which may have only a single vertex). Each line has a well-defined on-mass-shell 4-momentum, and energy-momentum must be conserved at each vertex of $\mathscr{D}_B$. An internal line b of $\mathscr{D}_B$ is called a positive line, a negative line, or a zero line if it is oriented in space-time in the direction of its 4-momentum, in the opposite direction, or has zero length. (For a line of zero length the vertex at which it begins and the vertex upon which it ends lie at the same space-time point. This is not allowed for positive or negative lines.) Any line l of $\mathscr{D}_B$ which is an internal line of a subdiagram $\mathscr{D}_b$ is required to be a positive line of b is a $+$ bubble, or a negative line if b is a $-$bubble. The line l associated with an original internal line of B is allowed to be a positive, negative, or zero line.

A vertex v such that all of its incoming and outgoing lines have parallel 4-momenta is called a parallel vertex and is allowed to be at infinity in space-time. In this case the trajectories incident upon v are not required to coincide, but they are required to satisfy angular-momentum conservation [11]. The trajectory of a line is the full line in space-time parallel to its 4-momentum and passing through all the vertices upon which the line begins or ends.

A $u=0$ *point* of a bubble diagram function f_B is by definition a point p such that there exists a $\mathscr{D}_B$ whose external trajectories carry momenta $p \equiv (p_1, \ldots, p_n)$ and pass through a common point, and whose internal trajectories do not all pass through this point.

The (usual) structure theorem gives no information at a $u=0$ point p (i.e., every direction could be a singular direction of f_B at such a point).

Remark. Macrocausality is the particular case of the structure theorem obtained when B is composed of a single $+$ bubble.

A diagram $\mathscr{D}_+$ is a diagram every internal line of which is a positive line or a zero line. Such a $\mathscr{D}$ is called a positive-α diagram. A diagram $\mathscr{D}_-$ is a diagram every internal line of which is either a negative line or a zero line. Such a $\mathscr{D}$ is called a negative-α diagram. A diagram $\mathscr{D}_\pm$ is a diagram with at least one positive internal line and at least one negative internal line. Such a $\mathscr{D}$ is called a mixed-α diagram.

The earlier proofs that r_{IJ} is the minus $i\varepsilon$ boundary value of an analytic function make use, as we have mentioned, of the "mixed-α cancellation ansatz". In the framework outlined above, this assumption asserts that the only possible singular directions of r_{IJ} are those associated with either positive-α diagrams or negative-α

diagrams; i.e., that all singularities of the bubble diagram functions f_B of r_{IJ} associated with mixed-α diagrams $\mathscr{D}_B$ must cancel among themselves in the sum r_{IJ}.

The consistency of this mixed-α cancellation assumption has been checked in many cases. That is, the mixed-α cancellation (and moreover the cancellation of the singularities of r_{IJ} associated with each individual mixed-α D) has been shown to the follow from the general discontinuity formula in the large number of cases that have been examined. On the other hand, the individual bubble diagram functions f_B of r_{IJ} do in general have mixed-α singularities that would disrupt the proof were they not cancelled by the mixed-α singularities from other terms: see Section 4.

The assumption that this cancellation must always occur, although apparently consistent with the discontinuity formula, has no a priori basis, and hence should be eliminated.

As a first step in this direction we shall prove here, without using this mixed-α cancellation assumption, the simplest of all discontinuity formulas, namely the pole-factorization theorem for a $3\to3$ equal-mass process below the 4-particle threshold. The result will be derived from macrocausality, unitarity and two weak regularity assumptions on scattering and bubble diagram functions.

These two new assumptions are described in Section 2. In Section 3 we recall the earlier derivation of the pole-factorization theorem, and in Section 4 we describe some properties of the mixed-α diagrams. These properties are used in the new proof which is given in Section 5.

The techniques that are used to obtain the results described in Section 4 are discussed in Appendix I. In Appendix II the approach to the pole-factorization theorem used in [12] is described, and contrasted with that used here. We shall see that it offers no advantages, and has certain disadvantages: to convert it to a proof one would need the present work and more.

2. Two Assumptions on Scattering Functions and Bubble Diagram Functions

In this section we describe two new assumptions on scattering functions and bubble diagram functions. The first assumption deals with $u=0$ points. It is a special case of a conjecture proposed in [13] on the basis of a general solution of the $u=0$ problem for phase-space integrals, and, for single bubbles, of both the implications of macrocausality at $u=0$ points and the holomic structure of Feynman integrals [15]. It is also discussed in [9d]. The second assumption is a slight strengthening of an analyticity property that can be derived from macrocausality (and Assumption 1).

(a) Assumption on $u=0$ Points

The first assumption concerns the $u=0$ points p. As noted in Section 1 neither macrocausality (in its original version) nor the corresponding structure theorem, which is derived from macrocausality and unitarity, gives any information about the singular directions at $u=0$ points. Let us define second kind $u=0$ points $p=\{p_k\}$ of f_B as those such that there exists a $\mathscr{D}_B$ the external trajectories of which pass through the origin and carry the 4-momenta p_k, while at least one internal line joins vertices that are not both parallel vertices and does not pass through the origin.

These second kind $u=0$ points are not encountered in the present work. We now state:

Assumption 1. *If p is not a second kind $u=0$ point of f_B, then the only singular directions of f_B at p are those corresponding to configurations of external trajectories of diagrams $\mathscr{D}_B$.*

(In other words, the rules for $u \neq 0$ points hold also at $u=0$ points, provided they are not second kind $u=0$ points.)

At second kind $u=0$ points certain limiting procedures must be considered. These are discussed in [9d] and [13].

(b) Analyticity Assumption (No Sprout Assumption)

The second assumption is a slight extension of an analyticity property that can be derived from macrocausality and assumption 1. This property will be described first. Let us fix the notation. The $+\alpha$-Landau surface $L(D_+)$ is the set of points p of the physical region such that there exists at least one diagram $\mathscr{D}$ with only $+$ lines whose topological structure is D and whose set of external 4-momenta is p. If p lies on the subset $L_1(D_+)$ of $L(D_+)$ then this diagram $\mathscr{D}$ is unique (up to space-time translations and dilations). As mentioned before, the surface $L_1(D_+)$ is a real analytic submanifold of $\mathscr{M}$ of codimension 1 [6]. For any point $p \in L_1(D_+)$ the well-defined direction u in $T_p^*\mathscr{M}$ corresponding to the configuration of external trajectories of $\mathscr{D}$ will be denoted by $u_+(p)$. This direction is also the direction that is conormal to $L_1(D_+)$ at p and oriented toward the physical side of $L_1(D_+)$.

Let Ω be a real domain of the physical region. Assume that Ω contains no point of $L(D_+)$ minus $L_1(D_+)$, and no point lying both in $L_1(D_+)$ and in the closure of a surface $L_1(D'_+)$ for a $D' \neq D$ that is related to D. (Related graphs are graphs that are contractions of a common parent.) Assume finally that Ω contains no $u=0$ points p of the second kind, relative to the scattering function f.

The following result is a consequence of macrocausality and Assumption 1. It is completely analogous to results discussed in [14].

Proposition 1. *The scattering function f can be decomposed in Ω into a sum of two distributions f_1, f_2 such that:*

(i) *f_1 is analytic in Ω outside $L_1(D_+)$ and the only possible singular direction of f_1 at any point p of $L_1(D_+)$ in Ω is $u_+(p)$.*

(ii) *the only possible singular directions of f_2 at any point p of $\Omega \cap L(D_+)$ correspond to connected diagrams D'_+ such that D' and D are unrelated.*

Property (i) is equivalent to the property that f_1 is, in Ω, the boundary value of a function $\boldsymbol{f}_1$ analytic in a domain of the complexification $\tilde{\mathscr{M}}$ of $\mathscr{M}$ that contains all real points p of Ω outside $L_1(D_+)$ and whose profile at each point p of $L_1(D_+)$ in Ω is the open half space dual to $u_+(p)$. This last statement means that, being given a system of real analytic coordinates $z \equiv (z_1, \ldots, z_{3n-4})$ of $\tilde{\mathscr{M}}$ at p and any open cone Γ in $\mathbb{R}^{3n-4}$ with apex at the origin whose closure is contained apart from the origin in the open half space dual to $u_+(p)$, there exists in $\mathbb{C}^{3n-4}$ a complex neighborhood ω of $z(p)$ such that all points in ω whose imaginary part $\operatorname{Im} z$ lies in Γ belong to the analyticity domain of f_1. Since $L_1(D_+)$ is a real analytic submanifold of $\mathscr{M}$ of codimension 1, it can be represented in the neighborhood of any point P of $L_1(D_+)$ by an equation of the form $\psi=0$, where ψ is a real analytic function with nonzero gradient which can be chosen such that $\psi>0$ on the physical side of $L_1(D_+)$ (this side, as mentioned, is always well determined). If ψ is taken as one of the real analytic coordinates of $\mathscr{M}$ at P then the open half-space dual to $u_+(p)$ corresponds to the set $\operatorname{Im}\psi>0$ (in the local coordinate system considered) at all points in the neighborhood of P.

The set $\hat{L}(D_+)$ is defined to be the subset of points p of $L_1(D_+)$ such that no connected $\mathscr{D}'_+$ has external trajectories corresponding to $(p, u_+(p))$ unless the corresponding graph D' contracts to D (i.e., unless D' is D or can be reduced to D by contracting to points some of the internal lines of D'.) If we consider a point P of Ω in $\hat{L}(D_+)$, then f_1 and f_2 are uniquely determined locally, modulo the addition of a real analytic function. It is sufficient to restrict our attention to this case and state:

Assumption 2. *For any point P of $\hat{L}(D_+)$ there is*

(i) *a system of local real analytic coordinates $z=z_1, \ldots, z_{3n-4}$ at P,*

(ii) *a complex neighborhood ω in $\mathbb{C}^{3n-4}$ such that $\hat{L}(D_+)$ is represented in ω by $\{z_1=0, z \text{ real}\}$, and*

(iii) *an open curve c in $\mathbb{C}^1$ that starts at the origin, such that the function f_1 of Proposition 1 can be analytically continued (single valuedly) into the set $\omega \cap \{z_1 \in c\}$.*

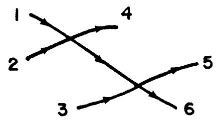
Assumption 2 could be replaced by the stronger Assumption 2′, which is the same as Assumption 2 except that the set $\{z_1 \in c\}$ is replaced by $\{\operatorname{Im} z_1>0\}$, where $z_1=\psi$.

This Assumption 2′ is similar to what is implied by macrocausality, but slightly stronger: it implies that for some sufficiently small complex neighborhood ω of $z(p)$ the open cone Γ discussed above can be expanded to the full half-space $\operatorname{Im} z_1>0$, in some system of local real analytic coordinates that has $z_1=\psi$. Macrocausality allows the cone Γ to be taken arbitrarily close to the half space, but the neighborhood ω may be forced to shrink as Γ expands. The Assumption 2′ asserts that for some sufficiently small ω this neighborhood ω can be held fixed as Γ expands to the half space.

Assumption 2 or 2′ has the effect of excluding from f_1 certain singularity surfaces (called sprouts) that are not excluded by macrocausality alone. These surfaces are certain special surfaces that touch the physical region only at points P lying in the union L^+ of the positive-α Landau surfaces, but are not confined to the union of the local complexifications of these surfaces. Thus Assumption 2 (or 2′) can be regarded as a precise formulation of some aspect of the idea that a scattering function have, in some complex neighborhood of each physical point P, no singularities that do not lie on the local complexifications of the positive-α Landau surfaces that pass through P. In other applications one may wish to use this stronger assumption (see [11]). But here we have tried to find the weakest analyticity assumption that would allow us to prove the pole-factorization theorem. Assumption 2′ is guaranteed true also if the S-matrix is holonomic [15].

3. Pole-Factorization Theorem (Proof Using the Mixed-α Cancellation Assumption)

In the remainder of this work, we consider a theory with only one type of particle, a boson of mass m, a $3 \to 3$ process below the 4-particle threshold ($s = (p_1 + p_2 + p_3)^2 = (p_4 + p_5 + p_6)^2 < 16m^2$, where the indices 1, 2, 3, resp. 4, 5, 6, label the initial and final particles respectively), and the graph D:

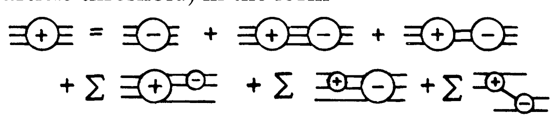


In this case $L(D_+)$ is defined by the equation $k^2 = m^2$, $k_o > 0$, where $k = p_1 + p_2 - p_4$ ($= p_5 + p_6 - p_3$). The corresponding space-time diagram with positive internal line is denoted by $\mathscr{D}$.

The direction $u_+(p)$ corresponds to the configuration of external trajectories such that 1, 2, 4 meet at a common space-time point A, and 3, 5, 6 meet at a common point B, with $(AB)_0 > 0$. The plus $i\varepsilon$ directions at p of $L(D_+)$ are those of the open half space dual to $u_+(p)$. If $\psi = k^2 - m^2$ is chosen to be one coordinate in a system of real analytic local coordinates of $\mathscr{M}$ at p, this open half space is represented (as already mentioned in Section 2) by $\operatorname{Im}\psi > 0$. The minus $i\varepsilon$ directions are those opposite to the plus $i\varepsilon$ directions.

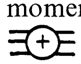
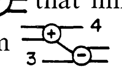
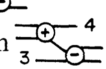
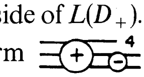
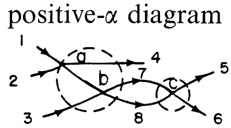
We first illustrate the general procedure outlined in Section 1, following the presentation of [5], which is a simple adaptation to the particular case of interest of the general algebraic methods of [2].

By using the (cluster) decompositions of the S-matrix and of $S^{-1} = S^\dagger$ into connected parts, one can write the unitarity equation $SS^\dagger = SS^{-1} = \mathbb{1}$ (below the 4-particle threshold) in the form

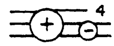
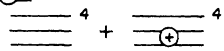


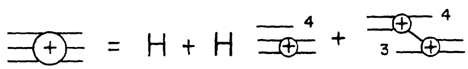
(4)

where the bubbles + and − denote here the momentum-space connected kernels of the S-matrix and of minus $S^{-1} = S^\dagger$, respectively. The sums $\sum$ refer to the various ways of assigning external momenta to the various bubbles involved.

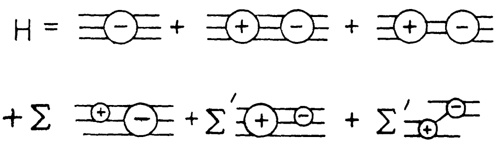
The decomposition of  that immediately arises from (4) by separating in the right-hand side the term  from the others does not yet provide a decomposition of the form (1). The term , which contains a factor $\delta(k^2 - m^2) \times \theta(k_0)$, $k = p_1 + p_2 - p_4$, does vanish, as required, on the non physical side of $L(D_+)$. (Here $k^2 < m^2$.) But among the remaining terms, one finds the term  which at any point P of $L_1(D_+)$ does admit $u_+(P)$ as a possible singular direction associated with a positive-α diagram $\mathscr{D}_B$. This $\mathscr{D}_B$ is a spacetime representation of the graph $D_B =$ 

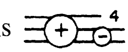
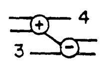
(where the subgraphs D_b are shown inside circles) in which the original internal lines 7, 8 of B have zero length. This representation coincides with $\mathscr{D}$ after the contraction of these two lines.

By transferring the term  on the left-hand side, multiplying (see below) both sides on the right by  and using two-particle unitarity below the three-particle threshold, (4) is transformed (see [5] for details) to the form:

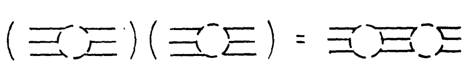
 (5)

where

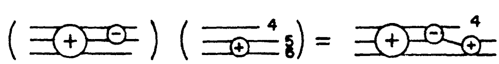
 (6)

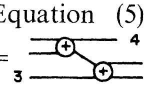
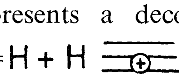
The sums $\sum'$ in (6) denote the sums $\sum$ of (3) from which the terms  and  respectively are removed.

The "multiplication" of bubble diagrams is here defined by the rule

 (7)

For instance



Equation (5) now represents a decomposition of the form (1) with $D_{IJ}=$  and $R_{IJ}=$ . For any B occurring in R_{IJ} one checks by direct inspection that there is no positive-α $\mathscr{D}'_B$ such that the corresponding D' contracts to D. Thus if P lies on $L_1(D_+)$ but on no $+\alpha$-Landau surface $L_1(D'_+)$ with $D' \neq D$ then there can be no positive-α $\mathscr{D}'_B$ with external momenta P. Moreover, the directions defined by the configurations of external trajectories of the possible negative-α $\mathscr{D}_B$ consist solely of $u_-(p)$. Thus the assumption of mixed-α cancellations allows one to conclude that $u_-(P)$ is the only possible singular direction in the essential support of r at P, and that r at P is, correspondingly, the boundary value of an analytic function from minus $i\varepsilon$ directions.

In this particular case, this result implies, in turn, that f can be written locally in the form:

$$f_{3,3}(p_1 \ldots p_6) = \frac{a(p_1 \ldots p_6)}{k^2 - m^2 + i\varepsilon} \tag{8}$$

where $k = p_1 + p_2 - p_4$, and a is a locally analytic function whose value at $k^2 = m^2$ is equal to:

$$a(p_1 \ldots p_6)|_{k^2=m^2} = f_{2,2}(p_1, p_2; p_4, k) f_{2,2}(p_3, k; p_5, p_6) . \tag{9}$$

i.e., the singularity of $f_{3,3}$ is a pole and its residue is the product of the two scattering functions associated with the two vertices of D.

This set of results is called the pole-factorization theorem.

At a general point P of $\hat{L}(D_+)$ (which may lie on other surfaces $L_1(D'_+)$ there is, by virtue of the same argument and the definition of $\hat{L}(D_+)$ (see Section 2) no positive-α $\mathscr{D}'_B$ whose configuration of external trajectories correspond to $(p, u_+(p))$. (B is here, as before, any one of the bubble diagrams involved in R.) If there is, moreover, no negative-α $\mathscr{D}_B$ whose configuration of external trajectories is specified by $(p, u_+(p))$ [1] then the assumption of mixed-α cancellations allows one to conclude that $u_+(P)$ does not belong to the essential support of $f-d(=r)$.

Using Lemma 3 of Section 5, one can show that this requirement on the negative-α diagrams $\mathscr{D}_B$ actually holds at any point of $\hat{L}(D_+)$. (This follows from the same methods as those used in the proof of Proposition 4 in Section 5.)

Remarks. (1) This result that $u_+(p)$ does not belong to the essential support of $f-d$ is sufficient to ensure the pole-factorization theorem [in the form (8), (9)], provided P lies on no surface $L_1(D'_+)$ with $D' \neq D$. This is because the essential support of f is then known to contain at most the direction $u_+(p)$, and the essential support of d contains only, as easily checked, the directions $u_+(p)$ and its opposite direction $u_-(p)$. Hence for these P the essential support of $r=f-d$ can contain only the directions $u_+(p)$ and $u_-(p)$ and the result that it does not contain $u_+(p)$ implies that it contains at most $u_-(p)$. Thus r is the minus $i\varepsilon$ boundary value of an analytic function.

(2) The methods of Appendix I allow one to obtain the following result on $\hat{L}(D_+)$:

Lemma 1. *All points of $L_1(D_+)$ that do not belong to $\hat{L}(D_+)$ lie in submanifolds of $L_1(D_+)$ of codimension 2 or more.*

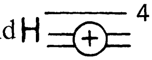
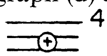
In particular, $L_1(D_+)$ minus $\hat{L}(D_+)$ is contained in the set of points $p=\{p_k\}$ of $L_1(D_+)$ such that two of the 4-momenta p_k are equal or such that $p_1+p_2-p_4$ lies in the same plane as p_3 and p_4 or as p_4 and p_5, or as p_5 and p_6, or as p_1 and p_3, or as p_2 and p_3.

4. Analysis of Mixed-α Landau Diagrams

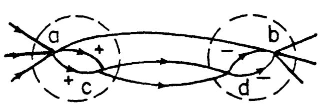
In view of the final remark (1) of Section 3, the mixed-α cancellation assumption would be unnecessary in the proof of that section, i.e. macrocausality and unitarity would directly provide the results at a point P of $L_1(D_+) - \bigcup_{D' \neq D} L_1(D'_+)$, provided no mixed-$\alpha$ diagram $\mathscr{D}'_B$ could have a configuration of external trajectories corresponding to $(p, u_+(p))$, where f_B is, as before one of the bubble diagram functions contributing to r, and provided P was not a $u=0$ point of any f_B that contributes to r. [If P were such a $u=0$ point then all directions, including $u_+(P)$, should a priori have to be allowed for f_B: see Section 1.] However, if Assumption 1 is accepted, then only $u=0$ points of the second kind would have to be absent, since this assumption asserts that the usual rules hold at the remaining $u=0$ points.

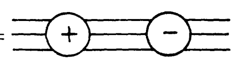
---

[1] The direction $u_+(p)$ would be obtained from a $\mathscr{D}_B$ with all lines negative if for instance P were a point of $\hat{L}(D_+)$ such that $P_1=P_6$, $P_2=P_4$, and $P_3=P_5$

The term R is the sum of H and  . We first outline in Paragraphs (a), (b), (c) typical examples of the problems that arise already for H, because of mixed-α diagrams, if the mixed-α cancellation assumption is dropped, and we then give in Paragraph (d) an example of a further difficulty arising from the multiplication of H by  .

(a) Consider the graph

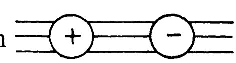
$D_B=$ 

which comes from the term $B=$  . (The circles indicate the subgraphs associated with the two bubbles.)

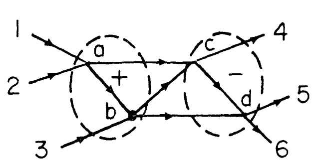
For any given values $P=(P_1 \dots P_6)$ of the external momenta, there always exists a nontrivial corresponding diagram $\mathscr{D}'_B$ such that A lies on B, and C and D are at infinity. (Here A, B, C, D are the space-time representative points of a, b, c, d.) One can, in fact, always find on-mass-shell 4 momenta k_1, k_2, k_3 such that $k_1=k_2$, and $k_1+k_2+k_3=p_1+p_2+p_3$ $(=p_4+p_5+p_6)$. In $\mathscr{D}'_B$ the trajectories of the two particles going from a to c and of the two particles going from d to b are taken parallel to $k_1=k_2$ and passing through $A=B$. The trajectories of the two lines going from C to D must also be parallel to $k_1=k_2$, but since the vertices C, D are at infinity, these two internal trajectories can be made noncoincident with the previous ones, even though angular-momentum conservation is assumed.

In this diagram $\mathscr{D}'_B$ all external trajectories pass through a common point, but some of the internal trajectories do not pass through this point. Hence P is a $u=0$ point of B and all directions can a priori be singular at P, according to the results described in Section 1.

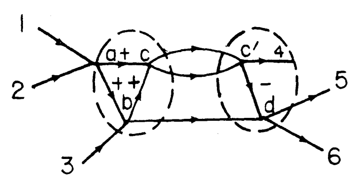
The assumption 1 on $u=0$ points presented in Section 2 will remove this difficulty.

(b) Other difficulties occur for the same term  .

Consider the graphs:



or



where the circles again indicate the subgraphs associated with the two bubbles. Let us consider the space-time representations $\mathscr{D}_B$ in which the line bd has zero length, as do the two lines going from c to c', in the second case.

In such a representation, if it exists, the lines bc and cd have parallel (and equal) 4-momenta (since $B=D$). Therefore the lines ac and 4 also have parallel (and equal) 4-momenta, and the line ab has the 4-momentum $p_1+p_2-p_4$.

Let us define a subset Ω_+ of $L_1(D_+)$ as follows. Being given any point $p=(p_1, \dots, p_6)$ of $L(D_+)$ we choose a space-time point A, through which we draw two lines parallel, respectively, to p_4 and $p_1+p_2-p_4$, and choose a point B on the second one such that $(AB)_0>0$. Then, by definition, p belongs to Ω_+ if and only if it is possible to find two on-mass-shell 4-momenta k_1, $k_2(k_1^2=k_2^2=m^2, k_{10}>0, k_{20}>0)$ such that $k_1+k_2=p_1+p_2-p_4+p_3(=p_5+p_6)$ and such that the line passing through B and parallel to k_1 meets the line parallel to p_4 and passing through A at some point, called C, that is later in time than A or B $((AC)_0>0, (BC)_0>0)$.

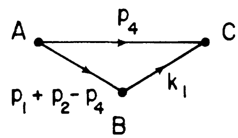
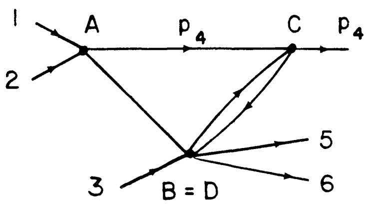


Fig. 1

See Figure 1, where the 4-momentum of each line is indicated. The point C can lie at infinity and hence the points where p_4 equals p_5 or p_6 lie in Ω_+. If and only if $p\in\Omega_+$ then one can construct a space-time representation $\mathscr{D}_B$ whose external trajectories correspond to $(p, u_+(p))$:

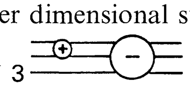


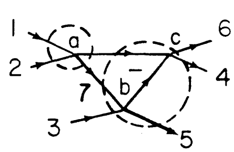
In fact, the trajectories 1, 2, 4 resp. 3, 5, 6 pass through A, resp. through $B=D$, and AB is directed along $p_1+p_2-p_4$.

An elemetary analysis shows that Ω_+ is not composed of isolated, accidental, points, but is a full open subset (of strictly positive measure) of $L_1(D_+)$.

(c) Paragraphs (a) and (b) have exhibited cases in which $u_+(p)$ is a possible singular direction of a bubble diagram function f_B included in r, either at all points p of $L_1(D_+)$ or at all points p of a full open subset of $L_1(D_+)$.

In a number of other cases, $u_+(p)$ appears as a possible singular direction of certain bubble diagram functions, associated with various other mixed-α diagrams, provided p belongs to certain lower dimensional subsets of $L_1(D_+)$.

Consider the instance for term  and the graph:

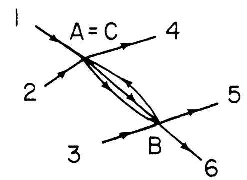


It is easily checked that there is a space-time representation $\mathscr{D}'_B$ whose external trajectories correspond to $(p, u_+(p))$ if one of the following conditions is satisfied:

(i) $p_3=p_5$.

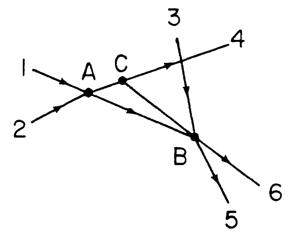
If we again denote by A, B, C the space-time representations of a, b. c, the required $\mathscr{D}'_B$ is obtained by putting $A=C$ and AB along the direction of $p_1+p_2-p_4$.

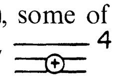
(The trajectories of 1, 2, 4 pass through A, those of 3, 5, 6 pass through B, since $p_6 = p_1 + p_2 - p_4$, and AB is oriented along the direction of $p_1 + p_2 - p_4$.)

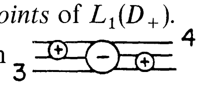


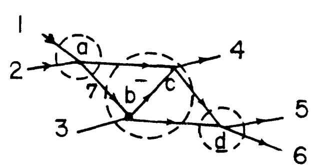
(ii) $p_4, p_6, p_1 + p_2 - p_4$ lie in a common plane.

The space-time diagram $\mathscr{D}'_B$ is constructed by attributing the 4-momenta $p_4, p_1 + p_2 - p_4$ and p_6 to the lines ac, ab, and bc respectively:

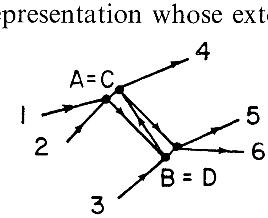


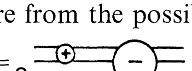
(d) Besides situations which are the direct analogs of those of Paragraphs (a) and (b), some of the situations of Paragraph (c) may lead, after multiplications of H by  to cases where $u_+(p)$ is now a possible singular direction of some bubble diagram functions at *all points* of $L_1(D_+)$.

Consider for instance the term  and the graph



By putting $A = C$, $B = D$, AB along the direction of $p_1 + p_2 - p_4$, $p_8 = p_3$, $p_7 = p_1 + p_2 - p_4$, one obtains, for any point $p = (p_1 \ldots p_6)$ of $L_1(D_+)$, a space-time representation whose external trajectories correspond to $(p, u_+(p))$:



This situation arises here from the possibility of choosing $p_8 = p_3$ in the sub-diagram associated with $B =$ . This corresponds to the situation (i) of Paragraph (c), but now after multiplication by the small bubble the low dimensional singularity surface of the original B is converted to a singularity surface that fills $L_1(D_+)$.

We conclude this section by collecting some results that will be used in Section 5:

First, the methods of Appendix I allow one to conclude that the features described in the above Paragraphs (b) and (c) are general:

Proposition 2. *If P is a point of $L_1(D_+)$ that does not lie in Ω_+, then $(P, u_+(P))$ cannot correspond to the configuration of external trajectories of any diagram $\mathscr{D}_B$ associated with one of the bubble diagram functions of h [as defined by Eq. (6)], except possibly when P lies in the union of a finite number of submanifolds N_t of $L_1(D_+)$ whose codimension in $L_1(D_+)$ is larger than one.*

Examples of submanifolds N_t have been given in Paragraph (c) above. All others are of the same type.

The proof of this result follows from a complete analysis of all contributing mixed-α diagrams. Since a very large number of cases have to be systematically considered, we shall omit the details here for the sake of conciseness (some details are given in Appendix I).

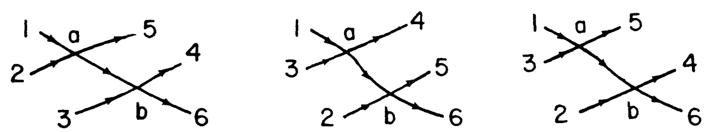
The next result refers to a set of manifolds N_t', which (in contrast to the manifolds N_t) depend only on the variables p_1, p_2, p_3, p_4: $\{p_1 = p_2\}$, $\{p_2 = p_3\}$, $\{p_1 = p_3\}$, $\{p_1 = p_4\}$, $\{p_2 = p_4\}$, $\{p_3 = p_4\}$, $\{p_5 = p_6\}$ [or equivalently $(p_1 + p_2 + p_3 - p_4)^2 = 4m^2$], $\{p_3, p_4, p_1 + p_2 - p_4\}$, $\{p_1, p_3, p_1 + p_2 - p_4\}$, $\{p_2, p_3, p_1 + p_2 - p_4\}$, $\{(p_5 + p_6 - p_4)^2 = m^2, (p_5 + p_6 - p_4)_0 > 0\}$ [or equivalently $(p_1 + p_2 + p_3 - 2p_4)^2 = m^2$, $(p_1 + p_2 + p_3 - 2p_4)_0 > 0$]. (The notation $\{q, r, s\}$ means the set of p such that the three 4-vectors q, r, s lie in a common plane.)

Different choices of possibly smaller manifolds might still allow one to prove this result. This would not, however, improve the final results of Section 5 (Proposition 5) and would complicate the analysis of mixed-α diagram in Lemmas 2, 3 below.

Lemma 2. *All points p of $L_1(D_+)$ that lie outside both Ω_+ and $\cup N_t'$ belong to $\hat{L}(D_+)$.*

Proof. It is sufficient to consider each possible diagram $\mathscr{D}'$ with only + lines and check that the configuration of its external trajectories cannot correspond to $(p, u_+(p))$ unless $D' = D$. The condition $p \notin \cup N_t'$ prevents any of the 4-momenta in the set (p_1, p_2, p_3) or in the set (p_4, p_5, p_6) from being parallel. Thus, Theorem 1 of Appendix I then shows than the only diagrams to be considered are diagrams with one internal line and triangle diagrams.

First consider the case of one internal line. The graphs D' to be considered are



a b c

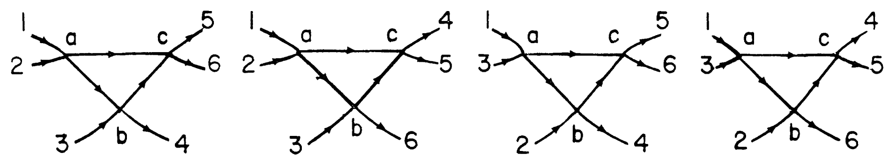
and graphs in which 1, 2 in b, or 5, 6 in c are exchanged. These are treated similarly.

In Case a, the meeting point of trajectories 1, 2, 4 in any $\mathscr{D}'$ whose configuration of external trajectories is $u_+(p)$ has to be the representative point A of a. Since trajectory 4 passes through this point, the 4-momentum of line ab is equal to p_4 and hence $p_3 = p_6$ and $p_5 = p_1 + p_2 - p_4$. Since trajectories 3, 5, 6 must meet, one concludes that the three 4-momenta p_3, p_4, $p_1+p_2-p_4$ lie in a common plane, i.e. $p\in N_t'$.

In Case b, $p_1 \neq p_4$ because $p \notin N_t'$. Hence the meeting point of 1, 2, 4 in any $\mathscr{D}'$ whose configuration of external trajectories is $u_+(p)$ has to be the representative point A of a. Since trajectory 2 passes through this point, trajectories 1, 2, 3 pass through a common point. This is excluded (since $p_5 \neq p_6$).

Finally in Case c, the meeting point of 1, 2, 4 in any $\mathscr{D}'$ giving $u_+(p)$ has to be the representative point B of b (since $p_2 \neq p_4$). An argument analogous to that given in Case a then shows that $p_1, p_3, p_1+p_2-p_4$ would have to lie in a common plane, i.e., $p \in \cup N_t'$.

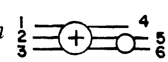
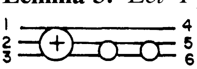
Next consider the triangle diagram. The graphs D' to be considered are:

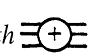
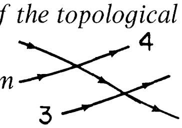


a b c d

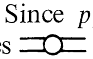
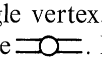
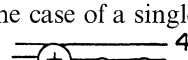
and graphs in which 5, 6 or 1, 2 are exchanged, which are treated similarly.

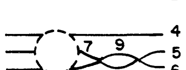
Arguments similar to those given above show that in any representative $\mathscr{D}'$ giving $u_+(p)$, p_3, p_4, $p_1+p_2-p_4$ would have to lie in the same plane in Case a, p would have to be in Ω_+ in Case b, $p_1, p_3, p_1+p_2-p_4$ would lie in the same plane in Case c. Finally in view of Lemma 3 of Appendix I, the meeting points A', resp. B', of 1, 2, 4, resp. 3, 5, 6 in any $\mathscr{D}_B$ giving $u_+(p)$ in Case d would have to lie at one of the representative points A, B, C of a, b, c. A' cannot be at A since 1, 2, 3 would meet. It cannot be at C since $(CB)_0<0$, $(CA)_0<0$. If it were at B, then arguments similar to those given above show that p_1 and p_2 would be equal to p_4 (B' would have to be at C, hence, $p_1+p_2-p_4=p_4$).

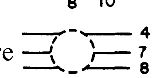
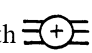
Lemma 3. *Let F_B be a bubble diagram function of the form*  *or*  , *where the small bubbles may be + or −, and let $\mathscr{D}_B$ be any corresponding diagram.*

If P is a point of $L_1(D_+)$ that does not lie in Ω_+ or in $\cup N_t'$ and if the topological structure of the subdiagram $\mathscr{D}_b$ associated with  *is different from* 

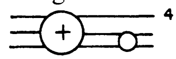
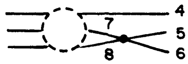
then the configuration of external trajectories of $\mathscr{D}_B$ cannot correspond to $(P, u_+(P))$.

Proof. Since $p_5 \neq p_6$ and $(p_5+p_6)^2<9m^2$ the subdiagrams associated with the bubbles  have necessarily a single vertex. It is therefore sufficient to prove Lemma 3 in the case of a single bubble . In fact any diagram $\mathscr{D}_B$ associated with the term  is a space-time representation of the graph:

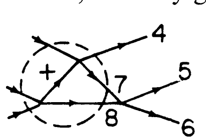


where  stands for a subgraph D_b associated with .

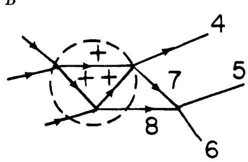
Since $p_5 \neq p_6$, the lines 9, 10 are necessarily zero lines in this representation, and hence do not change the arguments.

If we consider the term  then any corresponding graph D_B has the form  . For the same reason as above, we need not consider the case when 7, 8 originate on the same vertex of the subgraph D_b associated with  (This case is covered by Lemma 2). Finally, graphs D_B in which 1, 2, 3 end at a common vertex do not have to be considered (Lemma 1 of Appendix I). In view of Theorem 1 of Appendix I the only subgraphs D_b to be considered are graphs with one internal line or triangle graphs. (The condition $p \notin \cup N'_t$ prevents two of the 4-momenta p_1, p_2, p_3 resp. p_4, p_5, p_6 from being equal.)

Hence, the only graphs D'_B that need to be considered are of the form:

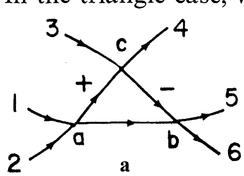


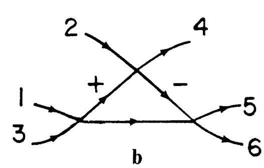
or



where the circles indicate the subgraphs D_b associated with , and where all ways of attributing 1, 2, 3 to the initial external lines are a priori possible. Moreover, we have to consider here only cases in which line 7 in any representation $\mathscr{D}_B$ is a negative line. If line 7 were a positive line, line 8 would also have to be a positive line. For all internal lines positive the fact that the configuration of external trajectories cannot correspond to $u_+(p)$ was shown in the proof of Lemma 2, for the case of the triangle. And this result follows from Theorem 1 of Appendix I in the case of the double triangle. On the other hand, if line 7 were a zero line, it would follow that two of the 4-momenta p_1, p_2, p_3 would be equal.

In the triangle case, we are therefore left with the following cases:

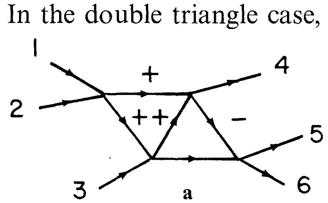


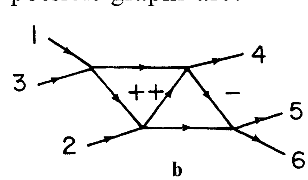


(and the graph obtained from b by exchanging 1, 2).

In case a, the meeting points A', B', of 1, 2, 4 and 3, 5, 6 in any $\mathscr{D}'_B$ giving $u_+(p)$ must be the representative points A, B of a, b. Hence by arguments similar to those used in the proof of Lemma 2, $p_3, p_4, p_1+p_2-p_4$ would have to lie in a common plane. In Case b, one cannot get $u_+(p)$ since the meeting points C, B of 2, 4 and 5, 6 are in the wrong time order $((CB)_0 < 0)$.

In the double triangle case, the possible graphs are:





(and the graph obtained from b by exchanging 1, 2).

In any corresponding $\mathscr{D}_B$, the line cb has to be a negative or zero line, in view of Lemma 4 of Appendix I. In Case a, the line ad would have to be parallel to p_4 in any $\mathscr{D}'_B$ giving $u_+(p)$ since 1, 2, 4 meet. Hence $p \in \Omega_+$. In Case b, it follows from Lemma 3 of Appendix I that the meeting point A' of 1, 2, 4 in any $\mathscr{D}_B$ giving $u_+(p)$ should be one of the representative points A, C, D of a, c, d, the meeting point of 3, 5, 6 being necessarily the representative point B of b.

But A' cannot be at D or at C since $(DB)_0 < 0$. It cannot be at A since then 1, 2, 3 could pass through a common point (Lemma 1 of Appendix I). The proof is therefore completed.

5. New Proof of the Pole-Factorization Theorem

We prove in this section the following result:

Theorem. *Being given any point P of $\hat{L}(D_+)$, macrocausality, unitarity and Assumptions 1 and 2 of Section 2 imply that the essential support of $f-d$ at P does not contain $u_+(P)$.*

The main steps of the proof are outlined in Subsection (a). Two preliminary mathematical results are then described in Subsection (b) and details on the proofs are given in Subsection (c).

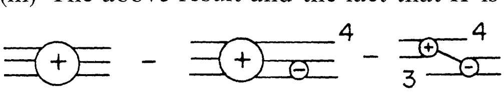
(a) Main Steps of the Proof

The result is obtained by the following steps:

(i) Assumption 1 eliminates the $u=0$ problems.

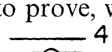
(ii) Proposition 2 of Section 4 then asserts that if P is a point of $L_1(D_+)$ that does not lie in Ω_+ or in $\cup N_t$ (where the manifolds N_t are those involved in the statement of Proposition 2), then $u_+(P)$ is not a singular direction of h at P.

(iii) The above result and the fact that H is equal also to



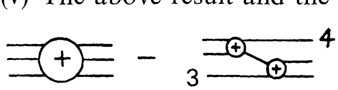
[see Eqs. (4) and (5)] allows one to prove, by using Assumption 2 of Section 2:

Proposition 3. *If P is a point of $L_1(D_+)$ that does not lie in Ω_+ or in the union of the manifolds N'_t introduced in Section 4 (see Lemma 2 and 3), then $u_+(P)$ is not a singular direction of h at P.*

(iv) The fact that the manifolds N'_t, and also Ω_+ are defined only in terms of p_1, p_2, p_3, p_4, allows one to avoid the problems of Paragraph (d) of Section 4, and to prove, with the aid of Lemma 3 of Section 4, that the "multiplication" of H by  does not modify this result; i.e., to prove:

Proposition 4. *If P is a point of $L_1(D_+)$ that lies outside Ω_+ and $\cup N'_t$, then $u_+(P)$ is not a singular direction of r at P.*

(v) The above result and the fact that R is equal also to



allows one to prove, by using again Assumption 2 of Section 2:

Proposition 5. *$u_+(p)$ is not a singular direction of r at any point P of $\hat{L}(D_+)$.* *Q.E.D.*

An alternative proof of this last step not depending on Assumption 2 will also be given.

Remark. For reasons which will become clear in the proof, this result is certainly not expected to hold for h itself: $u_+(p)$ is expected to be a singular directions of h at least at the points of Ω_+.

(b) Preliminary Mathematical Results

The following two mathematical results will be used.

Theorem 1. *Let Ω be a real domain of a real analytic manifold and let L be a connected real analytic submanifold of Ω of codimension 1. Let f_1 be a function that has, in the complexification $\mathscr{M}$ of $\mathscr{M}$, a schlicht domain of holomorphy that contains all points of Ω outside L, and that contains one point p_o of Ω in L. Suppose for any point p of L in Ω there is* (i) *a system of local real analytic coordinates* $(z=z_1, \ldots, z_{3n-4})$ *at p,* (ii) *a convex complex neighborhood ω in $\mathbb{C}^{3n-4}$ such that L is represented in ω by* $\{z_1=0$, *z real*$\}$, *and* (iii) *an open curve c in $\mathbb{C}^1$ that starts at the origin such that f_1 can be analytically continued (single valuedly) into the set $\omega \cap \{z_1 \in c\}$. Then f_1 is analytic at all points of Ω.*

Proof. Let p be an arbitrary point of L in Ω. Then p is connected to p_o by a compact curve in $L \cap \Omega$. This curve is covered by a finite subset of the neighborhoods ω of the theorem. One of these neighborhoods, ω_o, contains $z_o \equiv z(p_o)$. Let $\bar{z}$ be any other fixed real point in ω_o with $\bar{z}_1=0$. By a nonsingular linear change of coordinates one may obtain a new real local coordinate system, with components indicated by primes, such that $z'_1=z_1$, $z'(P_o)$ lies at the origin $(0, \ldots, 0)$, and $\bar{z}'_1=(0, 0, \ldots, 1)$. The original neighborhoods ω are required to be convex. Thus the straight line between $z'(p_o)$ and $\bar{z}'$ lies in the image ω'_o of ω_o. Hence for some neighborhood β (in $\mathbb{C}^1$) of the line from 0 to 1 the disc $d(\lambda)=\{z'; z'_1=\lambda, z'_2=0, \ldots, z'_{3n-5}=0, z'_{3n-4} \in \beta\}$ lies in ω'_o for $\lambda=0$. Thus for some sufficiently small initial segment c' of the curve c of the theorem the disc $d'(z_1)$ must lie in ω'_o for $z_1 \in c'$. By the assumption of the theorem these discs $d'(z_1)$, $z_1 \in c'$ lie in the domain of holomorphy. Thus the limiting disc $d'(0)$ must also lie in the domain of holomorphy, by virtue of Bremermann's continuity theorem [16]. Therefore, the point $\bar{z}$, which was an arbitrary point of $L \cap \omega_o$, lies in the domain of holomorphy. Using the same argument for the rest of the finite number of neighborhoods that connect p to p_o one concludes that any p in $L \cap \Omega$ lies in the domain of holomorphy. Thus, by virtue of the assumption of the theorem that the domain of holomorphy contains $\Omega - L$, we conclude that any p in Ω lies in the domain of holomorphy of f_1. Q.E.D.

Theorems 2 and 2′ below are slight adaptations of the structure theorem. They cover bubble diagram functions G_B that are more general than those introduced in Section 1. In these functions G_B the bubbles b represent no longer simply the momentum-space kernels $S^c_{I_b J_b}$ or $(S^c_{I_b J_b})^-$ but rather kernels G_b of bounded operators, defined again on the space of all real on-mass-shell initial and final 4-momenta associated with the sets I_b, J_b of incoming and outgoing particles of b, and satisfying energy-momentum conservation $\left(G_b = g_b \times \delta^4\left(\sum_{i \in I_b} p_i - \sum_{j \in J_b} p_j\right)\right)$.

A diagram $\mathscr{E}_B$ is now in general a set in space-time of oriented external and internal lines associated with the *original* external and internal lines of B. Each line has, as before, a given on-mass-shell 4-momentum to which it must be parallel. Finally the configuration of all lines associated with a given bubble b must correspond to a point of the closed cone with apex at the origin in $T^*_{p_b}\mathscr{M}_b$ containing the directions of the essential support of g_b at p_b (p_b is the set of external 4-momenta of b).

A $u=0$ point of B is such that there exists an $\mathscr{E}_B$ all of whose external lines pass through a common point, while at least one internal line does not pass through this point.

We then state:

Theorem 2. *The only possible directions in the essential support of g_B at a non $u=0$ point P are those corresponding to the configurations of external trajectories of some $\mathscr{E}_B$. (P is here a set of external 4-momenta.)*

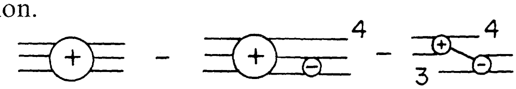
The proof is given in [9] and is in fact the first step of the proof of the structure theorem for the usual bubble diagram functions. The latter theorem arises from the supplementary information, coming from macrocausality, on the essential support of each f_b when b is a usual bubble.

Theorem 2′. *If all the directions in the essential support of each g_b are those corresponding to a certain class of diagrams $\mathscr{D}_b$ then the singular directions of G_B are constructed in the same way as in the structure theorem of Section 1 except that the subdiagrams $\mathscr{D}_b$ occurring in the diagrams $\mathscr{D}_B$ are now allowed to be only those of the admitted class at b.*

(c) Proof of Propositions 3, 4, 5

Proof of Proposition 3. Let Ω be a real open set of $\mathscr{M}$ that contains all points of $L_1(D_+)$ except those of Ω_+ and of $\cup N_t'$. For simplicity, Ω will be chosen such that if it contains $p=(p_1, p_2, p_3; p_4, p_5, p_6)$, then it contains all $p'=(p_1, p_2, p_3; p_4, p_5', p_6')$ (where $p_5'+p_6'=p_5+p_6$). [This property is obviously satisfied for the set of p that lie inside $L_1(D_+)$ but outside Ω_+ and $\cup N_t'$.]

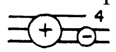
The set Ω contains no point of $L(D_+)$ minus $L_1(D_+)$ and no point lying on both $L_1(D_+)$ and on the closure of an $L_1(D_+')$ where D_+' and is related to D_+ but unequal to D_+. (These D_+' can only be pole diagrams D_+ with self-energy additions, but the latter are excluded by the conditions on Ω that $p_1 \neq p_2$ and $p_5 \neq p_6$.) Thus the conditions of Proposition 1 are met, and f has a decomposition f_1+f_2 of the kind specified in that proposition.

Using the fact that $H=$  (see Section 3), we now show that h can be written in Ω as a sum of two distributions h_1, h_2, with the following properties:

(i) h_1 is the boundary value in Ω of an analytic function $\boldsymbol{h}_1$ that is analytic at all points of Ω outside $L_1(D_+)$ and satisfies the same property as $\boldsymbol{f}_1$ in Proposition 1: The only possible direction in the essential support of h_1 at a point p of $L_1(D_+)$ in Ω is $u_+(p)$.

(ii) Being given any point p of $L_1(D_+)$ in Ω, the essential support of h_2 at p does not contain $u_+(p)$.

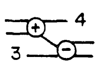
The fact that the first term of h, namely f, has a decomposition as a sum of two terms f_1, f_2 which satisfy respectively properties (i) and (ii) follows from Proposition 1 of Section 2 and Lemma 2 of Section 4. This latter lemma implies that all points of $L_1(D_+)$ in Ω belong to $\hat{L}(D_+)$. This result and (ii) of Proposition 1 yield (ii) above. It also means that the hypotheses of Assumption 2 (no sprouts) is satisfied. This Assumption 2 will be used presently.

Next, let g denote the term , after factorization of $\delta^4(\sum p_i - \sum p_j)$. The first factor is the scattering function. Thus the above mentioned decomposition of f induces a corresponding decomposition of g as a sum of two terms g_1, g_2. The term g_2 satisfies property (ii) in view of Lemma 3 of Section 4, and of Theorem 2′ of subsection (b).

The term g_1 is equal to:

$$g_1(p_1, \ldots, p_6) = \int f_1(p_1, p_2, p_3; p_4, p_7, p_8) f_{2,2}(p_7, p_8; p_5, p_6) \cdot \frac{1}{2} \delta^4(p_7 + p_8 - p_5 - p_6) \prod_l \frac{d\boldsymbol{p}_l}{2(\boldsymbol{p}_l^2 + m_l^2)^{1/2}}.$$

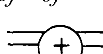
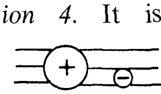
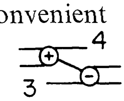
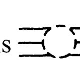
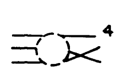
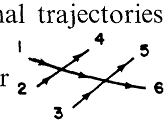
All points $(p_1, p_2, p_3; p_4, p_7, p_8)$ in the integration domain belong to Ω if $(p_1, \ldots, p_6)$ does. Since the analyticity properties of f_1 are expressed only in terms of the variables (p_1, p_2, p_3, p_4), and since the integration domain is compact, g_1 still satisfies property (i).

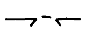
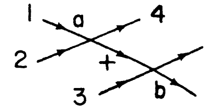
Finally, let d' denote the term  after factorization of $\delta^4(\sum p_i - \sum p_j)$. By decomposing $\delta(k^2 - m^2)$, it can obviously be decomposed as a sum of terms d_1', d_2' which respectively satisfy properties (i) and (ii).

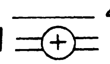
Hence, the announced decomposition of h follows, with $h_1 = f_1 + g_1 + d_1'$, $h_2 = f_2 + g_2 + d_2'$.

Now consider a point P of $L_1(D_+)$ that lies outside Ω_+ and outside the union $\cup N_t$ of the submanifolds N_t of Proposition 2. Proposition 2 ensures that $\boldsymbol{h}_1$ is analytic at P. Then Assumption 2 of Section 2 and Theorem 1 of Subsection (b) above, ensure that $\boldsymbol{h}_1$ (and h_1) is analytic at all points p of $L_1(D_+)$ outside Ω_+ and $\cup N_t'$.

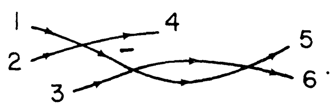
This fact, combined with property (ii) of h_2, completes the proof of Proposition 3.

Proof of Proposition 4. It is convenient to use again the fact that $H =$  $-$  $-$ . From this expression, it is known (structure theorem) that the only possible directions in the essential support of h are those corresponding to configurations of external trajectories of diagrams $\mathscr{D}_B$ whose topological structure is ,  or 

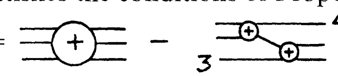
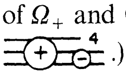
where  denotes a subgraph D_b associated with . On the other hand, Proposition 3, just proved, ensures that $u_+(p)$ is excluded from the essential support of h at all points p of $L(D_+)$ that lie outside Ω_+ and $\cup N_t'$. Hence, in view of Lemmas 2 and 3 of Section 4, and Theorem 2′ of Subsection (b) the part of the essential support of h that corresponds to the diagram  is absent at these points p.

Consider now the term $R' = H$  $= r' \times \delta^4(\sum p_i - \sum p_j)$ where by definition r' is given by the formula:

$$r'(p_1, \ldots, p_6) = \int h(p_1, p_2, p_3; p_4, p_7, p_8) f_{2,2}(p_7, p_8; p_5, p_6)$$
$$\cdot \frac{1}{2} \delta^4(p_7 + p_8 - p_5 - p_6) \prod_{l=7,8} \frac{d^3 p_l}{2(p_l^2 + m_l^2)^{1/2}}.$$

As already noticed, all points $(p_1, \ldots; p_4, p_7, p_8)$ that satisfy $p_7 + p_8 = p_5 + p_6$ must lie in $L_1(D_+)$ but outside Ω_+ and $\cup N_t'$, if $(p_1, \ldots, p_6)$ has this property. Thus the required property of r is a direct consequence of Lemma 3 and Theorem 2′, plus the fact that $u_+(p)$ cannot be obtained from a diagram whose topological structure is .

Proof of Proposition 5. Now let Ω be a real domain of $\mathscr{M}$ containing $\hat{L}(D_+)$, but no other point of $L(D_+)$. This domain Ω includes no points of $L(D_+)$ where $p_1 = p_2$ or $p_5 = p_6$. Consequently it satisfies the conditions of Proposition 1.

Using the fact that $R =$  and Proposition 1, one sees that r can be decomposed in Ω as a sum of two terms r_1, r_2, that satisfy respectively the same properties as those mentioned for h_1 and h_2 in the proof of Proposition 3, but in a domain Ω which is now much larger: it contains all points of Ω_+ and $\cup N_t'$ which lie in $\hat{L}(D_+)$. (This is not expected for h because of the term .)

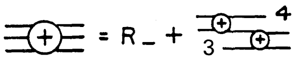
In view of Proposition 4, r_1 is moreover analytic at all points of $\hat{L}(D_+)$ apart from those of Ω_+ and $\cup N_t'$. Thus Assumption 2 of Section 2 and Theorem 1 of Subsection (b) (together with Lemma 1 stated at the end of Section 3) ensures that r_1 is analytic at all points of $\hat{L}(D_+)$.

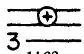
This fact, combined with the essential support property of r_2, completes the proof of Proposition 5.

Alternative Proof. To conclude the section we present also an alternative proof of Proposition 5 that does not rely on Assumption 2, and that allows one to prove the result at all points of $\hat{L}(D_+)$ apart possibly from the points of $\cup N_t'$ and of $\cup N_t''$, where the submanifolds N_t'' are defined in a way similar to the manifolds N_t', by exchanging p_3 and p_4, p_1 and p_5 and p_2 and p_6.

The starting point of the analysis that has been carried out above is the equation $SS^{-1} = 1$, which leads to (3) and to (5). If one starts instead from the equation

$S^{-1}S=1$, the analogous analysis allows one to write



where $R_-=H_-+$  H_- and H_- is an appropriate sum of bubble diagram functions (which is different from H). The same methods and assumptions now lead to:

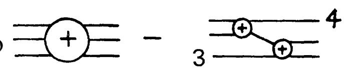
Proposition 4′. *$u_+(p)$ is not a singular direction of r_- at p if p lies in $L(D_+)$ but does not lie in Ω_- or in $\cup N_t''$.*

The set Ω_- is defined in a way similar to Ω_+, but with an exchange of the roles of particles 1, 2 and 5, 6, and of particles 4 and 3. In particular, being given $p=(p_1, \ldots, p_6)$ in $L_1(D_+)$, we choose a point D in space-time through which we draw two lines respectively parallel to p_3 and to $p_5+p_6-p_3(=p_1+p_2-p_4)$, and choose a point C on the second one, such that $(CD)_0>0$.

Then $p\in\Omega_-$ if one can find two on-mass-shell 4-momenta k_1', k_2' such that $k_1'+k_2'=p_5+p_6-p_3+p_4=p_1+p_2$ and such that the line passing through C and parallel to k_1' meets the line passing through D and parallel to p_3 at a point B earlier in time than C and D.

We then notice that

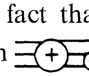
$$R=R_-$$

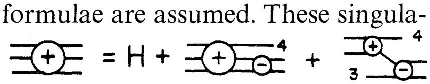
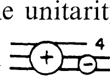
since they are both equal to  . Therefore $u_+(P)$ is not a singular direction of $r=r_-$ at any point P of $\hat{L}(D_+)$ except possibly if P lies in $\cup N_t'$ or in $\cup N_t''$ or in the intersection of Ω_+ and Ω_-.

The following lemma provides the announced result.

Lemma. *$\Omega_+\cap\Omega_-$ is empty apart possibly from points p that lie in $\cup N_t'$ or $\cup N_t''$.*

This lemma is a consequence of the definitions of Ω_+, Ω_- and of Theorem 1 of Appendix I.

Remark. It is important to note that Proposition 5 does not hold for h itself. The failure of this property for h arises in the first proof from the fact that H, in its expression used in the proof of Proposition 3, contains the term , and in the second proof from the fact that $H\neq H_-$ (whereas $R=R_-$).

As a matter of fact, $u_+(P)$ is certainly expected to be a singular direction of h at points P of Ω_+, even if the discontinuity formulae are assumed. These singularities will cancel in the unitarity equation  with those of the term , as easily checked, if the discontinuity formulae are assumed to hold (for graphs with one internal line and for triangle graphs).

Acknowledgements. The first-named author would like to thank Professor S. Mandelstam, Professor G. F. Chew, and Professor D. Jackson for support at the Physics Department of the University of California and hospitality at the Lawrence Berkeley Laboratory, where part of this work was done. The second-named author would like to thank Professor De Dominicis for support at Saclay where part of this work was done. We would also like to thank Professor J. Bros for a useful discussion related to Theorem 1, and Professor T. Kawai for discussions concerning holonomic functions.

Appendix I: Treatment of Mixed-α Diagrams

We outline below the main tools used in the analysis described in Section 4 and 5 of mixed-α Landau diagrams.

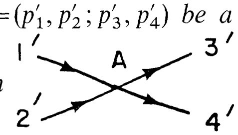
Theorem 1. *The only* $+\alpha$*-Landau surfaces occurring in the region* $9m^2 < s < 16m^2$ $(s=(p_1+p_2+p_3)^2)$ *and away from points p such that two initial or two final 4-momenta are equal, are those associated with graphs with only one internal line, or with triangle graphs.*

This theorem is proved in Appendix III.

Lemma 1. *Let* $p=(p_1, \ldots, p_6)$ *be a point of* $L_1(D_+)$*. Consider a space-time diagram* $\mathscr{D}_B$ *whose external lines carry momenta p. Thus,* $p_1 \neq p_2$ *and* $p_5 \neq p_6$*. If the trajectories of the initial particles 1, 2, 3, or of the final particles 4, 5, 6 pass through a common point, then the configuration of external trajectories cannot correspond to* $u_+(p)$.

Proof. We prove this result in the case where 1, 2, 3 meet at a common point A. In any $\mathscr{D}_B$ giving $u_+(p)$, the trajectory of 3 would have to pass through the meeting point B of 5, 6 and hence be directed along the direction of $p_1+p_2-p_4$, i.e., in the equal-mass case $p_1+p_2-p_4=p_3$, or $2p_3=p_5+p_6$. This is not possible since $p_5 \neq p_6$.

Lemma 2. *Let* $p'=(p'_1, p'_2; p'_3, p'_4)$ *be a point of* $\mathscr{M}_{2,2}$ *such that* $p'_1 \neq p'_2$*. In any space-time diagram*



the projection of any one of the trajectories onto the plane determined by any two other nonparallel trajectories must lie in the doublecone V_A *limited by these other two and composed of the points that are time-like with respect to A. If the projection lies on the boundary of* V_A*, then the trajectory coincides with this projection. If all trajectories lie in a common plane then each final trajectory has to coincide with one of the initial trajectories.*

An example is given in Figure 2, in which the heavy dotted lines represent the projections of the trajectories 3′, 4′ in the (1′, 2′) plane

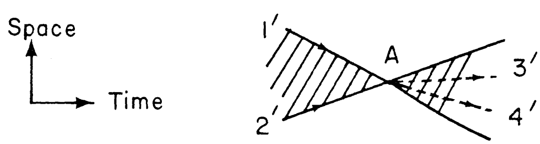
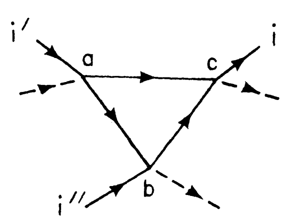


Fig. 2

This result follows from simple kinematical arguments.

Lemma 3. *Let* D_B *be a graph containing a part of the form*



where each index i′, i″, i‴ represents one of the particles 1, 2, 4. Let p be such that $p_1 \nparallel p_2$.

Then in any space-time representation $\mathscr{D}_B$ *whose representative points A, B, C of a, b, c are different from each other and are not at infinity, and are such that the trajectories i′, i″, i‴ meet at common point A′, this point A′ has to coincide with A, or B, or C″. The analogous result holds with (1, 2; 4) replaced by (5, 6; 3).*

Proof. We consider the case where each index i', i'', i''' represents one of the particles 1, 2, 4, and assume initially that A, B, C are not lined up.

Lemma 2, applied respectively to trajectories 1, 2, 4 at A, B, C ensures that the projection of A' in the plane ABC lies in $V_A \cap V_B \cap V_C$ where V_A, V_B, V_C are the respective doublecones limited by the lines AB, AC or AB, BC, or AC, BC. An example is shown in Figure 3 in the particular case $(AB)_0 > 0$, $(AC)_0 > 0$, $(BC)_0 > 0$.

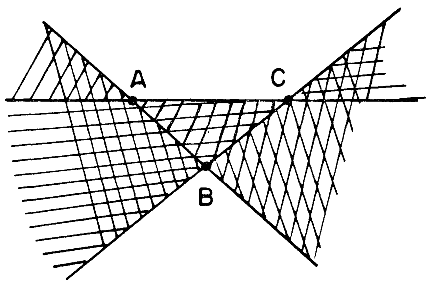
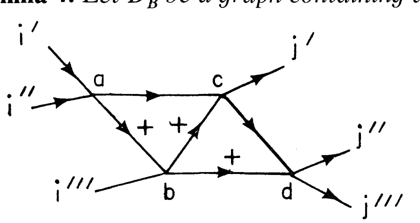


Fig. 3

The intersection is always limited to a certain part of the boundary of these cones (see Fig. 3), and a more extensive use of the last part of Lemma 2 provides the announced result.

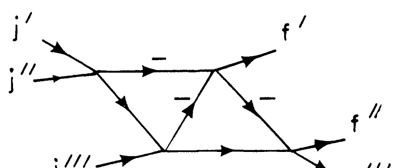
The proof when A, B, C are lined up (but all different from each other) follows easily from the fact that $p_1 \nparallel p_2$.

Lemma 4. *Let* D_B *be a graph containing a part of the form:*



where each index i′, i″, i‴ represents one of the initial particles 1, 2, 3. Then there is no representation $\mathscr{D}_B$ *of this graph (in which the lines ab, bc, bd are positive lines) in the region* $9m^2 < s < 16m^2$, *if* $p_i' \nparallel p_i''$ *and* $p_j'' \nparallel p_j'''$.

The same result holds for a graph containing a part of the form



where each index f', f'', f''' represents one of the final particles 3, 5, 6, if $p''_f \neq p'''_f$ and $p'_j \neq p''_j$.

Proof. We prove here the first part.

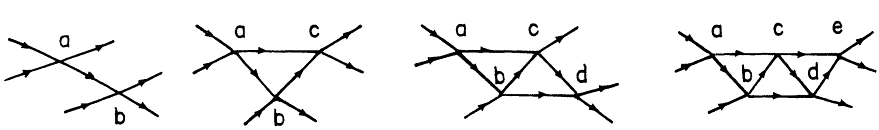
In any $\mathscr{D}_B$ of the required form the line ac has to be a positive line [since $(AB)_0 > 0, (BC)_0 > 0$] and thus the line cd must be a negative line or a zero line, in view of Theorem 1. (The needed slight refinement of Theorem 1 also is proved in Appendix III.)

Let A, B, C, D, be the representative points of a, b, c, d. We know that AC is not parallel to AB (since $p'_i \neq p''_i$), On the other hand, Lemma 2 ensures that the projection of D in the plane ABC must lie in $V_B \cap V_C$ where V_B and V_C are the double-cones limited respectively by the lines BC, AB and AC, BC. This fact, together with the above-mentioned sign conditions, implies that D must lie in the interior of the segment BC, or possibly at C. Hence BC and BD have to be parallel, and parallel also to AB, which is contrary to the fact that AB and AC are not parallel.

Examples of Applications. We now show how these results can be applied to the study of the bubble diagram functions involved in H. We exclude points p such that two initial, or two final, 4-momenta are equal.

First, it is not difficult to check that if any corresponding graph D_B contains two internal lines that start from the same vertex v_{in} and end at the same vertex v_f then the space-time representatives V_{in} and V_f of v_{in} and v_f must coincide: otherwise the 4-momenta of these lines would have to be equal and one checks that this would imply in turn the equality of some of the initial, or of some of the final 4-momenta. On the other hand, if $V_{in} = V_f$, these two internal lines are not seen in the space-time representation and it is sufficient to consider the diagram obtained after contraction of these lines (i.e. by identifying v_{in} and v_f and then removing these lines).

The only graphs we need to consider then, in view of Theorem 1, Lemma 1, and the form of H [see Eq. (6) of Section 3] are of the form:

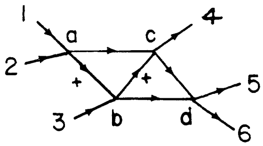


a b c d

with moreover certain specifications of signs for certain lines and different possible specifications of the external lines.

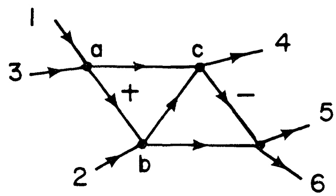
In Cases a, b, d, the detailed analysis shows that certain of the corresponding diagram $\mathscr{D}_B$ can give rise to $u_+(p)$, but only when p belongs to lower dimensional subsets of $L_1(D_+)$. These subsets correspond to cases when three or more 4-momenta, obtained by linear combinations of $p_1, \ldots, p_6$, lie in a common plane, or when some initial 4-momenta are parallel to some final 4-momenta.

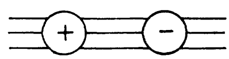
Similar conclusions are obtained in Case c, apart from the graph



already mentioned, which gives rise to $u_+(p)$ on Ω_+ (when the representatives B, D of b, d coincide in space-time).

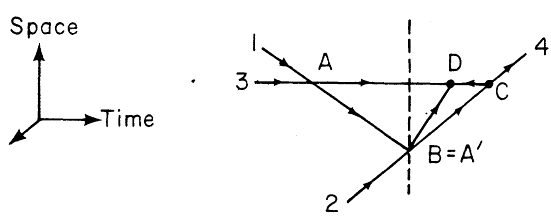
An example of an application of Lemma 3 concerns the graph:



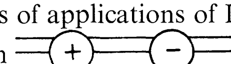
which occurs for instance in 

. Let us consider, for example, space-time representations $\mathscr{D}_B$ in which the representatives A, B, C of a, b, c are all different. Then Lemma 3 ensures that if $\mathscr{D}_B$ gives $u_+(p)$, then the point A' where the trajectories 1, 2, 4 meet is A, B, or C. On the other hand, the trajectories 3, 5, 6 must meet at the representative D of $d(p_5+p_6)$. A' cannot be at C since $(CD)_0<0$. It cannot be at A, by virtue of Lemma 1, since 1, 2, 3 would meet there.

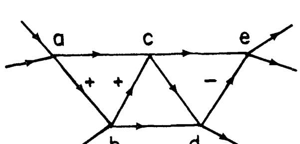
Finally if $A'=B$, the trajectory of particle 4 must be parallel to BC and therefore CD is parallel to AC. The representation, in view of the sign conditions has therefore the form:



This is not possible (Lemma 2) unless B lies on AC, and hence $p_1=p_3$.

Examples of applications of Lemma 4 are the diagrams of the form d, above, that occur in 

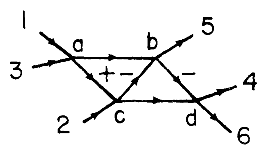
. In such diagrams the right-hand or left-hand triangle has to be a subdiagram associated with the bubble $+$ or $-$, and therefore either all lines ab, bc, ac are positive lines, or all lines ce, ed, cd are negative lines. Both cases are treated similarly and we consider here the case:

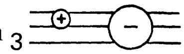


Lemma 4 ensures that the line bd cannot be a positive line in any space-time representation, if the initial 4-momenta are not parallel. If it is a negative or zero line then the line cd has to be a negative line [since $(BC)_0 > 0$, $(CD)_0 \leqq 0$]. The second part of Lemma 4 allows one to exclude the case when bd is a negative line (if the final 4-momenta are not parallel), and the problem is then reduced to a study of the situation in which bd is a zero line. (We shall not reproduce here this study, which obliges one to consider a number of different cases depending on the specification of the external lines.)

Apart from the type of arguments already described, it is in some cases useful to use an obvious extension of Lemma 2 that specifies, being given the projection plane defined two initial trajectories, the possible relative sides, of this plane on which the two final trajectories can be found.

Consider for instance the graph:



which occurs for instance in , and a space-time representation $\mathscr{D}_B$ corresponding to $u_+(p)$. In the BCD plane, we know from Lemma 2 itself that the projection of the point A' where 2, 4 meet (with 1) must be in the triangle BCD ($V_C \cap V_D$) [region (A') of Fig. 3], that the projection of the point B' where 5, 6 (with 3) meet must lie in the shaded region (B') of Figure 3′

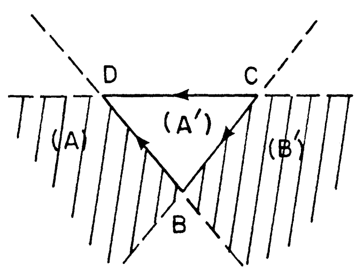


Fig. 3′

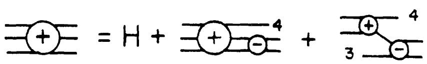
i.e., in the part of $V_B \cap V_D$ that does not lie earlier in time than the triangle BCD, over which A' lies, and finally that the projection of the representative A of a must lie in the shaded region (A), i.e., in the part of $V_B \cap V_C$ which is earlier in time than B or C.

Suppose that A is not in the plane BCD but is on one side of the plane, which we call "up", by convention. At B, energy-momentum conservation implies that trajectory 5 must point down and hence that B' is down. At C, on the other hand, the same argument shows that trajectory 2 comes from down and hence A' is down. At D, trajectory 4 must point down (in order to meet A') and trajectory 6 must therefore point up. But then it cannot pass through B' which is down (see above). Thus we may conclude that A must be in the plane BCD, and hence trajectories 2 and 5 must also lie in this plane. Thus A' and B' must also lie in the plane. But then trajectories 1 and 3 must also lie in this same plane, since $A' \neq B'$. And trajectories 4 and 6 must lie in this same plane, for the same reason. Thus p lies in $\cup N'_t$.

Appendix II: Comparison to Treatment of [12]

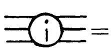
The proof of the pole factorization theorem, as given in [12], consists first of a check of internal consistency, and then a claim of an actual derivation of the result from "$i\varepsilon$" assumptions. Our purpose here is to see to what extent the method used in [12] could provide a simplification of, or improvement upon, the work presented in Sections 2 to 5.

[12] starts from Equation (3) written in the form:



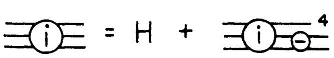
(1)

It is then *assumed* that the scattering function f does have a minus $i\varepsilon$ analytic continuation around $L(D_+)$, and the minus $i\varepsilon$ boundary value is denoted $f^{(i)}$. We shall put

 $= f^{(i)} \times \delta^4(\sum p_i - \sum p_j)$.

It is, moreover, claimed (without proof) that h is the minus $i\varepsilon$ boundary value of an analytic function in the neighborhood of any point p of $L(D_+)$.

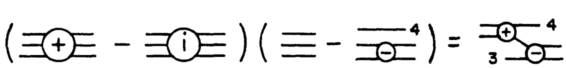
Equation (1) is first considered on the nonphysical side of $L(D_+)$ $(k^2 < m^2)$ where the last term vanishes. Then by taking the minus $i\varepsilon$ continuations of all terms around $L(D_+)$ and (minus $i\varepsilon$) boundary values, one obtains, according to [12]:



(2)

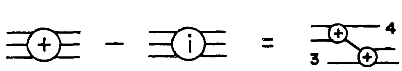
where H is unchanged in view of the claim on h.

A comparison of (1) and (2) gives:



(3)

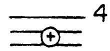
and one obtains, after "multiplication" on the right by  the usual discontinuity formula, namely:



(4)

As it stands, this proof is certainly not correct. In order to get a proof, one should first check the above mentioned property of h, i.e., that $u_-(p)$ is the only singular direction of h at any point p of $L(D_+)$. We know from our present work that this is not expected to be always true: in fact $u_+(p)$ is definitely expected to be a singular direction of h at every point p of the set Ω_+ (See the remark at the end of Section 5.).

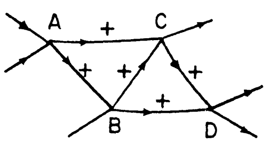
A correct analysis of this problem would require work similar to that presented here, and even more. [We only had to prove that $u_+(p)$ was not a singular direction, under appropriate conditions on p.]

One would for instance have to prove that the result holds at appropriate points p such that $p_3 = p_5$, since such points always occur in the integration domain after multiplication by  . [This step is needed to derive (4) from (3): see above.]

To summarize, this type of approach requires additional assumptions, such as the assumption on the existence of a minus $i\varepsilon$ analytic continuation of f, which is, in the present work, a corollary of the discontinuity formula. And in order to get an actual proof, it leads both to the same problems as those treated in the present work, and moreover to some apparently more difficult ones.

Appendix III: Three Equal-Mass Particles Can Collide Only Thrice

It has been proved before in the literature that equal-mass particles below the four-particle threshold can collide at most three times. That is, there is no space-time diagram $\mathscr{D}$ of the form



unless AB lies on AC and DC lies on BD.

This result plays an important role in our arguments. Since we have been unable to locate in the literature the original proof of this result, we give one here.

Note first that if AB lies on AC then DC lies on BD, by equal-mass kinematics. So we can assume that AB does not lie on AC, and that the triangle ABC is not degenerate. The question is whether line CD can emerge from C and meet BD emerging from B. To see that this is impossible we go to the brick-wall frame of reaction B. Taking A, B, and C to lie in the x, t plane one has, in the brick-wall frame $p^o_{AB}=p^o_{BC}$, and $p^x_{AB}=-p^x_{BC}=p^x_{BD}<0$. Since $(p^o_{BD})^2-(p^x_{BD})^2-(p^y_{BD})^2-(p^z_{BD})^2=m^2$, and $p=mv$, one condition on the point $D=(x, y, z, t)$, is, with $B=(0,0,0,0)$,

$$t^2=\alpha x^2+r^2\,, \qquad x<0, t<0$$

where $r^2=y^2+z^2$ and $\alpha=1+m^2/(p^x_{AB})^2$. For any fixed t this curve is represented by the outer left-half ellipse in Figure 3a. The projection of D onto the plane of ABC must lie in the shaded region of Figure 3b.

The condition on D coming from line CD is represented by the inner ellipse in Figure 3a, where intersections of these two ellipses with the x axis have the arrangement indicated there.

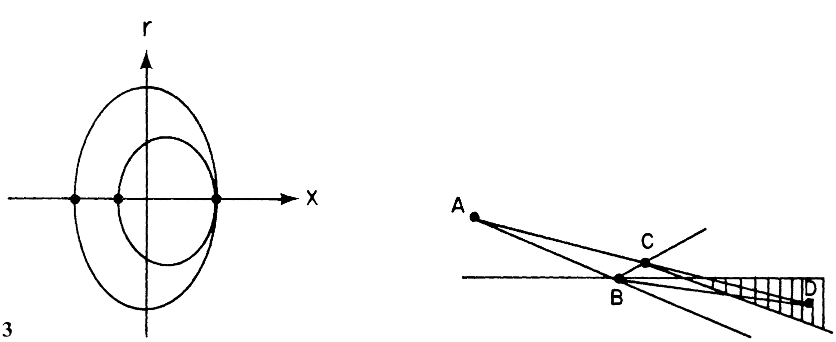


Fig. 3 **a** **b**

As t grows from $t_0 = \overline{BC}^0$ the inner ellipse grows from a point on the x axis at $x = t/\sqrt{\alpha}$. It can never intersect the two points on the outer ellipse at $x=0$, since these points represent particles traveling at the velocity of light away from point B, and the particles from C start later. Therefore, by the properties of ellipses the inner ellipse can never intersect the outer left-half ellipse. Thus there can be no meeting point D. Q.E.D.

References

1. Iagolnitzer, D., Stapp, H. P.: Commun. math. Phys. **14**, 15—55 (1969) and references quoted therein
2. Coster, J., Stapp, H. P.: J. Math. Phys. **10**, 371—396 (1969); **11**, 1441—1463 (1970); **11**, 2743—2763 (1970)
3. Bloxham, M. J. W., Olive, D. I., Polkinghorne, J. C.: J. Math. Phys. **10**, 494—502, 545—552, 553—561 (1969)
4. Stapp, H. P.: In: Structural analysis of collision amplitudes (Les Houches 1975 June Institute) (ed. R. Balian, D. Iagolnitzer), Amsterdam, New York, Oxford; North-Holland Publ. Co. 1976
5. Iagolnitzer, D.: In: The S-matrix. Amsterdam, New York, Oxford; North-Holland Publ. Co., to be published
6. Chandler, C., Stapp, H. P.: J. Math. Phys. **10**, 826—859 (1969) Theorem 5
7. Pham, F.: Ann. Inst. Henri Poincaré **6**, 89—204 (1967)
8. Iagolnitzer, D.: In: Structural analysis of collision amplitudes, Part III (Les Houches 1975 June Institute) (ed. R. Balian, D. Iagolnitzer), Amsterdam, New York, Oxford; North-Holland Publ. Co. 1976)
9. Iagolnitzer, D.: (a) Commun. math. Phys. **41**, 39—53 (1975). (b) Structural analysis of collision amplitudes (Ref. [8]) Part II, Course 1. (c) Proceedings of the Oji Seminar in Algebraic Analysis, Kyoto, Japan (1976). (d) Structure theorem in S-matrix theory – Some complementary remarks (unpublished)
10. Stapp, H. P.: J. Math. Phys. **9**, 1548—1592 (1968); J. Math. Phys. **11**, 2743—2763 (1970), Appendix A and [9]. The version of the theorem presented here is due to [9]
11. Kawai, T., Stapp, H. P.: In: Lecture notes in physics, Vol. 39, p. 36. Berlin-Heidelberg-New York: Springer 1975 and in Proceedings of the Oji Seminar in Algebraic Analysis, Kyoto, Japan (1976) (Also RIMS-214, Research Inst. for Math. Sciences, Kyoto University, Kyoto, Japan, 1976)
12. Eden, R. J., Landshoff, P. V., Olive, D. I., Polkinghorne, J. C.: The analytic S-matrix, p. 216. Cambridge: Univ. Press 1966
13. Kashiwara, M., Kawai, T., Stapp, H. P.: In: Proceedings of the Oji Seminar in Algebraic Analysis, Kyoto, Japan (1976) (Also RIMS-213, Research Inst. for Math. Sciences, Kyoto University, Kyoto, Japan, 1976)
14. Iagolnitzer, D.: In: Lecture notes in mathematics, Vol. 449. Berlin-Heidelberg-New York: Springer 1973
15. Kashiwara, M., Kawai, T.: In: Proceedings of the Oji Seminar in Algebraic Analysis, Kyoto, Japan (1976) (Also RIMS-211 Research Inst. for Math. Sciences, Kyoto University, Kyoto, Japan, 1976)
16. Bros, J., Epstein, H., Glaser, V.: Nuovo Cimento **31**, 1265—1302 (1964), p. 1289

Communicated by R. Stora

Received February 10, 1977

Note added in proof: A $u=0$ structure theorem that validates the $u=0$ assumption has recently been proved (D. Iagolnitzer, CERN preprint 2279) on the basis of a refined macrocausality condition that may also yield the no sprout property.